The group measure space / crossed product construction

Def: An *action* of a gp G on a vNa M is a gp. homom. $\alpha: G \longrightarrow \mathrm{Aut}(M) = \{\text{unital } *\text{-alg. iso's } M \to M\}$.

Exercise: Every unital $*$-alg. iso of M is σ-WOT cts.

Example: Suppose $u: G \longrightarrow \mathcal{U}(M)$ s.t. $\forall g \in G$, $u_g M u_g^* = M$. Then $\alpha: G \longrightarrow \mathrm{Aut}(M)$ by $\alpha_g = \mathrm{Ad}(u_g)$ is an action.

Def: An *inner* automorphism of M is an aut Φ of M s.t. $\exists\, u \in \mathcal{U}(M)$ w/ $\Phi(x) = uxu^*$ $\forall x \in M$. In this case, we say Φ is *implemented* by the unitary u. If Φ is not inner, Φ is called *outer*. An action $\alpha: G \to \mathrm{Aut}(M)$ is called *outer* if α_g inner $\Rightarrow g = e$.

Example: Let (X, μ) be a measure space and T a bijection of X preserving the measure class of μ.

$\Leftrightarrow$ $\mu(A) = 0 \iff \mu(T^{-1}A) = 0$ $\forall A$ m'ble.

Then T gives an automorphism α_T of $L^\infty(X, \mu)$ by

$$(\alpha_T f)(x) := f(T^{-1}x).$$

Moreover, if T preserves μ, then α_T is implemented by $(u_T \xi)(x) := \xi(T^{-1}x)$ for $\xi \in L^2(X, \mu)$. Observe:

$$(u_T f u_T^* \xi)(x) = (f u_T^* \xi)(T^{-1}x) = f(T^{-1}x)(u^*\xi)(T^{-1}x) = f(T^{-1}x)\,\xi(x) = [\alpha_T(f)\xi](x).$$

Def: A bijection T of (X, μ) as above is called *ergodic* if A m'ble w/ $T(A) = A \implies \mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Prop: T ergodic $\iff L^\infty(X, \mu)^{\alpha_T} = \mathbb{C}1$.

Pf: Note that $T(A) = A \iff \alpha_T(\chi_A) = \chi_A$. Hence T is ergodic $\iff P(L^\infty(X, \mu)^{\alpha_T}) = \{0, 1\} \iff L^\infty(X, \mu)^{\alpha_T} = \mathbb{C}1$.

Def: Say an action $\alpha: G \to \mathrm{Aut}(M)$ is *ergodic* if $M^G = \mathbb{C}1$.

Exercises:

① Any gp action $\alpha: G \to \mathrm{Aut}(M)$ is *implementable* if M is a II_1 factor acting on $L^2(M, tr)$. [Hint: α_g preserves tr!]

② Do ① for a finite vNa M w/ σ-WOT cts tracial state tr s.t. $\forall g \in G$, $tr \circ \alpha_g = tr$. [This is automatic by !ness of tr in ①.]

③ All automorphisms of $B(H)$ are *inner*.

④ Let $\mathbb{F}_2 = \langle a, b \rangle$. Show $a \mapsto b;\ b \mapsto a$ extends to an automorphism of $L\mathbb{F}_2$. Prove it is *outer*.

Def: If G is topological, say an action $\alpha: G \to \mathrm{Aut}(M)$ is:
____ cts if $\forall x \in M$, $\forall g_\lambda \to g$ in G, $\alpha_{g_\lambda}(x) \to \alpha_g(x)$ in ____.
Here, the blank could be norm, SOT, *-SOT. usually best!

⑤ Is the translation action of $\mathbb{R}$ on $L^\infty(\mathbb{R})$ norm, SOT, *-SOT cts?

Let G be a loc. cpt gp w/ dg its left translation invariant Haar meas. Suppose $\alpha: G \to \mathrm{Aut}(M)$ is an action where $M \subseteq B(H)$. Form the Hilbert space $L^2(G, H) = \{\xi: G \to H \mid \int \|\xi(g)\|^2 dg < \infty\} \cong L^2 G \otimes H$.
Then G and M act on $L^2(G, H)$ via:

- $u_g = \lambda_g \otimes 1$, i.e., $(u_g \xi)(h) = \xi(g^{-1}h)$
- $[\pi(m)\xi](h) = \alpha_{h^{-1}}(m)\xi(h)$.

Exercise: $\pi: M \to B(L^2(G, H))$ is a an injective σ-WOT cts unital $*$-homom, so $\pi(M) \cong M$ as vNa's. Moreover, $\forall g \in G$, $u_g \pi(m) u_g^* = \pi(\alpha_g(m))$, i.e., the G-action is implemented by the u_g.

Def: The *crossed product* $M \rtimes_\alpha G = (\pi(M) \cup \{u_g\})'' \subseteq B(L^2(G, H))$.
– contains M and unitaries implementing G-action $\alpha: G \to \mathrm{Aut}(M)$.

Exercise: Finite linear combinations $\sum x_g u_g$ form a σ-WOT dense unital $*$-subalg of $M \rtimes_\alpha G$.

We'll be interested in the case M finite w/ faithful normal tracial state tr and Γ a discrete gp w/ $\mathrm{tr}\circ\alpha_g = \mathrm{tr}$ $\forall g$

$\hookrightarrow$ 2^{nd} equivalent defn of $M \rtimes_\alpha \Gamma$:

Form $H = L^2M \otimes \ell^2\Gamma$. Get amplified left M-action and left Γ-action by $u_g(m\Omega \otimes \delta_h) = \alpha_g(m)\Omega \otimes \delta_{gh}$, i.e., if $v_g \in \mathcal{U}(L^2M)$ s.t. $v_g(m\Omega) = \alpha_g(m)\Omega$ $\forall m \in M$, $u_g = v_g \otimes \lambda_g$.

Defn $M \rtimes_\alpha \Gamma = \{m \otimes 1, u_g = v_g \otimes \lambda_g \mid m \in M, g \in \Gamma\}'' \subseteq B(H = L^2M \otimes \ell^2\Gamma)$

Right action: If $x \in M$, $g \in \Gamma$, get right action of M and Γ on $L^2M \otimes \ell^2\Gamma$ by $(m\Omega \otimes \delta_h)x = m\alpha_h(x)\Omega \otimes \delta_h$ and $(m\Omega \otimes \delta_h)\cdot g = m\Omega \otimes \delta_{hg}$ (right Γ-action is $1 \otimes \rho$).

Note the left + right actions of $M, \Gamma \curvearrowright L^2M \otimes \ell^2\Gamma \curvearrowleft M, \Gamma$ commute. We'll eventually show that $L^2M \otimes \ell^2\Gamma \cong L^2(M \rtimes_\alpha \Gamma)$ and that these L/R actions match the usual L/R actions.

Lemma: If M a finite vNa w/ faithful normal tracial state tr, then $M' \cap B(L^2M)$ is JMJ where $Jm\Omega = m^*\Omega$ conj. lin. unitary. $\hookrightarrow$ prove for HW. For $M = L\Gamma$, $JMJ = R\Gamma$.

Claim 1: $\forall x \in M \rtimes_\alpha \Gamma$ $\exists!(x_g) \in \ell^2(\Gamma, M) = \{m : \Gamma \to M \mid \sum \|m_g\Omega\|^2_{L^2M} < \infty\}$ s.t. $x(\Omega \otimes \delta_e) = \sum x_g\Omega \otimes \delta_g$.

Pf: For $g \in \Gamma$, define $P_g : L^2M \otimes \ell^2\Gamma \to L^2M$ by $m\Omega \otimes \eta \longmapsto \langle \eta, \delta_g \rangle \alpha_{g^{-1}}(m)\Omega$.
Then $P_g^* : L^2M \to L^2 \otimes \ell^2\Gamma$ is given by $m\Omega \longmapsto \alpha_g(m) \otimes \delta_g$. Notice that P_g^* is right M-linear $\forall g \Rightarrow$ so is P_g:

$$P_g^*(Jx^*J m\Omega) = P_g^*(mx\Omega) = \alpha_g(mx)\Omega \otimes \delta_g = \alpha_g(m)\alpha_g(x)\Omega \otimes \delta_g$$
$$= (\alpha_g(m)\Omega \otimes \delta_g)\cdot x = P_g^*(m\Omega)\cdot x$$

Thus $P_g x P_e^* \in (JMJ)' = M \subseteq B(L^2M)$. Define $x_g = \alpha_g(P_g x P_e^*) \in M$. Then:

$$\langle x(\Omega \otimes \delta_e), m\Omega \otimes \delta_g \rangle = \langle xP_e^*\Omega, P_g^*\alpha_{g^{-1}}(m)\Omega \rangle = \langle P_g x P_e^*\Omega, \alpha_{g^{-1}}(m)\Omega \rangle$$
$$= \mathrm{tr}(\alpha_{g^{-1}}(m)^*[P_g x P_e^*]) = \mathrm{tr}(m^* x_g) = \langle \sum_h x_h \Omega \otimes \delta_h, m\Omega \otimes \delta_g \rangle \quad \forall m \in M, g \in \Gamma.$$

Hence $\sum x_h\Omega \otimes \delta_h \in L^2M \otimes \ell^2\Gamma$ and equals $x(\Omega \otimes \delta_e)$.

Claim 2: $\forall g \in \Gamma$, $m \in M$, $x(m\Omega \otimes \delta_g) = \sum_h x_h \alpha_h(m)\Omega \otimes \delta_{hg}$

Pf: Note $m\Omega \otimes \delta_g = (\Omega \otimes \delta_e)\cdot m \cdot g$. Since right + left actions commute,

$$x(m\Omega\otimes\delta_g) = [x(\Omega\otimes\delta_e)]\cdot m\cdot g = \Big[\sum_h x_h \Omega\otimes\delta_h\Big]\cdot m\cdot g$$

$$= \sum_h x_h \alpha_h(m)\Omega\otimes\delta_{hg}.$$

Claim 3: $\Omega\otimes\delta_e$ is cyclic + separating for $M \rtimes_\alpha \Gamma$.

Pf: $x=0 \iff x_g = 0\ \forall g \implies \Omega\otimes\delta_e$ separating.

If $\xi = \sum m_g \Omega\otimes\delta_g$ is a finite sum in $L^2M\otimes\ell^2\Gamma$, then $\xi = (\sum m_g u_g)(\Omega\otimes\delta_e)$, so $\xi \in (M\rtimes_\alpha\Gamma)(\Omega\otimes\delta_e) \Rightarrow \Omega\otimes\delta_e$ cyclic.

Claim 4: $\omega_{\Omega\otimes\delta_e}$ is a faithful normal tracial state on $M\rtimes_\alpha\Gamma$.

Pf: Tracial: For a finite linear comb's $\sum x_g u_g, \sum y_h u_h \in M\rtimes_\alpha\Gamma$,

$$\langle(\sum x_g u_g)(\sum y_h u_h)(\Omega\otimes\delta_e), \Omega\otimes\delta_e\rangle = \sum\langle x_g\alpha_g(y_h)u_g u_h(\Omega\otimes\delta_e), \Omega\otimes\delta_e\rangle$$

$$= \sum\langle x_g\alpha_g(y_h)(\Omega\otimes\delta_{gh}), \Omega\otimes\delta_e\rangle$$

$$= \sum_{gh=e}\langle x_g\alpha_g(y_h)\Omega, \Omega\rangle$$

$$= \sum_g tr(x_g\alpha_g(y_{g^{-1}}))$$

$$= \sum_g tr\circ\alpha_{g^{-1}}(x_g\alpha_g(y_{g^{-1}}))$$

$$= \sum_g tr(\alpha_{g^{-1}}(x_g)y_{g^{-1}})$$

$$= \sum_g tr(y_{g^{-1}}\alpha_{g^{-1}}(x_g))$$

$$= \langle(\sum y_h u_h)(\sum x_g u_g)(\Omega\otimes\delta_e), \Omega\otimes\delta_e\rangle.$$

Now use $\omega_{\Omega\otimes\delta_e}$ is normal and σ-WOT density of the unital $*$-alg of finite linear combinations.

Claim 5: $L^2M\otimes\ell^2\Gamma \cong L^2(M\rtimes_\alpha\Gamma)$ w.r.t. $tr = \omega_{\Omega\otimes\delta_e}$.

Pf: The map $m\Omega\otimes\delta_g \longmapsto m u_g\hat{1} \in L^2(M\rtimes_\alpha\Gamma)$ where $\hat{1}$ is image of 1 extends to a L/R M/Γ-linear unitary isomorphism.

Def: An automorphism $\alpha \in \mathrm{Aut}(M)$ is <u>free</u> or <u>properly outer</u> if
$y \in M$ and $y\alpha(x) = xy$ $\forall x \in M \implies y = 0$.

An action $\alpha: G \to \mathrm{Aut}(M)$ is <u>free</u> if α_g not free $\implies g = e$.

<u>Exercise:</u> Suppose X cpt Hausd. and μ is a finite non-neg. regular Borel measure on X. Let $T: (X,\mu) \to (X,\mu)$ be a homeom. preserving the measure class of μ. Show α_T free $\iff \mu(\{x \mid Tx = x\}) = 0$.

<u>Prop:</u> If M is a factor, then every outer automorphism is free.

<u>Pf:</u> Suppose $\alpha \in \mathrm{Aut}(M)$ and $\exists y \in M \setminus \{0\}$ s.t. $y\alpha(x) = xy$ $\forall x \in M$. We'll show α <u>inner</u>. If $y \in \mathcal{U}(M)$, we'd be finished. Taking adjoints, we have $y^* x = \alpha(x) y^*$ $\forall x \in M$. Thus $yy^* x = y\alpha(x) y^* = xyy^*$ $\forall x \in M$, so $yy^* \in Z(M)$. Similarly, $y^*y \in Z(M)$: $y^*yx = y^*\alpha^{-1}(x)y = xy^*y$.
Recall $\mathrm{sp}(yy^*) \cup \{0\} = \mathrm{sp}(y^*y) \cup \{0\}$. Since $Z(M) = \mathbb{C}1$, yy^* and y^*y are two elts of $[0,\infty)$, and $yy^* = 0 \iff y = 0 \iff y^*y = 0$. Thus $yy^* = y^*y =: r \in [0,\infty)$. If $y \neq 0 \iff r \neq 0$, $u = r^{-1/2} y \in \mathcal{U}(M)$ and $\alpha = \mathrm{Ad}(u)$, so α is inner. So α outer $\implies y = 0 \implies \alpha$ free.

<u>Lem:</u> If $\alpha: \Gamma \to \mathrm{Aut}(M)$ is <u>free</u>, $M' \cap M \rtimes_\alpha \Gamma \subseteq Z(M)$.

<u>Note:</u> This implies: (a) $Z(M) = M' \cap M \subseteq M' \cap M \rtimes_\alpha \Gamma \subseteq Z(M)$, and
(b) $Z(M \rtimes_\alpha \Gamma) \subseteq M' \cap M \rtimes_\alpha \Gamma \subseteq Z(M)$.

<u>Pf:</u> Suppose $y \in M' \cap M \rtimes_\alpha \Gamma$ and let $(y_g) \subset M$ s.t. $y(\Omega \otimes \delta_e) = \sum y_g \Omega \otimes \delta_g$.
Then for $x \in M$, we calculate $y(x\Omega \otimes \delta_e)$ in 2 ways:

$$y(x\Omega \otimes \delta_e) = \sum_g y_g \alpha_g(x) \otimes \delta_g \quad \text{by } \underline{\text{Claim 2}}.$$
$$\shortparallel$$
$$yx(\Omega \otimes \delta_e) = xy(\Omega \otimes \delta_e) = \sum x y_g \otimes \delta_g$$

The only way these two vectors in $L^2 M \otimes \ell^2 \Gamma$ can agree is if $y_g \alpha_g(x) = x y_g$. Since x is arbitrary and α is free, we must have $y_g = 0$ for $g \neq e$. Now for $g = e$, $y_e(\Omega \otimes \delta_e) = y_e \Omega \otimes \delta_e = y(\Omega \otimes \delta_e)$. Since $\Omega \otimes \delta_e$ is separating by <u>Claim 3</u>, $y = y_e \in M$.
Since $y \in M'$, $y \in M' \cap M = Z(M)$.

Cor: Suppose $\alpha: \Gamma \to \mathrm{Aut}(M)$ is an action.

① If M is a factor and α is outer, then $M \rtimes_\alpha \Gamma$ is a factor.

② If α is free and ergodic, then $M \rtimes_\alpha \Gamma$ is a factor.

Pf: ① M a factor and α outer $\Rightarrow$ α free, so by the Lemma,
$Z(M \rtimes_\alpha \Gamma) \subseteq M' \cap M \rtimes_\alpha \Gamma \subseteq Z(M) = \mathbb{C}1$.

② $Z(M \rtimes_\alpha \Gamma) \subseteq M' \cap M \rtimes_\alpha \Gamma \subseteq Z(M)$. Suppose $x \in Z(M)$ and $[x, u_g] = 0$ $\forall g$. Then $u_g x u_g^* = \alpha_g(x) = x$ $\forall g \in \Gamma$, so $x \in M^\Gamma = \mathbb{C}1$.

Def: When $M = L^\infty(X, \mu)$ and $\alpha: \Gamma \to \mathrm{Aut}(M)$ comes from an action of Γ on (X, μ) preserving the measure class of μ, we call $L^\infty(X,\mu) \rtimes_\alpha \Gamma$ the group measure space construction.

Prop: $\alpha: \Gamma \to \mathrm{Aut}(L^\infty(X,\mu))$ free $\Rightarrow$ $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is maximal abelian.

Pf: If $L^\infty(X,\mu) \subseteq A \subseteq L^\infty(X,\mu) \rtimes \Gamma$ w/ A abelian, then $A \subseteq Z(L^\infty(X,\mu) \rtimes \Gamma) \subseteq L^\infty(X,\mu)$.

Examples: The following are examples of free + ergodic actions.

① Consider $(X,\mu) = (\mathbb{Z}, \nu)$ counting measure, $\mathbb{Z}$ acts by translation.
$L^\infty(X,\mu) \rtimes \mathbb{Z} \cong B(\ell^2\mathbb{Z})$.

② $(X,\mu) = (\mathbb{T}, d\theta)$, $\Gamma = \mathbb{Z}$ generated by $T(z) = e^{i\alpha}z$ where $\alpha/2\pi \notin \mathbb{Q}$. [HW!]

③ (Bernoulli) Γ infinite countable, (X,μ) probability space. Consider $(X,\mu)^\Gamma$ w/ product measure. $\Gamma \curvearrowright (X,\mu)^\Gamma$ by $(g \cdot A)(h) = A(g^{-1}h)$, where $A: \Gamma \to (X,\mu)$ m'ble.

- Also, can do $\Gamma \curvearrowright \bigotimes^\Gamma (M, \mathrm{tr})$ by $g \cdot (x_{g_1} \otimes x_{g_2} \otimes \cdots) = x_{gg_1} \otimes x_{gg_2} \otimes \cdots$.

④ $SL(2,\mathbb{Z}) \curvearrowright \mathbb{R}^2$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$.

⑤ The "$ax+b$" gp $\mathbb{Q} \rtimes \mathbb{Q}^\times$ acts on $\mathbb{R}$ by $\begin{bmatrix} a & b \\ & 1 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} ax+b \\ 1 \end{bmatrix}$.

Type of the crossed product:

Suppose now $\alpha: \Gamma \to \mathrm{Aut}(M)$ is free + ergodic, so $M \rtimes_\alpha \Gamma$ a factor.

- free: α_g free $\forall g \neq e$, i.e., $y \in M$ s.t. $x\alpha_g(y) = yx$ $\forall x \in M \Rightarrow y = 0$.
- ergodic: $M^\Gamma = \mathbb{C}1$.

Special case: $M = L^\infty(X, \mu)$ for α_g coming from $g \curvearrowright (X, \mu)$.

4 types of free + ergodic actions of countable discrete gps $\Gamma \curvearrowright (X, \mu)$:

type I: Γ acts freely transitively (X a Γ-torsor) [ex ①]

type II_1: Γ preserves a finite measure on X. [ex ② + ③]

type II_∞: Γ preserves an infinite measure on X. [ex ④]

type III: Γ preserves no measure equivalent to μ [ex ⑤]

We'll now show the type of action gives the type of $L^\infty(X, \mu) \rtimes \Gamma$.

Thm I: If $\Gamma \overset{\alpha}{\curvearrowright} (X, \mu)$ is transitive, $L^\infty(X, \mu) \rtimes_\alpha \Gamma$ is type I.

Pf: Since α transitive and Γ countable, X is countable. Hence we may remove all measure zero pts so μ is just weighted counting measure. Consider $p = \mathbb{1}_{\{x\}}$ for any $x \in X$. Then p is a minimal projection in $L^\infty(X, \mu) \rtimes_\alpha \Gamma$, a factor, which must be type I. [[Suppose $y \in L^\infty(X, \mu) \rtimes_\alpha \Gamma$ and let $(y_g) \subseteq M$ be its ℓ^2-sequence s.t. $y(\Omega \otimes \delta_e) = \sum_g y_g \Omega \otimes \delta_g$. Then $pyp(\Omega \otimes \delta_e) = p(\sum_g y_g p \Omega \otimes \delta_g) = \sum_g \underbrace{p y_g \alpha_g(p)}_{\mathbb{1}_{\{x\} \cap \{gx\}} y_g = \delta_{g=e} y_e} \Omega \otimes \delta_g$

$\Rightarrow pyp = y_e p \in \mathbb{C}p$.]]

Thm II_1: Suppose $\Gamma \overset{\alpha}{\curvearrowright} M$ free + ergodic w/ (M, tr) a tracial vNa and $\mathrm{tr} \circ \alpha_g = \mathrm{tr}$ $\forall g$. Then $M \rtimes_\alpha \Gamma$ is either finite dim'l or type II_1.

Pf: It suffices to show $M \rtimes_\alpha \Gamma$ has a normal faithful tracial state [[observe: If $\exists$ an infinite projection in a vNa N, then N cannot have a faithful trace. Indeed if $uu^* = p$ and $u^*u = q \leq p$, then $\forall$ traces $\mathrm{tr}(p - q) = \mathrm{tr}(uu^* - u^*u) = 0$.]] Recall we showed that $\omega_{\Omega \otimes \delta_e}$ is such a normal faithful tracial state on $M \rtimes_\alpha \Gamma$.

Cor: If $\Gamma \overset{\alpha}{\curvearrowright} (X,\mu)$ free, ergodic, non-transitive and μ a finite measure s.t. $\mu(gA)=\mu(A)$ $\forall$ m'ble A, then $L^\infty(X,\mu)\rtimes_\alpha \Gamma$ is type II_1.

Pf: The action of Γ on $L^\infty(X,\mu)$ preserves the faithful normal tracial state $\int \cdot \, d\mu$. Hence $L^\infty(X,\mu)\rtimes_\alpha \Gamma$ is either finite dim'l or type II_1. It suffices to prove $L^\infty(X,\mu)\rtimes_\alpha \Gamma$ finite dim'l and α free + ergodic $\Rightarrow$ α essentially transitive.
Note $L^\infty(X,\mu)$ is a finite dim'l vNa, which has minimal proj's.
Thus $(X,\mu)\cong(Y,\nu)$ for some finite meas. space Y w/ ν a weighted counting measure. [By a maximality argument, can write $1=\sum_{i=1}^n \chi_{A_i}$ where each χ_{A_i} is minimal in $L^\infty(X,\mu)$, $A_i \subseteq X$ disjoint m'ble sets. Define $Y=\{1,\dots,n\}$ w/ $\nu(\{i\})=\mu(A_i)$.]

Exercise: Show that a free + ergodic action of Γ on a finite measure space is transitive.

Obs: A factor M is type II_∞ $\iff$ 1_M infinite and $\exists\, p\in P(M)\setminus\{0\}$ finite s.t. pMp is type II_1.

Lemma: If M type II_∞, $\exists$ a II_1 factor N s.t. $M\cong N\,\overline{\otimes}\,B(\ell^2(I))$.

Pf: Let $p\in P(M)\setminus\{0\}$ be finite and set $N=pMp$. Let $\{p_i\}_{i\in I}$ be a maximal family of $\perp$ proj's s.t. $p_i\approx p$ $\forall i$.

Claim: $\sum p_i \approx 1$.

Pf: Set $q=1-\sum p_i$. By maximality, $p\not\precsim q$, so $q\precsim p$. Since 1_M is infinite, I is infinite, so $\exists\, i_0\in I$ and a bijection $I \hookrightarrow I\setminus\{i_0\}$.
Then $1=q+\sum_{i\in I} p_i \precsim p_{i_0}+\sum_{i\in I\setminus\{i_0\}} p_i = \sum p_i \precsim 1$.

Now since $\sum p_i \approx 1$, $\exists$ p.i. $u\in M$ s.t. $uu^*=\sum p_i$, $u^*u=1$.
$\forall i\in I$, define $q_i=u^*p_iu$. Then $\sum q_i=1$, and $q_i\approx p$ $\forall i$.
Now form a s.m.u. $\{e_{ij}\}$ s.t. $e_{ii}=q_i$ $\forall i$ in the usual way.

Exercise: If $\{e_{ij}\}_{i,j\in I}$ a s.m.u. in a vNa M, then $\exists$ a spatial $*$-iso f
$M\cong e_{11}Me_{11}\,\overline{\otimes}\,B(\ell^2(I))$.
implemented by a unitary on underlying space.

Thm II_∞: If μ is infinite and σ-finite on X and Γ acts freely + ergodically preserving μ but not transitively, then $L^\infty(X,\mu)\rtimes\Gamma$ is type II_∞.

Pf. There are no minimal proj's in $L^\infty(X,\mu)$, so there are subsets $Y\subseteq X$ of arbitrary positive finite measure. Let $Y\subseteq X$ s.t. $0<\mu(Y)<\infty$ and let $\xi:\Gamma\to L^2(X,\mu)$ by $\xi(g)=\delta_{g=e}\,\mu(Y)^{-1/2}\chi_Y$. [Here, $L^\infty(X,\mu)\rtimes\Gamma$ acts on $L^2(\Gamma, L^2(X,\mu))$ by $[u_g\eta](h)=\eta(g^{-1}h)$ and $(f\eta)(h)=\alpha_{h^{-1}}(f)\eta(h)$.]

Now let $p=\chi_Y\in L^\infty(X,\mu)$. We have ω_ξ on the ∞-dim'l factor $p[L^\infty(X,\mu)\rtimes\Gamma]p$ is a normal tracial state, so the compression is type II_1. But $L^\infty(X,\mu)\rtimes\Gamma$ is not type II_1 as it contains an infinite family of mutually $\perp$ equivalent projections [Look in $L^\infty(X,\mu)$.]. Hence $L^\infty(X,\mu)\rtimes\Gamma$ is type II_∞.

We'll skip the proof that if Γ preserves no measure equivalent to μ, then $L^\infty(X,\mu)\rtimes\Gamma$ is type III.

Conditional expectation: Let (M,tr) be a tracial vNa and $\alpha:\Gamma\to\mathrm{Aut}(M)$ an action s.t. $\mathrm{tr}\circ\alpha_g=\mathrm{tr}$ $\forall g\in\Gamma$. Recall that $L^2(M\rtimes_\alpha\Gamma)\cong L^2M\,\overline{\otimes}\,\ell^2\Gamma$ as (normal) $M\rtimes_\alpha\Gamma$ rep'ns, and we have a unital inclusion of tracial vNa's $M\subseteq M\rtimes_\alpha\Gamma$:

$$\begin{aligned}\forall x\in M,\quad \mathrm{tr}_{M\rtimes_\alpha\Gamma}(x)&=\langle x(\Omega\otimes\delta_e),\Omega\otimes\delta_e\rangle\\&=\langle x\Omega\otimes\delta_e,\Omega\otimes\delta_e\rangle\\&=\langle x\Omega,\Omega\rangle_{L^2M}=\mathrm{tr}(x).\end{aligned}$$

Get a (normal) inclusion map $i:L^2M\to L^2M\otimes\ell^2\Gamma$ by $m\Omega\mapsto m\Omega\otimes\delta_e$.

Lem: i is a right M-linear isometry.

Pf: $m\Omega \cdot x = mx\Omega \mapsto mx\Omega \otimes \delta_e = (m\Omega \otimes \delta_e)\cdot x$

$\|m\Omega\|_2^2 = tr_M(m^*m) = tr_{M \rtimes_\alpha \Gamma}(m^*m) = \|m\Omega \otimes \delta_e\|_2^2$.

Fact: If $\pi_H : N \to B(H)$ and $\pi_K : N \to B(K)$ are normal rep'ns of a vNa and $x \in B(H,K)$ s.t. $x\pi_H(n) = \pi_K(n)x$ $\forall n \in N$, then $x^* \in B(K,H)$ satisfies $x^*\pi_K(n) = \pi_H(n)x^*$ $\forall n \in N$. ← just take adjoints!

Cor: $i^* : L^2M \otimes \ell^2\Gamma \to L^2M$ by $m\Omega \otimes \delta_g \mapsto \delta_{g=e}\, m\Omega$ is also right M-linear.

Def: For $x \in M \rtimes_\alpha \Gamma$, define $E_M(x) = i^* x i \in B(L^2M)$. Observe $E_M(x) \in JMJ' = M$. E_M is called a conditional expectation.

(you'll do general case for HW.)

Prop: The conditional expectation $E_M : M \rtimes_\alpha \Gamma \to M$ enjoys the following properties:

① For a finite sum $\sum x_g u_g \in M \rtimes_\alpha \Gamma$, $E_M(\sum x_g u_g) = x_e$.

② As $M \subseteq M \rtimes_\alpha \Gamma$, $E_M^2 = E_M$.

③ E_M is a normal unital completely positive map

④ $\forall x \in M \rtimes_\alpha \Gamma$, $\|E(x)\| \le \|x\|$, $E_M(x^*) = E_M(x)^*$, $E_M(x)^* E_M(x) \le E_M(x^*x)$, and $E_M(x^*x) = 0 \iff x = 0$.

⑤ $\forall a,b \in M$, $x \in M \rtimes_\alpha \Gamma$, $E_M(axb) = a E_M(x) b$.

Pf: You'll prove ②–⑤ in general for HW! For ①:

$$
\begin{aligned}
[i^*(\textstyle\sum x_g u_g) i] m\Omega &= i^* \textstyle\sum x_g u_g (m\Omega \otimes \delta_e) \\
&= i^* \textstyle\sum x_g \alpha_g(m)\Omega \otimes \delta_g \\
&= x_e m\Omega.
\end{aligned}
$$