General Banach Algebras

Def: A Banach alg. A is a complete normed algebra, i.e., $(A, \|\cdot\|)$ is a Banach space s.t. $\|ab\| \le \|a\| \cdot \|b\|$ $\forall a, b \in A$.

Last time: we saw that if $J \subseteq A$ is a closed 2-sided ideal, A/J is Banach w/ $\|a+J\| = \inf\{\|a+j\| \mid j \in J\}$.

Examples:

0. Let X be any Banach space and define $xy = 0$ $\forall x, y \in X$.
1. $\mathbb{C}$ / $M_n(\mathbb{C})$ is Banach alg.
2. $C(X)$ for X cpt Hausd.
3. $C_0(X)$ X locally cpt Hausd.
4. $C_b(X)$ cts bdd fcts, X any top. space.
5. $L^\infty(X, \mu)$
6. $D \subset \mathbb{C}$ a domain, simply conn. w/ simply conn. cpt closure K.
   $A(K) = \{f \in C(K) \mid f|_D \text{ is holo}\}$.
   $D = \mathbb{D}$ is disk algebra
7. $C^n[0,1]$ w/ $\|f\| = \sum_{k=0}^{n} \|\frac{1}{k!} f^{(k)}\|_\infty$.

(8) $\ell^1(\mathbb{Z})$ w/ convolution:

$$(x*y)(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, y(k)$$

- unital w/ unit $\delta_0(n) = \delta_{n=0}$.

(9) $\ell^1(\mathbb{Z}_{\geq 0})$ w/ $(x*y)(n) = \sum_{k=0}^{n} x(n-k)\, y(k)$.

(10) $L^1(\mathbb{R}^n)$ w/ convolution

$$(f*g)(x) = \int f(x-y)\, g(y)\, dy.$$

- non unital

(11) $B(X)$ for ~~X~~ Banach

(12) $K(X)$ cpt ops

(13) $B(X)/K(X)$ Calkin alg.

Adjoining a unit:

For A non-unital: adjoin a unit by setting
$A_1 = A \oplus \mathbb{C}1$, mult. $(a,w)(b,z) = (ab + wb + za, wz)$,
$\|(a,z)\|_{A_1} := \|a\|_A + |z|$, ℓ^1 norm.

Note: another norm might be better, e.g.,

$$\|(a,z)\|_{A_1} := \sup_{\|b\| \neq 0} \frac{\|ab + zb\|}{\|b\|}.$$

Q: what is $\|1\|_A$?

<u>Left regular Rep'n</u>: Given A a Banach alg, we define $\lambda : A \to B(A)$ by $\lambda_a b = ab$.

<u>Exercise</u>: λ is a <u>norm decreasing</u> homom.

Suppose A unital. Then $\forall a \in A$,

$$\|a\| \le \|\lambda_a\|_{B(A)} \cdot \|1\| \le \|a\| \cdot \|1\| \quad \text{so } \|1\| \ge 1.$$

$\Rightarrow$ $\|\cdot\|_{B(A)}$ on $\lambda(A)$ is <u>strongly equivalent</u> to $\|\cdot\|_A$.

Can use equivalent $\|\cdot\|_{B(A)}$ on $\lambda(A)$ where <u>$\|1\| = 1$</u>.

<u>Def</u>: An <u>approximate unit</u> for A is a net $(e_\lambda) \subseteq B_1^A[0]$ s.t. $\forall a \in A$,

$$\lim e_\lambda a = \lim a e_\lambda = a.$$

<u>Exercise</u>: Find an approximate unit in:

1. $C_0(X)$
2. $L^1(\mathbb{R}^n)$
3. $K(H)$, H a Hilbert space.

<u>Exercise</u>: Show if $\exists$ a.u. $(e_\lambda) \subset A$, then $\lambda : A \to B(A)$ is an isometry. (assume $\|1\| = 1$.)

Q: when do a.u.'s exist? $\Rightarrow$ functional <u>calculus</u>.

<u>Spectrum</u>: A is a unital Banach alg., $a \in A$.

$Sp_A(a) = \{z \in \mathbb{C} \mid a - z1 \text{ not invertible}\}$.

- it depends on A. When A is understood, simply write $Sp(a)$.

For A non-unital, we define $Sp_A(a) = Sp_{A_1}(a+0)$.

<u>Prop</u>: If A unital, $\|a\| < 1$, then $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$.

<u>Cor</u>: If A unital, $GL(A) = \{a \in A \mid 0 \notin Sp(a)\}$ is open.

- So are set of L/R-inv. elts.
- map $a \mapsto a^{-1}$ is <u>cts</u>.

<u>Thm</u>: If A unital, $Sp(a) \subset \mathbb{C}$ is nonempty + cpt.

<u>Rems</u>:

- $|z| > \|a\| \Rightarrow (a-z)^{-1} = -\sum_{n=0}^{\infty} z^{-n-1} a^n = \frac{-1}{z} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$.
- If $z_0 \notin Sp(a)$, then for $|z - z_0| < \|(a-z_0)^{-1}\|^{-1}$,

$$(a-z) = (a-z_0) - (z-z_0) = (a-z_0)\left(1 - \frac{z-z_0}{a-z_0}\right)$$

$$\Rightarrow (a-z)^{-1} = (a-z_0)^{-1} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{a-z_0}\right)^n = \sum_{n=0}^{\infty} (z-z_0)^n (a-z_0)^{-n-1}.$$

- $\forall \varphi \in A^*$, $z \mapsto \varphi((a-z)^{-1})$ is <u>holo</u> on $\mathbb{C} \setminus Sp(a)$.
- We will see that for $z \notin Sp(a)$, $z \mapsto (a-z)^{-1}$ is <u>Banach-valued holomorphic</u>.

Examples:

① $C(X)$; $sp(f) = f(X)$. $\Leftarrow$ Same for disk alg, $C^n[0,1]$.

② $C_0(X)$; $sp(f) = \overline{f(X)}$.

③ $C_b(X)$; $sp(f) = \overline{f(X)}$.

④ $L^\infty(X)$; $sp(f) = \text{ess range}(f) = \{z_0 \in \mathbb{C} \mid z \mapsto \frac{1}{f(z) - z_0}$ is in $L^\infty(X)\}$

$= \{z_0 \in \mathbb{C} \mid \forall \varepsilon > 0,\ \mu(f^{-1}(B_\varepsilon(z_0))) > 0\}$.

⑤ $\ell^1(\mathbb{Z}), \ell^1(\mathbb{Z}_{\geq 0}), L^1(\mathbb{R})$??? Fourier Analysis

Spectral Radius: For $a \in A$, we define the spectral radius of a by $r(a) := \lim_{n \to \infty} \|a^n\|^{1/n}$.

Prop: $r(a)$ is well-defined and equals $\inf_n \|a^n\|^{1/n}$.

Pf: Fix $m \in \mathbb{N}$. For $n \in \mathbb{N}$, write $n = qm + r$ with $q, r \in \mathbb{Z}_{\geq 0}$ and $r < m$. Then $\|a^n\| \leq \|a^m\|^q \|a\|^r$.

$\Rightarrow \|a^n\|^{1/n} \leq \|a^m\|^{q/n} \cdot \|a\|^{r/n} \to \|a^m\|^{1/m}$ as $n \to \infty$.

Thus $\limsup \|a^n\|^{1/n} \leq \inf_m \|a^m\|^{1/m} \leq \liminf \|a^n\|^{1/n}$.

Prop: For $a \in A$, $sp(a) \subset \{|z| \leq r(a)\}$, and $\exists\, z \in sp(a)$ s.t. $|z| = r(a)$.

Proof: WLOG, A is unital.

$\Rightarrow$

Step 1: for $|z| > r(a)$, consider $\frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$.

Since $|z| > r(a)$, $\lim \|a^n\|^{1/n} < |z|$. Thus $\exists N \geq 0$ s.t. $\forall n > N$, $\|a^n\|^{1/n} < |z|$. $\iff \frac{\|a^n\|^{1/n}}{|z|} < 1$.

Hence $\exists r > 0$ s.t. $\frac{\|a^n\|^{1/n}}{|z|} \leq r < 1$ $\forall n > N$.

$\Rightarrow \frac{\|a^n\|}{|z|^n} \leq r^n$. Hence $\underbrace{-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n}_{=(a-z)^{-1}}$ converges.

Step 2: Enough to show that if $sp(a) \subset \{|z| < r\}$, then $r \geq r(a)$. Fix such an r and a $\varphi \in A^*$. For $|z| > \|a\|$, $(a-z)^{-1} = -\frac{1}{z}\sum_n\left(\frac{a}{z}\right)^n \Rightarrow \varphi((a-z)^{-1}) = -\sum_0^{\infty}\varphi(a^n)z^{-n-1}$.

Now $z \mapsto \varphi((a-z)^{-1})$ is holo. on $\mathbb{C}\setminus sp(a)$.

Thus, the following Laurent series converges on a nbhd of $|z| \geq r$:

$$\boxed{\varphi((a-z)^{-1}) = -\sum_0^{\infty}\varphi(a^n)z^{-n-1} \quad \forall |z| \geq r}$$

Thus $\sup_n |r^{-n-1}\varphi(a^n)| = \frac{1}{r}\sup_n\left|\varphi\left[\left(\frac{a}{r}\right)^n\right]\right| < \infty$ $\forall \varphi \in A^*$.

By Uniform Bddness Principle, $\sup_n \|\left(\frac{a}{r}\right)^n\| < M < \infty$

$\Rightarrow \|a^n\|^{1/n} \leq r M^{1/n}$ $\forall n$ $\Longrightarrow r(a) \leq r$.

Def: $a \in A$ is called *quasinilpotent* if $sp(a) = \{0\} \iff r(a) = 0$.

Thm (Gelfand-Mazur): The only normed division alg$/\mathbb{C}$ is $\mathbb{C}$.

Pf: Take $a \in A$. $\bar{A}$ is a Banach alg. $sp_{\bar{A}}(a) \subseteq sp_A(a) \neq \emptyset$. Let $\lambda \in sp_{\bar{A}}(a)$. Then $a-\lambda$ not inv $\Rightarrow a-\lambda = 0$.

Holomorphic Functional Calculus

Let A be a unital Banach alg.

__Note:__ If $a \in A$ and p is a rational fct $p(z) = \prod_{j=1}^{k} (z-z_j)^{m_j}$ without poles in $sp(a)$, we can define

$$p(a) = \prod_{j=1}^{k} (a - z_j)^{m_j}$$

$\leftarrow$ $[(a-z_1)^{-1}, (a-z_2)] = 0$, $[(a-z_1)^{-1}, (a-z_2)^{-1}] = 0$ } Not entirely obvious! use 1st Resolvent Formula, and power series

__Notation:__

For $\mathcal{U} \subseteq \mathbb{C}$ open, let $H(\mathcal{U})$ be the alg. of holomorphic fcts on $\mathcal{U}$. Recall $H(\mathcal{U})$ is a closed subspace of $C(\mathcal{U})$ w/ a Frechet TVS structure.

For $K \subseteq \mathbb{C}$ cpt, let $\mathcal{O}(K) = \bigcup \{H(\mathcal{U}) \mid K \subset \mathcal{U}\}$.

__HFC:__

Let $a \in A$. The HFC is a __unital__ __alg. homom.__

$$\begin{aligned} \mathcal{O}(sp(a)) &\longrightarrow A \\ f &\longmapsto f(a) \end{aligned}$$

Such that:

1. If $p \in \mathcal{O}(sp(a))$ is rational, $p(a)$ is as above.
2. (Spectral mapping thm) $sp(f(a)) = f(sp(a))$.
3. If $g \in \mathcal{O}(sp(f(a)))$, then $g(f(a)) = (g \circ f)(a)$.

\+ Additional properties!

Preliminaries from complex analysis:

① If $\underset{\text{cpt}}{K} \subseteq \underset{\text{opn}}{\mathcal{U}} \subseteq \mathbb{C}$, $\exists$ a simple closed contour $\gamma \subset \mathcal{U}\setminus K$, s.t. $\operatorname{ind}_\gamma(z) = \begin{cases} 1 & z \in K \\ 0 & z \notin \mathcal{U}. \end{cases}$

② If γ_1, γ_2 are two contours as in ① and $f \in H(\mathcal{U}\setminus K)$, $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

③ If γ a contour as in ①, $f \in H(\mathcal{U})$, then $\forall z \in K$, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$.

Banach valued differentiation + Integration:

For $\mathcal{U} \subseteq \mathbb{C}$ open and $\mathfrak{X}$ a Banach space we say $f: \mathcal{U} \to \mathfrak{X}$ is

① weakly holomorphic if $\varphi \circ f : \mathcal{U} \to \mathbb{C}$ is holo. on $\mathcal{U}$ $\forall \varphi \in \mathfrak{X}^*$, and

② strongly holomorphic if $\forall z \in \mathcal{U}$,
$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$
exists, where the limit is taken in $\mathfrak{X}$.

Note: Obviously strongly holo. $\Rightarrow$ weakly holo.

Q: what about other direction?

The trick is to define a version of the Cauchy integral formula for cts fcts $f: \mathcal{U} \to \mathfrak{X}$.

<u>Banach-valued Riemann path integral:</u>

Let $\gamma: [a,b] \longrightarrow \mathfrak{X}$ be a cts path.

Want to define $\int_a^b \gamma(t)\,dt$.

<u>Lemma</u>: There is at most one $x \in \mathfrak{X}$ s.t.

$$\varphi(x) = \int_a^b (\varphi \circ \gamma)(t)\,dt \quad \forall \varphi \in \mathfrak{X}^*.$$

<u>Pf</u>: Cor. to Hahn-Banach

<u>For HW</u>: you'll define $\int_a^b \gamma(t)\,dt = \lim_{\|P\| \to 0} x_{(P,u)}$

where $$x_{(P,u)} = \sum_{i=1}^n \gamma(u_i)\underbrace{(t_i - t_{i-1})}_{\Delta_i}$$

where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a <u>partition</u>,

$\|P\| = \max_{1 \le i \le n} (\Delta_i = t_i - t_{i-1})$, and $u \in [a,b]^n$ s.t.

$t_{i-1} \le u_i \le t_i \quad \forall i = 1, \ldots, n$.

<u>Facts</u>: ① This limit exists and satisfies
$$\varphi\left(\int_a^b \gamma(t)\,dt\right) = \int_a^b (\varphi \circ \gamma)(t)\,dt \quad \forall \varphi \in \mathfrak{X}^*.$$

② $\int_a^b : C([a,b], \mathfrak{X}) \to \mathfrak{X}$ is a ~~bdd~~ linear trans.

Thm: If $\mathfrak{X}$ is a Banach space and $f: \mathcal{U} \to \mathfrak{X}$ is weakly holo., then

① f is strongly cts

② Cauchy-Goursat + Cauchy integral formula hold: If $\gamma \subset \mathcal{U}$ is a simple closed contour w/ $\operatorname{ins}(\gamma) \subset \mathcal{U}$ ($\operatorname{ind}_\gamma(z) = 0 \ \forall z \notin \mathcal{U}$),

a) $\displaystyle\int_\gamma f(z)\,dz = 0.$

b) $\forall z \in \operatorname{ins}(\gamma)$, $\displaystyle f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$

③ f is strongly holo.

Pf: WLOG, we may assume $0 \in \mathcal{U}$, and we need only prove ① + ③ at 0. (translation is ok.)

① Since $0 \in \mathcal{U}$, $\exists r > 0$ s.t. $\overline{B_{2r}(0)} \subset \mathcal{U}$.
Let $\varphi \in \mathfrak{X}^*$. Since $\varphi \circ f$ holo.,

$$(*) \quad \frac{\varphi f(z) - \varphi f(0)}{z} = \frac{1}{2\pi i}\int_{|w|=2r} \frac{\varphi f(w)}{(w-z)w}\,dw \quad \forall\, 0<|z|<2r.$$

Let $M_\varphi = \max|\varphi \circ f|$ on $\overline{B_{2r}(0)}$. By $(*)$ above,

$$0<|z|\le r \implies \left|\varphi\left(\frac{f(z)-f(0)}{z}\right)\right| \le \frac{M_\varphi}{r}.$$

Hence $\left\{ \frac{f(z)-f(0)}{z} \;\middle|\; 0<|z|\le r \right\}$ bdd weakly $\Rightarrow$ bdd in $\|\cdot\|$ by UBP. Thus $\exists R > 0$ s.t.

$0<|z|\le r \Rightarrow \|f(z)-f(0)\| \le |z| R \to 0$ as $z \to 0$.

② If $f: D \to X$ cts, $\gamma \subset D$ a path,

define $\int_\gamma f(z)\,dz$ as the ! elt. in X s.t.

$$(*) \quad \boxed{\varphi\left(\int_\gamma f(z)\,dz\right) = \int_\gamma \varphi f(z)\,dz \quad \forall \varphi \in X^*.}$$

Indeed, $\int_\gamma f(z)\,dz$ is the Riemann path integral that you'll define in your HW.

Now (a) + (b) follow immediately by $(*)$, as they hold for the RHS of $(*)$. (uses HB.)

③ Choose r as in ①, i.e., $\overline{B_{2r}(0)} \subseteq \mathcal{U}$.

For $0 < |z| < 2r$, $\dfrac{f(z) - f(0)}{z} = \dfrac{1}{2\pi i} \displaystyle\int_{|w|=2r} \frac{f(w)}{(w-z)w}\,dw$. Now note

that for $0 < |z| \le r$, the fcts $w \mapsto \dfrac{f(w)}{(w-z)w}$ in $C([0,1], X)$

converge uniformly to $w \mapsto \dfrac{f(w)}{w^2}$.

Hence $\displaystyle\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \frac{1}{2\pi i} \int_{|w|=2r} \frac{f(w)}{w^2}\,dw \in X$,

as $\int_a^b : C([0,1], X) \to X$ is a bdd lin. trans.

Cor: If $f: \mathcal{U} \to X$ Banach-valued holo., and $\gamma_1, \gamma_2 \subset \mathcal{U}$ are closed paths in $\mathcal{U}$ s.t. $\operatorname{ind}_{\gamma_1}(z) = \operatorname{ind}_{\gamma_2}(z)$ $\forall z \notin \mathcal{U}$,

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz. \quad \text{(use Cauchy-Goursat)}$$

Construction of HFC:

For $a \in A$, $f \in \mathcal{O}(sp(a))$, $\exists$ open $\mathcal{U} \supset sp(a)$ s.t. $f \in H(\mathcal{U})$. Let γ be a simple closed contour in $\mathcal{U} \setminus sp(a)$ s.t. $\operatorname{Ind}_\gamma(z) = \begin{cases} 1 & \forall z \in sp(a) \\ 0 & \forall z \notin \mathcal{U} \end{cases}$.

Define $f(a) = \frac{1}{2\pi i} \int_\gamma (z-a)^{-1} f(z)\,dz$, the ! elt in A

s.t. $\quad \varphi(f(a)) = \frac{1}{2\pi i} \int_\gamma \varphi((z-a)^{-1}) f(z)\,dz \quad \forall \varphi \in A^*$.
$\uparrow$ holo. $\forall \varphi \in A^*$!

Note $f(a)$ is independent of choice of γ, since the RHS is indep. of choice of γ as $z \mapsto (z-a)^{-1}$ is weakly holo.

Thm: The HFC $\mathcal{O}(sp(a)) \ni f \longmapsto f(a) \in A$ satisfies:

1. $f \mapsto f(a)$ is an algebra hom.
2. If $sp(a) \subseteq \mathcal{U}$ and $(f_n) \subseteq H(\mathcal{U})$ w/ $f_n \to f$ loc. unif., then $f_n(a) \longrightarrow f(a)$ in A.
3. If $f(z) = \sum \alpha_k z^k$ has radius of convergence $> r(a)$, then $f(a) = \sum \alpha_k a^k$.
   In particular, this holds $\forall$ poly's p.

Moreover, these properties uniquely characterize the HFC.

Lemma: (First resolvent formula) $\forall w, z \notin sp(a)$,

$$(z-a)^{-1} - (w-a)^{-1} = (w-z)(w-a)^{-1}(z-a)^{-1}$$
$$= (w-z)(z-a)^{-1}(w-a)^{-1}$$

$\Big\}$ directly verify.

$\leftarrow$ commute! $\circlearrowright$

Pf of Thm: (1) Pick simple closed curves γ, λ s.t. $sp(a) \subseteq ins(\gamma)$, $\gamma \cup ins(\gamma) \subseteq ins(\lambda)$. Then

$$f(a)g(a) = \frac{-1}{4\pi^2} \int_\gamma \frac{f(z)}{z-a}\, dz \int_\lambda \frac{g(w)}{w-a}\, dw$$

$$= \frac{-1}{4\pi^2} \int_\gamma \int_\lambda f(z) g(w) \underbrace{(z-a)^{-1}(w-a)^{-1}}_{\frac{(z-a)^{-1} - (w-a)^{-1}}{w-z}}\, dw\, dz$$

$$= \frac{-1}{4\pi^2} \int_\gamma \frac{f(z)}{z-a} \Bigg[\underbrace{\int_\lambda \frac{g(w)}{w-z}\, dw}_{2\pi i\, g(z)} \Bigg] dz + \frac{1}{4\pi^2} \int_\lambda \frac{g(w)}{w-a} \Bigg[\underbrace{\int_\gamma \frac{f(z)}{w-z}\, dz}_{w \in out(\gamma) \Rightarrow \boxed{=0}} \Bigg] dw$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f(z) g(z)}{z-a}\, dz$$

$$= (fg)(a)$$

(2) Suppose $f_n \to f$ loc unif on $\mathcal{U} \supseteq sp(a)$.

Pick cpt $K \subseteq \mathcal{U}$ s.t. $sp(a) \subset K^\circ$. By HW,

$\exists\, M_K > 0$ s.t. $\forall g \in H(\mathcal{U})$, $\|g(a)\| \le M_K \|g\|_{C(K)}$

(3) Hence $\|(f - f_n)(a)\| \le M_K \|f - f_n\|_{C(K)} \to 0$.

Pick $\gamma = \{|z| = R\}$, pos. oriented, w/

$r(a) < R <$ radius of conv. of f.

Note $\sum_{k=0}^{n} \alpha_k z^k \longrightarrow f(z)$ unif on $\overline{B_R(0)}$.

Then: $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\int_\gamma \left[\sum_{k=0}^\infty \alpha^k z^k\right]\frac{dz}{z-a}$

$$= \frac{1}{2\pi i}\sum_{k=0}^\infty \alpha^k \int_\gamma z^{k-1}\underbrace{\left[\sum_{n=0}^\infty \left(\frac{a}{z}\right)^n\right]}_{\text{conv. unif in } C[r,A]} dz \qquad \frac{1}{z}\sum_{n=0}^\infty \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{2\pi i}\sum_{k,n=0}^\infty \alpha^k a^n \Big[\int_\gamma \underbrace{z^{k-n-1}\,dz}_{2\pi i\,\delta_{k=n}}\Big]$$

$$= \sum_{k=0}^\infty \alpha^k a^k.$$

Unique Characterization of HFC

First, recall Runge's Thm: $\forall$ cpt $K \subset \mathcal{U} \subset \mathbb{C}$ s.t. $\mathcal{U}$ is simply conn., every $f \in H(\mathcal{U})$ can be unif. approximated by a poly on K.

Stronger version: Suppose K cpt, and $E \subseteq \hat{\mathbb{C}} \setminus K$ which intersects every conn. component of $\hat{\mathbb{C}} \setminus K$. If $K \subseteq \mathcal{U}$ open and $f \in H(\mathcal{U})$, then f may be unif. approx. by a rational fct whose poles lie in E.

Suppose now $\Phi: \mathcal{O}(sp(a)) \longrightarrow A$ is a homom. s.t.

1. $\Phi(z \mapsto 1) = 1$ and $\Phi(z \mapsto z) = a$, and
2. if $f_n \to f$ loc. unif on $\mathcal{U} \supseteq sp(a)$, $\Phi(f_n) \to \Phi(f)$.

Then $\Phi(f) = f(a)$ $\forall f \in \mathcal{O}(sp(a))$.

Pf: Notice $\Phi(q) = q(a)$ $\forall$ poly's q. If q is a poly s.t. $q(z) \neq 0$ $\forall z \in sp(a)$, then $\frac{1}{q} \in \mathcal{O}(sp(a))$. Since

$$1 = \Phi\left(q\cdot\frac{1}{q}\right) = \Phi(q)\cdot\Phi\left(\frac{1}{q}\right) = q(a)\,\Phi\left(\frac{1}{q}\right),$$

$$\Phi\left(\frac{1}{q}\right) = \frac{1}{q(a)} \implies \Phi(r) = r(a) \quad \forall \text{ rational } r.$$

Now for an arbitrary $f \in \mathcal{O}(sp(a))$, $\exists$ open $\mathcal{U} \supseteq sp(a)$ and a seq. (r_n) of rational fcts s.t. $r_n \to f$ unif. on $sp(a)$ by Runge's Thm. Then

$$\hat{\Phi}(f) = \lim_n \Phi(r_n) = \lim_n r_n(a) = f(a).$$

Spectral Mapping Thm: If $f \in \mathcal{O}(sp(a))$,

$$sp(f(a)) = f(sp(a)).$$

Pf: $\subseteq$: If $\lambda \notin f(sp(a))$, then $g(z) = \dfrac{1}{f(z) - \lambda}$ is in $\mathcal{O}(sp(a))$. Then $g(a)[(f-\lambda)(a)] = g(a)(f(a) - \lambda) = 1$, so $\lambda \notin sp(f(a))$.

$\supseteq$: If $\lambda \in sp(a)$, $\exists g \in \mathcal{O}(sp(a))$ s.t. $f(z) - f(\lambda) = (z - \lambda) g(z)$.
If $f(\lambda) \notin sp(f(a))$, $1 = (z - \lambda) g(z) \dfrac{1}{f(z) - f(\lambda)}$

$$\Rightarrow \quad 1 = (a - \lambda) g(a) \frac{1}{f(a) - f(\lambda)}, \text{ a contradiction.}$$

Cor: If $f \in \mathcal{O}(sp(a))$ and $g \in \mathcal{O}(f(sp(a)))$, $(g \circ f)(a) = g(f(a))$.

Pf: Consider $\Phi : \mathcal{O}(sp(f(a))) \to \mathcal{O}(sp(a))$ by $g \mapsto g \circ f$.
We claim it's a homom s.t.

① $(z \mapsto 1) \mapsto (z \mapsto 1)$; $(z \mapsto z) \mapsto (z \mapsto f(z))$

② $g_n \to g$ loc. unif on $\mathcal{U} \supseteq sp(f(a)) = f(sp(a))$ $\Rightarrow$ $g_n \circ f \to g \circ f$ loc. unif on $f^{-1}(\mathcal{U}) \supseteq sp(a)$.

Now consider the composite $\mathcal{O}(sp(f(a))) \to \mathcal{O}(sp(a)) \to A$
$g \longmapsto g \circ f \longmapsto (g \circ f)(a)$

Clearly it's a homom. s.t.

① $(z \mapsto 1) \mapsto 1_A$; $(z \mapsto z) \mapsto f(a)$

② if $g_n \to g$ loc. unif, $(g_n \circ f)(a) \to (g \circ f)(a)$.

By !ness of HFC for $f(a)$, $(g \circ f)(a) = g(f(a))$.

Applications: Let A be a unital Banach alg., $a \in A$.

① If $0_{\mathbb{C}}$ is in the unbdd component of $\mathbb{C} \setminus Sp(a)$, then a has a log in A.

Pf: Take a simple curve γ in $\mathbb{C} \setminus Sp(a)$ connecting $0_{\mathbb{C}}$ and ∞. Then $\mathbb{C} \setminus \gamma$ is a simply conn. domain and $0 \notin \mathbb{C} \setminus \gamma$. Hence $\exists f \in H(\mathbb{C} \setminus \gamma)$ s.t. $e^{f(z)} = z$. Then $f(a) \in A$ and $e^{f(a)} = a$.

Cor: A complex invertible matrix has a log.

② Suppose $Sp(a) = K_1 \cup K_2$, both cpt, $\neq \emptyset$, $K_1 \cap K_2 = \emptyset$. Take disjoint open $\mathcal{U}_1 \supset K_1$ and $\mathcal{U}_2 \supset K_2$. Set $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, and define $f(z) = \begin{cases} 1 & z \in \mathcal{U}_1 \\ 0 & z \in \mathcal{U}_2 \end{cases}$.

Let $P_1 = f(a)$ and $P_2 = 1 - P_1$.

Then $P_1^2 = P_1$, $P_2^2 = P_2$ and $P_1 + P_2 = 1$ ← all true for f.

Moreover, $[a, P_i] = 0$ ($z f(z) = f(z) z$), $Sp(a P_1) = K_1 \cup \{0\}$, $Sp(a P_2) = K_2 \cup \{0\}$ by the Spectral Mapping Thm.

③ If $T \in \mathcal{B}(\mathfrak{X})$, $Sp(T) = K_1 \cup K_2$ as in ②, set $E = P_1 \mathfrak{X}$, $F = P_2 \mathfrak{X}$, so (E, F) complementary. Then $T P_i = P_i T = P_i T P_i \Rightarrow E, F$ T-invariant.

Invariant Subspace Problem: Does every op. in $\mathcal{B}(\mathcal{H})$ have a nontrivial invariant subspace? (Fails for Banach spaces by Enflo.)

Dependence of Spectrum on the Algebra

Let $\mathrm{id}: z \mapsto z$, and $\mathbb{D} = B_1(0) \subset \mathbb{C}$.

Example: $\mathrm{sp}_{C(\partial\mathbb{D})}(\mathrm{id}) = \partial\mathbb{D}$. Let $A = \overline{\{\text{poly's on } \partial\mathbb{D}\}} \subseteq C(\partial\mathbb{D})$.

Claim: $\mathrm{sp}_A(\mathrm{id}) = \overline{\mathbb{D}}$.

Pf: $\|\mathrm{id}\|_A = 1 \Rightarrow \mathrm{sp}_A(\mathrm{id}) \subseteq \overline{\mathbb{D}}$. Clearly $\mathrm{sp}_{C(\partial\mathbb{D})}(\mathrm{id}) \subseteq \mathrm{sp}_A(\mathrm{id})$.

Suppose $|\lambda| < 1$, and suppose $\lambda \notin \mathrm{sp}_A(\mathrm{id})$. Then $\exists f \in A$ s.t. $(\mathrm{id} - \lambda) f = 1$, and a seq. of poly's (p_n) s.t. $p_n \to f$ unif on $\partial\mathbb{D}$. By the Maximum Modulus Principle, (p_n) is unif. Cauchy on $\overline{\mathbb{D}}$! Hence $\exists g \in C(\overline{\mathbb{D}})$ s.t. $p_n \to g$ unif, and $g|_{\partial\mathbb{D}} = f$. Since $p_n(\mathrm{id}-\lambda) \to 1$ unif on $\partial\mathbb{D}$, $p_n(\mathrm{id}-\lambda) \to 1$ unif on $\overline{\mathbb{D}}$ as well, so $g(\mathrm{id}-\lambda) = 1$. But $g(\lambda)(\lambda-\lambda) = 0$, $\Rightarrow\Leftarrow$.

Def: For $K \subseteq \mathbb{C}$ cpt, the polynomially convex hull of K
$$\mathrm{PConv}(K) = \{z \in \mathbb{C} \mid |p(z)| \leq \|p\|_K \ \forall \text{ poly } p\}$$
K is polynomially cnvx if $K = \mathrm{PConv}(K)$.

Ex: $\mathrm{PConv}(\partial\mathbb{D}) = \overline{\mathbb{D}}$.

Prop: $K \subseteq \mathbb{C}$ cpt. Then $\mathbb{C} \setminus \mathrm{PConv}(K)$ is the unbdd component of $\mathbb{C} \setminus K$. K poly cnvx $\Leftrightarrow$ $\mathbb{C} \setminus K$ conn.

Pf: Enumerate $\pi_0(\mathbb{C} \setminus K)$ by $(U_n)_{n \geq 0}$ w/ U_0 unbdd.
Let $L = K \cup \bigcup_{n \geq 1} U_n = \mathbb{C} \setminus U_0$. Clearly $K \subset \mathrm{PConv}\, K$.

$\hookrightarrow$

If $n \geq 1$, $\mathcal{U}_n$ is a bdd open set, and $\partial \mathcal{U}_n \subseteq K$. By the Maximum Modulus Principle, $\mathcal{U}_n \subseteq \mathrm{PConv}K$, so $L \subseteq \mathrm{PConv}K$.

Now if $\lambda \in \mathcal{U}_0$, $z \mapsto (z-\lambda)^{-1} \in \mathcal{O}(L)$. Since $\mathcal{U}_0$ is conn., by Runge's Thm, can unif approx $z \mapsto (z-\lambda)^{-1}$ by poly's, i.e. $\exists (p_n)$ w/ $p_n(z) \longrightarrow (z-\lambda)^{-1}$ unif on L. Then $p_n(z)(z-\lambda) \longrightarrow 1$ unif on L. Set $q_n = p_n(z)(z-\lambda)$. Pick $N > 0$ s.t. $\|q_n - 1\| < \frac{1}{2}$ $\forall n \geq N$. Since $K \subset L$ and $|q_n(\lambda) - 1| = 1$, we have $\lambda \notin \mathrm{PConv}K$. Thus $\mathrm{PConv}K = L$.

[[The poly $q_N - 1$ satisfies $|(q_N - 1)(\lambda)| > \|q_N - 1\|_K$]]

<u>Thm</u>: Suppose $1_A \in B \subseteq A$ is a unital inclusion of Banach alg's. Then $\forall b \in B$,

① $sp_A(b) \subseteq sp_B(b)$ and $\partial sp_B(b) \subseteq \partial sp_A(b)$.

② $\mathrm{PConv}[sp_A(b)] = \mathrm{PConv}[sp_B(b)]$.

③ If $G \subseteq \mathbb{C} \setminus sp_A(b)$ is a <u>hole</u> (bdd conn. component), either $G \subseteq sp_B(b)$ or $G \cap sp_B(b) = \emptyset$.

④ If $B = \overline{\{\text{poly's in } b\}} \subseteq A$, $sp_B(b) = \mathrm{PConv}[sp_A(b)]$.

<u>Pf</u>: ① The first inclusion is trivial. Suppose $\lambda \in \partial sp_B(b)$. Since $sp_A(b)^{\circ} \subseteq sp_B(b)^{\circ}$, it suffices to show $\lambda \in sp_A(b)$. Suppose for contradiction $\lambda \notin sp_A(b)$. Then $\exists a \in A$ s.t. $a(b-\lambda) = (b-\lambda)a = 1$. Since $\lambda \in \partial sp_B(b)$, $\exists (\lambda_n) \subset \mathbb{C} \setminus sp_B(b)$ s.t. $\lambda_n \to \lambda$. Then $(b-\lambda_n)^{-1} \in B$ $\forall n$, so $(b-\lambda_n)^{-1} \in A$. Since $b - \lambda_n \to b - \lambda$, and inversion is <u>cts</u> on $G(A)$, $(b-\lambda_n)^{-1} \to (b-\lambda)^{-1} = a \in A$. Then $a \in B$ since B is complete, $\Rightarrow\Leftarrow$.

② Immediate from ① and Max. Mod. Principle.

③ Let G be a hole of $sp_A(b)$. Set $G_1 = G \cap sp_B(b)$ and $G_2 = G \setminus sp_B(b)$. Thus $G = G_1 \cup G_2$, $G_1 \cap G_2 = \emptyset$. Clearly G_2 is open. Since $\partial sp_B(b) \subseteq sp_A(b)$ and $G \cap sp_A(b) = \emptyset$, we must have $G_1 = G \cap sp_B(b)^\circ$, which is also open. Since G is conn., G_1 or $G_2 = \emptyset$.

④ Suppose $B = \overline{\{\text{poly's in } b\}} \subseteq A$. By ①+②, we know $sp_A(b) \subseteq sp_B(b) \subseteq \mathrm{PConv}(sp_A(b))$. Fix λ in $\mathrm{PConv}(sp_A(b))$. Suppose for contradiction $\lambda \notin sp_B(b)$. Then $(b-\lambda)^{-1} \in B \subseteq A$. Then $\exists\, (p_n)$ seq. of poly's w/ $p_n(b) \longrightarrow (b-\lambda)^{-1}$. Define $q_n(z) = (z-\lambda)p_n(z)$, so $\|q_n(b) - 1\| \to 0$. But then we have

$$
\begin{aligned}
\|q_n(b) - 1\| &\geq r(q_n(b) - 1) \\
&= \sup\{|z-1| \mid z \in sp_A(q_n(b))\} \\
&= \sup\{|q_n(z) - 1| \mid z \in sp_A(b)\} \\
&= \|q_n - 1\|_{C(sp_A(b))} \\
&\geq |q_n(\lambda) - 1| \\
&= 1,
\end{aligned}
$$

Spectral Mapping Thm.

$\lambda \in \mathrm{PConv}(sp_A(b))$

which is a contradiction.

Gelfand Theory

Given a unital <u>commutative</u> Banach alg A, we'll construct a canonical cpt. Hausd. space $\hat{A}$, together w/ a cts unital alg. homom.

$$\wedge : A \longrightarrow C(\hat{A})$$

For now, A need not be unital.

<u>Def</u>: A <u>multiplicative linear fcl</u> or (algebra) <u>character</u> on A is a nonzero linear $\varphi : A \to \mathbb{C}$ s.t. $\varphi(ab) = \varphi(a)\varphi(b)$ $\forall a,b \in A$.
The set of characters is denoted $\hat{A}$.

<u>Example</u>: For X cpt. Hausd., $ev_x : C(X) \to \mathbb{C}$ by $f \longmapsto f(x)$ is a character. $(X \subseteq \widehat{C(X)})$

<u>Remarks</u>: ① If A unital, $\varphi \in \hat{A} \Rightarrow \varphi(1) = 1$.

② If A nonunital, $A_1 = A \oplus \mathbb{C}1$, then $\forall \varphi \in \hat{A}$, $\exists! \tilde{\varphi} \in \hat{A}_1$ s.t. $\tilde{\varphi}|_A = \varphi$. Indeed, we must have

$$\tilde{\varphi}(a + \lambda 1) = \tilde{\varphi}(a) + \lambda \tilde{\varphi}(1) = \varphi(a) + \lambda \in \mathbb{C}.$$

Then
$$\begin{aligned}
\tilde{\varphi}((a+\lambda 1)(b+\mu 1)) &= \tilde{\varphi}(ab + \mu a + \lambda b + \lambda\mu 1) \\
&= \varphi(ab) + \mu\varphi(a) + \lambda\varphi(b) + \lambda\mu \\
&= \varphi(a)\varphi(b) + \mu\varphi(a) + \lambda\varphi(b) + \lambda\mu \\
&= (\varphi(a) + \lambda)(\varphi(b) + \mu) \\
&= \tilde{\varphi}(a+\lambda 1)\,\tilde{\varphi}(b+\mu 1).
\end{aligned}$$

Conversely, if $\varphi \in \hat{A}_1$, $\varphi|_A$ is either 0 or in $\hat{A}$.

Lemma: If A is a unital Banach alg. and $J \subseteq A$ is a proper ideal, then $\forall a \in J$, $\|a - 1\| \geq 1$.

Pf: If $\|1 - a\| < 1$, $a \in G(A)$, a contradiction.

Applications: ① If $J \subseteq A$ proper ideal, $\overline{J} \subseteq A$ is too.

② All maximal ideals are closed.

Back to A commutative.

Prop: If A unital and $\varphi \in \hat{A}$, $\|\varphi\| = 1$.

Pf: Let $a \in A$ w/ $\varphi(a) \neq 0$. Then $1 - \varphi(a)^{-1} a \in \ker(\varphi)$,

so $1 \leq \|1 - (1 - \varphi(a)^{-1} a)\| = \dfrac{\|a\|}{|\varphi(a)|} \Rightarrow |\varphi(a)| \leq \|a\|$.

Since $\varphi(1) = 1$, $\|\varphi\| = 1$.

Cor: $\ker \varphi \subseteq A$ is a maximal ideal.

Recall the Gelfand-Mazur Thm tells us that the only normed division alg. / $\mathbb{C}$ is (iso. to) $\mathbb{C}$. This means if A is a unital comm. Banach alg. w/ no nontrivial ideals, then $A = \mathbb{C}$.

Exercise: $\forall a \notin G(A)$, Aa is an ideal.

Prop: If A is unital and commutative, each max. ideal is the kernel of a unique $\varphi \in \hat{A}$.

Pf: Let $J \subseteq A$ be a max. ideal, so A/J is a unital Banach alg. w/ no nontrivial ideals, i.e., $A/J = \mathbb{C}$.

Hence $\varphi : A \to A/J$ the canonical surjection is in $\hat{A}$.

Prop: For A a unital commutative Banach alg and $a \in A$, TFAE:

1. $a \notin G(A)$
2. $\exists$ max ideal $a \in J \subseteq A$
3. $\exists \varphi \in \hat{A}$ s.t. $\varphi(a) = 0$.

Pf: ① ⇒ ②: Standard Zorn's Lemma argument; Aa is an ideal containing a. Take max elt. of $\{$ideals $a \in J \subseteq A\}$ under inclusion.

② ⇒ ③: $\varphi : A \to A/J$ works.

③ ⇒ ①: If $a \in G(a)$, $\forall \varphi \in \hat{A}$, $\varphi(a)\varphi(a^{-1}) = 1$, so $\varphi(a) \neq 0$.

Cor: For $a \in A$, $\lambda \in sp(a) \iff \exists \varphi \in \hat{A}$ s.t. $\varphi(a) = \lambda$. so $sp(a) = \{\varphi(a) \mid \varphi \in \hat{A}\}$.

Gelfand topology: Note that $\hat{A}$ is contained in the unit sphere of A^*, i.e., $\{\varphi \in A^* \mid \|\varphi\| = 1\}$.

Claim: $\hat{A} \subset A^*$ is weak* closed, hence cpt by Banach-Alaoglu.

Pf: Suppose $(\varphi_\lambda) \subset \hat{A}$ w/ $\varphi_\lambda \to \varphi$ weak*. Then $\forall a, b \in A$,

$$\varphi(ab) = \lim \varphi_\lambda(ab) = \underbrace{\lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)}_{\text{use } (\varphi_\lambda(a)) \text{ and } (\varphi_\lambda(b)) \text{ are eventually bdd}}.$$

Exercise: Suppose $A_1 = A \oplus \mathbb{C}1$ is the unitization of A. Recall $\forall \varphi \in \hat{A}$, $\exists!$ extension $\tilde{\varphi} \in \hat{A}_1$ s.t. $\tilde{\varphi}|_A = \varphi$, and $\exists! \varphi_0 : A_1 \to \mathbb{C}$ s.t. $\varphi_0|_A = 0$ $[\varphi_0 : A_1 \to A_1/A = \mathbb{C}]$. We get an inclusion $\hat{A} \xhookrightarrow{i} \hat{A}_1$ by $\varphi \mapsto \tilde{\varphi}$ whose image is $\hat{A}_1 \setminus \{\varphi_0\}$. Show that

1. the relative top. on $i(\hat{A})$ is the weak* top on $\hat{A}$
2. $\hat{A}_1$ is the 1 pt compactification of $\hat{A}$.

Gelfand Transform: For $a \in A$, define $ev_a : \hat{A} \to \mathbb{C}$ by $ev_a(\varphi) = \varphi(a)$. Then $\hat{a}$ is a cts fct on $\hat{A}$ (recall $\hat{A}$ has the relative weak* top.), and $\|ev_a\|_{C(\hat{A})} \leq \|a\|$. Moreover,

$$\Gamma : A \to C(\hat{A}) \quad \text{is an alg. homom.}$$
$$a \mapsto ev_a$$

Remarks: ⓪ usually $\Gamma(A)$ is not closed in $C(\hat{A})$

① If A is not unital, and $\hat{A}$ not cpt, then $\Gamma(A) \subseteq C_0(\hat{A})$.

② The Gelfand top. on $\hat{A}$ is the weakest top. on $\hat{A}$ s.t. the fcts ev_a in $\Gamma(A) \subset C(\hat{A})$ are cts.

Pf: WLOG, A is unital. The identity is a map

$$\underbrace{(\hat{A}, \text{Gelfand top.})}_{\text{Cpt.}} \xrightarrow{id} \underbrace{(\hat{A}, \text{top. induced by } \Gamma(A))}_{\text{Hausd.}}$$

Hence it is closed, and thus an iso.

Note also: $\varphi_\gamma \to \varphi \iff \varphi_\gamma(a) \to \varphi(a) \ \forall a$
$\iff ev_a(\varphi_\gamma) \to \hat{a}(\varphi) \ \forall a$

- defines both the weak* and top. induced by $\Gamma(A)$.

③ $\ker \Gamma = \{a \in A \mid sp(a) = \{0\}\} = \{\text{quasi nilpotent elts}\}$.

Γ inj $\iff \nexists$ quasi nilpotent elts. (call A semi-simple)

Prop: Suppose A is unital and commutative. Let $a \in A$ and $f \in \mathcal{O}(sp(a))$. Then $ev_{f(a)} = f \circ ev_a : \hat{A} \to \mathbb{C}$.

Pf: First, for $\lambda \in \mathbb{C} \setminus sp(a)$ and $\varphi \in \hat{A}$,

$$1 = \varphi(1) = \varphi[(a-\lambda)(a-\lambda)^{-1}] = [\varphi(a) - \lambda]\,\varphi[(a-\lambda)^{-1}]$$

So $\varphi[(a-\lambda)^{-1}] = \dfrac{1}{\varphi(a)-\lambda}$.

Suppose $f \in H(\mathcal{U})$ for $\mathcal{U} \supset sp(a)$, and let γ be a simple closed contour as in the defn of $f(a)$. Then

$$ev_{f(a)}(\varphi) = \varphi(f(a)) = \varphi\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz\right) = \frac{1}{2\pi i}\int_\gamma f(z)\,\varphi[(z-a)^{-1}]\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\varphi(a)}\,dz = f(\varphi(a)).$$

$(f \circ ev_a)(\varphi)$.

Exercise: Show that if the unital comm. Banach alg A is generated by $a \in A$, i.e., poly's in A are dense in A, then $\hat{A}$ is homeomorphic to $sp_A(a)$.

Examples of Gelfand Transform

① X cpt. Hausd. Clearly $X \hookrightarrow \widehat{C(X)}$ by $x \mapsto ev_x$. In fact, this map is surj. + cts $\Rightarrow$ homeom.

Exercise: Suppose $J \subseteq C(X)$ is a max ideal. Show $\exists x \in X$ s.t. $f(x) = 0$ $\forall f \in J$. Thus $J = \ker(ev_x)$.

[[If not, can build invertible elt in J.]]

② X loc. cpt. Hausd., X not cpt. Still have $\widehat{C_0(X)} = X$.

[[pass to unitization, apply ①, $\widehat{C_0(X)} = C(X \cup \{\infty\}) \setminus \{\infty\}$, $\infty = \mathscr{C}_0$]]

③ If G is a loc. cpt ab. gp, define $\hat{G} =$ cts homom. $G \to U(1) = \mathbb{T}$.

In general, $\hat{G}$ is also a LCA gp w/ $\hat{\hat{G}} \cong G$. Then $L^1(G)^\wedge \cong \hat{G}$, and $\Gamma: L^1(G) \to C_0(\hat{G})$ is known as the Fourier Transform.

(a) when $G = \mathbb{Z}$, $\ell^1(\mathbb{Z})$ has conv. mult. $(a*b)(n) = \sum_{-\infty}^{\infty} a(n-k)\, b(k)$.

Define $\delta_n(k) = \delta_{n=k}$, so $\delta_n * \delta_k = \delta_{n+k}$. Thus $\delta_1^n = \delta_n$ $\forall n \in \mathbb{Z}$.

By the exercise, $\ell^1(\mathbb{Z})^\wedge \cong sp(\delta_1)$.

Claim: $\ell^1(\mathbb{Z})^\wedge \cong U(1) = \mathbb{T}$.

Pf: For $\varphi \in \ell^1(\mathbb{Z})^\wedge$, let $\lambda = \varphi(\delta_1)$. Then since $\|\varphi\| = 1$ and $\|\delta_n\| = 1$ $\forall n$, $|\lambda| \le \|\varphi\| \cdot \|\delta_1\| = 1$, and $|\lambda^{-1}| = |\varphi(\delta_{-1})| \le 1$.

Can identify φ w/ $(\varphi(\delta_n))_{n \in \mathbb{Z}} = (\lambda^n)_{n \in \mathbb{Z}} \in \ell^\infty(\mathbb{Z})$

Conversely, for $\lambda \in U(1)$, can define $\varphi_\lambda \in \ell^1(\mathbb{Z})^\wedge \subset \ell^\infty(\mathbb{Z})$ by $\varphi_\lambda(\delta_n) = \lambda^n$. Then $\varphi_\lambda \in \ell^1(\mathbb{Z})^\wedge$:

$$\varphi_\lambda(a*b) = \sum_{-\infty}^{\infty} (a*b)(n)\lambda^n = \sum_{-\infty}^{\infty} \sum_{j+k=n} a(j)\, b(k)\, \lambda^n$$

$$= \sum_j a(j)\lambda^j \sum_k b(k)\lambda^k = \varphi_\lambda(a)\varphi_\lambda(b).$$

If $a \in \ell^1(\mathbb{Z})$, $\hat{a}(e^{i\theta}) = \sum_{-\infty}^{\infty} a(n) e^{in\theta}$ $\Rightarrow$ $\Gamma(\ell^1(\mathbb{Z})) \subseteq C_0(\mathbb{T})$ is alg of abs. conv. Fourier series. Gelfand top is usual one.

(b) Consider $L^1(\mathbb{T})$ w/ conv. mult: $(f*g)(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i(\theta-t)})g(e^{it})dt$

Define $z_n(e^{i\theta}) = e^{in\theta}$ for $n \in \mathbb{Z}$. Then:

$$(z_n * z_k)(e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} e^{in(\theta-t)} e^{ikt}\,dt$$

$$= \frac{e^{in\theta}}{2\pi}\int_0^{2\pi} e^{i(k-n)t}\,dt = \begin{cases} e^{in\theta} & n=k \\ 0 & n\neq k. \end{cases}$$

$\Rightarrow$ $z_n * z_k = \delta_{n=k}\, z_n$, idempotents.

$\forall \varphi \in L^1(\mathbb{T})^\wedge$, $\varphi(z_n) \in \{0,1\}$ $\forall n \in \mathbb{Z}$

Note $\varphi(z_n) = 1$ for exactly one $n \in \mathbb{Z}$, since

$$\varphi(z_n)\varphi(z_k) = \varphi(z_n * z_k) \quad \forall n,k \in \mathbb{Z}.$$

– gives a map $L^1(\mathbb{T})^\wedge \to \mathbb{Z}$.

Conversely, for $n \in \mathbb{Z}$, define $\varphi_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\,d\theta$.

Then clearly $\varphi_n(z_k) = \delta_{n=k}$, and φ_n is multiplicative on trigonometric poly's, which are dense in $L^1(\mathbb{T})$.

– Gelfand top on $\mathbb{Z}$ is the discrete one.

– For $f \in L^1(\mathbb{T})$, $(ev_f(n))_{n\in\mathbb{Z}}$ = seq. of Fourier coeffs of f

(c) Consider $L^1(\mathbb{R})$ w/ conv. mult. $(f*g)(x) = \int f(x-y)g(y)dy$.

We know $L^1(\mathbb{R}) \cong L^\infty(\mathbb{R})$, so let's find the $\kappa \in L^\infty(\mathbb{R})$ which induce $\varphi \in L^1(\mathbb{R})^\wedge$.

<u>Step 1</u>: $\kappa(x+y) = \kappa(x)\kappa(y)$ a.e.

<u>Pf</u>: Fix $f,g \in L^1(\mathbb{R})$. Then $\varphi(f*g) = \iint \kappa(x)f(x-y)g(y)\,dy\,dx$

$$= \iint \kappa(x)f(x-y)g(y)\,dx\,dy = \iint \kappa(x+y)f(x)g(y)\,dx\,dy$$

$$\varphi(f)\varphi(g) = \iint \kappa(x)\kappa(y)f(x)g(y)\,dx\,dy$$

$$\Rightarrow \iint [\kappa(x+y) - \kappa(x)\kappa(y)]\, \underbrace{f(x)g(y)}_{\text{span a dense subspace of } L^1(\mathbb{R}^2).}\,dx\,dy \quad \forall f,g \in L^1(\mathbb{R})$$

<u>Step 2</u>: κ is cts (up to a null set)

<u>Pf</u>: $\varphi \neq 0$, so $\exists a<b$ s.t. $\int_a^b \kappa(x)dx \neq 0$. Then a.e. $x \in \mathbb{R}$,

$$\kappa(x)\int_a^b \kappa(y)dy = \int_a^b \kappa(x)\kappa(y)\,dy = \int_a^b \kappa(x+y)dy$$

This means that $\chi(x) = \int_a^b \chi(x+y)\,dy \Big/ \int_a^b \chi(y)\,dy$ a.e.

$\leadsto \chi(x) = \int_{a+x}^{b+x} \chi(y)\,dy \Big/ \int_a^b \chi(y)\,dy$, which is cts.

<u>Step 3</u>: Assume χ cts. Then $|\chi(x)| = 1$ $\forall x$, so $\chi : \mathbb{R} \to U(1) = \mathbb{T}$ a cts. homom.

<u>Pf</u>: For $x \in \mathbb{R}$, $\sup_{n \in \mathbb{Z}} |\chi(x)|^n = \sup_{n \in \mathbb{Z}} |\chi(nx)| < \infty \Rightarrow |\chi(x)| \in \{0,1\}$.
$\nwarrow$ $\chi \in L^\infty(\mathbb{R})$.

Since $\mathbb{R}$ conn., $|\chi(x)| = 1$ $\forall x \in \mathbb{R}$. [$\chi(0) = 1$, so $\chi(-x) = \chi(x)^{-1}$!]

<u>Step 4</u>: $\exists t \in \mathbb{R}$ s.t. $\chi(x) = e^{-itx}$.

<u>Pf</u>: Since $\chi : \mathbb{R} \to U(1)$ cts, $\exists \eta : \mathbb{R} \to \mathbb{R}$ cts s.t. $\chi(x) = e^{i\eta(x)}$.

[$\mathbb{R}$ is simply conn, so every cts map $\mathbb{R} \to \mathbb{T}$ lifts to $\mathbb{R}$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\exp(i-)"] \\ \mathbb{R} \arrow[ur, dashed] \arrow[r] & \mathbb{T} \end{tikzcd}$$

As $\chi(0) = 1$, we may assume $\eta(0) = 0$.]

By Step 1, we have $\eta(x+y) = \eta(x) + \eta(y)$ mod 2π.
By continuity, $\eta(x+y) = \eta(x) + \eta(y)$ in a nbhd of $0_{\mathbb{R}}$.
The same argument shows $S := \{(x,y) \in \mathbb{R}^2 \mid \eta(x+y) = \eta(x) + \eta(y)\}$ is open and non-empty. But η cts $\Rightarrow S$ is closed, so $\eta(x+y) = \eta(x) + \eta(y)$ $\forall x, y \in \mathbb{R}$. Then setting $t = \eta(-1)$, we have $\eta(x) = -tx$.

[For $n \in \mathbb{Z}$, $0 = \eta(0) = \eta(n + (-n)) = \eta(n) + n\eta(-1) \leadsto \eta(n) = -tn$. For $p, q \in \mathbb{Z}$, $q \neq 0$, $-tp = \eta(p) = \eta(q \cdot \frac{p}{q}) = q\eta(p/q)$, so $\eta(p/q) = -tp/q$. The result now follows by continuity.]

We can now identify $L^1(\mathbb{R})^\wedge$ w/ $\mathbb{R}$ via $\chi_t \leftrightarrow t$ where $\chi_t(x) := e^{-itx}$. Now the Gelfand Transform is exactly the Fourier Transform

$$\Gamma(f)(t) = \int_{\mathbb{R}} f(x) e^{-itx}\,dx$$

The Gelfand top. is induced by $\Gamma(f)$ for $f \in L^1(\mathbb{R})$.

If $f(x) = e^{iax} e^{-x^2/2}$, $\Gamma(f)(t) = \sqrt{2\pi}\, e^{-(x-a)^2/2}$

$\leadsto$ Gelfand top on $\mathbb{R}$ is the usual one.

<u>Exercise</u>: Compute $\hat{A}$, the Gelfand top. on $\hat{A}$, and $\Gamma : A \longrightarrow C(\hat{A})$ for

① $\ell^1(\mathbb{Z}_{\geq 0})$ $(a * b)(n) := \sum_{k=0}^{n} a(n-k)\, b(k)$.

② Disk alg $A(\mathbb{D}) = \{ f \in C(\overline{\mathbb{D}}) \mid f|_{\mathbb{D}} \text{ hol.} \}$

<u>C^*-algebras</u>

An <u>involution</u> on a $\mathbb{C}$-alg. is a fct $* : A \longrightarrow A$; $a \mapsto a^*$ s.t.

① $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$ $\forall \lambda \in \mathbb{C}$, $a, b \in A$

② $(ab)^* = b^* a^*$ $\forall a, b \in A$

③ $a^{**} = a$ $\forall a \in A$.

$\rightarrow$ $(A, *)$ called a <u>complex $*$-alg.</u>

<u>Examples</u>:

① $M_n(\mathbb{C})$ w/ adjoint / conjugate transpose.

② $\mathbb{C} \oplus \mathbb{C}$ w/ componentwise addition + multiplication, and $(a, b)^* := (\bar{b}, \bar{a})$.

③ $C(X)$ X cpt. Haus., $f^* = \bar{f}$

④ $C_0(X)$ X loc. cpt. Haus.

⑤ $L^1(\mathbb{R})$ w/ $f^*(x) := \overline{f(-x)}$

⑥ $L^\infty(\mathbb{R})$ w/ $f^* = \bar{f}$.

⑦ Disk alg $A(\mathbb{D})$ w/ $f^*(z) = \overline{f(\bar{z})}$.

⑧ A $*$-closed subalg of $B(H)$, H a Hilb. space (next week).

<u>Note</u>: If A unital, <u>$1^* = 1$</u>: $1^* = 1^* 1 = (1^* 1)^{**} = (1^* 1)^* = 1^{**} = 1$

<u>Def</u>: A <u>C^*-alg</u> is a normed $\mathbb{C}$ $*$-alg $(A, *, \|\cdot\|)$ s.t.

① $(A, \|\cdot\|)$ is a Banach alg

② $\|a^* a\| = \|a\|^2$ $\forall a \in A$.

<u>Exercise</u>: Determine which of ①–⑧ above are C^*-alg's.

Operators: $a \in A$ is called

① self-adjoint if $a = a^*$

② positive ($a \geq 0$) if $a = b^*b$ for some $b \in A$ [positive $\Rightarrow$ s.a.!]

③ normal if $a^*a = aa^*$

④ a projection if $a = a^* = a^2$

⑤ an isometry if $a^*a = 1_A$ (and A is unital)

⑥ a unitary if $a^*a = 1_A = aa^*$ (and A is unital).

Note: unitary $\iff$ invertible isometry
$\iff$ normal isometry

⑦ a partial isometry if a^*a is a projection.

We'll study these in great detail, especially when $A = B(H)$.

Elementary Properties: Let A be a C^*-alg.

① $*$ is isometric: Suppose $a \neq 0$. $\|a\|^2 = \|a^*a\| \leq \|a\| \cdot \|a^*\|$
$\|a^*\|^2 = \|aa^*\| \leq \|a\| \cdot \|a^*\|$

② $\|a\| = \|L_a\|$ for $L_a \in B(A)$.

Pf: $\|ab\| \leq \|a\| \cdot \|b\| \Rightarrow \|L_a\| \leq \|a\|$. Now $\|aa^*\| = \|a\|^2 = \|a\| \cdot \|a^*\|$.

③ (Adjoining a unit) $A_1 = A \oplus \mathbb{C}1$ w/ usual addition + mult., w/
$(a+\lambda)^* := a^* + \overline{\lambda}$, and $\|a+\lambda\|_{A_1} = \sup\{\|ab+\lambda b\| \mid b \in A\} = \|L_a + \lambda 1\|_{B(A)}$
is a C^*-alg which contains A isometrically as a codim 1 ideal.

Pf: By ②, suffices to prove C^*-axiom: For $a \in A$, $\lambda \in \mathbb{C}$, $\varepsilon > 0$,
let $b \in A$ s.t. $\|b\| = 1$ and

$$\|a+\lambda\|_{A_1}^2 - \varepsilon \leq \|ab+\lambda b\|_A^2 = \|(ab+\lambda b)^*(ab+\lambda b)\|_A = \|(b^*a^* + \overline{\lambda}b^*)(ab+\lambda b)\|_A$$

$$= \|L_{b^*}(a^*+\overline{\lambda})(a+\lambda)b\|_{A_1} \leq \|(a^*+\overline{\lambda})(a+\lambda)b\|_{A_1} \leq \|(a^*+\overline{\lambda})(a+\lambda)\|_{A_1}$$

So $\|a+\lambda\|_{A_1}^2 \overset{(*)}{\leq} \|(a^*+\overline{\lambda})(a+\lambda)\|_{A_1} \overset{(*)}{\leq} \|a^*+\overline{\lambda}\|_{A_1} \cdot \|a+\lambda\|_{A_1}$, so

$\|a+\lambda\|_{A_1} \leq \|a^*+\overline{\lambda}\|_{A_1}$ and by symmetry, $\|a+\lambda\|_{A_1} \geq \|a^*+\overline{\lambda}\|_{A_1}$.

Thus the above inequalities $(*)$ are equalities.

④ If $a \in A$ normal, then $\|a\| = r(a)$.

Pf: $\|a^2\|^2 = \|(a^2)^*(a^2)\| = \|\underbrace{(a^*a)}_{\text{self-adjoint}}{}^2\| = \|a^*a\|^2 = \|a\|^4 \Rightarrow \|a^2\| = \|a\|^2$.

Iterating, $\forall n \in \mathbb{N}$, $\|a^{2^n}\| = \|a\|^{2^n} \Rightarrow r(a) = \lim_n \|a^{2^n}\|^{1/2^n} = \|a\|$.

⑤ $\forall b \in A$, $\lambda \in sp_A(b) \iff \bar{\lambda} \in sp_A(b^*)$.

⑥ If A unital and $a \in G(A)$, $(a^*)^{-1} = (a^{-1})^*$, and $\lambda \in sp(a) \iff \lambda^{-1} \in sp(a^{-1})$.

⑦ If A unital and $u \in A$ unitary, then $sp(u) \subset \partial\mathbb{D}$.

Pf: $u^* = u^{-1} \Rightarrow \forall \lambda \in sp(u)$, $\bar{\lambda}^{-1} \in sp(u)$. Since $\|u\| = 1$, $|\lambda|, |\lambda^{-1}| \leq 1 \Rightarrow \lambda \in \partial\mathbb{D}$.

⑧ If $a = a^*$, then e^{ia} is unitary (defined using HFC, pass to unitization A_1 if necessary)

Pf: $(e^{ia})^* = \left(\sum_{n \geq 0} \frac{(ia)^n}{n!}\right)^* = \sum \frac{(-ia)^n}{n!} = e^{-ia}$. By HFC, $\overbrace{e^{ia}e^{-ia}}^{\text{commute.}} = 1$.

⑨ If $a = a^*$, $sp_A(a) \subset \mathbb{R}$. (Note A need not be unital!)

Pf: by ⑦, $\underbrace{sp(e^{ia})}_{\text{pass to } A_1 \text{ if necessary.}} \subset \partial\mathbb{D}$. Now $sp(e^{ia}) \overset{\text{Spectral Mapping Thm}}{=} e^{i\,sp(a)} \Rightarrow \underbrace{sp(a) \subset \mathbb{R}}_{\text{holds in } A_1 \text{ and} \text{ holds in } sp_A(a).}$.

⑩ Every $a \in A$ can be written as $a = Re(a) + i\,Im(a)$ where $Re(a) = \frac{a + a^*}{2}$ and $Im(a) = \frac{a - a^*}{2i}$ both self adjoint.

⑪ If A comm., every $\varphi \in \hat{A}$ is a $*$-hom.

Pf: $\varphi(a^*) = \varphi(Re(a) - i\,Im(a)) = \underbrace{\varphi(Re(a))}_{\in sp(Re(a))} - i\,\underbrace{\varphi(Im(a))}_{\in sp(Im(a))} = \overline{\varphi(Re(a)) + i\varphi(Im(a))} = \overline{\varphi(a)}$ ← both $\in \mathbb{R}$.

<u>Gelfand-Naimark Thm 1:</u> If A is a unital comm. C^*-alg, then A is isometrically $*$-isomorphic via Γ to $C(\hat{A})$.

Pf: <u>Γ is an isometry</u>: Every elt in A is normal, so

$$\|\Gamma(a)\|_{C(\hat{A})} = \sup\{ev_a(\varphi) \mid \varphi \in \hat{A}\} = \sup\{\varphi(a) \mid \varphi \in \hat{A}\}$$
$$= \sup\{\lambda \mid \lambda \in sp_A(a)\} = r(a) = \|a\|_A.$$

<u>Γ is $*$-preserving</u>: Every $\varphi \in \hat{A}$ is $*$-preserving, so if $a \in A$,

$$\Gamma(a^*)(\varphi) = ev_{a^*}(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{ev_a(\varphi)} = \overline{\Gamma(a)}(\varphi).$$

<u>Γ is onto</u>: $\Gamma(A)$ is a s.a. Banach subalg of $C(\hat{A})$ which contains 1 and separates pts of $\hat{A}$. Stone-Weierstrass $\Rightarrow \Gamma(A) = C(\hat{A})$.

<u>Cor:</u> A non-unital comm. C^*-alg A is isometrically $*$-isomorphic via Γ to $C_0(\hat{A})$.

<u>Pf:</u> Let A_1 be the unitization, and recall $\hat{A}_1 = \hat{A} \cup \{\infty\}$ where $\infty = \varphi_0$ s.t. $\varphi_0|_A = 0$. Then $\Gamma : A \to C(\hat{A}_1)$ is isometric onto a subalg of $C_0(\hat{A})$ which separates pts and does not vanish at any $\varphi \in \hat{A}$. By Stone-Weierstrass, Γ is onto.

<u>Lemma (Spectral Permanence)</u>: Suppose $1_A \in B \subseteq A$ is a unital inclusion of C^*-alg's and $b \in B$. Then $sp_B(b) = sp_A(b)$.

<u>Pf:</u> It's enough to prove $b \in G(A) \Rightarrow b \in G(B)$.
Suppose $b \in G(A)$. Then $b^* \in G(A)$, and $(b^*)^{-1} = (b^{-1})^*$.
Hence $b^*b \in G(A)$. Since b^*b is sa., $sp_A(b^*b) \subseteq \mathbb{R}$.
Since $\partial sp_A(b^*b) = \partial sp_B(b^*b)$, we have $sp_B(b^*b) = sp_A(b^*b)$.
Hence $b^*b \in G(B)$, so b has a <u>left</u> inverse in B, namely $(b^*b)^{-1}b^*$. A similar argument applied to bb^* shows that b is <u>right</u> invertible in B.

<u>Continuous Functional Calculus</u>

Suppose A is a unital C^*-alg and $a \in A$ is <u>normal</u>.
Let $C^*(a)$ be the unital C^*-subalg of A generated by a, i.e., the <u>smallest</u> unital C^*-subalg containing a.
Then the inverse of the Gelfand Transform $C^*(a) \cong C(sp_A(a))$ is an isometric $*$-iso $\Phi : C(sp_A(a)) \to C^*(a) \subseteq A$ s.t.

① $\Phi(1) = 1$; $\Phi(\overset{id}{z \mapsto z}) = a$

② $\forall f \in \mathcal{O}(sp(a))$, $\Phi(f) = f(a)$ from the HFC

<u>Step 1:</u> There is a homeom. $sp_A(a) \cong C^*(a)^\wedge$.

Pf: By spectral permanence, $sp_A(a) = sp_{C^*(a)}(a)$. Note that $C^*(a)^\wedge \ni \varphi \longmapsto \varphi(a) \in sp_A(a)$ is surjective + cts. since $C^*(a)^\wedge$ has the relative wk-* top. Now we claim this map is injective. Suppose $\varphi(a) = \psi(a)$ for $\varphi, \psi \in C^*(A)^\wedge$. Then since φ, ψ are *-homs, $\varphi(a^*) = \overline{\varphi(a)} = \overline{\psi(a)} = \psi(a^*)$. Hence $\varphi = \psi$ on all poly's in a and a^*, so $\varphi = \psi$.

Step 2: Since $C^*(a) \xrightarrow[\sim]{\Gamma} C(C^*(a)^\wedge) \xrightarrow[\sim]{\wedge} C(sp_A(a))$ is an isometric *-iso, we define for $f \in C(sp_A(a))$

$\Phi(f) = (\Gamma^{-1} \circ \wedge^{-1})(f)$. Now we verify $\forall \varphi \in \hat{A}$,

a) $\Gamma(1)(\varphi) = \varphi(1) = 1 = \wedge^{-1}(1)(\varphi) \implies \Phi(1) = 1$

b) $\Gamma(a)(\varphi) = \varphi(a) = id(\varphi(a)) = \wedge^{-1}(id)(\varphi) \implies \Phi(id) = a$

Step 3: To verify the CFC extends the HFC, by the uniqueness property of the HFC, it suffices to show that if $(f_n), f$ are holo. on $\mathcal{U} \supseteq sp_A(a)$ s.t. $f_n \to f$ loc. unif., then $\Phi(f_n) \to \Phi(f)$ in A. Since $sp_A(a)$ cpt, $f_n \to f$ unif. in $C(sp_A(a)) \cong C^*(a)$, and we are finished.

Notation: For A a unital C^*-alg, $a \in A$ normal, we denote by the CFC $C(sp_A(a)) \xrightarrow{\sim} C^*(a) \subseteq A$

$$f \longmapsto f(a)$$

Note: If A non-unital and $a \in A$ normal, can define CFC $\underbrace{C_0(sp_A(a))}_{\text{vanish at } 0 \in sp_{A_1}(a).} \longrightarrow C^*(a)$, the nonunital C^*-subalg of A gen by a.

We'll now prove $\exists$ a contravariant equivalence of categories $\left\{ \begin{array}{c} \text{cpt Hausd.} \\ \text{spaces} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{unital comm.} \\ C^*\text{-alg's} \end{array} \right\}$.

$*$-homomorphisms of C^*-alg's:

Lemma: If A, B are unital C^*-alg's and $\phi: A \to B$ is a unital $*$-hom, then ϕ is norm decreasing. If ϕ is injective, then ϕ is an isometry.

Pf: First, note if $a \in G(A)$, then $\phi(a) \in G(B)$. Hence $sp_B(\phi(a)) \subseteq sp_A(a)$. Thus $r_B(\phi(a)) \leq r_A(a)$. Then $\forall a \in A$,

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \|\phi(a^*a)\| = r_B(\phi(a^*a))$$
$$\leq r_A(a^*a) = \|a^*a\| = \|a\|^2.$$

Now suppose ϕ is injective.

Claim: $sp_A(a) = sp_B(\phi(a))$ $\forall$ normal $a \in A$.

Pf: Suppose $\exists \lambda \in sp_A(a) \setminus sp_B(\phi(a))$. Then since $sp_A(a)$ is normal, $\exists$ cts $f: sp_A(a) \to [0,1]$ s.t. $f \neq 0$ ~~on~~ & $f|_{sp_B(\phi(a))} = 0$. Then $f(a) \neq 0$, but $\underbrace{\phi(f(a)) = f(\phi(a))}_{\text{holds } \forall \text{ poly's in } a, a^*, \text{ dense in } C(sp_A(a)) \text{ by Stone-Weierstrass!}} = 0$, $\Rightarrow \Leftarrow$.

Now, $\forall a \in A$, $\|\phi(a)\|^2 = r_B(\phi(a^*a)) = r_A(a^*a) = \|a\|^2$.

Thm: $\exists$ a contravariant eq. of cats:

$$\left\{ \begin{matrix} \text{cpt. Hausd. top. spaces} \\ \text{w/ cts maps} \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \text{unital comm. } C^*\text{-alg's} \\ \text{w/ unital } *\text{-hom's.} \end{matrix} \right\}$$

Given X cpt Hausd. $\rightsquigarrow C(X)$.
Given A unital comm. C^*-alg, $\rightsquigarrow \hat{A}$
] must extend these to functors.

We'll then need to construct natural iso's from composites each way to the appropriate id functors.

Step ①: Suppose X, Y cpt Hausd, and $f : X \to Y$ cts. Define $C(f) : C(Y) \to C(X)$ by $g \mapsto g \circ f$. Then $C(f)$ is a unital $*$-hom. Indeed, $1 \circ f = 1$, $\bar{g} \circ f = \overline{g \circ f}$, and $(\lambda g_1 + g_2 g_3) \circ f = \lambda (g_1 \circ f) + (g_2 \circ f)(g_3 \circ f)$.

Step ②: (a) $C(\mathrm{id}_X : X \to X) = \mathrm{id}_{C(X)}$.

(b) If $X \xrightarrow{f} Y \xrightarrow{g} Z$, $C(g \circ f) = C(f) \circ C(g)$, since if $h \in C(Z)$, $h \circ (g \circ f) = C(f)(h \circ g) = C(f)(C(g)(h))$

Step ③: If $\phi : A \to B$ is a unital $*$-hom, then $\hat{\phi} : \hat{B} \to \hat{A}$ by $\varphi \mapsto \varphi \circ \phi$ is cts.

Indeed, if $\varphi_\gamma \to \varphi$ wk$*$ in $\hat{B}$, then $\forall a \in A$, $\varphi_\gamma(\phi(a)) \to \varphi(\phi(a))$, so $\varphi_\gamma \circ \phi \to \varphi \circ \phi$ wk$*$ in $\hat{A}$, and thus $\hat{\phi}$ is cts.

Step ④: (a) $(\mathrm{id}_A)^\wedge = \mathrm{id}_{\hat{A}}$

(b) If $A \xrightarrow{\phi} B \xrightarrow{\psi} C$, $(\psi \circ \phi)^\wedge = \phi^\wedge \circ \psi^\wedge$.

Step ⑤: We've already seen that $X \cong C(X)^\wedge$. This iso. is natural:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ C(X)^\wedge \arrow[r, "C(f)^\wedge"'] & C(Y)^\wedge \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ ev_x \arrow[r, mapsto] & ev_{f(x)} \end{tikzcd}$$

$C(f)^\wedge(ev_x)(g)$
$= [ev_x \circ C(f)](g)$
$= ev_x(g \circ f)$
$= g(f(x)) = ev_{f(x)}(g)$ ✓

Step ⑥: We've already seen $A \cong C(\hat{A})$. This iso is natural:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ C(\hat{A}) \arrow[r, "C(\hat{\phi})"'] & C(\hat{B}) \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & \phi(a) \arrow[d, mapsto] \\ ev_a \arrow[r, mapsto] & ev_{\phi(a)} \end{tikzcd}$$

$C(\hat{\phi})(ev_a)(\varphi)$
$= (ev_a \circ \hat{\phi})(\varphi)$
$= ev_a(\varphi \circ \phi)$
$= \varphi(\phi(a)) = ev_{\phi(a)}(\varphi)$ ✓

Exercise: What happens for non-unital comm. C^*-alg's?

Application: Stone–Čech compactification: To each Tychonoff top. space X (T_1 and completely regular, i.e., if $F \subseteq X$ closed, $x \notin F$, $\exists$ cts $f: X \to [0,1]$ s.t. $f(x)=0$, $f|_F = 1$), $\exists$ a cpt Hausd. space $\beta X \supseteq X$ s.t.

① $\overline{X} = \beta X$ (βX is a compactification of X)

② $\forall$ cpt K and cts $f: X \to K$, can extend f to a map $\beta f: \beta X \to K$, i.e.

$$\begin{tikzcd} \beta X \arrow[dr, "\exists \beta f"] & \\ X \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "f"'] & K \end{tikzcd}$$

Pf: Define $\beta X = C_b(X)^\wedge$ where $C_b(X)$ is the unital comm. C^*-alg of cts bdd fcts on X w/ $\|\cdot\|_\infty$. Since $ev_x \in C_b(X)^\wedge$ $\forall x \in X$, get natural inclusion $X \subseteq \beta X$. Since X is Tychonoff and top. on βX is induced by $\Gamma(C_b(X))$, the inclusion $X \subseteq \beta X$ is cts wrt original topology on X. Since $C(\beta X) \cong C_b(X)$, we have X is dense in βX. [βX cpt. Haus. $\Rightarrow$ normal. So if X not dense, $\exists$ cts $f: \beta X \to [0,1]$, $f \neq 0$, s.t. $f|_X = 0$. But this is a contradiction.] Now every $f \in C_b(X)$ extends to βX.

Suppose now $f: X \to K$ cpt. Define $C_b(f): C(K) \to C_b(X)$ by $g \mapsto g \circ f$. Now $C_b(f)$ is cts since if $g_n \to g$ unif on K, then $g_n \circ f \to g \circ f$ unif on X.
Finally, we define $\beta f = C_b(f)^\wedge : \beta X \to K$, which we know is cts. For every $x \in X$ and $g \in C(K)$,

$$[\beta f(ev_x)](g) = [C_b(f)^\wedge(ev_x)](g) = [ev_x \circ C_b(f)](g)$$
$$= ev_x(g \circ f) = g(f(x)) = ev_{f(x)}(g),$$

and thus βf extends f.

Cor: $\forall$ cptification K of X, $\exists$ cts surj $\beta X \to K$ which restricts to 1 on X.