C^*-algebras

An *involution* on a $\mathbb{C}$-alg. is a fct $*: A \to A$; $a \mapsto a^*$ s.t.

① $(\lambda a+b)^* = \bar{\lambda} a^* + b^*$ $\forall \lambda \in \mathbb{C}$, $a,b \in A$

② $(ab)^* = b^* a^*$ $\forall a,b \in A$

③ $a^{**} = a$ $\forall a \in A$.

↳ $(A,*)$ called a *complex $*$-alg.*

Examples:

① $M_n(\mathbb{C})$ w/ adjoint / conjugate transpose.

② $\mathbb{C} \oplus \mathbb{C}$ w/ componentwise addition + multiplication, and $(a,b)^* := (\bar{b}, \bar{a})$.

③ $C(X)$ X cpt. Haus., $f^* = \bar{f}$

④ $C_0(X)$ X loc. cpt. Haus.

⑤ $L^1(\mathbb{R})$ w/ $f^*(x) := \overline{f(-x)}$

⑥ $L^\infty(\mathbb{R})$ w/ $f^* = \bar{f}$.

⑦ Disk alg $A(\mathbb{D})$ w/ $f^*(z) = \overline{f(\bar{z})}$.

⑧ A $*$-closed subalg of $B(H)$, H a Hilb. space (next week).

Note: If A unital, $\underline{1^* = 1}$: $1^* = 1^* 1 = (1^* 1)^{**} = (1^* 1)^* = 1^{**} = 1$

Def: A *C^*-alg* is a normed $\mathbb{C}$ $*$-alg $(A, *, \|\cdot\|)$ s.t.

① $(A, \|\cdot\|)$ is a Banach alg

② $\|a^* a\| = \|a\|^2$ $\forall a \in A$.

Exercise: Determine which of ① – ⑧ above are C^*-alg's.

Operators: $a \in A$ is called

① self-adjoint if $a = a^*$

② positive ($a \geq 0$) if $a = b^*b$ for some $b \in A$ [positive $\Rightarrow$ s.a.!]

③ normal if $a^*a = aa^*$

④ a projection if $a = a^* = a^2$

⑤ an isometry if $a^*a = 1_A$ (and A is unital)

⑥ a unitary if $a^*a = 1_A = aa^*$ (and A is unital).

Note: unitary $\iff$ invertible isometry
$\iff$ normal isometry

⑦ a partial isometry if a^*a is a projection.

We'll study these in great detail, especially when $A = B(H)$.

Elementary Properties: Let A be a C^*-alg.

① $*$ is isometric: Suppose $a \neq 0$. $\|a\|^2 = \|a^*a\| \leq \|a\| \cdot \|a^*\|$
$\|a^*\|^2 = \|aa^*\| \leq \|a\| \cdot \|a^*\|$

② $\|a\| = \|L_a\|$ for $L_a \in B(A)$.

Pf: $\|ab\| \leq \|a\| \cdot \|b\| \Longrightarrow \|L_a\| \leq \|a\|$. Now $\|aa^*\| = \|a\|^2 = \|a\| \cdot \|a^*\|$.

③ (Adjoining a unit) $A_1 = A \oplus \mathbb{C}1$ w/ usual addition + mult., w/ $(a+\lambda)^* := a^* + \overline{\lambda}$, and $\|a+\lambda\|_{A_1} = \sup\{\|ab + \lambda b\| \mid b \in A\} = \|L_a + \lambda 1\|_{B(A)}$ is a C^*-alg which contains A isometrically as a codim 1 ideal.

Pf: By ②, suffices to prove C^*-axiom: For $a \in A$, $\lambda \in \mathbb{C}$, $\varepsilon > 0$, let $b \in A$ s.t. $\|b\| = 1$ and

$$\|a+\lambda\|_{A_1}^2 - \varepsilon \leq \|ab + \lambda b\|_A^2 = \|(ab+\lambda b)^*(ab+\lambda b)\|_A = \|(b^*a^* + \overline{\lambda}b^*)(ab+\lambda b)\|_A$$
$$= \|L_{b^*}(a^*+\overline{\lambda})(a+\lambda)b\|_{A_1} \leq \|(a^*+\overline{\lambda})(a+\lambda)b\|_{A_1} \leq \|(a^*+\overline{\lambda})(a+\lambda)\|_{A_1}$$

So $\|a+\lambda\|_{A_1}^2 \overset{(*)}{\leq} \|(a^*+\overline{\lambda})(a+\lambda)\|_{A_1} \overset{(*)}{\leq} \|a^*+\overline{\lambda}\|_{A_1} \cdot \|a+\lambda\|_{A_1}$, so

$\|a+\lambda\|_{A_1} \leq \|a^*+\overline{\lambda}\|_{A_1}$ and by symmetry, $\|a+\lambda\|_{A_1} \geq \|a^*+\overline{\lambda}\|_{A_1}$.

Thus the above inequalities $(*)$ are equalities.

(4) If $a \in A$ normal, then $\|a\| = r(a)$.

Pf: $\|a^2\|^2 = \|(a^2)^*(a^2)\| = \|\underbrace{(a^*a)}_{\text{self-adjoint}}{}^2\| = \|a^*a\|^2 = \|a\|^4 \Rightarrow \|a^2\| = \|a\|^2$.

Iterating, $\forall n \in \mathbb{N}$, $\|a^{2^n}\| = \|a\|^{2^n} \Rightarrow r(a) = \lim_n \|a^{2^n}\|^{1/2^n} = \|a\|$.

(5) $\forall b \in A$, $\lambda \in sp_A(b) \iff \bar{\lambda} \in sp_A(b^*)$.

(6) If A unital and $a \in G(A)$, $(a^*)^{-1} = (a^{-1})^*$, and $\lambda \in sp(a) \iff \lambda^{-1} \in sp(a^{-1})$.

(7) If A unital and $u \in A$ unitary, then $sp(u) \subset \partial\mathbb{D}$.

Pf: $u^* = u^{-1} \Rightarrow \forall \lambda \in sp(u)$, $\bar{\lambda}^{-1} \in sp(u)$. Since $\|u\| = 1$, $|\lambda|, |\bar{\lambda}^{-1}| \le 1 \Rightarrow \lambda \in \partial\mathbb{D}$.

(8) If $a = a^*$, then e^{ia} is unitary (defined using HFC, pass to unitization A_1 if necessary)

Pf: $(e^{ia})^* = \left(\sum_{n\geq 0} \frac{(ia)^n}{n!}\right)^* = \sum \frac{(-ia)^n}{n!} = e^{-ia}$. By HFC, $\overbrace{e^{ia}e^{-ia}}^{\text{commute.}} = 1$.

(9) If $a = a^*$, $sp_A(a) \subset \mathbb{R}$. (Note A need not be unital!)

Pf: by (7), $\underbrace{sp(e^{ia})}_{\text{pass to } A_1 \text{ if necessary.}} \subset \partial\mathbb{D}$. Now $sp(e^{ia}) = e^{i\,sp(a)}$ $\Rightarrow$ $\underbrace{sp(a) \subset \mathbb{R}}_{\text{holds in } A_1 \text{ and! } \Rightarrow \text{ holds in } sp_{A_1}(a).}$.

Spectral mapping Thm

(10) Every $a \in A$ can be written as $a = Re(a) + i\,Im(a)$ where $Re(a) = \frac{a+a^*}{2}$ and $Im(a) = \frac{a-a^*}{2i}$ both self adjoint.

(11) If A comm., every $\varphi \in \hat{A}$ is a $*$-hom.

Pf: $\varphi(a^*) = \varphi(Re(a) - i\,Im(a)) = \underbrace{\varphi(Re(a))}_{\in sp(Re(a))} - i\,\underbrace{\varphi(Im(a))}_{\in sp(Im(a))} = \overline{\varphi(Re(a)) + i\varphi(Im(a))} = \overline{\varphi(a)}$ ← both $\in \mathbb{R}$.

__Gelfand-Naimark Thm 1:__ If A is a unital comm. C^*-alg, then A is isometrically $*$-isomorphic via Γ to $C(\hat{A})$.

Pf: __Γ is an isometry__: Every elt in A is normal, so

$$\|\Gamma(a)\|_{C(\hat{A})} = \sup\{ev_a(\varphi) \mid \varphi \in \hat{A}\} = \sup\{\varphi(a) \mid \varphi \in \hat{A}\}$$
$$= \sup\{\lambda \mid \lambda \in sp_A(a)\} = r(a) = \|a\|_A.$$

__Γ is $*$-preserving__: Every $\varphi \in \hat{A}$ is $*$-preserving, so if $a \in A$,

$$\Gamma(a^*)(\varphi) = ev_{a^*}(\varphi) = \varphi(a^*) = \overline{\varphi(a)} = \overline{ev_a(\varphi)} = \overline{\Gamma(a)}(\varphi).$$

__Γ is onto__: $\Gamma(A)$ is a s.a. Banach subalg of $C(\hat{A})$ which contains 1 and separates pts of $\hat{A}$. Stone-Weierstrass $\Rightarrow \Gamma(A) = C(\hat{A})$.

Cor: A non-unital comm. C^*-alg A is isometrically $*$-isomorphic via Γ to $C_0(\hat{A})$.

Pf: Let A_1 be the unitization, and recall $\hat{A}_1 = \hat{A} \cup \{\infty\}$ where $\infty = \varphi_0$ s.t. $\varphi_0|_A = 0$. Then $\Gamma : A \to C(\hat{A}_1)$ is isometric onto a subalg of $C_0(\hat{A})$ which separates pts and does not vanish at any $\varphi \in \hat{A}$. By Stone-Weierstrass, Γ is onto.

Lemma (Spectral Permanence): Suppose $1_A \in B \subseteq A$ is a unital inclusion of C^*-alg's and $b \in B$. Then $sp_B(b) = sp_A(b)$.

Pf: It's enough to prove $b \in G(A) \Rightarrow b \in G(B)$.
Suppose $b \in G(A)$. Then $b^* \in G(A)$, and $(b^*)^{-1} = (b^{-1})^*$.
Hence $b^*b \in G(A)$. Since b^*b is sa., $sp_A(b^*b) \subseteq \mathbb{R}$.
Since $\partial sp_A(b^*b) = \partial sp_B(b^*b)$, we have $sp_B(b^*b) = sp_A(b^*b)$.
Hence $b^*b \in G(B)$, so b has a left inverse in B, namely $(b^*b)^{-1}b^*$. A similar argument applied to bb^* shows that b is right invertible in B.

Continuous Functional Calculus

Suppose A is a unital C^*-alg and $a \in A$ is normal. Let $C^*(a)$ be the unital C^*-subalg of A generated by a, i.e., the smallest unital C^*-subalg containing a.
Then the inverse of the Gelfand Transform $C^*(a) \cong C(sp_A(a))$ is an isometric $*$-iso $\Phi : C(sp_A(a)) \to C^*(a) \subseteq A$ s.t.

1. $\Phi(1) = 1$; $\Phi(\overset{id}{z \mapsto z}) = a$
2. $\forall f \in \mathcal{O}(sp(a))$, $\Phi(f) = f(a)$ from the HFC

Step 1: There is a homeom. $sp_A(a) \cong C^*(a)^\wedge$.

Pf: By spectral permanence, $sp_A(a) = sp_{C^*(a)}(a)$. Note that $C^*(a)^\wedge \ni \varphi \longmapsto \varphi(a) \in sp_A(a)$ is surjective + cts. since $C^*(a)^\wedge$ has the relative wk-* top. Now we claim this map is injective. Suppose $\varphi(a) = \psi(a)$ for $\varphi, \psi \in C^*(A)^\wedge$. Then since φ, ψ are *-homs, $\varphi(a^*) = \overline{\varphi(a)} = \overline{\psi(a)} = \psi(a^*)$. Hence $\varphi = \psi$ on all poly's in a and a^*, so $\varphi = \psi$.

Step 2: Since $C^*(a) \xrightarrow[\sim]{\Gamma} C(C^*(a)^\wedge) \xrightarrow{\hat{\ }} C(sp_A(a))$ is an isometric *-iso, we define for $f \in C(sp_A(a))$

$\Phi(f) = (\Gamma^{-1} \circ \Lambda^{-1})(f)$. Now we verify $\forall \varphi \in \hat{A}$,

a) $\Gamma(1)(\varphi) = \varphi(1) = 1 = \Lambda^{-1}(1)(\varphi) \Rightarrow \Phi(1) = 1$

b) $\Gamma(a)(\varphi) = \varphi(a) = id(\varphi(a)) = \Lambda^{-1}(id)(\varphi) \Rightarrow \Phi(id) = a$

Step 3: To verify the CFC extends the HFC, by the uniqueness property of the HFC, it suffices to show that if $(f_n), f$ are holo. on $\mathcal{U} \supseteq sp_A(a)$ s.t. $f_n \to f$ loc. unif., then $\Phi(f_n) \to \Phi(f)$ in A. Since $sp_A(a)$ cpt, $f_n \to f$ unif. in $C(sp_A(a)) \cong C^*(a)$, and we are finished.

Notation: For A a unital C^*-alg, $a \in A$ normal, we denote by the CFC $C(sp_A(a)) \xrightarrow{\sim} C^*(a) \subseteq A$

$$f \longmapsto f(a)$$

Uniqueness of CFC: If $\Phi : C(sp_A(a)) \longrightarrow A$ is a (cts) unital *-homom. s.t. $\Phi(1) = 1$ and $\Phi(id) = a$, then

$$\Phi(f) = f(a) \quad \forall f \in C(sp_A(a)).$$

Note: If A non-unital and $a \in A$ normal, can define CFC $\underbrace{C_0(sp_A(a))}_{\text{vanishes at } \varphi_0 \in sp_{A_1}(a).} \longrightarrow C^*(a)$, the nonunital C^*-subalg of A gen by a.

*-homomorphisms of C^*-alg's:

Lemma: If A, B are unital C^*-alg's and $\phi: A \to B$ is a unital *-hom, then ϕ is norm decreasing. If ϕ is injective, then ϕ is an isometry.

Pf: First, note if $a \in G(A)$, then $\phi(a) \in G(B)$. Hence $sp_B(\phi(a)) \subseteq sp_A(a)$. Thus $r_B(\phi(a)) \leq r_A(a)$. Then $\forall a \in A$,

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \|\phi(a^*a)\| = r_B(\phi(a^*a))$$
$$\leq r_A(a^*a) = \|a^*a\| = \|a\|^2.$$

Now suppose ϕ is injective.

Claim: $sp_A(a) = sp_B(\phi(a))$ $\forall$ normal $a \in A$.

Pf: Suppose $\exists \lambda \in sp_A(a) \setminus sp_B(\phi(a))$. Then since $sp_A(a)$ is normal, $\exists$ cts $f: sp_A(a) \to [0,1]$ s.t. $f \neq 0$ & $f|_{sp_B(\phi(a))} = 0$.
Then $f(a) \neq 0$, but $\underbrace{\phi(f(a)) = f(\phi(a))}_{\text{holds } \forall \text{ poly's in } a, a^*, \text{ dense in } C(sp_A(a)) \text{ by Stone-Weierstrass!}} = 0$, $\Rightarrow \Leftarrow$.

Now, $\forall a \in A$, $\|\phi(a)\|^2 = r_B(\phi(a^*a)) = r_A(a^*a) = \|a\|^2$.

Thm: $\exists$ a contravariant eq. of cats:

$$\left\{\begin{array}{l}\text{cpt. Hausd. top. spaces} \\ \text{w/ cts maps}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{l}\text{unital comm. } C^*\text{-alg's} \\ \text{w/ unital *-hom's.}\end{array}\right\}$$

Given X cpt Hausd. $\rightsquigarrow C(X)$.
Given A unital comm. C^*-alg, $\rightsquigarrow \hat{A}$
] must extend these to functors.

We'll then need to construct natural iso's from composites each way to the appropriate id functors.

Step ①: Suppose X, Y cpt Hausdf, and $f : X \to Y$ cts. Define $C(f) : C(Y) \to C(X)$ by $g \mapsto g \circ f$. Then $C(f)$ is a unital $*$-hom. Indeed, $1 \circ f = 1$, $\bar{g} \circ f = \overline{g \circ f}$, and
$(\lambda g_1 + g_2 g_3) \circ f = \lambda (g_1 \circ f) + (g_2 \circ f)(g_3 \circ f)$.

$\mathrm{Hom}_{\mathrm{CptHausTop}}(-, \mathbb{C})$

Step ②: (a) $C(\mathrm{id}_X : X \to X) = \mathrm{id}_{C(X)}$.

(b) If $X \xrightarrow{f} Y \xrightarrow{g} Z$, $C(g \circ f) = C(f) \circ C(g)$,
since if $h \in C(Z)$, $h \circ (g \circ f) = C(f)(h \circ g) = C(f)(C(g)(h))$

Step ③: If $\phi : A \to B$ is a unital $*$-hom, then $\hat{\phi} : \hat{B} \to \hat{A}$ by $\psi \mapsto \psi \circ \phi$ is cts.

Indeed, if $\psi_\lambda \to \psi$ wk$*$ in $\hat{B}$, then $\forall a$, $\psi_\lambda(\phi(a)) \to \psi(\phi(a))$, so $\psi_\lambda \circ \phi \to \psi \circ \phi$ wk$*$ in $\hat{A}$, and thus $\hat{\phi}$ is cts.

$\mathrm{Hom}_{\mathrm{UnitCommC^*alg}}(-, \mathbb{C})$

Step ④: (a) $(\mathrm{id}_A)^\wedge = \mathrm{id}_{\hat{A}}$

(b) If $A \xrightarrow{\phi} B \xrightarrow{\psi} C$, $(\psi \circ \phi)^\wedge = \phi^\wedge \circ \psi^\wedge$.

Step ⑤: We've already seen that $X \cong C(X)^\wedge$. This iso. is natural:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ C(X)^\wedge \arrow[r, "C(f)^\wedge"'] & C(Y)^\wedge \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & f(x) \arrow[d, mapsto] \\ ev_x \arrow[r, mapsto] & ev_{f(x)} \end{tikzcd}$$

$C(f)^\wedge(ev_x)(g)$
$= [ev_x \circ C(f)](g)$
$= ev_x(g \circ f)$
$= g(f(x)) = ev_{f(x)}(g)$ ✓

Step ⑥: We've already seen $A \cong C(\hat{A})$. This iso is natural:

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ C(\hat{A}) \arrow[r, "C(\hat{\phi})"'] & C(\hat{B}) \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & \phi(a) \arrow[d, mapsto] \\ ev_a \arrow[r, mapsto] & ev_{\phi(a)} \end{tikzcd}$$

$C(\hat{\phi})(ev_a)(\psi)$
$= (ev_a \circ \hat{\phi})(\psi)$
$= ev_a(\psi \circ \phi)$
$= \psi(\phi(a)) = ev_{\phi(a)}(\psi)$ ✓

Exercise: What happens for non-unital comm. C^*-alg's?

Application: Stone–Čech compactification: To each Tychonoff top. space X (T_1 and completely regular, i.e., if $F \subseteq X$ closed, $x \notin F$, $\exists$ cts $f: X \to [0,1]$ s.t. $f(x)=0$, $f|_F = 1$), $\exists$ a cpt Hausd. space $\beta X \supseteq X$ s.t.

① $\overline{X} = \beta X$ (βX is a compactification of X)

② $\forall$ cpt K and cts $f: X \to K$, can extend f to a map $\beta f : \beta X \to K$, i.e.

$$\begin{tikzcd} \beta X \arrow[dr, "\exists \beta f"] & \\ X \arrow[u, phantom, "\cup"] \arrow[r, "f"'] & K \end{tikzcd}$$

Pf: Define $\beta X = C_b(X)^\wedge$ where $C_b(X)$ is the unital comm. C^*-alg of cts bdd fcts on X w/ $\|\cdot\|_\infty$. Since $ev_x \in C_b(X)^\wedge$ $\forall x \in X$, get natural inclusion $X \subseteq \beta X$. Since X is Tychonoff and top. on βX is induced by $\Gamma(C_b(X))$, the inclusion $X \subseteq \beta X$ is cts wrt original topology on X. Since $C(\beta X) \cong C_b(X)$, we have X is dense in βX. [βX cpt. Hausd. $\Rightarrow$ normal. So if X not dense, $\exists$ cts $f: \beta X \to [0,1]$, $f \neq 0$, s.t. $f|_X = 0$. But this is a contradiction.] Now every $f \in C_b(X)$ extends to βX.

Suppose now $f: X \to K$ cpt. Define $C_b(f): C(K) \to C_b(X)$ by $g \mapsto g \circ f$. Now $C_b(f)$ is cts since if $g_n \to g$ unif on K, then $g_n \circ f \to g \circ f$ unif on X.
Finally, we define $\beta f = C_b(f)^\wedge : \beta X \to K$, which we know is cts. For every $x \in X$ and $g \in C(K)$,

$$[\beta f(ev_x)](g) = [C_b(f)^\wedge(ev_x)](g) = [ev_x \circ C_b(f)](g)$$
$$= ev_x(g \circ f) = g(f(x)) = ev_{f(x)}(g),$$

and thus βf extends f.

Cor: $\forall$ cptification K of X, $\exists$ cts surj $\beta X \to K$ which restricts to 1 on X.

Positive Cone: $\{a\in A \mid a\geq 0\}$ is a closed cone.

Thm: Let $J\subseteq A$ be a left ideal, not nec. closed. $\exists$ a right approx id for J consisting of elts of A_+. If J is separable, can choose approx id to be a seq.

Pf: WLOG, A is unital. Let $S\subseteq J$ be a dense subset. (Can take $S=J$ or S a countable dense subset if J sep.) Let Λ be the collection of finite subsets of S, ordered by inclusion. For $\lambda=\{a_1,\dots,a_n\}\in\Lambda$, define $b_\lambda=\sum_{i=1}^n a_i^*a_i$.

Note that $b_\lambda\in J\cap A_+$ [J L-ideal, A_+ closed cone.]

Define $e_\lambda=(\frac{1}{n}+b_\lambda)^{-1}b_\lambda$ by the CFC. Then $e_\lambda\in J\cap A_+$, and $\|e_\lambda\|\leq 1$.

Now, $\forall a\in S$, $\forall \lambda\in\Lambda$ w/ $a\in\lambda$, we have $\underbrace{a^*a\leq b_\lambda}_{b_\lambda-a^*a\geq 0}$.

$$(a-ae_\lambda)^*(a-ae_\lambda)=(1-e_\lambda)a^*a(1-e_\lambda)\leq(1-e_\lambda)b_\lambda(1-e_\lambda)$$
$$=f(b_\lambda)\quad\text{where}$$
$$f(t)=(1-(\tfrac{1}{n}+t)^{-1}t)^2t=(\tfrac{1}{n}+t)^{-2}\left[\tfrac{1}{n}+t-t\right]^2t$$
$$=\frac{1}{n}\,\frac{\frac{1}{n}}{\frac{1}{n}+t}\,\frac{t}{\frac{1}{n}+t}\leq\frac{1}{n}$$

Hence $\|a-ae_\lambda\|^2\leq\frac{1}{n}\to 0$ as $|\lambda|\to\infty$.

Rem: w/ more effort, can arrange $\lambda\leq\lambda'\Rightarrow e_\lambda\leq e_{\lambda'}$.

Cor: If $J\subseteq A$ a closed 2-sided ideal, $J=J^*$, so J a C^*-alg.

Pf: $\forall a\in J$, $\|e_\lambda a^*-a^*\|=\|ae_\lambda-a\|\to 0$, so $a^*\in J$.

Cor: A right approx id for a 2-sided closed ideal $J\subseteq A$ is also a left approx. id. for J.

Prop: If $J\subseteq A$ closed 2sided ideal and $I\subseteq J$ closed 2sided ideal, then $I\subseteq A$ is a closed 2sided ideal.

Pf: Let (e_λ) be a 2-sided approx id for J. Then $\forall a \in A$, $\forall x \in I$, $ax = \lim_\lambda a(e_\lambda x) = \lim_\lambda \underbrace{\underbrace{(ae_\lambda)}_{\in J} x}_{\in I}$.

Cor: If $I, J \subseteq A$ closed 2-sided ideals, $I \cap J = \underbrace{IJ}_{\text{closed linear span}}$.

Pf: $\supseteq$: obvious

$\subseteq$: Suppose $x \in I \cap J$. Let (e_λ) be an approx id for I. Then $x = \lim_\lambda e_\lambda x \in IJ$.

Cor: $J^2 = J$ $\forall$ closed 2-sided ideal $J \subseteq A$.

Quotients: We saw if A a Banach alg and $J \subseteq A$ a closed 2-sided ideal, then A/J is a Banach alg.

Lemma: Let (e_λ) be an approx unit for J. Then $\forall a \in A$,

$$\|a+J\|_{A/J} = \lim_\lambda \|a - a\underbrace{e_\lambda}_{\in J}\|.$$

Pf: WLOG, can assume A is unital. Now by CFC (on HW!), $\|1 - e_\lambda\| \leq 1$. It's obvious $\|a+J\| \leq \|a - ae_\lambda\|$ $\forall \lambda$. Let $\varepsilon > 0$. Let $j \in J$ s.t. $\|a - j\| \leq \|a+J\|_{A/J} + \varepsilon$. Then $\forall \lambda$,

$$\|a - ae_\lambda\| = \|a(1-e_\lambda)\| \leq \|(a-j)(1-e_\lambda)\| + \|j(1-e_\lambda)\|$$
$$\leq \|a-j\| \cdot \|1-e_\lambda\| + \|j(1-e_\lambda)\| \leq \|a-j\| + \|j - je_\lambda\|$$
$$\leq \|a-j\| + \varepsilon \quad (\text{large enough } \lambda) \leq \|a+J\| + 2\varepsilon.$$

Since ε arbitrary, the conclusion follows.

Prop (Segal): Suppose A a C^*-alg and $J \subseteq A$ a closed 2-sided ideal. Then A/J is a C^*-alg w/

$$(a+J)^* := a^* + J \quad \text{and} \quad \|a+J\| = \inf_{j \in J} \|a - j\|.$$

Pf: Suffices to prove $\|\overbrace{(a^*+J)(a+J)}^{a^*a+J}\| \geq \|a+J\|^2$. By the lemma,

$$\|a+J\|^2 = \lim_\lambda \|a - ae_\lambda\|^2 = \lim_\lambda \|(a-ae_\lambda)^*(a-ae_\lambda)\|$$
$$= \lim_\lambda \|(1-e_\lambda)a^*a(1-e_\lambda)\|$$

used $\|1-e_\lambda\| \leq 1$. $\rightarrow \leq \overline{\lim_\lambda} \|a^*a(1-e_\lambda)\| = \|a^*a + J\|$

Representing C^*-alg's on Hilbert Space

Basics on Hilbert Spaces:

Let H be a $\mathbb{C}$ v-sp. A sesquilinear form on H is a map $\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathbb{C}$ s.t.

$$\left.\begin{array}{l} (1)\ \langle \alpha\eta + \xi, \zeta \rangle = \alpha \langle \eta, \zeta \rangle + \langle \xi, \zeta \rangle \\ (2)\ \langle \eta, \alpha\xi + \zeta \rangle = \bar{\alpha} \langle \eta, \xi \rangle + \langle \eta, \zeta \rangle \end{array}\right\} \forall \alpha \in \mathbb{C},\ \eta, \xi, \zeta \in H.$$

A sesquilinear form is called

- self-adjoint if $\overline{\langle \eta, \xi \rangle} = \langle \xi, \eta \rangle$ $\forall \eta, \xi \in H$
- non-degenerate if $\langle \eta, \xi \rangle = 0$ $\forall \xi \Rightarrow \eta = 0$
- positive if $\langle \eta, \eta \rangle \geq 0$ $\forall \eta \in H$.
  ↳ A positive sesq. form is called definite if $\langle \eta, \eta \rangle = 0 \Rightarrow \eta = 0$.

An inner product is a positive definite sesq. form.

Exercises:

(1) (Polarization) $4\langle \eta, \xi \rangle = \sum_{k=0}^{3} i^k \langle \eta + i^k \xi, \eta + i^k \xi \rangle$.

(2) $\langle \cdot, \cdot \rangle$ s.a. $\iff \langle \eta, \eta \rangle \in \mathbb{R}$ $\forall \eta$.

(3) positive $\Rightarrow$ s.a.

(4) positive $\Rightarrow$ [definite $\iff$ non-degenerate].

Let $\langle \cdot, \cdot \rangle$ be a positive sesq. form. Define $\|\cdot\| : H \to [0, \infty)$ by $\|\xi\|^2 = \langle \xi, \xi \rangle$. Then $\forall \alpha \in \mathbb{C}$,

Cauchy-Schwarz: $|\alpha|^2 \|\xi\|^2 + 2\operatorname{Re} \alpha \langle \xi, \eta \rangle + \|\eta\|^2 = \|\alpha\xi + \eta\|^2 \geq 0$
$\Rightarrow |\langle \xi, \eta \rangle| \leq \|\xi\| \cdot \|\eta\|$.

↳ $\|\cdot\|$ is a seminorm on H.

Exercise: ① $\langle \cdot , \cdot \rangle$ an inner prod $\iff \|\cdot\|$ a norm on H.

② In this case, $|\langle x, y\rangle| = \|x\| \cdot \|y\| \iff x, y$ proportional.

③ $\|\cdot\|$ satisfies parallelogram id:

$$\|\eta + \xi\|^2 + \|\eta - \xi\|^2 = 2\left(\|\eta\|^2 + \|\xi\|^2\right).$$

④ Conversely if a norm on H satisfies the parallel. axiom, defining $\langle \cdot , \cdot \rangle$ as in polarization gives a well-defined inner prod.

Def: A Hilbert space is a $\mathbb{C}$ $\vec{v}$.sp. w/ an inner prod. $\langle \cdot , \cdot \rangle : H \times H \longrightarrow \mathbb{C}$ s.t. $(H, \|\cdot\|)$ Banach w/ $\|\xi\|^2 = \langle \xi, \xi \rangle$.

Exercise: A Hilb. space is uniformly cnvx.

Cor: reflexive, strictly cnvx, every closed cnvx set has an elt of min norm.

Recall if $K \subseteq H$, then $K^\perp = \{\xi \in H \mid \langle \xi, k\rangle = 0 \ \forall k \in K\}$.

Exercise: ① If K a subspace, $\overline{K} = K^{\perp\perp}$.

② If K a closed subspace, $H = K \oplus K^\perp$.

Thm (Riesz-Rep.): The map $\xi \longmapsto \langle \cdot , \xi \rangle$ is a conj. linear isometric iso $H \longrightarrow H^*$.

Pf: $\langle \cdot , \xi \rangle : H \longrightarrow \mathbb{C}$ is in H^* and $\|\langle \cdot , \xi\rangle\| \leq \|\xi\|$ by CS. Since $\langle \xi, \xi \rangle = \|\xi\|^2$, the map is isometric.

Conversely, if $\varphi \in H^* \setminus (0)$, $K = \ker \varphi \subseteq H$ is proper closed subspace. Let $\xi \in K^\perp$ s.t. $\varphi(\xi) = 1$. $\forall \eta \in H$, $\eta - \varphi(\eta)\xi \in K$, and thus

$$\langle \eta, \xi \rangle = \langle \eta - \varphi(\eta)\xi + \varphi(\eta)\xi, \xi \rangle = \varphi(\eta)\|\xi\|^2.$$

Hence $\varphi(\eta) = \frac{1}{\|\xi\|^2}\langle \eta, \xi \rangle \quad \forall \eta \in H.$

A subset $B \subset H \setminus \{0\}$ is called:

- <u>orthogonal</u> if $\eta, \xi \in B$ distinct $\Rightarrow \langle \eta, \xi \rangle = 0$
- <u>orthonormal</u> if B orthogonal and $\|\eta\| = 1 \; \forall \eta \in B$.
- an <u>orthonormal basis</u> if B orthonormal and $\overline{\operatorname{span}(B)} = H$.

<u>Lemma</u>: Every H has an ONB. $\forall \xi \in H$, $\xi = \sum \langle \xi, e_i \rangle e_i$,
(parseval identity) $\|\xi\|^2 = \sum |\langle \xi, e_i \rangle|^2$.

<u>Prop</u>: $\exists$ bijective, isometric correspondence btwn
① bdd sesquilinear forms $B: H \times H \to \mathbb{C}$ w/
$$\|B\| = \sup\{ |B(\eta, \xi)| \mid \|\eta\|, \|\xi\| \le 1 \}$$
② bdd ops $H \to H$.

This bijective correspondence maps s.a./pos. forms to s.a./pos ops.

<u>Pf</u>: HW!

<u>Def</u>: Suppose $T \in B(H)$. For $\eta \in H$, the map $\xi \mapsto \langle T\xi, \eta \rangle$ is in H^*, so $\exists! \, T^*\eta \in H$ s.t. $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \; \forall \xi$. The assignment $\eta \mapsto T^*\eta$ is <u>linear</u> + <u>bdd</u>. We call $T^* \in B(H)$ the (Hilbert) <u>adjoint</u> of T.

<u>Exercise</u>: For a Hilbert space H, can define the <u>conjugate</u> Hilb. space $\overline{H}$ by $\overline{H} = \{ \overline{\xi} \mid \xi \in H \}$ w/ $\lambda \overline{\xi} + \overline{\eta} = \overline{\overline{\lambda}\eta + \xi}$.
For $T \in B(H, K)$, get $\overline{T} \in B(\overline{H}, \overline{K})$ by $\overline{T}\,\overline{\xi} = \overline{T\xi}$.
Then since $H^* \cong \overline{H}$ via Riesz Rep'n Thm, the Banach adjoint T^t and the Hilb adjoint T^* are related by $T^* = \overline{T^t}$.

<u>Basic Properties of $B(H)$</u>:

① $\ker T^* = (TH)^\perp$: $\langle T\eta, \xi \rangle = \langle \eta, T^*\xi \rangle$, so $\xi \perp TH \Leftrightarrow T^*\xi \perp H$
$\Leftrightarrow T^*\xi = 0$.

② $S = T \Leftrightarrow \langle S\xi, \xi \rangle = \langle T\xi, \xi \rangle \quad \forall \xi \in H$.

Pf: $\Rightarrow$: trivial. $\Leftarrow$: $\underbrace{\text{Polarization}}_{\text{for } \langle S\cdot,\cdot\rangle, \langle T\cdot,\cdot\rangle!} \Rightarrow \langle S\eta, \xi \rangle = \langle T\eta, \xi \rangle \; \forall \eta, \xi \in H$
$\Rightarrow (S - T)\eta \perp H \quad \forall \eta \in H \Rightarrow S = T$.

③ T normal $\iff \|T\xi\| = \|T^*\xi\| \quad \forall \xi \in H$.

Pf: $\|T\xi\|^2 = \langle T^*T\xi, \xi\rangle = \langle TT^*\xi, \xi\rangle = \|T^*\xi\|^2 \iff T^*T = TT^*$.

④ $T = T^* \iff \langle T\xi, \xi\rangle \in \mathbb{R} \quad \forall \xi \in H$. HW!

Now suppose $T \in B(H)$ normal.

⑤ $\exists \xi \in H$ s.t. $T\xi = \lambda\xi \iff T^*\xi = \bar{\lambda}\xi$.

Pf: Immediate from ③.

⑥ If $T\xi_1 = \lambda_1\xi_1$ and $T\xi_2 = \lambda_2\xi_2$ w/ $\lambda_1 \neq \lambda_2$, then $\xi_1 \perp \xi_2$.

Pf: $\lambda_1\langle \xi_1, \xi_2\rangle = \langle T\xi_1, \xi_2\rangle = \langle \xi_1, T^*\xi_2\rangle = \lambda_2\langle \xi_1, \xi_2\rangle \Rightarrow$ result.

⑦ Every $\lambda \in sp(T)$ is an approx eigenvalue of T.

Pf: Suppose not. Then $\exists \varepsilon > 0$ s.t. $\|(T-\lambda)\xi\| \geq \varepsilon\|\xi\| \quad \forall \xi \in H$.
Then $T-\lambda$ is injective w/ closed range, and so is $T^* - \bar{\lambda}$ by ③.
But $\ker(T^* - \bar{\lambda}) = [(T-\lambda)H]^\perp = 0 \Rightarrow T-\lambda$ is surj., so
$T-\lambda$ is invertible and $\lambda \notin sp(T)$.

⑧ $\|T\| = \sup\{|\langle T\xi, \xi\rangle| \mid \|\xi\| = 1\}$.

Pf: Since $\|T\| = r(T)$, $\exists \lambda \in sp(T)$ s.t. $|\lambda| = \|T\|$. Then since λ is an approx eigenvalue, $\exists \xi_n \to \xi$ s.t. $\|\xi_n\| = 1 \ \forall n$ and
$T\xi_n - \lambda\xi_n \to 0$. Thus

$$|\langle T\xi_n, \xi_n\rangle - \lambda| = |\langle T\xi_n, \xi_n\rangle - \lambda\overbrace{\langle \xi_n, \xi_n\rangle}^{1}|$$
$$= |\langle T\xi_n - \lambda\xi_n, \xi_n\rangle|$$
$$\leq \|T\xi_n - \lambda\xi_n\| \cdot \underbrace{\|\xi_n\|}_{1} \to 0 \text{ as } n \to \infty.$$

⑨ If $T = T^*$, $\sup\{\langle T\xi, \xi\rangle \mid \|\xi\| = 1\} = \max\{\lambda \mid \lambda \in sp(T)\}$ and
$\inf\{\langle T\xi, \xi\rangle \mid \|\xi\| = 1\} = \min\{\lambda \mid \lambda \in sp(T)\}$.

Pf: Let $M = \max\{\lambda \mid \lambda \in sp(T)\}$. By ⑦,

$$\underbrace{M + \|T\|}_{=\max\{\lambda \mid \lambda \in sp(T+\|T\|)\} \text{ by Spectral mapping.}} = \sup\{\underbrace{\langle (T+\|T\|)\xi, \xi\rangle}_{\geq 0} \mid \|\xi\| = 1\}$$
$$= \sup\{\langle T\xi, \xi\rangle \mid \|\xi\| = 1\} + \|T\|.$$

Similar argument for min/inf subtracting $\|T\|$.

⑩ $T \geq 0 \iff \langle T\xi, \xi\rangle \geq 0 \quad \forall \xi \in H$. HW!

Basic spectral theory for normal ops in $B(\mathcal{H})$:

Recall that if $T \in B(\mathcal{H})$ is normal, the cts fnl calculus gives us an isometric $*$-iso $C(sp(T)) \xrightarrow{\sim} C^*(T) \subseteq B(\mathcal{H})$.

$$T \longmapsto f(T)$$

It is uniquely determined by $(z \mapsto 1) \longmapsto 1$ and $(z \mapsto t) \mapsto T$, since poly's in T and T^* are dense in $C^*(T)$, and any unital $*$-homom btwn C^*-alg's is cts.

Fuglede's Thm: If $T \in B(H)$ normal and $S \in B(H)$ s.t. $ST = TS$, then $ST^* = T^*S$.

Pf: (Due to Rosenblum) For $z \in \mathbb{C}$, since $ST = TS$, $Se^{i\bar{z}T} = e^{i\bar{z}T}S$, so $S = e^{i\bar{z}T} S e^{-i\bar{z}T}$. Define $f: \mathbb{C} \to B(H)$ by

$$f(z) = e^{izT^*} S e^{-izT^*} = e^{izT^*} e^{i\bar{z}T} S e^{-i\bar{z}T} e^{-izT^*} = e^{i(zT^* + \bar{z}T)} S e^{-i(zT^* + \bar{z}T)}.$$

Now $zT^* + \bar{z}T$ is s.a., so $e^{i(zT^* + \bar{z}T)}$ is unitary. Hence f is a bdd $B(H)$ valued holo. fct, and thus constant.

[! weakly holo $\iff$ strongly holo., so Liouville applies.!]

Now taking $\underbrace{\frac{d}{dz}\Big|_{z=0} f(z) = T^*S - ST^*}_{\text{power series expansion, first order.}} = 0 \Rightarrow$ result.

Cor: If $T \in B(H)$ normal and $ST = TS$, then $Sf(T) = f(T)S$ for all $f \in C(sp(T))$.

Pf: By Fuglede's thm, the result holds $\forall$ poly's in $z, \bar{z}$, and thus for all $f \in C(sp(T))$ by Stone-Weierstrass.

Cor: If $S, T \in B(H)$ are positive and $S^2 = T$, then $S = \sqrt{T}$.

Pf: S and T commute, so S and $\sqrt{T}$ commute. Now $\forall \lambda > 0$,

$(\sqrt{T} - S + \lambda)(\sqrt{T} + S + \lambda) = \sqrt{T}^2 - S^2 + 2\lambda\sqrt{T} + \lambda^2 = \lambda(2\sqrt{T} + \lambda)$,

and $2\sqrt{T} + \lambda$ and $\sqrt{T} + S + \lambda$ are invertible [$a + \lambda$ inv. $\forall a \geq 0, \lambda > 0$]

Thus $\sqrt{T} - S + \lambda$ invertible $\forall \lambda > 0 \Rightarrow sp(\sqrt{T} - S) \subset [0, \infty)$ so $\sqrt{T} - S \geq 0$.

Similarly, $S - \sqrt{T} \geq 0$, so $S = \sqrt{T}$.

Remark: All results on this page apply for a unital C^*-alg, not just $B(H)$!

Projections: $p\in B(H)$ is a projection if $p=p^2=p^*$.

Lemma: There is a bijective correspondence btwn closed subspaces and projections $p\in B(H)$ by $p\mapsto pH$.

Pf: It's clear $pH\subseteq H$ is a closed subspace since p is cts and $p^2=p$. Moreover, if $p=p^*$, then $pH^\perp=\ker p^*=\ker p=(1-p)H$. Conversely, every $K\subseteq H$ has an $\perp$ complement $K^\perp$ s.t. $H=K\oplus K^\perp$, and proj onto K is an idempotent. We claim it is self-adjoint, since $\ker p^*=pH^\perp=K^\perp=\ker p \Rightarrow p^*(1-p)=0 \Rightarrow p^*=p^*p$ $\Rightarrow p=p^*$ as p^*p is self-adjoint.
It's clear these two constructions were mutually inverse.

Def: We say p,q are mutually $\perp$, denoted $p\perp q$, if $pH\perp qH$.

Exercise: Show $p\perp q \iff pq=0$.

Def: For $p,q\in B(H)$ projections, define $p\wedge q$ to be the projection onto $pH\cap qH$ and $p\vee q$ to be the projection onto $\overline{pH+qH}$.

Exercise: prove $p\vee q = 1-(1-p)\wedge(1-q)$.

Exercise: ① $K\subseteq H$ is T-invariant $\iff P_KTP_K=TP_K$
② $K\subseteq H$ T-invariant $\iff K^\perp$ is T^*-invariant
③ $K\subseteq H$ T and T^*-invariant $\iff TP_K=P_KT$.

Partial Isometries: $u\in B(H)$ is a partial isometry if u^*u is a proj.

Exercise: TFAE: ① u a p.i. ② $u=uu^*u$ ③ u^* a p.i. ④ $u^*=u^*uu^*$ ∃HW!

Polar Decomposition: For each $T\in B(H,K)$, $\exists!$ op $|T|\in B(H)$ w/ $|T|\geq 0$ s.t. $\|T\eta\|=\||T|\eta\|$ $\forall \eta\in H$, and $|T|^2=T^*T$. Moreover, $\exists!$ p.i. $u\in B(H,K)$ s.t. $\ker u=\ker T$ and $u|T|=T$. In particular, $u^*T=|T|$.

Pf: If $S\geq 0$ s.t. $\|T\eta\|=\|S\eta\|$ $\forall\eta\in H$, then $\|T\eta\|^2=\langle T^*T\eta,\eta\rangle=\langle S^2\eta,\eta\rangle=\|S\eta\|^2$, so $S^2=T^*T$ and $S=\sqrt{T^*T}$ by !ness of pos. sq. rt. Define $u:|T|H\to K$ by $u|T|\eta=T\eta$, and note $\|u|T|\eta\|=\|T\eta\|=\||T|\eta\| \Rightarrow u$ an isometry and thus well-defined. We can then extend u to $\overline{|T|H}$ by continuity, and define $u=0$ on $\overline{|T|H}^\perp$. Then u is a p.i. and $u|T|=T$. If $v\in B(H,K)$ is another such p.i. w/ $\ker v=\ker T=\ker u$ and $v|T|=T$, then $u|T|\eta=v|T|\eta$ $\forall\eta\in H$, so $u=v$ on $|T|H$. Both ops are 0 on $\overline{|T|H}^\perp$, so $u=v$.
Finally, u^*u is proj onto $\overline{|T|H}$, so $u^*T\eta=u^*u|T|\eta=|T|\eta$ $\forall\eta\in H$.

Cpt ops on Hilb. sp.: Let $K(H) \subseteq B(H)$ be the cpt ops.
We get some additional nice properties on Hilb. sp.:

Lemma: $\exists$ a net (P_λ) of finite rank proj's s.t. $\forall \xi \in H$, $P_\lambda \xi \to \xi$.

[I we say $P_\lambda \to 1$ in the strong operator topology.]

Pf: Let $(e_i)_{i \in I}$ be an ONB for H. Let Λ be the set of finite subsets of I ordered by inclusion. For $\lambda \in \Lambda$, let P_λ be proj onto span$\{e_i \mid i \in \lambda\}$. By Parseval's id, $\|P_\lambda \xi - \xi\| \to 0$ $\forall \xi \in H$.

Prop: $K(H)$ is the closure of the finite rank ops

Pf: It's clear that all finite rank ops are contained in $K(H)$, which is closed. Let $T \in K(H)$, and let (P_λ) be as in the lemma. We claim $P_\lambda T \to T$. If not, $\exists \varepsilon > 0$ s.t. (passing to subnets if necessary) $\forall \lambda$, $\exists \xi_\lambda \in H$ w/ $\|\xi_\lambda\| = 1$ s.t. $\varepsilon \le \|(1-P_\lambda)T\xi_\lambda\|$, and $T\xi_\lambda \to \eta \in H$ in norm. Then

$$\varepsilon \le \|(1-P_\lambda)T\xi_\lambda\| \le \|(1-P_\lambda)(T\xi_\lambda - \eta)\| + \|(1-P_\lambda)\eta\|$$
$$\le \|T\xi_\lambda - \eta\| + \|(1-P_\lambda)\eta\| \to 0, \quad \text{a contradiction.}$$

Notation: We write $\langle \eta | \xi \rangle = \langle \xi, \eta \rangle$ $\forall \eta, \xi \in H$, so $\langle \cdot | \cdot \rangle$ is a sesquilinear form which is linear on the right.

[I keep both $\langle \cdot | \cdot \rangle$ and $\langle \cdot, \cdot \rangle$, so have cake and eat it too.]

For $\xi \in H$, write $|\xi\rangle = \xi \in H$ and $\langle \eta | = \langle \cdot, \eta \rangle \in H^*$ so that $\langle \eta | \xi \rangle$ also means $\langle \eta | (|\xi\rangle)$. Now swap to write $|\eta\rangle\langle \xi| \in B(H)$ is $\zeta \mapsto |\eta\rangle\langle \xi | \zeta \rangle = \langle \zeta, \xi \rangle \eta$.

Exercise: Show (1) $|\eta\rangle\langle \xi|^* = |\xi\rangle\langle \eta|$.
(2) $(|\eta\rangle\langle \xi|)(|\xi\rangle\langle \zeta|) = \|\xi\|^2 (|\eta\rangle\langle \zeta|)$.
(3) If $\|\xi\| = 1$, $|\xi\rangle\langle \xi|$ is proj onto $\mathbb{C}\xi \subseteq H$.

Def: $T \in B(H)$ is *diagonalizable* if $\exists$ an ONB (e_i) of eigenvectors of T.

Prop: A diagonalizable op is cpt $\iff$ eigenvalue seq. (λ_i) for (e_i) is in $c_0(I)$.

Pf: We already know the spectrum of a cpt op is a countable set whose only possible acc pt is 0, and all $\lambda \in \mathrm{sp}$ are eval's w/ fin'l diml eigenspaces. Conversely, if $(\lambda_i) \in c_0(I)$, $\sum \lambda_i |e_i\rangle\langle e_i|$ converges in $\|\cdot\|$ to T, so T is a limit of finite rank ops.

Spectral thm for normal cpt ops: Suppose $T \in K(H)$ normal. Then $\exists (e_i)$ an ONB of eigenvectors for T, i.e., $T = \sum \lambda_i \underbrace{|e_i\rangle\langle e_i|}_{\text{mutually } \perp \text{ proj's.}}$

Pf: It suffices to prove H is the direct sum of eigenspaces of T, where if $(H_i)_{i \in I}$ are Hilbert spaces, the direct sum is

$$\bigoplus_{i \in I} H_i = \left\{ (\xi_i)_{i \in I} \;\middle|\; \xi_i \in H_i \ \forall i \text{ and } \sum \|\xi_i\|^2 < \infty \right\}.$$

[[we already know non-zero eval's of T have f. dim'l eigenspaces and that the evals $\lambda_n \to 0$]]

Let (λ_n) be the non-zero eigenvalues of T. Let (E_n) be the corresponding eigenspaces. By the spectral characterization of cpt ops, either $\exists$ finitely many eval's, or countably many s.t. $\lambda_n \to 0$. Moreover, $\dim(E_n) < \infty$ $\forall n$. Observe that E_n is also the eigenspace for T^* corresp. to $\overline{\lambda_n}$, and $E_n \perp E_m$ for $m \neq n$. Since each E_n is T and T^*-invariant, so is $\bigoplus_{n=1}^{\infty} E_n$. Let $E_0 = (\bigoplus E_n)^\perp$. Since E_0 is T and T^* invariant, $T|_{E_0}$ is normal and cpt. But $T|_{E_0}$ has no non-zero eigenvalues, so $T|_{E_0} = 0$. Thus $H = \bigoplus_{n \geq 0} E_n$.

Cor: Suppose $T \in B(H,K)$ s.t. T^*T is cpt. Then T is cpt.

Pf: Let $T = u|T|$ be the polar decomposition. It's enough to prove $|T|$ is cpt. We know $|T|^2 = T^*T = \sum \lambda_n P_n$ is cpt. Define $S \in B(H)$ by $\sum \lambda_n^{1/2} P_n$. Then S is cpt, $S \geq 0$ and $S^2 = T^*T$, so $S = |T|$ by !ness of pos. sq. rt.

Schmidt Rep'n: Suppose $T \in K(H)$, so $|T| = (T^*T)^{1/2}$ is cpt. Enumerate the e.val's of $|T|$ by $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$ w/ multiplicity as necessary. Note that $\lambda_0 = \|T\|$. $\lambda_n = s_n(T)$, the n^{th} singular value of T. By the spectral thm $\exists$ ON v's $(f_n)_{n \geq 0}$ s.t. $|T| f_n = \lambda_n f_n$, and thus $|T| = \sum \lambda_n |f_n\rangle\langle f_n|$ converges in op. norm. Let u be the p.i. s.t. $T = u|T|$, $\ker T = \ker |T| = \ker u$. Setting $e_n = u f_n$ for $n \geq 0$, the e_n's are ON and $T = \sum \lambda_n |e_n\rangle\langle f_n|$ converges in op. norm. Then $T^* = \sum \lambda_n |f_n\rangle\langle e_n|$, so $s_n(T) = s_n(T^*)$ $\forall n \geq 0$.

Minimax: Let $T \geq 0$, $T \in K(H)$, $T \neq 0$. Then $\forall n \geq 0$ w/ $n \leq \dim H$,

$$s_n(T) = \min_{\substack{E \subseteq H \\ \operatorname{codim} E = n}} \max_{\substack{\xi \in E \\ \|\xi\|=1}} \langle T\xi, \xi \rangle =: m_n$$

Pf: We know $n=0$ holds. Assume $n>0$ and let (f_n) be ON s.t. $Tf_n = \lambda_n f_n$ $\forall n \geq 0$. Let $E_n = \operatorname{span}\{f_0, \dots, f_{n-1}\}^\perp$. Then $f_n \in E_n$ and $\lambda_n = \langle Tf_n, f_n \rangle$, so $\lambda_n \geq m_n$. Conversely, if $E \subseteq H$ has codim n, then $\exists \xi \in \operatorname{span}\{f_0, \dots, f_n\} \cap E \neq (0)$ w/ $\|\xi\| = 1$. Then writing $\xi = \sum_{i=0}^n \alpha_i f_i$ w/ $\alpha_i = \langle \xi, f_i \rangle$, $\sum |\alpha_i|^2 = 1$, we have

$$\langle T\xi, \xi \rangle = \sum_{i=0}^n |\alpha_i|^2 \lambda_i \geq \lambda_n$$ as $\lambda_i \searrow 0$. Hence $\lambda_n \leq m_n$.

Cor: If $T \in K(H)$, $s_n(T) = \min_{\substack{E \subseteq H \\ \operatorname{codim} E = n}} \max_{\substack{\xi \in E \\ \|\xi\|=1}} \|T\xi\|$.

Pf: $s_n(T) = \sqrt{s_n(T^*T)}$ and $\langle T^*T\xi, \xi \rangle = \|T\xi\|^2$. Apply Minimax for $T^*T \geq 0$.

Cor: If $T \in K(H)$ and $S \in B(H)$, then both $s_n(ST), s_n(TS) \leq \|S\| s_n(T)$.

Pf: $s_n(ST) = \min\max \|ST\xi\| \leq \min\max \|S\| \cdot \|T\xi\| = \|S\| \cdot \min\max \|T\xi\| = \|S\| s_n(T)$.

$s_n(TS) = s_n(S^*T^*) \leq \|S^*\| s_n(T^*) = \|S\| s_n(T)$.

Thm: If $T \in K(H)$, $s_n(T) = \operatorname{dist}(T, F_n)$ where $F_n = \{F \in B(H) \mid \operatorname{rank} F = n\}$.

Pf: Write $T = \sum_i \lambda_i |e_i\rangle\langle f_i|$ in Schmidt rep'n. Then the operator $F = \sum_{i=0}^{n-1} \lambda_i |e_i\rangle\langle f_i| \in F_n$, and $T - F = \sum_{i=n}^\infty \lambda_i |e_i\rangle\langle f_i|$ has norm λ_n, hence $\operatorname{dist}(T, F_n) \leq \lambda_n$. Now $\forall F \in F_n$, $\dim(\operatorname{span}\{f_0, \dots, f_n\}) = n+1$, so $\exists \xi \in \operatorname{span}\{f_0, \dots, f_n\}$ w/ $\|\xi\| = 1$ and $F\xi = 0$. Then

$$\|T - F\| \geq \|(T-F)\xi\| = \|T\xi\| \geq \lambda_n.$$

Cor: If $S, T \in K(H)$, $s_{m+n}(S+T) \leq s_m(S) + s_n(T)$.

Pf: Let $\varepsilon > 0$. Take $F_1 \in F_m$ s.t. $\|S - F_1\| < s_m(S) + \varepsilon$ and $F_2 \in F_n$ s.t. $\|T - F_2\| < s_n(T) + \varepsilon$. Then $\exists\, 0 \leq k \leq m+n$ s.t. $F_1 + F_2 \in F_k$, and thus

$$s_{m+n}(S+T) \leq s_k(S+T) = \operatorname{dist}(S+T, F_k) \leq \|(S+T) - (F_1+F_2)\|$$
$$\leq \|S - F_1\| + \|S - F_2\| < s_m(S) + s_n(T) + 2\varepsilon.$$

Since ε was arbitrary, the result follows.

The trace: Let (e_i) be an ONB of H. Define $\mathrm{Tr}: B(H)_+ \to [0,\infty]$ by $\mathrm{Tr}(T) = \sum_{i\in I} \langle Te_i, e_i\rangle$.

Lemma: $\mathrm{Tr}(T^*T) = \mathrm{Tr}(TT^*)$ $\forall T \in B(H)$.

Pf: $\forall i,j \in I$, $\langle Te_i, e_j\rangle\langle e_j, Te_i\rangle = \langle T^*e_j, e_i\rangle\langle e_i, T^*e_j\rangle \geq 0$. Hence the sum is order independent:

$$\mathrm{Tr}(T^*T) = \sum_i \langle T^*Te_i, e_i\rangle = \sum_i \langle Te_i, Te_i\rangle = \sum_i\sum_j \langle \langle Te_i, e_j\rangle e_j, Te_i\rangle$$
$$= \sum_i\sum_j \langle Te_i, e_j\rangle\langle e_j, Te_i\rangle = \sum_j\sum_i \langle T^*e_j, e_i\rangle\langle e_i, T^*e_j\rangle = \cdots = \mathrm{Tr}(TT^*).$$

Cor: $\forall T \geq 0$ and unitary $u \in B(H)$, $\mathrm{Tr}(T) = \mathrm{Tr}(u^*Tu)$.

Pf: $\mathrm{Tr}(u^*Tu) = \mathrm{Tr}(u^*T^{1/2}T^{1/2}u) = \mathrm{Tr}(T^{1/2}uu^*T^{1/2}) = \mathrm{Tr}(T^{1/2}T^{1/2}) = \mathrm{Tr}(T)$.

Well-defined: If (f_i) another ONB, $\exists$ a unitary $u \in B(H)$ s.t. $ue_i = f_i$ $\forall i$, and $\mathrm{Tr}(T) = \mathrm{Tr}(u^*Tu) = \sum_i \langle Tue_i, ue_i\rangle = \sum_i \langle Tf_i, f_i\rangle$.

Cor: If $T \geq 0$, $\mathrm{Tr}(T) \geq \|T\|$. [$\|T\| = \sup_{\|\xi\|=1} \langle T\xi, \xi\rangle \leq \sum \langle Te_i, e_i\rangle$.]

Lemma: If $T \in B(H)$ s.t. $\mathrm{Tr}(|T|^p) < \infty$ for some $p > 0$, then T cpt.

Pf: Given an ONB $(e_i)_{i\in I}$ and an $\varepsilon > 0$, $\exists$ a finite subset $\lambda \subseteq I$ s.t. $\sum_{i\notin\lambda} \langle |T|^p e_i, e_i\rangle < \varepsilon$. Let P_λ denote proj onto span$\{e_i \mid i \in \lambda\}$, so then $\| |T|^{p/2}(1-P_\lambda)\|^2 = \|(1-P_\lambda)|T|^p(1-P_\lambda)\| \leq \mathrm{tr}[(1-P_\lambda)|T|^p(1-P_\lambda)] < \varepsilon$.

Thus we may approx $|T|^{p/2}$ by finite rank ops. Thus $|T|^{p/2}$ cpt, so $|T|^p = |T|^{p/2}|T|^{p/2}$ is also cpt. We may choose a suitable ONB (f_i) so that $|T|^p = \sum \lambda_i |f_i\rangle\langle f_i|$, $\lambda_i \geq 0$ and $\lambda_i \to 0$. Then by CFC, $|T| = \sum \lambda_i^{1/p} |f_i\rangle\langle f_i|$ and $\lambda_i^{1/p} \to 0$, so $|T|$ is cpt. $\iff$ T cpt.

Cor: If $T \in K(H)$, $\mathrm{Tr}(|T|^p) = \sum s_n(T)^p$.

Pf: We have $|T| = \sum \lambda_n |e_n\rangle\langle e_n|$, so $|T|^p = \sum \lambda_n^p |e_n\rangle\langle e_n|$, and thus $\mathrm{Tr}(|T|^p) = \sum \lambda_n^p = \sum s_n(T)^p$.

Def: Schatten p-class $p \geq 1$ $\mathcal{L}^p(H) = \{T \in B(H) \mid \mathrm{Tr}(|T|^p) = \sum_n s_n(T)^p < \infty\}$

$\hookrightarrow$ Trace class $= \mathcal{L}^1(H)$ and Hilbert-Schmidt $= \mathcal{L}^2(H)$.

Lemma: $\mathcal{L}^p(H)$ is a $*$-closed 2-sided ideal in $B(H)$, which is algebraically spanned by the positive ops in $\mathcal{L}^p(H)$.

Pf: $*$-closed: $s_n(T) = s_n(T^*)$ $\forall n \geq 0$.

$\cdot$-closed: $s_n(ST), s_n(TS) \leq \|S\| \cdot s_n(T)$ $\forall n \geq 0$ $\forall S \in B(H)$.

$+$-closed: $s_{2n}(S+T) \leq s_n(S) + s_n(T)$. Since $(s_n(S)), (s_n(T)) \in \ell^p$, so is $(s_n(S) + s_n(T))$. Similarly, $s_{2n+1}(S+T) \leq s_n(S) + s_{n+1}(T)$, and $(s_n(S) + s_{n+1}(T)) \in \ell^p \Rightarrow (s_n(S+T)) \in \ell^p$.

Spanned by pos. part:

Step 1: Every s.a. T can be written as $T = T_+ - T_-$ w/ $T_\pm \geq 0$ and $T_+T_- = T_-T_+ = 0$.

Pf: Set $T_+ = \chi_{[0,\infty)}(T)T$ and $T_- = \chi_{(0,\infty]}(T)T$.

Step 2: Every $T \in B(H)$ is a linear comb. of 4 pos. elts.

Pf: $T = \mathrm{Re}(T) + i\,\mathrm{Im}(T)$ w/ $\mathrm{Re}(T) = \frac{T+T^*}{2}$, $\mathrm{Im}(T) = \frac{T-T^*}{2i}$ s.a.

Step 3: If $T \in J$ a 2-sided $*$-closed ideal, then T is a lin. comb. of 4 pos elts from J.

Pf: $\mathrm{Re}(T)_\pm$ and $\mathrm{Im}(T)_\pm \in J$.

Cor: $\mathcal{L}^1(H) = \mathrm{span}\{T \geq 0 \mid \mathrm{Tr}(T) < \infty\}$.

Prop: Tr extends to a linear map $\mathcal{L}^1(H) \longrightarrow \mathbb{C}$.

Pf: For $T \in \mathcal{L}^1(H)$, can write $T = \sum_{k=0}^{3} i^k T_k$ w/ $T_k \geq 0$. Define $\mathrm{Tr}(T) = \sum_{k=0}^{3} i^k \mathrm{Tr}(T_k) \in \mathbb{C}$. Show well-defined.
If $T = \sum_{k=0}^{3} i^k S_k$ w/ $S_k \geq 0$, $\mathrm{Re}(T) = S_0 - S_2$ and $\mathrm{Im}(T) = \frac{S_1 - S_3}{2i}$. Then $\sum_{k=0}^{3} i^k \mathrm{Tr}(S_k) = \mathrm{Tr}(\mathrm{Re}(T)) + i\,\mathrm{Tr}(\mathrm{Im}(T))$, which is independent of choice of S_0, S_1, S_2, S_3.

Prop: For $S, T \in \mathcal{L}^2(H)$, $T^*S \in \mathcal{L}^1(H)$. The space $\mathcal{L}^2(H)$ is a Hilbert space w/ inner product $\langle S, T\rangle_{\mathcal{L}^2} = \mathrm{Tr}(T^*S)$.

Pf: Note $\mathcal{L}^2(H) = \{T \in B(H) \mid \mathrm{Tr}(|T|^2) = \mathrm{Tr}(T^*T) < \infty\}$ is a $*$-closed 2-sided ideal. Hence $T \in \mathcal{L}^2(H) \Rightarrow T^*T \in \mathcal{L}^1(H)$. By polarization,
$$T^*S = \frac{1}{4}\sum_{k=0}^{3} i^k \underbrace{(S+i^kT)^*\underbrace{(S+i^kT)}_{\in \mathcal{L}^2}}_{\in \mathcal{L}^1}.$$
It's now easy to see $\langle S,T\rangle_{\mathcal{L}^2}$ is positive. Now for $T \in \mathcal{L}^2$, $\|T\|_2^2 = \mathrm{Tr}(T^*T) \geq \|T^*T\| = \|T\|^2$, so $\langle\cdot,\cdot\rangle_{\mathcal{L}^2}$ an inner prod. Moreover, every $\|\cdot\|_2$-Cauchy seq. is $\|\cdot\|$-Cauchy. To show $\mathcal{L}^2$ is $\|\cdot\|_2$ complete, it suffices to show if (T_n) $\|\cdot\|_2$-Cauchy s.t. $T_n \to T$ in $\|\cdot\|$, then $T_n \to T$ in $\|\cdot\|_2$. Note $T \in K(H)$. Note $\forall$ finite rk proj P,
$$\mathrm{Tr}(P(T-T_n)^*(T-T_n)P) = \lim_m \mathrm{Tr}(P(T_m-T_n)^*(T_m-T_n)P)$$
$$\leq \limsup_m \mathrm{Tr}((T_m-T_n)^*(T_m-T_n)) = \limsup_m \|T_m - T_n\|_2^2.$$
Since P arbitrary, $\mathrm{Tr}((T-T_n)^*(T-T_n)) \leq \limsup_m \|T_m-T_n\|_2^2$. Hence $T \in \mathcal{L}^2$ and $T_n \to T$ in $\|\cdot\|_2$.

Lemma: For $T\in\mathcal{L}^1$, $|\mathrm{Tr}(T)|\le\mathrm{Tr}(|T|)$.

Pf: Let $T=u|T|$ be polar decomposition. Then $|T|^{1/2}, u|T|^{1/2}\in\mathcal{L}^1$, and by C-S, $|\mathrm{Tr}(T)|^2=|\langle |T|^{1/2}, u|T|^{1/2}\rangle|^2\le \||T|^{1/2}\|_2^2\cdot\|u|T|^{1/2}\|_2^2$

$$= \mathrm{Tr}(|T|)\cdot\mathrm{Tr}(|T|^{1/2}u^*u|T|^{1/2})$$
$$\le \mathrm{Tr}(|T|)\cdot\|u^*u\|\cdot\mathrm{Tr}(|T|) = \mathrm{Tr}(|T|)^2.$$

Cor: $\forall S\in B(H)$, $T\in\mathcal{L}^1$, $|\mathrm{Tr}(ST)|, |\mathrm{Tr}(TS)|\le\|S\|\,\mathrm{Tr}(|T|)$.

Pf: Have $ST\in\mathcal{L}^1$, so $|\mathrm{Tr}(ST)|\le\mathrm{Tr}(|ST|)$. Now $s_n(ST)\le\|S\|s_n(T)$ $\forall n\ge 0$.
Hence $\mathrm{Tr}(|ST|)=\sum_n s_n(ST)\le\|S\|\sum_n s_n(T)=\|S\|\,\mathrm{Tr}(|T|)$.

Lemma: For $S,T\in\mathcal{L}^2$, $\mathrm{Tr}(ST)=\mathrm{Tr}(TS)$. Also true for $S\in B(H)$, $T\in\mathcal{L}^1$!

Pf: By polarization,
$$4\,\mathrm{Tr}(T^*S)=\sum_{k=0}^{3} i^k\,\mathrm{Tr}((S+i^kT)^*(S+i^kT))$$
$$=\sum_{k=0}^{3} i^k\,\mathrm{Tr}((S+i^kT)(S+i^kT)^*)$$
$$=\sum_{k=0}^{3} i^k\,\mathrm{Tr}((S^*+i^{-k}T^*)^*(S^*+i^{-k}T^*))$$
$$=\sum_{k=0}^{3} i^k\,\mathrm{Tr}((T^*+i^kS^*)^*(T^*+i^kS^*)) = 4\,\mathrm{Tr}(ST^*).$$

For second part, we may assume $T\ge 0$ by linearity of the eq'n. Then
$\mathrm{Tr}(ST)=\mathrm{Tr}(\underbrace{ST^{1/2}T^{1/2}}_{\text{in }\mathcal{L}^2})=\mathrm{Tr}(T^{1/2}ST^{1/2})=\mathrm{Tr}(T^{1/2}ST^{1/2})=\mathrm{Tr}(T^{1/2}T^{1/2}S)=\mathrm{Tr}(TS)$.
a associate

Prop: $\mathcal{L}^1(H)$ is a Banach $*$-algebra w/ $\|T\|_1=\mathrm{Tr}(|T|)$.

Pf: We know $\|\lambda T\|_1=\mathrm{Tr}(|\lambda T|)=\mathrm{Tr}(|\lambda|\,|T|)=|\lambda|\cdot\|T\|_1$. Moreover, $\mathrm{Tr}(|T|)=0\Rightarrow |T|=0\Rightarrow T=0$. To prove subadditivity, let $S+T=u|S+T|$ be polar decomp. Then $u^*(S+T)=|S+T|$, so

$$\|S+T\|_1=\mathrm{Tr}(u^*(S+T))=\mathrm{Tr}(\underbrace{u^*S}_{\in\mathcal{L}^1}+\underbrace{u^*T}_{\in\mathcal{L}^1})\le|\mathrm{Tr}(\underbrace{u^*S}_{\in\mathcal{L}^1})|+|\mathrm{Tr}(\underbrace{u^*T}_{\in\mathcal{L}^1})|$$
$$\le\|u^*\|\,\mathrm{Tr}(|S|)+\|u^*\|\,\mathrm{Tr}(|T|)\le\|S\|_1+\|T\|_1.$$

To prove submultiplicativity, note
$\|ST\|_1=\mathrm{Tr}(|ST|)\le\|S\|\,\mathrm{Tr}(|T|)=\||S|\|\,\mathrm{Tr}(|T|)\le\mathrm{Tr}(|S|)\,\mathrm{Tr}(|T|)=\|S\|_1\cdot\|T\|_1$.

Note $\|T\|_1=\|T^*\|_1$, since $s_n(T)=s_n(T^*)$ $\forall n\ge 0$.

Now if (T_n) $\|\cdot\|_1$-Cauchy, (T_n) $\|\cdot\|$-Cauchy, so $\exists T\in K(H)$ s.t. $T_n\to T$.

Now $\forall$ finite rk proj P, $\forall$ p.i. u, by the above Corollary,
$$|\mathrm{Tr}(pu^*(T-T_n))|=\lim_m|\mathrm{Tr}(Pu^*(T_m-T_n))|\le\limsup_m\|T_m-T_n\|_1.$$

Hence for the polar decomposition $T-T_n=u|T-T_n|$, we have
$|\mathrm{Tr}(p|T-T_n|)|\le\limsup_m\|T_m-T_n\|_1$. Since p arbitrary, we have
$\mathrm{Tr}(|T-T_n|)\le\limsup_m\|T_m-T_n\|_1$, so $T\in\mathcal{L}^1$ and $T_n\to T$ in $\|\cdot\|_1$.

Prop: $\mathcal{L}^p(\mathcal{H})$ is Banach under $\|T\|_p := \mathrm{Tr}(|T|^p)^{1/p} = \|s_n(T)\|_{\ell^p}$.

Pf: Similar to those for $\mathcal{L}^2$ and $\mathcal{L}^1$ and is omitted.

[[See E.3.4.3 in Analysis Now for a sketch.]]

Thm: For $1 < q \le 2 \le p < \infty$ s.t. $\frac{1}{p}+\frac{1}{q}=1$, $\forall S \in \mathcal{L}^p$ and $T \in \mathcal{L}^q$, we have $ST \in \mathcal{L}^1$ and $|\mathrm{Tr}(ST)| \le \|S\|_p \cdot \|T\|_q$. The bilinear form $(S,T) = \mathrm{Tr}(ST)$ implements a duality which exhibits $\mathcal{L}^p$ and $\mathcal{L}^q$ as isometrically isomorphic to e/o dual spaces. [[Assume $\mathcal{H}$ separable here for convenience.]]

Pf: **Step 0:** Recall that for $1 \le q \le p$, $\ell^q \subseteq \ell^p$ w/ $\|\cdot\|_q \ge \|\cdot\|_p$. True for $\mathcal{L}^q \subseteq \mathcal{L}^p$!

For $\frac{1}{p}+\frac{1}{q}=1$ and $(x_n) \in \ell^p$, $(|x_n|^{p-1}) \in \ell^q$ w/ $\|(|x_n|^{p-1})\|_q^q = \|(x_n)\|_p^p = \|(x_n)\|_p \cdot \||x_n|^{p-1}\|_q$

Hence $(\ell^p)^* \cong \ell^q$ via the bilinear pairing $((x_n),(y_n)) = \sum_{n\ge1} x_n y_n$

[[Note $(x,y) = \int_{\mathbb{N}} xy$ w/ counting measure. Tr is a non-commutative $\int$.]]

Step 1: If $S \ge 0$, $S \in \mathcal{L}^p$, $p \ge 2$, $\forall \|\xi\| = 1$, $\langle S^2\xi,\xi\rangle^{p/2} \le \langle S^p \xi,\xi\rangle$.

Pf: Let $(e_n) \subset \mathcal{H}$ ONB s.t. $S = \sum \lambda_n |e_n\rangle\langle e_n|$. Then $\forall \xi \in \mathrm{span}\{e_1,\dots,e_k\}$,

$$\langle S^2\xi,\xi\rangle = \sum_{i,j} \langle \langle \xi,e_i\rangle S^2 e_i, \langle \xi,e_j\rangle e_j\rangle = \sum_{i,j} |\langle\xi,e_i\rangle|^2 \langle \underbrace{S^2 e_i}_{\lambda_i^2 e_i}, e_j\rangle$$
$$= \sum_i \lambda_i^2 |\langle \xi,e_i\rangle|^2.$$

Now since $t \mapsto (t)^{p/2}$ is cnvx and $\overbrace{\sum_{i=1}^k |\langle\xi,e_i\rangle|^2}^{=\|\xi\|^2} = 1$, we have

$$\langle S^2\xi,\xi\rangle^{p/2} = \left(\sum_{i=1}^k \lambda_i^2 |\langle\xi,e_i\rangle|^2\right)^{p/2} \le \sum_{i=1}^k \lambda_i^p |\langle\xi,e_i\rangle|^2 = \langle S^p\xi,\xi\rangle.$$

Hence the $\le$ holds on the algebraic span of (e_i), which is dense in $\mathcal{H}$.

Since $\xi \mapsto \langle S^2\xi,\xi\rangle^{p/2}$ and $\xi \mapsto \langle S^p\xi,\xi\rangle$ are cts, the result follows.

Step 2: If $S,T \ge 0$, $ST \in \mathcal{L}^1$ and $\mathrm{Tr}(|ST|) \le \|S\|_p \cdot \|T\|_q$.

Pf: Pick (f_n) ONB for T s.t. $T = \sum \mu_n |f_n\rangle\langle f_n|$. Then $\forall n$,

$$|\langle |ST| f_n, f_n\rangle|^2 \le \||ST|f_n\|^2 \cdot \overbrace{\|f_n\|^2}^{1} = \langle |ST|^2 f_n,f_n\rangle = \langle T^*S^*ST f_n,f_n\rangle$$
$$= \langle S^*S Tf_n, Tf_n\rangle = \mu_n^2 \langle |S|^2 f_n,f_n\rangle$$

$\Rightarrow \forall n$, $\langle |ST| f_n,f_n\rangle \le \mu_n \langle |S|^2 f_n,f_n\rangle^{1/2} \overset{\text{Step 1}}{\le} \mu_n \langle |S|^p f_n,f_n\rangle^{1/p}$.

Now setting $x_n = \langle |S|^p f_n,f_n\rangle^{1/p}$, $(x_n) \in \ell^p$ as

$$\|(x_n)\|_{\ell^p}^p = \sum_n \langle |S|^p f_n,f_n\rangle = \mathrm{Tr}(|S|^p) < \infty \text{ as } S \in \mathcal{L}^p.$$

Also, $(\mu_n) \in \ell^q$ as $\sum \mu_n^q = \mathrm{Tr}(|T|^q) < \infty$ as $T \in \mathcal{L}^q$.

By Hölder $\le$, $\mathrm{Tr}(|ST|) = \sum_n \langle |ST| f_n,f_n\rangle \le \sum_n \mu_n \langle |S|^p f_n,f_n\rangle^{1/p}$

$$\le \|(x_n)\|_{\ell^p} \cdot \|(\mu_n)\|_{\ell^q} = \|S\|_p \cdot \|T\|_q.$$

Step 3: $\forall S \in \mathcal{L}^p$, $T \in \mathcal{L}^q$, $ST \in \mathcal{L}^1$ and $|Tr(ST)| \leq \|S\|_p \cdot \|T\|_q$.

Pf: using Polar decomposition, write $S = u|S|$ and $T^* = v|T^*|$, and note $|S|, |T^*| \geq 0$, $|S| = u^* S \in \mathcal{L}^p$, and $|T^*| = v^* T^* \in \mathcal{L}^q$. Then

$$ST = S(T^*)^* = u|S|(v|T^*|)^* = u|S||T^*|v^* \in \mathcal{L}^1 \text{ by Step 2.}$$

Finally, $|Tr(ST)| = |Tr(u|S||T^*|v^*)| \leq \|u\| \cdot \|v^*\| \cdot Tr(|\,|S| \cdot |T^*|\,|)$
$$\leq \||S|\|_p \cdot \||T^*|\|_q = \|S\|_p \cdot \|T\|_q.$$

Step 4: The map $T \longmapsto Tr(\cdot T)$ is an isometry $\mathcal{L}^q \longrightarrow (\mathcal{L}^p)^*$.

Pf: Map well-defined + norm decreasing by Step 3. Let $T = u|T|$ be polar dec. Then $|T| = u^* T \in \mathcal{L}^q$ (ideal).

Claim: $\forall r > 0$, $s_n(|T|)^r = s_n(|T|^r) = s_n(u|T|^r) = s_n(|T|^r u^*)$.

pf: Let $|T| = \sum \lambda_n |e_n\rangle\langle e_n|$ be Schmidt rep'n w/ $\lambda_n = s_n(T)$. Let $f_n = ue_n$, so By CFC, $s_n(|T|^r) = \lambda_n^r = s_n(|T|)^r$. $u|T|^r = \sum \lambda_n^r |f_n\rangle\langle e_n|$. Then $|T|^r u^* u |T|^r = \sum \lambda_n^{2r} |e_n\rangle\langle e_n| = |T|^{2r}$, so $s_n(u|T|^r) = s_n(|T|^r u^* u |T|^r)^{1/2} = \lambda_n^r$. [note that for $R \in K(H)$, $s_n(R) := s_n(|R|) = s_n(R^*R)^{1/2}$.] The last equality follows immediately, since adjoint preserves singular values.

By the claim, we have $s_n(|T|^{q-1}) = s_n(|T|)^{q-1}$, so $|T|^{q-1} \in \mathcal{L}^p$ by Step 0. Let $S = |T|^{q-1} u^* \in \mathcal{L}^p$ (ideal). By Step 0, w/ $\lambda_n = s_n(T)$:

$$Tr(ST) = Tr(|T|^{q-1} u^* T) = Tr(|T|^q) = \|T\|_q^q = \|(\lambda_n)\|_q^q$$
$$= \|(\lambda_n)\|_q \cdot \|(|\lambda_n|^{q-1})\|_p = \|T\|_q \cdot \|S\|_p$$

by the claim. Hence the map is an isometry.

Step 5: The map in Step 4 is surjective.

Pf: Since $q \leq 2 \leq p$, $\mathcal{L}^q \subseteq \mathcal{L}^2 \subseteq \mathcal{L}^p$ w/ $\|\cdot\|_q \geq \|\cdot\|_2 \geq \|\cdot\|_p$ by Step 0. Thus if $\varphi \in (\mathcal{L}^p)^*$, $\varphi|_{\mathcal{L}^2} \in (\mathcal{L}^2)^*$, and $\exists T \in \mathcal{L}^2$ s.t. $\varphi|_{\mathcal{L}^2} = Tr(\cdot T)$ by Riesz Rep. It remains to prove $T \in \mathcal{L}^q$ and $\varphi = Tr(\cdot T)$ on $\mathcal{L}^p$. Let $T = u|T|$ be the polar decomposition and P an arbitrary finite rank proj. Then $S := |T|^{q-1} P u^*$ is finite rank, and thus in $\mathcal{L}^2$. We calculate:

$$\varphi(S) = Tr(ST) = Tr(|T|^{q-1} P \underbrace{u^* T}_{|T|}) = Tr(|T|^q P). \text{ Hence}$$

$\forall$ finite rk proj P, $Tr(|T|^q P) \leq \|\varphi\| \cdot \||T|^{q-1}\|$, and $T \in \mathcal{L}^q$.

Finally, $\varphi = Tr(\cdot T)$ on $\mathcal{L}^p$ since finite ranks are $\|\cdot\|_p$-dense in $\mathcal{L}^p$, so $\mathcal{L}^2 \subseteq \mathcal{L}^p$ is $\|\cdot\|_p$-dense. [observe that if $T \geq 0$ in $\mathcal{L}^p$ w/ Schmidt rep $T = \sum_i \lambda_i |e_i\rangle\langle e_i|$, then setting $T_n = \sum_{i=1}^n \lambda_i |e_i\rangle\langle e_i|$, $T - T_n = \sum_{i>n} \lambda_i |e_i\rangle\langle e_i| \geq 0$, and $|T - T_n|^p = \sum_{i>n} \lambda_i^p |e_i\rangle\langle e_i|$. Thus $\|T - T_n\|_p^p = \sum_{i>n} \lambda_i^p \longrightarrow 0$ as $n \to \infty$.]

Step 6: $(\mathcal{L}^q)^* \cong \mathcal{L}^p$ is omitted.

More on $B(\mathcal{H})$ next semester!

Representing $\mathbb{C}$ $*$-alg's on Hilb. space:

Def: A representation of a $\mathbb{C}$ $*$-alg A is a pair $(\mathcal{H},\pi)$ w/ $\mathcal{H}$ a Hilb. space and $\pi : A \to B(\mathcal{H})$ a $*$-hom. If A is unital, we usually discuss unital representations.

- A rep'n $(\mathcal{H},\pi)$ is called nondegenerate if the subspace $\{\pi(a)\xi \mid a \in A, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

Example: Let Γ be a discrete gp. Recall that the group algebra

$$\mathbb{C}[\Gamma] = \{\sum_g x_g \cdot g \mid x_g \in \mathbb{C}, \text{ all but finitely many } x_g = 0\} \text{ w/}$$

addition: $\sum_g x_g \cdot g + \sum_g y_g \cdot g = \sum_g (x_g + y_g) g$

multiplication: $(\sum_g x_g g)(\sum_h y_h h) = \sum_k (\sum_{gh=k} x_g y_h) k$

$*$: $(\sum_g x_g g)^* = \sum_g \overline{x_g} g^{-1}$

Define $\ell^2\Gamma = \{\xi : \Gamma \to \mathbb{C} \mid \sum_{g\in\Gamma} |\xi(g)|^2 < \infty\}$. We define $\forall g \in \Gamma$ $\lambda_g : \ell^2\Gamma \to \ell^2\Gamma$ by $(\lambda_g \xi)(h) = \xi(g^{-1}h)$.

Notice that λ_g is invertible w/ inverse $\lambda_{g^{-1}}$.

Moreover, λ_g is an isometry: $\|\lambda_g\xi\|_2^2 = \sum_h |\xi(g^{-1}h)|^2 = \|\xi\|_2^2$.

Thus λ_g is unitary, and $\lambda_g^* = \lambda_{g^{-1}}$, so $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$

Since Γ is a basis for $\mathbb{C}\Gamma$, we get $\lambda : \mathbb{C}[\Gamma] \to B(\ell^2\Gamma)$ by extending linearly, which is a $*$-homomorphism.

Def'n: A linear functional φ on a unital $\mathbb{C}$ $*$-alg A is called positive if $\varphi(a^*a) \geq 0$ $\forall a \in A$. A state on A is a positive linear fnl φ on A s.t. $\varphi(1) = 1$.

Exercise: If φ is a state on the unital $\mathbb{C}$ $*$-alg A, $\varphi(a^*) = \overline{\varphi(a)}$ $\forall a \in A$.

Exercise: If A is a unital C^*-alg, then φ is a state and iff $\varphi \in A^*$ w/ $\|\varphi\| = \varphi(1)$.

[For both: Adapt pf for $C(X)$ when we calculated $\partial_{ext}(P(X))$.]

Example: Let $A \subseteq B(H)$ be a $*$-subalg, and $\xi \in H$. Then $a \mapsto \langle a\xi, \xi \rangle$ is positive. It is a state iff $\|\xi\| = 1$.

Non-example: Let $A = \mathbb{C} \oplus \mathbb{C}$ w/ usual addition and multiplication, but w/ $(x,y)^* := (\bar{y}, \bar{x})$. Then A has no states!

Given a C^*-alg A and a state φ on A, get a sesquilinear form on A by $\langle a, b \rangle_\varphi := \varphi(b^* a)$. [can also do $\varphi(ab^*)$.]

By Cauchy-Schwarz, $N_\varphi := \{ a \in A \mid \langle a, a \rangle_\varphi = 0 \}$
$= \{ a \in A \mid \langle a, b \rangle_\varphi = 0 \ \forall b \in A \}$

is a left ideal. Thus $\langle \cdot, \cdot \rangle_\varphi$ is an inner product on A/N_φ, and A acts on A/N_φ by left multiplication: $L_a(b + N_\varphi) = ab + N_\varphi$.

Q: When is the action of A on A/N_φ bdd?

Prop: Let A be a unital normed $*$-alg and $\varphi \in A^*$ a pos. lin. ftl.

1. $\|\varphi\| = \varphi(1)$
2. Left regular action of A on A/N_φ is bdd w/ $\|L_a\| \le \|a\| \ \forall a$.

Lemma: If $a = a^*$, $\|a\| < 1$, $\exists b = b^*$ s.t. $b^2 = 1 - a$.

Pf: Can write $\sqrt{1-z}$ as a power series w/ radius of convergence 1.

Pf of Prop: ① If $a = a^*$ in A w/ $\|a\| < 1$, $\varphi(1-a) = \varphi(b^2) \ge 0$, and $\varphi(a) \le \varphi(1)$.
Similarly, $\varphi(1+a) \ge 0$, so $|\varphi(a)| \le \varphi(1)$. Hence $\forall a = a^*$, $|\varphi(a)| \le \varphi(1)\|a\|$.
For arbitrary a, $|\varphi(a)|^2 = |\langle 1, a \rangle_\varphi|^2 \le \langle 1, 1 \rangle_\varphi \cdot \langle a, a \rangle_\varphi = \varphi(1)\varphi(a^* a) \le \varphi(1)\|a^* a\| \le \varphi(1)\|a\|^2$.

② $\forall b \in A$, $\varphi_b : A \to \mathbb{C}$ by $a \mapsto \varphi(b^* a b)$ is positive, so $|\varphi(b^* a b)| \le \|a\| \varphi_b(1) = \|a\| \varphi(b^* b)$.
So $\|ab\|_\varphi^2 = \varphi(b^* a^* a b) = \varphi_b(a^* a) \le \|a^* a\| \varphi(b^* b) \le \|a\|^2 \cdot \|b\|_\varphi^2$.

Cor: If A is a unital normed $*$-alg and pos. linear ftl φ, $\exists$ $*$-rep'n (H_φ, π_φ) and a cyclic vector $\Omega_\varphi \in H_\varphi$ s.t. $\pi_\varphi(A)\Omega_\varphi$ dense in H_φ and $\varphi(a) = \langle \pi_\varphi(a)\Omega_\varphi, \Omega_\varphi \rangle$.

Pf: Let $H_\varphi = \overline{A/N_\varphi}^{\|\cdot\|_\varphi}$, $\pi_\varphi(a) = L_a$, and Ω_φ = image of 1_A in H_φ.

Note: We also denote H_φ as $L^2(A, \varphi)$, the $\boxed{\text{GNS}}$ rep. assoc. to φ.

Def'n: If (H, π) is a $*$-rep'n of the $*$-alg A and $\Omega \in H$, we define the cyclic subspace gen by Ω to be $K_\Omega = \overline{\{\pi(a)\Omega \mid a \in A\}} \subseteq H$.

A triple (H,π,Ω) is called a cyclic rep'n if (π,H) is a $*$-rep and $\Omega\in H$ is a cyclic vector, i.e., $K_\Omega = H \iff \pi(A)\Omega\subseteq H$ dense.

Prop: Suppose (H_i,π_i,Ω_i) $i=1,2$ are two cyclic rep'ns of A s.t. $\varphi_1=\varphi_2$ w/ $\varphi_i(a)=\langle \pi_i(a)\Omega_i,\Omega_i\rangle$. Then $\exists\, u\in B(H_1,H_2)$ unitary s.t. $\pi_2 = u\pi_1(\cdot)u^*$.

Pf: For $a\in A$, we want $u(\pi_1(a)\Omega_1)=\pi_2(a)\Omega_2$. Note this u is isometric, and thus well-defined: $\|\pi_2(a)\Omega_2\|_{H_2}^2 = \varphi_2(a^*a)=\varphi_1(a^*a)=\|\pi_1(a)\Omega_1\|_{H_1}^2$. Since $\pi_1(A)\Omega_1$ is dense in H_1, $u\in B(H_1,H_2)$. Since $\pi_2(A)\Omega_2$ dense in H_2, u is onto, so u is an invertible isometry, i.e., a unitary.

Direct Sums: If $(H_i,\pi_i)_{i\in I}$ is a family of rep'ns, can define $H=\bigoplus H_i$ and $\pi=\bigoplus \pi_i$ by $\pi(a)_i := \pi_i(a)$ on H_i. This rep'n is bdd iff $\forall a\in A$, $\|\pi_i(a)\|$ is uniformly bdd for $i\in I$.

Def'n: For A a unital C^*-alg, the universal rep'n of A is $\bigoplus_{\text{states }\varphi} L^2(A,\varphi)$. So it's a $\bigoplus$ of cyclic rep's. We'll see all rep's are $\bigoplus$'s of cyclic reps.

Lemma: If $B\subseteq A$ is a unital C^*-subalg, any state on B extends to a state on A.

Pf: Apply HB, noting that $\|\tilde\varphi(1)\| = \|\varphi(1)\| = \|\varphi\| = \|\tilde\varphi\| \Rightarrow \tilde\varphi$ a state (by exercise)

Prop: Let A be a unital C^*-alg and $a\in A$ s.a. Then $\forall \lambda\in sp(a)$, $\exists$ state φ_λ on A s.t. $\varphi_\lambda(a)=\lambda$. [The associated GNS cyclic rep'n $(L^2(A,\varphi_\lambda),\pi_\lambda,\Omega_\lambda)$ satisfies $\|\Omega_\lambda\|=1$ and $\varphi_\lambda(a)=\langle\pi_\lambda(a)\Omega_\lambda,\Omega_\lambda\rangle$.]

Pf: Recall $C(sp(a))\cong C^*(a) =: B\subseteq A$. Define $\varphi: B\to\mathbb{C}$ by ev_λ, which is a state on $C(sp(a))$. Use the lemma to extend to A.

Thm (Gelfand-Naimark 2): The universal rep'n is isometric (norm-preserving). If $\lambda=\|a^*a\|$, $\exists\, \varphi_\lambda\in A^*$ s.t. $\varphi_\lambda(a^*a)=\lambda$.

Pf: Let $a\in A$. Then $\|\pi(a)\|^2=\|\pi(a^*a)\|$.
Then π_λ satisfies $\|\pi_\lambda(a^*a)\| = \lambda = \|a^*a\|$.

We can also do all this for non-unital C^*-alg's.

Irreducibility / Indecomposability

A rep'n (H,π) is called
- irreducible if $\nexists$ a nontrivial $\pi(A)$ invariant subspace $K\subseteq H$.
- indecomposable if $\nexists$ nontrivial $K_1,K_2\subseteq H$ $\pi(A)$-invariant s.t. $H=K_1\oplus K_2$.

Prop: If $K \subseteq H$ is $\pi(A)$-invariant, then so is $K^\perp$. Hence (H,π) is irred. iff it is indecomposable.

Pf: If $\xi \in K^\perp$ and $\eta \in K$, $\langle \pi(a)\xi, \eta\rangle = \langle \xi, \pi(a)^*\eta\rangle = \langle \underbrace{\xi}_{\in K^\perp}, \underbrace{\pi(a^*)\eta}_{\in K}\rangle = 0$.
Hence $\pi(a)\xi \in K^\perp$.

Cor: Every finite dim'l rep'n of a unital normed $*$-alg is completely reducible or semisimple (a finite $\oplus$ of irreducibles).

Cor: Every $*$-rep'n is a $\oplus$ of cyclic rep'ns.
Pf: imitate pf that H has an ONB.

Schur's Lemma: (H,π) is irreducible $\iff \pi(A)' = \{x \in B(H) \mid x\pi(a) = \pi(a)x \ \forall a \in A\} = \mathbb{C}1$ commutant of $\pi(A)$.
[Note $\pi(A)'$ is a C*-alg as $[x\pi(a) = \pi(a)x \ \forall a]$ is a $*$-closed $\|\cdot\|$-closed condition.]

Pf: If H is not irreducible, $\exists$ nontrivial $\pi(A)$-inv. subspace $K \subseteq H$, and $K^\perp$ is also $\pi(A)$-invariant. Then $P_K \in \pi(A)' \neq \mathbb{C}1$.
Conversely, if $\pi(A)' \neq \mathbb{C}$, then by Gelfand-Mazur, $\exists\, x, y \in \pi(A)'$ s.t. $x,y \geq 0$, $x,y \neq 0$, and $xy = yx = 0$ i.e., x,y are commuting positive zero-divisors. [If B is a unital C*-alg s.t. $B \neq \mathbb{C}$, as every elt is a linear comb. of 4 pos. elts, $\exists$ a $b \in B$ w/ $b \geq 0$ s.t. $|sp(b)| \geq 2$. Use CFC to get $x,y \in C^*(b)$.] Now $\ker(x) \neq (0)$ is $\pi(A)$-invariant, so H reducible.
[$\xi \in \ker x \iff x\xi = 0$. $\forall a \in A$, $x\pi(a)\xi = \pi(a)x\xi = 0$, so $\pi(a)\xi \in \ker x$.]

Def: For A a unital C*-alg and $\varphi, \psi \in A^*_+$, say $\varphi \geq \psi$ if $\varphi - \psi$ is positive.

Example: Suppose φ is a state and $x \in \pi(A)'$ w/ $0 \leq x \leq 1$. Define $\psi = \psi_x$ by
$\psi(a) = \langle \pi_\varphi(a) x \Omega_\varphi, \Omega_\varphi\rangle_{H_\varphi}$. Note $\psi \geq 0$, since $\forall a \in A$,
$\psi(a^*a) = \langle \pi_\varphi(a^*a) x \Omega_\varphi, \Omega_\varphi\rangle = \langle \pi_\varphi(a) x^{1/2}\Omega_\varphi, \pi_\varphi(a) x^{1/2}\Omega_\varphi\rangle$.
Since $1 - x \geq 0$, $\varphi - \psi \geq 0$, so $\psi \leq \varphi$. Conversely, if $\varphi, \psi \geq 0$ w/ $\varphi \geq \psi$, then $|\psi(b^*a)|^2 \leq \psi(b^*b)\psi(a^*a) \leq \varphi(b^*b)\varphi(a^*a)$, so $N_\varphi \subseteq N_\psi$. For a fixed $b \in A$, the map $\pi_\varphi(a)\Omega_\varphi \mapsto \psi(b^*a)$ on $\pi_\varphi(A)\Omega_\varphi$ is well-defined and bdd by the above. By Riesz Rep., $\exists\, x \in B(H)$ s.t.

$$\psi(b^*a) = \langle x\pi_\varphi(a)\Omega_\varphi, \pi_\varphi(b)\Omega_\varphi\rangle.$$

Now since $0 \leq \psi \leq \varphi$, $0 \leq x \leq 1$. Moreover, $x \in \pi_\varphi(A)'$:

$$\begin{aligned}
\langle \pi_\varphi(c)\, x\, \pi_\varphi(a)\Omega_\varphi, \pi_\varphi(b)\Omega_\varphi\rangle &= \langle x\pi_\varphi(a)\Omega_\varphi, \pi_\varphi(c^*b)\Omega_\varphi\rangle \\
&= \psi((c^*b)^*a) \\
&= \psi(b^*ca) \\
&= \langle x\pi_\varphi(ca)\Omega_\varphi, \pi_\varphi(b)\Omega_\varphi\rangle.
\end{aligned}$$

Hence $\pi_\varphi(c)x = x\pi_\varphi(c)$ by density of $\pi_\varphi(A) \subseteq L^2(A,\varphi)$.

We summarize the previous example w/ the following:

Prop: The map $x \mapsto \psi_x$ is a bijection btwn

$$\{x \in \pi(A)' \mid 0 \le x \le 1\} \longleftrightarrow \{\psi \ge 0 \mid 0 \le \psi \le \varphi\}.$$

Def: A state φ is called <u>pure</u> if $0 \le \psi \le \varphi \Rightarrow \psi = r\varphi$ for some $r \in [0,1]$.

Thm: Let $\varphi \in A^*_+$. Then φ is pure $\iff$ $L^2(A,\varphi)$ is irred repn.

Pf: If φ not pure, $\exists\, 0 \le \psi \le \varphi$ w/ $\psi \ne r\varphi$. Then $\exists\, x \in \pi_\varphi(A)'$ s.t. $x \notin \mathbb{C}1$. Then $\pi_\varphi(A)' \ne \mathbb{C}1$, so $L^2(A,\varphi)$ not irred. Conversely, if $\exists\, K \subseteq L^2(A,\varphi)$ nontrivial $\pi_\varphi(A)$-invariant, $P_K \in \pi(A)'$ is a projection, and $\varphi_{P_K} \ne r\varphi$.

Let $S(A)$ be the set of states of A, a convex wk* cpt subset of A^*.

Prop: $\varphi \in S(A)$ is pure $\iff$ φ is extreme in $S(A)$.

Pf: If φ not extreme, $\exists\, \psi_1, \psi_2 \in S(A)$ s.t. $\varphi = t\psi_1 + (1-t)\psi_2$, $t \in (0,1)$, w/ $\psi_1 \ne \varphi \ne \psi_2$. Then $\varphi \ge t\psi_1$, so φ is not pure. Conversely, if φ not pure, $\exists\, 0 \le \psi \le \varphi$ w/ $\psi \ne r\varphi$, $r \in [0,1]$. Then setting $0 < t = \|\psi\| = \underbrace{\psi(1) < \varphi(1)}_{(\varphi-\psi)(1)=0 \Rightarrow \|\varphi-\psi\|=0.} = 1$, $\varphi = t\dfrac{\psi}{\|\psi\|} + (1-t)\dfrac{(\varphi-\psi)}{\|\varphi-\psi\|}$, so φ not extreme.