Compact Operators:

Lemma: X a loc. cpt. Banach space $\iff$ X f.dim'l.

Pf: $\Leftarrow$: easy; Bolzano–Weierstrass.

$\Rightarrow$: Suppose $\overline{B_1(0)}$ is cpt. Then $\{B_{\frac{1}{2}}(x) \mid x \in B_1(0)\}$ is an open cover of $\overline{B_1(0)}$, so $\exists x_1, \dots, x_n \in B_1(0)$ s.t.

(adapted from mathSE)

$$B_1(0) \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{2}}(x_i) \subseteq \bigcup_{i=1}^{n} x_i + B_{\frac{1}{2}}(0) \subseteq \underbrace{\operatorname{span}\{x_1, \dots, x_n\}}_{Y} + B_{\frac{1}{2}}(0)$$

$\leadsto B_1(0) \subseteq Y + \frac{1}{2}[B_1(0)] \subseteq Y + \frac{1}{2}[Y + \frac{1}{2}B_1(0)] = Y + \frac{1}{2^2}B_1(0)$

$\leadsto B_1(0) \subseteq Y + \frac{1}{2^n}B_1(0)$ $\forall n \in \mathbb{N}$ by induction.

If $x \in B_1(0)$, write $x = y_n + x_n$ w/ $y_n \in Y$, $x_n \in \frac{1}{2^n}B_1(0)$.

Then $x_n \to 0$, so $y_n \to x$. But Y closed, so $x \in Y$.

Let X, Y be Banach spaces.

Def: $K \in B(X, Y)$ is called cpt if K maps bdd subsets in X to precpt (rel. cpt.) subsets of Y.

Observe: K cpt $\iff$ $\overline{K B_1^X(0)}$ cpt $\iff$ $\forall$ bdd seq. $(x_n) \subseteq X$, (Kx_n) has a conv. subseq.

Properties:

1. K finite rk (KX f.dim'l) $\Rightarrow$ K cpt.
2. $1 \in B(X)$ cpt $\iff$ X f.dim'l.
3. K_1, K_2 cpt $\Rightarrow$ $K_1 + K_2$ cpt (pass to 2 conv. subseq's.)
4. Composite of bdd op + cpt op is cpt. (apply bdd to conv. subseq., or find conv. subseq. after applying bdd op.)
5. (co)restrictions of cpt ops are cpt.

$\hookrightarrow$ idempotent $p \in B(X)$ cpt $\iff$ $\dim pX < \infty$.

Prop: $\{\text{cpt ops}\} \subseteq B(X, Y)$ is a norm closed subspace.

pf. $\to$

Pf: Let $T \in B(X, Y)$ be a closure of cpts. Recall a subset of a cplete metric space is precpt $\iff$ it is totally bdd.

Let $\mathcal{U} = B_1^X(0)$, $\varepsilon > 0$. Take cpt $K \in B(X, Y)$ s.t. $\|T-K\| < \frac{\varepsilon}{3}$.

$K\mathcal{U}$ is precpt, so can be coverd by finitely many $\frac{\varepsilon}{3}$-balls.

$\exists x_1, \dots, x_n \in \mathcal{U}$ s.t. $K\mathcal{U} \subseteq \bigcup_{i=1}^{n} B^Y_{\frac{\varepsilon}{3}}(Kx_i)$. Fix $x \in \mathcal{U}$.

Then $\exists x_j$ s.t. $\|Kx - Kx_j\| < \frac{\varepsilon}{3}$. Then

$$\|Tx - Tx_j\| \le \|Tx - Kx\| + \|Kx - Kx_j\| + \|Kx_j - Tx_j\|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $T\mathcal{U} \subseteq \bigcup_{i=1}^{n} B_\varepsilon(Tx_i)$, so $T\mathcal{U}$ is ε-bdd.

Examples of cpt ops:

① $X = \ell^p$ $1 \le p \le \infty$. Consider a seq. (λ_n) of $\mathbb{C}$ #s. Define ~~T~~

$T: X \to X$ by $(Tx)_n = \lambda_n x_n$.

a) If $(\lambda_n) \in c_{00}$ (all but finitely many λ_n are zero), T is finite rank, and thus cpt.

b) If $\lambda_n \to 0$, T is a norm limit of cpt ops, so is cpt; as $\|T\| = \sup_n |\lambda_n|$.

c) If $\lambda_n \not\to 0$, T is not cpt. Pick subseq. n_j s.t. $|\lambda_{n_j}| \ge \varepsilon > 0$ $\forall j$.

Let $e_n \in X$ be $e_n(m) = \delta_{n=m}$. Then $(Te_{n_j})_{j=1}^\infty$ has no norm-conv. subseq.

② $X = C[0,1]$, $F \in C([0,1]^2)$. $T \in B(C[0,1])$ by

$(Tf)(x) = \int_0^1 F(x,y) f(y)\, dy$. $\|T\| \le \|F\|_\infty$. T is cpt (Hw).

Hw

③ $X = C^1[0,1]$, $Y = C[0,1]$. $\|f\|_X = \|f\|_\infty + \|f'\|_\infty$. Let

$I: X \to Y$ be injection. I is cpt., $\|I\| \le 1$.

④ $X = L^p[0,1]$ $1 < p \le \infty$. $V \in B(X)$ is Volterra op:

$(Vf)(x) = \int_0^x f(y)\, dy$. $\|V\| \le 1$, and V cpt.

Prop: $K \in B(\mathfrak{X}, Y)$ cpt $\iff$ $K^* \in B(Y^*, \mathfrak{X}^*)$ cpt.

Pf $\Rightarrow$: Suppose K cpt, and let $(\varphi_n) \subset Y^*$ be bdd. Let $\mathcal{U}$ be open unit ball of $\mathfrak{X}$. Note that (φ_n) is equicts on bdd subsets of Y, and on $\overline{K\mathcal{U}}$ in particular, which is cpt. By the Arzela-Ascoli Thm,

[$\mathfrak{X}$ cpt Hausd. • $F \subseteq C(\mathfrak{X})$ equicts if $\forall x \in \mathfrak{X}$ and $\varepsilon > 0$, $\exists$ open nbhd $\mathcal{U}_x$ of x s.t. $\forall y \in \mathcal{U}_x$, $\forall f \in F$, $|f(x) - f(y)| < \varepsilon$.
• F is ptwise bdd if ~~$\forall x \in \mathfrak{X}$~~, $\sup_{f \in F} \{|f(x)|\} < \infty$.
AA Thm: F is precpt (in $\|\cdot\|_\infty$-norm top?) $\iff$ F is equicts + ptwise bdd.]

$\exists$ subseq. (φ_{n_j}) s.t. (φ_{n_j}) conv. unif on $\overline{K\mathcal{U}}$.

For $x \in \mathcal{U}$, $\varphi_{n_j}(Kx) = (K^*\varphi_{n_j})(x)$, so $(K^*\varphi_{n_j})$ conv. unif. on $\mathcal{U}$. This means $(K^*\varphi_{n_j})$ conv. in $\|\cdot\|_{\mathfrak{X}^*}$.

Hence K^* is cpt.

$\Leftarrow$: K^* cpt $\Rightarrow$ K^{**} cpt $\Rightarrow$ $K^{**}|_{\mathfrak{X}} = K$ is cpt.

Prop: Suppose $K \in B(\mathfrak{X}, Y)$ cpt and $Z \subseteq KY$ closed subspace. Then Z is f.dim'l.

Pf: $K|_{K^{-1}Z} \in B(K^{-1}Z, Z)$ is cpt, and an open map by OMT. A nbhd of 0 in $K^{-1}Z$ is mapped to a nbhd of 0 in Z. Thus Z locally cpt $\Rightarrow$ Z f. dim'l.

Spectral Analysis of cpt ops

$\mathfrak{X}$ a Banach space $/\mathbb{C}$.

Spectrum basics: For $T \in B(\mathfrak{X})$, define

$$Sp(T) = \{z \in \mathbb{C} \mid T - z1 \text{ not inv.}\}$$

$\mathbb{C} \setminus Sp(T)$ is the *resolvent set* of T.

<u>Thm:</u> $sp(T) \subseteq \mathbb{C}$ is a <u>nonempty</u> <u>cpt</u> subset.

<u>Lem:</u> If $\|T\|<1$, $1-T$ invertible w/ $(1-T)^{-1} = \sum_{n=0}^{\infty} T^n$, <u>norm</u> conv.

<u>Pf:</u> $\sum T^n$ conv. since $\sum \|T^n\| \le \sum \|T\|^n = \frac{1}{1-\|T\|}$.

Can directly verify $(1-T)\sum T^n = (\sum T^n)(1-T) = 1$.

<u>Cor:</u> $GL(\mathfrak{X}) = \{T \in B(\mathfrak{X}) \mid T \text{ inv.}\} \subseteq B(\mathfrak{X})$ is <u>open</u>.

<u>Pf:</u> If $T \in B(\mathfrak{X})$ inv., $T - S = T(1-T^{-1}S)$ $\forall S \in B(\mathfrak{X})$.

Hence if $\|T^{-1}S\|<1$, $T-S$ is inv.

Thus $T-S$ is invertible $\forall S$ s.t. $\|S\| < \frac{1}{\|T^{-1}\|}$.

$[\, \|T^{-1}S\| \le \|T^{-1}\| \cdot \|S\| < \|T^{-1}\| \cdot \frac{1}{\|T^{-1}\|} = 1 \,!]$

<u>Pf of thm:</u> By the cor., the resolvent set of T is open, so $sp(T)$ is closed. If $|z| > \|T\|$, we have

$T - z1 = -z(1 - \frac{1}{z}T)$, and $\|\frac{T}{z}\| < 1 \Rightarrow T - z1$ <u>inv.</u>

w/ $(T-z1)^{-1} = -\sum_{n=0}^{\infty} z^{-n-1} T^n$. $(*)$

Thus $sp(T)$ is closed subset of $B^{\mathbb{C}}_{\|T\|}(0) \Rightarrow$ <u>cpt</u>.

Must show $sp(T)$ is <u>nonempty</u>.

<u>Claim 1:</u> If $z_0 \in \mathbb{C} \setminus sp(T)$, then $\forall |z-z_0| < \frac{1}{\|(T-z_0 1)^{-1}\|}$,

$(T-z1)^{-1} = \sum_{n=0}^{\infty} (z-z_0)^n (T-z_0 1)^{-n-1}$. & conv. in $\|\cdot\|$.

<u>Claim 2:</u> $\forall \varphi \in B(\mathfrak{X})^*$, $f_\varphi(z) := \varphi[(T-z1)^{-1}]$ is <u>holo.</u> on $\mathbb{C} \setminus sp(T)$, <u>$|f_\varphi(z)| \to 0$ as $|z| \to \infty$</u>

<u>Pf:</u> $f_\varphi(z) = \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n$ $\forall |z-z_0| < \frac{1}{\|(T-z_0 1)^{-1}\|}$

where $\alpha_n = \varphi[(T-z_0 1)^{-n-1}]$.

By $(*)$, for $|z| > \|T\|$, $|f_\varphi(z)| \le \sum |z|^{-n-1} \|T\|^n \|\varphi\| = \frac{\|\varphi\|}{|z|} \cdot \frac{1}{1 - \frac{\|T\|}{|z|}} \to 0$ as $|z| \to \infty$.

If $sp(T) = \emptyset$, $\forall \varphi \in B(\mathfrak{X})^*$, f_φ is <u>entire</u>, and <u>bdd</u>. Hence $f_\varphi = 0$.

So $\varphi((T-z1)^{-1}) = 0$ $\forall \varphi \in B(\mathfrak{X})^* \Rightarrow (T-z1)^{-1} = 0$, a <u>contradiction</u>.

(Sorenson) (F. Riesz)

Spectrum of cpt ops: Let $K \in B(X)$ be cpt, $\dim X = \infty$.

Lem 1: $\ker(K-1)$ is f. dim'l.

Pf: $\ker(K-1) \subseteq KX$ is a subspace.

Lem 2: $(K-1)X$ closed and has finite codim.

Pf: Let $y \in \overline{(K-1)X}$, so $\exists (x_n) \subset X$ w/ $y = \lim Kx_n - x_n$. ← cites (∗)

Case 1: If (x_n) has a bdd subseq., $\exists (x_{n_k})$ s.t. (Kx_{n_k}) converges, (∗) $\boxed{x_{n_k} \to x}$, so $Kx - x = \lim_k Kx_{n_k} - x_{n_k} = y \in (K-1)X$.

(∗∗) Also applies when $\operatorname{dist}(x_n, \ker(K-1))$ has bdd subseq.!

Case 2: Assume for contradiction $\underbrace{\|x_n\| \to \infty}_{\text{should be } \operatorname{dist}(x_n, \ker(K-1)) \to \infty!}$ as $n \to \infty$.

WLOG, we may assume $K-1$ injective [I can replace X by $X/\ker K-1$] [I replace $K \leadsto \tilde{K}$, still cpt.]

See (∗∗)!

If $(\operatorname{dist}(x_{n_k}, \ker(K-1)))_{k \in \mathbb{N}}$ bdd, can choose $z_k \in \ker(K-1)$ s.t. $\|x_{n_k} - z_k\|$ bdd $\forall k$ and $(K-1)(x_{n_k} - z_k) \to y$. Repeat Case 1.

Claim: K preserves $\ker(K-1)$ $\leadsto$

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[d, "Q"'] & X \arrow[d, "Q"] \\ X/\ker(K-1) \arrow[r, "\tilde{K}"] & X/\ker(K-1) \end{tikzcd}$$

descends to $\tilde{K}$.

K cpt $\Rightarrow \tilde{K}$ is cpt. $\quad x + \underline{\ker} \longmapsto Kx + \underline{\ker}$.

Now $\tilde{K} - \tilde{1}$ is injective.

Let $x_n' = \frac{x_n}{\|x_n\|}$. Since K cpt, we may assume (Kx_n') converges, say to z. But $Kx_n' - x_n' = \frac{Kx_n - x_n}{\|x_n\|} \to 0$ as $\frac{Kx_n - x_n \to y}{\|x_n\| \to \infty}$,

so $x_n' \to z$, so $\|z\| = 1$. But $Kz - z = \lim Kx_n' - x_n' = 0$, and $z \in \ker(K-1) \setminus \{0\}$, a contradiction.

We conclude by the cases that $(K-1)X$ is closed. Now since K^* is cpt, and $[(K-1)X]^\perp = \ker(K^* - 1)$, we have $[(K-1)X]^\perp$ is f. dim'l. But $[(K-1)X]^\perp \cong \left[X/(K-1)X\right]^*$, and thus $X/(K-1)X$ is f. dim'l.

Def: For $j = 0, 1, 2, \ldots$, define $M_j = (K-1)^j X$, $N_j = \ker(K-1)^j$. ← Both cpt. By Lem's 1+2,

$\dim N_j < \infty \ \forall j$, M_j's all closed w/ finite codim.

Moreover, $X = M_0 \supset M_1 \supset M_2 \supset \cdots$ <u>All</u> M_j, N_j are
$(0) = N_0 \subset N_1 \subset N_2 \subset \cdots$ <u>K-invariant</u> subspaces.

<u>Prop:</u> $\exists j \geq 0$ s.t. $M_j = M_{j+1}$ and $N_j = N_{j+1}$, and $X = M_j \oplus N_j$
(complementary $\underset{\text{subspaces}}{K\text{-invariant}}$). Moreover:

a) $(K-1)|_{M_j}$ is invertible in $B(M_j)$

b) If $j>0$, $(K-1)|_{N_j}$ is nilpotent w/ $(K-1)^j_{N_j} = 0$.

$\hookrightarrow$ part (b) is <u>trivial</u> as $N_j = \ker(K-1)^j$.

<u>Pf:</u>

<u>Step ①</u>: $\exists j_1 \geq 0$ s.t. $M_{j_1} = M_{j_1+1}$.

<u>Pf:</u> If not, $M_{j+1} \subsetneq M_j$ $\forall j \geq 0$. $\forall j$, pick $x_j \in M_j$ s.t. $\mathrm{dist}(x_j, M_{j+1}) = 1$, and $\|x_j\| \leq 2$. $[\,!\ \mathrm{dist}(x_j, M_{j+1}) = \|x + M_{j+1}\|_{M_j/M_{j+1}}$ open map... $!\,]$

For $j < i$, $Kx_j - Kx_i = x_j + \underbrace{(K-1)x_j}_{\in M_{j+1}} - \underbrace{Kx_i}_{\in KM_i \subseteq M_i \subseteq M_{j+1}} \in x_j + M_{j+1}$

$\Rightarrow \|Kx_j - Kx_i\| \geq 1$, so $(Kx_j)_j$ has <u>no</u> conv. subseq., $\Rightarrow\!\Leftarrow$.

<u>Step ②</u>: $\exists j_2$ s.t. $N_{j_2} = N_{j_2+1}$.

<u>Pf:</u> $N_j = \ker(K-1)^j = [(K^*-1)^j X^*]_\perp$. Apply Step ① to $(K^*-1)^j X^*$.

Let m be <u>minimal</u> s.t. $M_m = M_{m+1}$. Let n be <u>minimal</u> s.t. $N_n = N_{n+1}$.

<u>Step ③</u>: $M_n \cap \ker(K-1) = (0)$. Thus $(K-1)M_n = M_n$, and $n \leq m$.

<u>Pf:</u> Let $x \in M_n \cap \ker(K-1)$, and let $y \in X$ s.t. $x = (K-1)^n y$. Then $0 = (K-1)x = (K-1)^{n+1} y \Rightarrow y \in N_{n+1} = N_n \Rightarrow (K-1)^n y = 0 = x$.

Now if $\exists z \in M_n \setminus (K-1)M_n$, since $K-1$ acts injectively on M_n, $(K-1)z \in (K-1)M_n \setminus (K-1)^2 M_n$. Iterating, we see $(K-1)^j z \in (K-1)^j M_n \setminus (K-1)^{j+1} M_n$, which contradicts <u>Step ①</u>.

<u>Step ④</u>: $M_m \cap \ker(K-1) = (0)$. Thus $N_{m+1} = N_m$, and $n \leq m$, so <u>$m=n$</u>.

Let $x \in M_m \cap \ker(K-1)$. Since $(K-1)^n M_m = M_m$, $\exists y \in M_m$ s.t. $x = (K-1)^n y$. Then $0 = (K-1)x = (K-1)^{n+1} y \Rightarrow y \in N_{n+1} = N_n$.
So $x = (K-1)^n y = 0$.

Now if $z \in N_{m+1}$, $(K-1)^{m+1} z = 0$, so $(K-1)^m z \in M_m \cap \ker(K-1) = (0)$, so $x \in N_m$

<u>Note</u>: as a corollary to Step ④, we see that $(K-1)|_{M_n}$ is <u>injective</u> on M_n. Since $(K-1)M_n = M_n$, $(K-1)|_{M_n}$ is a cts bij $M_n \to M_n$, and thus inv. by OMT, proving <u>a)</u>.

<u>Step ⑤</u>: $X = M_n \oplus N_n$

<u>Pf</u>: ~~$x \in M_n \cap N_n \implies x = (K-1)^n y$~~ Since $(K-1)|_{M_n} \in B(M_n)$ is invertible, so is $(K-1)^n|_{M_n}$. So if $x \in M_n \cap N_n$, then $(K-1)^n x = 0 \implies x = 0$.

Now for arbitrary $x \in X$, look at $(K-1)^n x \in M_n$. Since $(K-1)^n|_{M_n}$ inv., $\exists\, y \in M_n$ s.t. $\underset{M_n}{(K-1)^n y} = \underbrace{(K-1)^n x}_{\in M_n}$.

Then $x - y \in N_n$, and $x = \underset{M_n}{y} + \underset{N_n}{(x-y)}$.

<u>Cor</u>: $K-1$ not inv. $\iff \ker(K-1) \neq (0)$

($\exists$ an eigenvector!)

<u>Pf</u>: $K-1$ not inv. $\iff$ ~~$M_n \neq X = M_n \oplus N_n$~~ $K-1$ not bij. (OMT)

$\iff N_1 \neq (0)$ <u>or</u> $M_1 \neq X$

$\iff N_1 \neq (0)$ <u>and</u> $M_1 \neq X$.

<u>Cor</u>: $\dim(\ker(K-1)) \underset{①}{=} \operatorname{codim}_X((K-1)X) \underset{②}{=} \dim(\ker(K^*-1))$.

<u>Pf</u>: ① $\operatorname{codim}_X((K-1)X) = \operatorname{codim}_{N_n}((K-1)N_n)$

$= \dim(\ker(K-1)|_{N_n})$ (Rank-Nullity Thm)

$= \dim(\ker(K-1))$

② $[\underbrace{(K-1)X}_{\cong X/(K-1)X}]^\perp = \ker(K^*-1)$ $\rightsquigarrow$ result.

<u>Note that</u> results for $K-1$ <u>also</u> apply to $K - z1 = z(\tfrac{1}{z}K - 1)$ $\forall z \in \mathbb{C}\setminus\{0\}$!

Conclusion: $\forall z \neq 0$, $\exists$ pair of complementary subspaces, K-invariant, M, N (8)

s.t. (i) $\dim(N) < \infty$; $\operatorname{codim}(M) < \infty$

(2) $(K - z1)|_M$ is invertible in $B(M)$

(3) $(K - z1)|_N$ is nilpotent.

Moreover, $\dim(\ker(K - z1)) = \operatorname{codim}(\operatorname{range}(K - z1))$ $(z \neq 0)$.

Cor: The non-zero pts in $sp(K)$ are isolated eigenvalues. all eigenspaces are f.d.

There are only countably many of them, and 0 is the only possible acc. pt.

Pf: Suppose $\lambda \in sp(K) \setminus \{0\}$. Then $(K - z1)|_M$ is invertible for $|z - \lambda|$ small and $\cancel{z \neq 0}$, since $GL(M) \subseteq B(M)$ open. But since $(K - \lambda 1)|_N$ is nilpotent, $(K - z1)|_N$ is invertible for $|z - \lambda| > 0$. Thus $K - z1$ is invertible when $0 < |\lambda - z| < \varepsilon$ for some small ε.

Cor (Fredholm Alternative):

(1) For $\lambda \neq 0$, the eq'n $Kx - \lambda x = y$ has a ! sol'n x $\forall y \in X$.

$\iff$ the eq'n $K^* \varphi - \lambda \varphi = \psi$ has a ! sol'n φ $\forall \psi \in X^*$

Pf: wma $\lambda = 1$:

$(K - 1)X = X \iff \ker(K - 1) = (0) = [(K^* - 1)X^*]_\perp$

$\iff (K^* - 1)X^* = X^*$

$\iff \ker(K^* - 1) = (0)$.

$\ker(K^* - 1) = [(K - 1)X]^\perp$

(2) $y \in (K - 1)X \iff \forall \varphi \in \ker(K^* - 1) = [(K - 1)X]^\perp$, $\varphi(y) = 0$.

(3) $\varphi \in (K^* - 1)X^* \iff \forall x \in \ker(K - 1)$, $\varphi(x) = 0$

$\ker(K - 1)^\perp = \underbrace{(K^* - 1)X^*}_{\text{closed.}}$

Fredholm ops

Let X, Y be Banach spaces. ①

Def: $T \in B(X,Y)$ is a Fredholm op if $\dim \ker T < \infty$ and $\operatorname{codim} TX = \dim(Y/TX) < \infty$. In this case,

$$\operatorname{ind}(T) := \dim(\ker(T)) - \operatorname{codim}(TX), \quad \text{Fredholm index of } T$$

Examples:

① Invertible ops have index 0.

② Suppose $K \in B(X)$ cpt. Then $\forall \lambda \neq 0$, $K - \lambda 1$ Fredholm, w/ index 0.

③ $\pm$Shift $S_\pm$ on ℓ^p has index ∓ 1. $(Sx)_n = x_{n\pm 1}$

④ X, Y f. dim'l, the Rank-Nullity Thm says: [① $\dim(X) = \dim \ker T + \dim TX$. ② $\dim X = \dim \ker T + \dim Y - \operatorname{codim} TX$.] $\operatorname{index}(T) = \dim(X) - \dim(Y)$.

Prop: Suppose $T \in B(X,Y)$ Fredholm. Then $\forall$ closed subspace $Z \subseteq X$, $TZ \subseteq Y$ is closed.

~~TX~~ Y admits closed E s.t. $TX \oplus E = Y$ — used

seems we only used $\operatorname{codim} TX < \infty$.

not nec. closed!

Pf: Claim 1: If $Z, F \subseteq X$ closed subspaces w/ F fin. dim'l, $Z+F$ closed.

Pf: Consider $Q: X \to X/Z$. Then QF is f. dim'l, and thus closed in X/Z. Then $Z + F = Q^{-1}(QF)$, closed.

Let $X_0 = X/\ker T$, and pick $y_1, \dots, y_n \in Y$ s.t. $\{y_i + TX \mid i = 1, \dots, n\}$ is a basis for Y/TX. Let $F = \operatorname{span}\{y_1, \dots, y_n\} \subseteq Y$.

Now consider $X_0 \oplus F$ w/ norm $\|u + v\| := \|u\|_{X_0} + \|v\|_F$, ℓ^1 norm.

Define $\tilde{T}: X_0 \oplus F \to Y$ by $(x + \ker T) + y \mapsto Tx + y$.

Clearly $\tilde{T}$ is bdd and bijective, hence invertible by OMT.

If $Z \subseteq X$ closed subspace, by Claim 1, $Z + \ker T$ closed in X, so $(Z + \ker T)/\ker T$ closed in X_0, and also in $X_0 \oplus F$. Hence

$\tilde{T}[(Z + \ker T)/\ker T] = TZ$ is closed in Y.

Atkinson's

Thm: For $T \in B(X,Y)$ TFAE: (2)

(1) T Fredholm

(2) $\exists S \in B(Y,X)$ s.t. $ST-1$ and $TS-1$ have finite rank.

(3) $\exists$ ~~S~~ $\in B(Y,X)$ s.t. $ST-1$ and $TS-1$ are cpt.

(4) $\exists S_1,S_2$ - - - - - S_1T-1 - . TS_2-1 - - - .

Pf: (1)$\Rightarrow$(2): Let T be Fredholm. Take an idempotent $P \in B(X)$ st. $\ker T = \ker P$. Take an idempotent $Q \in B(Y)$ st. $TX = QY$.

Then $T|_{PX} \in B(PX, QY)$ is invertible. Let S_0 be inverse.

Define $S = S_0 Q \in B(Y, PX)$. Then $STx = S_0QTx = \begin{cases} 0 & x \in \ker T = \ker P \\ S_0Tx & x \in PX. \end{cases}$

$$T = \begin{bmatrix} QTP & QT(1-P) \\ (1-Q)TP & (1-Q)T(1-P) \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} PX \\ \oplus \\ \ker P = (1-P)X \end{pmatrix} \to \begin{bmatrix} QY \\ (1-Q)Y \end{bmatrix}$$

$$S = \begin{bmatrix} S_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} QY \\ (1-Q)Y \end{bmatrix} \to \begin{bmatrix} PX \\ (1-P)X \end{bmatrix}$$

$$\leadsto ST = \begin{bmatrix} S_0T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P, \quad \underbrace{1-ST}_{=\boxed{1-P}} \text{ has finite rank}$$

$$TS = \begin{bmatrix} TS_0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = Q, \quad 1-Q \text{ has finite rk.}$$

(2)$\Rightarrow$(3): Trivial, ~~(3)$\Rightarrow$(4)~~: Also trivial. S_1, S_2

(4)$\Rightarrow$(1): Suppose $S \in B(Y,X)$ s.t. $ST-1 = K_1$ cpt and $TS-1 = K_2$ cpt.
$S_1T-1 = K_1$ $TS_2-1 = K_2$

Then $\ker T \subseteq \ker(1+K_1)$ is f. dim'l.

$(1+K_2)Y \subseteq TX \Rightarrow \operatorname{codim}(TX) < \infty$.

same pf, better statement.

Def: An S as in (3) above is a Fredholm inverse of T.

Observe: two Fredholm inverses differ by a cpt: If S_1, S_2 two F.I.'s,

$$S_1 = S_1 1 = S_1(TS_2 - K_2) = (S_1T)S_2 - S_1K_2 = (1+K_1)S_2 - S_1K_2 = S_2 + \underbrace{K_1S_2 - S_1K_2}_{\text{cpt.}}$$

Lem: If $\exists S_1, S_2 \in B(Y,X)$ s.t. S_1T-1 cpt and TS_2-1 cpt, then T Fredholm.

(Can build into Thm!)

Def: Let $K(X)$ = closed 2-sided ideal of ~~cpt~~ ops in $B(X)$.
Quotient $B(X)/K(X)$ called the Calkin algebra on X.

Note: If A a Banach alg and $I \subseteq A$ a closed 2-sided ideal,
A/I is a Banach alg under quotient norm w/ $(a+I)(b+I) := ab+I$.

Pf: must show $\|\cdot\|_{A/I}$ submult.

$$\|ab+I\| = \inf_{i\in I} \|ab+i\| \le \inf_{i,j\in I} \|ab+\underbrace{aj+ib+ij}_{\in I}\| = \inf_{i,j\in I} \|(a+i)(b+j)\|$$

$$\le \inf_{i,j\in I} \|a+i\|\cdot\|b+j\| = \inf_{i\in I}\|a+i\| \cdot \inf_{j\in I}\|b+j\| = \|a+I\|\cdot\|b+I\|$$

Properties of Fredholm ops:

1. $T \in B(X)$ Fredholm $\iff$ $T+K(X)$ invertible in Calkin alg.
2. $T \in B(X)$ Fredholm, $K \in K(X)$ $\Rightarrow$ $T+K$ Fredholm.
3. A Fred. inv. of a Fred. op. is Fred.
4. Product of 2 Fredholms is Fredholm.
5. Adjoint of Fredholm is Fredholm, $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$.

Pf: $\ker T^* = (TX)^\perp \cong (Y/TX)^*$, f. dim'l. $\dim \ker T^* = \mathrm{codim}(TX)$.
~~[illegible]~~ Note TX closed $\Rightarrow T^*Y^*$ closed by Closed Range Thm.
Now: $X^*/T^*Y^* \cong \ (T^*Y^*)_\perp{}^\top \cong [(T^*Y^*)_\perp]^* = (\ker T)^*$.
Thus $\dim X^*/T^*Y^* = \mathrm{codim}\, T^*Y^* = \dim (\ker T)^* = \dim \ker T$.

Multiplication Thm: If $T \in B(X,Y)$, $S \in B(Y,Z)$ Fredholm,
then $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$.

Proof:
Case 1: X, Y, Z f. dim'l. $\mathrm{ind}(S) = \dim Y - \dim Z$, $\mathrm{ind}(T) = \dim X - \dim Y$ ✓
~~Case 2: Arbitrary X, Y, Z. Find complementary subspaces X_0, X_1 of X,~~
~~Y_0, Y_1 of Y. Z_0, Z_1 of Z s.t.~~

~~① X_0, Y_0, Z_0 f. dim'l.~~
~~② $TX_0 \subseteq Y_0$, $SY_0 \subseteq Z_0$~~
~~③ $TX_1 \subseteq Y_1$, $SY_1 \subseteq Z_1$~~

all Banach spaces

f. dim'l as ST Fredholm

Claim 1:

$$0 \to \ker T \xrightarrow{i} \ker ST \xrightarrow{T} \ker S \xrightarrow{Q} Y/TX \xrightarrow{S} Z/STX \to Z/SY \to 0$$

$$x \mapsto x \qquad x \mapsto Tx \qquad y \mapsto y+TX \qquad y+TX \mapsto Sy+STX \qquad z+STX \mapsto z+SY.$$

is exact.

Claim 2: If $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ is exact, then

$$\sum_j (-1)^j \dim V_j = 0.$$

Pf of Mult. Thm: Apply Claim 2 to the exact seq. in Claim 1:

$\operatorname{ind}(T) = \dim\ker T - \operatorname{codim} TX \qquad \operatorname{ind}(S) = \dim\ker S - \operatorname{codim} SY.$

$$0 = \dim\ker T - \dim\ker ST + \dim\ker S - \operatorname{codim} TX + \operatorname{codim} STX - \operatorname{codim} SY$$

$$0 = \operatorname{ind}(T) - \operatorname{ind}(ST) + \operatorname{ind}(S).$$

Cor: If $T \in B(X,Y)$ Fredholm has a Fredholm inverse S, then $\operatorname{ind}(S) = -\operatorname{ind}(T)$.

Stability Thm: If $T \in B(X,Y)$ Fredholm, $\exists \varepsilon > 0$ s.t. $\forall\, U \in B(X,Y)$ w/ $\|U\| < \varepsilon$, $T+U$ is Fredholm w/ $\operatorname{ind}(T+U) = \operatorname{ind}(T)$.

Pf: Let S be a Fredholm inv. of T, and let $\varepsilon = \|S\|^{-1}$. Then $\exists$ cpt K_1, K_2 s.t. $ST = 1+K_1$ and $TS = 1+K_2$. If $\|U\| < \varepsilon$, $S(T+U) = 1+K_1+SU$, and $\|SU\| \le \|S\|\cdot\|U\| < 1 \Rightarrow 1+SU$ is invertible. Thus:

$$(*) \quad \left.\begin{aligned} (1+SU)^{-1}S(T+U) &= 1+\underbrace{(1+SU)^{-1}K_1}_{\text{cpt}} \\ (T+U)S\underbrace{(1+US)^{-1}}_{\text{also invertible}} &= 1+\underbrace{K_2(1+US)^{-1}}_{\text{cpt}} \end{aligned}\right\} \Rightarrow T+U \text{ Fredholm.}$$

By Mult. Thm and $(*)$, $\underbrace{\operatorname{ind}(1+SU)^{-1}}_{0} + \underbrace{\operatorname{ind}(S)}_{-\operatorname{ind}(T)} + \operatorname{ind}(T+U) = 0 \Rightarrow$ result.

Cor: If $T \in B(X,Y)$ Fredholm, $K \in B(X,Y)$ cpt, then [$T+K$ Fredholm and] $\mathrm{ind}(T+K) = \mathrm{ind}(T)$.

Pf: By the Stability Thm, the fct $[0,1] \to \mathbb{Z}$ given by $t \mapsto \mathrm{ind}(T+tK)$ is cts, and thus constant.

Def: $T \in B(X,Y)$ is left (resp. right) Fredholm if $\exists S \in B(Y,X)$ s.t. $ST = 1+K$ for some cpt $K \in B(X)$.
(resp $TS = 1+K$ $\quad \cdot \quad \cdot \quad$ $K \in B(Y)$.)

Prop: (1) T left Fredholm $\iff$ TX closed and complemented and $\ker(T)$ is f. dim'l.
$\iff \exists S$ s.t. $ST = 1+K$ for a finite rk K.

(2) T right Fredholm $\iff$ $\ker T$ complemented and $\mathrm{codim}\, TX < \infty$.
$\iff \exists S$ s.t. $TS = 1+K$ for a finite rk K.

Pf: Can be extracted from pf. of Atkinson's Thm.

(1) $\Rightarrow$: If $\exists S$ cpt s.t. $ST - 1 = K$ cpt, $\ker T \subseteq \ker(1+K)$ f. dim'l.
Now write $ST = 1+K$, cpt. we have a pair of complementary $E, F \subset X$ s.t.

- $X = E \oplus F$
- $1+K$ inv. on F
- $1+K$ nilpotent on E, f.dim'l.

Now F closed $ST = 1+K$ inv. ~~on F $\Rightarrow$ bdd below, i.e. $\exists c > 0$ s.t.~~
~~$\|S\|\|Tx\| \geq \|STx\| \geq c\|x\|$ $\forall x \in F$. $\Rightarrow$ $\|Tx\| \geq \frac{c}{\|S\|}\|x\|$ $\forall x \in F$,~~

so $T|_F$ is ~~inj~~ left inv! $\Rightarrow$ $\boxed{TF \text{ is closed and complemented}}$. But $\underset{\text{f.dim'l}}{TX = TE} + TF$, closed. complemented.

$(ST)^{-1}|_F S$ is L-inv.

Claim: If $Z \subset X$ closed + complemented and $F \subseteq X$ f.dim'l, $Z+F$ closed + complemented.

Pf: Let P be an idempotent s.t. $Z = PX$. Then $(1-P)F \subseteq (1-P)X$ is f. dim'l, and thus complemented. Let $Q \in B((1-P)X)$ s.t. $Q(1-P)X = (1-P)F$. Extend Q to X by $Q = 0$ on $PX = Z$. Then $[P,Q] = 0$, since $PQ = QP = 0$.
Define idempotent $R = P + Q\cancel{(1-P)}$. Then $RX = PX + (1-P)F = Z+F$.

⑥

$\Leftarrow$: Let $P \in \mathcal{B}(X)$ be idempotent s.t. $\ker P = \ker T$, and let $Q \in \mathcal{B}(Y)$ be idempotent s.t. $QY = TX$. Then $T|_{PX}$ is invertible, w/ inverse S_0. Define $S = S_0 Q$

$$T \longleftrightarrow \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} PX \\ \underset{\substack{\| \\ (1-P)X}}{\ker T} \end{bmatrix} \longrightarrow \begin{bmatrix} QY = TX \\ (1-Q)Y \end{bmatrix}$$

$$S \longleftrightarrow \begin{bmatrix} S_0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} QY \\ (1-Q)Y \end{bmatrix} \longrightarrow \begin{bmatrix} PX \\ (1-P)X \end{bmatrix}$$

Then $ST = \begin{bmatrix} S_0T & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = P \in \mathcal{B}(X)$, and $1-P = 1-ST$ has finite rank. (onto $\ker T$)

---

② $\Rightarrow$: Suppose $TS = 1+K$, K cpt. Then $(1+K)Y \subseteq TX$, so TX has fin. codim, and is complemented by the lemma/prop.

Decompose $Y = E_1 \oplus E_2 \oplus F$ s.t.

- $E = E_1 \oplus E_2$, F are K-inv., $(1+K)|_E$ nilp. and $(1+K)|_F$ inv.
- $TX = E_2 \oplus F$ (note that $TS|_F$ inv. $\Rightarrow$ $F \subseteq TX$)

Now let $N = T^{-1}E_2$ and $M = T^{-1}F$, closed complementary subspaces for $X = N \oplus M$. Note $\ker T = \ker(T|_N) \oplus \ker(T|_M)$ so it suffices to show both are complemented. First, $T|_F$ has R-inv. $S(TS|_F)^{-1}$, so $\ker(T_F)$ complemented. Next, $T|_N : N \twoheadrightarrow E_2$, which is f. dim'l, so $\exists$ bdd right inv. $E_2 \to N$ for $T|_N$, and $\ker(T|_N)$ complemented.

$\Leftarrow$: use idempotents to ~~construct~~ construct an S s.t. $TS = 1-P$, P finite rk.