1

weak * top:

Let (X,τ) be a TVS, $\mathbb{X}^* = \{\varphi : \mathbb{X} \to \mathbb{F} \text{ linear cts}\}$.

Def. The weak* top on $\mathbb{X}^*$ is the top induced by the separating space of ftl's $\{ev_x \mid x \in \mathbb{X}\}$ on $\mathbb{X}^*$.

$ev_x : \mathbb{X}^* \to \mathbb{F}$ by $\varphi \mapsto \varphi(x)$.

$\varphi_\lambda \to \varphi \iff \varphi_\lambda(x) \to \varphi(x) \quad \forall x \in \mathbb{X}$.

Local Basis: $N(\varphi_0; x_1, \dots, x_n; \varepsilon) = \{\varphi \in \mathbb{X}^* \mid |\varphi(x_i) - \varphi_0(x_i)| < \varepsilon \ \forall i = 1, \dots, n\}$.

Fact: If $\mathbb{X}^*$ separates pts of $\mathbb{X}$, $(\mathbb{X}^*, wk^*)^* = \mathbb{X}$. ← wk top.

Pf: HW3! ($\underset{\text{wk top}}{\mathbb{X}}$ and $\underset{\text{wk* top}}{\mathbb{X}^*}$ are in alg. duality)

Annihilators!

Def: If $\mathbb{X}, Y$ TVS, $T : \mathbb{X} \to Y$ cts + linear, define

$T^* : Y^* \to \mathbb{X}^*$ by $\varphi \mapsto \varphi \circ T$.

T^* is wk* cts: Show $\forall \varphi_\lambda \to \varphi$ in Y^*, $\varphi_\lambda \circ T \to \varphi \circ T$ in $\mathbb{X}^*$.

Let $x \in \mathbb{X}$. $(\varphi_\lambda \circ T)(x) = \varphi_\lambda(\underbrace{Tx}_{\in Y}) \to \varphi(Tx) = (\varphi \circ T)(x)$ ✓

↳ we'll revisit this later on for Banach spaces after we have Baire cat + consequences!

Bidual: $\exists$ canonical injection $\mathbb{X} \xhookrightarrow{i} \mathbb{X}^{**}$ for normed space $\mathbb{X}$.

it's an isometry: $[\because \forall x \neq 0,\ \exists f \in \mathbb{X}^* \text{ s.t. } \|f\| = 1 \text{ and } f(x) = \|x\|]$

↳ complete $\mathbb{X}$ via $\tilde{\mathbb{X}} = \overline{i(\mathbb{X})}$ in $\mathbb{X}^{**}$. Reflexive: i is onto.

TODO: Annihilators + preannihilators here!

$Y \subseteq X$

$Y^{\perp} = \{\varphi \in X^* \mid \varphi|_Y = 0\}$.

$Z \subseteq X^*$

$Z_{\perp} = \{x \in X \mid \varphi(x) = 0 \ \forall \varphi \in Z\}$.

Exercises on annihilators:

① $(Y^{\perp})_{\perp}$ is closure of $Y \subseteq X$.

For Y closed,
② $\exists$ canonical SES $Y^* \cong X^*/Y^{\perp}$ and $(X/Y)^* \cong Y^{\perp}$.

$\varphi|_Y \longleftarrow\!\shortmid \varphi \in X^*$ $\qquad [x+Y \mapsto \varphi(x)] \longleftarrow\!\shortmid \varphi$

ker is $Y^{\perp}$.

Recall: $\underset{\text{bdd/cts}}{T: X \to Y}$ $\rightsquigarrow$ $T^*: Y^* \to X^*$ by $T^*\varphi = \varphi \circ T$.

Prop: If $T \in B(X,Y)$, X, Y normed, then $\|T^*\| = \|T\|$, so $T^* \in B(Y^*, X^*)$.

Pf: $\|T^*\varphi\| = \sup\limits_{\|x\|=1} |\varphi(Tx)| \le \sup\limits_{\|x\|=1} \|\varphi\| \cdot \|T\| \cdot \|x\| = \|\varphi\| \cdot \|T\|$,

so $\|T^*\| \le \|T\|$.

Let $\varepsilon > 0$. $\exists x \in X$ w/ $\|x\| = 1$ s.t. $\|Tx\| > \|T\| - \varepsilon$.

By cor to HB, $\exists \varphi \in Y^*$ s.t. $\varphi(Tx) = \|Tx\|$ and $\|\varphi\| = 1$.

So $\|T\| \le \varepsilon + \|Tx\| = \varepsilon + \varphi(Tx) = \varepsilon + |\varphi(Tx)| = \varepsilon + |(T^*\varphi)(x)|$

$\le \varepsilon + \|T^*\|$.

Since $\varepsilon > 0$ arbitrary, $\|T\| \le \|T^*\|$.

Exercise: X, Y Banach, $T: X \to Y$, $S: Y^* \to X^*$ linear, $\varphi(Tx) = (S\varphi)x$ $\forall x \in X$, $\varphi \in Y^*$.

Then T, S bdd w/ $S = T^*$.

Prop: Let X be normed and $Z \subseteq X^*$ a wk* closed subspace. $\forall \varphi \in X^* \setminus Z$, $\exists x \in Z^\perp$ s.t. $\varphi(x) \neq 0$.

Pf: Pick a cvx wk* opn $\mathcal{U} \in \mathcal{O}(\varphi)$ s.t. $\mathcal{U} \cap Z = \emptyset$. By HB 3 (SHT), $\exists f \in (X^*, wk*)^*$ and $t \in \mathbb{R}$ s.t.

$$\operatorname{Re} f(\varphi) \in \operatorname{Re} f(\mathcal{U}) < t \leq \operatorname{Re} f(Z).$$

By your HW, $(X^*, wk*)^* = X$. Since Z a subsp $t \leq 0$, and ~~thus~~ $f \in Z^\perp$.

Cor: Every wk* closed $Z \subseteq X^*$ is of the form $Y^\perp$ for some norm closed $Y \subseteq X$.

Pf: Set $Y = Z^\perp = \{x \in X \mid \varphi(x) = 0 \ \forall \varphi \in Z\}$. Obviously $Z \subseteq Y^\perp$. By the prop, we get equality.

Prop: X, Y Banach, $T \in B(X, Y)$. Then $T^*: Y^* \to X^*$ wk* cts. Conversely, every wk* cts linear $S: Y^* \to X^*$ is of the form $S = T^*$ for some $T \in B(X, Y)$. In particular, S is norm bdd.

Pf: Already saw T^* wk* cts ✓

Suppose $S: Y^* \to X^*$ wk* cts. Then $\forall x \in X$, $ev_x \circ S$ is a wk* cts fnl on Y^*, and thus $ev_x \circ S \in Y$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{ev_y}$

Define $T: X \to Y$ by $Tx = y$ if $ev_x \circ S = ev_y$.

Then $\forall x \in X$, $\forall \varphi \in Y^*$, $\varphi(Tx) = (ev_x \circ S)(\varphi) = ev_x(S\varphi) = (S\varphi)(x)$.

By HW, T is bdd and $T^* = S$.

Examples + properties of T^*:

Exs:

① Fin. dim'l.: $X = \mathbb{C}^n$, $Y = \mathbb{C}^m$. $T \in B(X, Y)$ is represented by an $m \times n$ matrix. T^* is T^t.

② Mult ops: (X, μ) finite meas space. Let $\varphi \in L^\infty(X, \mu)$. For $1 \le p < \infty$, define $T : L^p \to L^p$ by $Tf = \varphi f$. Then $T^* : L^{p'} \to L^{p'}$ w/ $\frac{1}{p} + \frac{1}{p'} = 1$ is given by $T^* g = \varphi g$.

③ Integral ops: (X, μ), (Y, ν) finite meas. spaces, Let K be a bdd m'ble fct on $(X \times Y, \mu \times \nu)$. Let $1 \le p, q < \infty$. Define $T \in B(L^p, L^q)$ by $(Tf)(y) = \int f(x) K(x,y)\, d\mu(x)$. T is int. op w/ kernel K. Then $T^* : L^{q'} \to L^{p'}$ is given by $(T^* g)(x) = \int g(y) K(x,y)\, d\nu(y)$.

④ Shifts: $T \in (\ell^p, \ell^p)$ by $(Tx)(n) = \begin{cases} 0 & n = 1 \\ x(n-1) & n > 1 \end{cases}$.

Then $(T^* y)(n) = y(n+1)$. $T \leftrightarrow \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ 0 & \ddots & 0 \end{bmatrix}$ $T^* = T^t$.

Properties: $T \in B(X, Y)$ X, Y Banach.

① $\ker T^* = (TX)^\perp$ — $T^*\varphi = 0 \iff (T^*\varphi)(x) = 0 \ \forall x \iff \varphi(Tx) = 0 \ \forall x \iff \varphi \in (TX)^\perp$.

$\ker T = (T^* Y^*)_\perp$ — $Tx = 0 \iff \varphi(Tx) = 0 \ \forall \varphi \iff (T^*\varphi)(x) = 0 \ \forall \varphi \iff x \in (T^* Y^*)_\perp$.

② T^* inj $\iff$ TX dense in Y

T inj $\iff$ $T^* Y^*$ wk* dense in X^*.

② $T^{**}|_X = T$. — $[T^{**}(ev_x)](\varphi) = ev_x(T^*\varphi) = (T^*\varphi)(x) = \varphi(Tx) = ev_{Tx}(\varphi)$.

③ $T \in B(X, Y)$, $S \in B(Y, Z)$, $(ST)^* = T^* S^*$. — $[(ST)^*\varphi](x) = \varphi(STx) = (\varphi S)(Tx) = (S^*\varphi)(Tx) = (T^* S^* \varphi)(x)$.

(4) T inv. $\Leftrightarrow$ T^* inv.

Pf: (a) T inv. $\Rightarrow$ $T^*(T^{-1})^* = (T^{-1}T)^* = (id)^* = id.$ $\Rightarrow T^*$ inv.

$(T^{-1})^*(T)^* = (T T^{-1})^* = (id)^* = id.$

(b) Suppose T^* inv. Then T^{**} inv., so $\forall x \in X$,

$$\|Tx\| = \|T^{**}x\| \geq \frac{\|(T^{**})^{-1}T^{**}x\|}{\|(T^{**})^{-1}\|} \geq \frac{\|x\|}{\|(T^{**})^{-1}\|}.$$

Exercise: TFAE:

(1) $\exists\, c>0$ s.t. $\|Tx\| \geq c\|x\|$ $\forall x \in X$.

(2) T is injective and TX is closed.

Now ~~T^{**} inj $\Rightarrow T$~~ T^* inj. inj $\Leftrightarrow$ TX dense in Y $\Rightarrow TX = Y$.

Thus T is a cts bij, so $T^{-1} \in$ Bdd.

(5) $T \in B(X)$. $Y \subseteq X$ closed subspace. Y T-inv. $\Leftrightarrow$ $Y^\perp$ is T^*-invariant.

Hint: use $x \notin Y \Rightarrow \exists\, \varphi \in Y^\perp$ s.t. $\varphi(x) = 1$. (HB!)

Prop: Suppose $Y \subseteq X$ closed, X normed. Let $J: Y \to X$ be inclusion and $Q: X \to X/Y$ be quotient. Then we may identify Q^* w/ inclusion $Y^\perp \hookrightarrow X^*$ and J^* w/ quotient map $X^* \to X^*/Y^\perp$.

[Note: Canonical isos $X^*/Y^\perp \cong Y^*$ and $(X/Y)^* \cong Y^\perp$.]

Pf: If $\varphi \in (X/Y)^*$, $Q^*\varphi \in Y^\perp$. Since $Q B_1^X(0) = B_1^{X/Y}(0)$,

$$\|Q^*\varphi\| = \sup_{\|x\|<1} (Q^*\varphi)(x) = \sup_{\|x\|<1} \varphi(Qx) = \|\varphi\|, \text{ so}$$

$(X/Y)^* \xrightarrow{Q^*} Y^\perp \subseteq X^*$ is isometric.

If $\psi \in Y^\perp$, $\exists!\, \varphi \in (X/Y)^*$ s.t. $\psi(x) = \varphi(Qx)$, so $Q^*\varphi = \psi$.

Thus Q^* surjective.

Prop: Let X, Y be a normed spaces, $Y \subseteq X$ closed subspace. There are canonical isometric isomorphisms $Y^* \cong X^*/Y^\perp$ and $(X/Y)^* \cong Y^\perp$. If

$$Y \xhookrightarrow{I} X \xrightarrow{Q} X/Y$$

are the canonical maps, then

$$\begin{tikzcd} Y^* \arrow[d, phantom, "\cong"] & X^* \arrow[l, "I^*"'] \arrow[dl] & (X/Y)^* \arrow[l, "Q^*"'] \arrow[d, phantom, "\cong"] \\ X^*/Y^\perp & & Y^\perp \arrow[ul, hook] \end{tikzcd}$$

Can identify I^* and Q^* w/ the canonical maps.

Pf: If $\varphi \in (X/Y)^*$, $Q^*\varphi \in Y^\perp$, so this iso is "restrict the codomain".

Since $Q B_1^X(0) = B_1^{X/Y}(0)$, $\|Q^*\varphi\| = \sup_{\|x\| \le 1} \underbrace{(Q^*\varphi)(x)}_{\varphi(Qx)} = \|\varphi\|$.

Thus Q^* is isometric. If $\psi \in Y^\perp$, $\exists!\, \varphi \in (X/Y)^*$ s.t. $\psi(x) = \varphi(Qx)$, and $Q^*\varphi = \psi$. Thus Q^* surj.

$QI = 0 \Rightarrow I^*Q^* = 0 \Rightarrow Y^\perp \subseteq \ker I^*$.

Also, $I^*\varphi = \varphi|_Y$, so $Y^\perp = \ker I^*$.

Denote $\tilde{I}^* : X^*/_{Y^\perp} \to Y^*$ I^* factors through Y^*; $\|\tilde{I}^*\| = \|I^*\| = 1$.

I^* surj: Use HB to extend $\psi \in Y^*$ to $\varphi \in X^*$ w/ $\|\psi\| = \|\varphi\|$. Then $\psi = I^*\varphi$.

Also, $\|\tilde{I}^*(\varphi + Y^\perp)\| = \|\psi\| = \|\varphi\| \ge \|\varphi + Y^\perp\| \Rightarrow \tilde{I}^*$ isometric.

Lemma: Let $\mathcal{U}, V$ be open unit balls of Banach spaces X, Y.

For $T \in B(X, Y)$, ~~and consider the following statements~~ TFAE:

1. $\exists \alpha > 0$ s.t. $\|T^*\psi\| \ge \alpha \|\psi\|$ $\forall \psi \in Y^*$.
2. $\alpha V \subseteq \overline{T\mathcal{U}}$
3. $\alpha V \subseteq T\mathcal{U}$
4. $TX = Y$

~~Thm (1) ⇒ (2) ⇒ (3) ⇒ (4).~~

~~Moreover, (4) ⇒ (1) for some $\alpha > 0$.~~

Pf: ①⇒②: Pick $y_0 \notin \overline{T\mathcal{U}}$. Now $\overline{T\mathcal{U}}$ cvx, closed, balanced.

use HB!

[Exercise: Suppose $\mathfrak{X}$ is a loc. cvx TVS and $B \subset \mathfrak{X}$ cvx, balanced, closed. $\forall x_0 \in \mathfrak{X} \setminus B$, $\exists \varphi \in \mathfrak{X}^*$ s.t. $|\varphi(x)| \leq 1$ $\forall x \in B$ but $\varphi(x_0) > 1$.]

Pick $\varphi \in \mathcal{Y}^*$ s.t. $|\varphi(y)| \leq 1$ $\forall y \in \overline{T\mathcal{U}}$, but $\varphi(y_0) > 1$.

If $x \in \mathcal{U}$, $|(T^*\varphi)(x)| = |\varphi(Tx)| \leq 1$, so $\|T^*\varphi\| \leq 1$.

By ①, $\alpha < \alpha|\varphi(y_0)| \leq \alpha \|\varphi\| \cdot \|y_0\| \leq \|y_0\| \cdot \|T^*\varphi\| \leq \|y_0\|$.

Hence $\|y\| \leq \alpha \Rightarrow y \in \overline{T\mathcal{U}}$.

②⇒③: Earlier lemma.

③⇒④: obvious

④⇒①: OMT

How? [③⇒①: $\|T^*\varphi\| = \sup\limits_{\|x\|=1} |(T^*\varphi)(x)| = \sup\limits_{\|x\|=1} |\varphi(Tx)| \geq \sup\limits_{y \in \alpha V} |\varphi(y)| = \alpha\|\varphi\|$.

[or $T\mathfrak{X} = \mathcal{Y} \Rightarrow T^*$ inj. Must show T^* has closed range ...]

Closed Range thm: $\mathfrak{X}, \mathcal{Y}$ Banach, $T \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$. TFAE:

① $T\mathfrak{X}$ closed in $\mathcal{Y}$

② $T^*\mathcal{Y}^*$ wk* closed in $\mathfrak{X}^*$

③ $T^*\mathcal{Y}^*$ norm closed in $\mathfrak{X}^*$

Pf: we can write $T: \mathfrak{X} \to \mathcal{Y}$ as

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "T"] \arrow[d, "Q"] & \mathcal{Y} \\ \mathfrak{X}_0 := \mathfrak{X}/\ker T \arrow[r, "T_0"] & \overline{T\mathfrak{X}} =: \mathcal{Y}_0 \arrow[u, hook, "I"] \end{tikzcd} \quad \overset{(\cdot)^*}{\rightsquigarrow} \quad \begin{tikzcd} \mathfrak{X}^* & \mathcal{Y}^* \arrow[l, "T^*"'] \arrow[d, "I^*"] \\ (\ker T)^\perp = (\mathfrak{X}/\ker T)^* \arrow[u, "Q^*"] & \overline{T\mathfrak{X}}^* = \mathcal{Y}^*/(T\mathfrak{X})^\perp = \mathcal{Y}^*/\ker T^* \arrow[l, "T_0^*"'] \end{tikzcd}$$

$\mathrm{im}(T) = \mathrm{im}(T_0)$. $\mathrm{im}\,T^* = \mathrm{im}(T_0^*)$.

so $T\mathfrak{X} = T_0\mathfrak{X}_0$, and $T^*\mathcal{Y}^* = T_0^*\mathcal{Y}_0^*$.

Note: $T_0(x + \ker T) = Tx$ and $T_0^*(\varphi + \ker T^*) = T^*\varphi$.

①⇒②: If $T\mathfrak{X}$ closed, then T_0 is bdd + bij, hence inv. by OMT. So T_0^* is also inv., and $\mathrm{im}\,T_0^* = (\ker T)^\perp$. Hence $T^*\mathcal{Y}^*$ is wk* closed.
$\mathrm{im}\,T^* =$

②⇒③: obvious.

(3)⟹(1): If T^* has norm closed, then $\mathrm{im}(T_0^*)^*$ is closed, and T_0^* is inj. By your HW, T_0^* is bdd below. By the lemma, $\mathrm{im}(T_0) = T\mathfrak{X} = \mathrm{im}\,T$, i.e., $T\mathfrak{X}$ is closed.

Idempotents + Complemented Subspaces:

We'll now study the Banach alg $B(\mathfrak{X})$, $\mathfrak{X}$ Banach space.

Def: $p \in B(\mathfrak{X})$ is an idempotent if $p^2 = p$.

Then $1-p$ is a idempotent, and $p\mathfrak{X} = \ker(1-p)$; $\ker p = (1-p)\mathfrak{X}$.

Also $\mathfrak{X} = p\mathfrak{X} + (1-p)\mathfrak{X}$ and $p\mathfrak{X} \cap (1-p)\mathfrak{X} = (0)$

$\Rightarrow$ $\mathfrak{X} = p\mathfrak{X} \oplus (1-p)\mathfrak{X}$

Operator decomposition: write $T \in B(\mathfrak{X})$ as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$ where $T_{11} \in B(p\mathfrak{X})$, $T_{12} \in B((1-p)\mathfrak{X}, p\mathfrak{X})$, $T_{21} \in B(p\mathfrak{X}, (1-p)\mathfrak{X})$, $T_{22} \in B((1-p)\mathfrak{X})$.

Sum in $B(\mathfrak{X})$ $\longleftrightarrow$ matrix sum

composition $\longleftrightarrow$ matrix multiplication.

Exercise: $Tp = pT \iff T(p\mathfrak{X}) \subseteq p\mathfrak{X}$ and $T((1-p)\mathfrak{X}) \subseteq (1-p)\mathfrak{X}$
$\iff T$ diagonal.

$T(p\mathfrak{X}) \subseteq p\mathfrak{X} \iff T_{21} = 0$

$T((1-p)\mathfrak{X}) \subseteq (1-p)\mathfrak{X} \iff T_{12} = 0$.

Def: We say a closed subspace $E \subseteq \mathfrak{X}$ is complemented if $\exists$ closed subspace $F \subseteq \mathfrak{X}$ s.t. $\mathfrak{X} = E \oplus F$. F is called a complementing subspace of E, not unique!

Examples: (0) $p\mathfrak{X} \oplus (1-p)\mathfrak{X}$.

(1) If E has finite codim, it is complemented.

pf: pick $x_1, \dots, x_n$ s.t. $\{x_1+E, \dots, x_n+E\}$ a basis for $\mathfrak{X}/E$.
Let $F = \operatorname{span}\{x_i \mid i=1,\dots,n\}$. $F \subseteq \mathfrak{X}$ closed, and $\mathfrak{X} = E \oplus F$.

(2) If $\dim E < \infty$, E is complemented.

pf: Let $\{e_1, \dots, e_n\}$ be a basis for E. By HB, $\exists \varphi_1, \dots, \varphi_n \in \mathfrak{X}^*$ s.t. $\varphi_j(e_k) = \delta_{j=k}$. Define $p: \mathfrak{X} \to \mathfrak{X}$ by $px = \sum_{j=1}^{n} \varphi_j(x) e_j$.
Then $p \in B(\mathfrak{X})$, $p^2 = p$, and $p\mathfrak{X} = E$.

(3) All subspaces of a Hilbert space are complemented.

HW: (1) K cpt metric space, $F \subseteq K$ closed, nonempty.
$I(F) = \{f \in C(K) \mid f|_F = 0\}$.
$I(F)$ is complemented in $C(K)$.

(2) c_0 is uncomplemented in ℓ^∞.

Prop: $\mathfrak{X}$ Banach, E, F complementary subspaces of $\mathfrak{X}$. Then $\exists$ an idempotent $p \in B(\mathfrak{X})$ w/ $p\mathfrak{X} = E$ and $(1-p)\mathfrak{X} = F$.

Pf: For $x \in \mathfrak{X}$, can write x uniquely as $x = e + f$ w/ $e \in E$, $f \in F$.
Define $px = e$. Clearly p linear, $p^2 = p$, $p\mathfrak{X} = E$, $(1-p)\mathfrak{X} = F$.

Claim: $p \in B(\mathfrak{X})$.

Pf: Show graph(p) closed. Suppose $x_n \to x$ and $x_n = e_n + f_n$, $x = e + f$.
Suppose $e_n \to y$. Since E closed, $y \in E$. $f_n = x_n - e_n \to x - y \in F$, since F is closed. Thus $x = y + (x-y) \Rightarrow y = e$.

Cor: If $E \subseteq \mathfrak{X}$ complemented, $E^\perp \subseteq \mathfrak{X}^*$ complemented.

Pf: Let $p \in B(\mathfrak{X})$ s.t. $E = p\mathfrak{X}$. Then $p^* \in B(\mathfrak{X}^*)$ is an idempotent, and
$\ker p^* = (1-p^*)\mathfrak{X} = E^\perp$.

Cor: $T \in B(X,Y)$ left invertible $\iff$ T inj. and TX closed + complemented.

Pf: $\Rightarrow$: Suppose $\exists\, S \in B(Y,X)$ s.t. $ST = 1$. (clearly T inj.) Define $p = TS \in B(Y)$. Then $p^2 = p$, so p is an idempotent. $TX = TSY = pX$ as S is surj.
Thus TX closed + complemented.

$\Leftarrow$: $T_0 \in B(X, TX)$ is inv. by OMT. Let $S_0 \in B(TX, X)$ be inverse.
Let $p \in B(Y)$ be idempotent s.t. $pY = TX$. Define $S = S_0 p \in B(Y,X)$.
Then $ST = S_0 pT = S_0 T_0 = 1$.

Cor: $T \in B(X,Y)$ right inv. $\iff$ T surj. and $\ker T$ complemented.

- you prove.

Banach - Alaoglu

Thm (BA) Let $\mathbb{X}$ be a TVS and $\mathcal{U} \in \mathcal{O}(0_{\mathbb{X}})$. Define

$$K = \{\varphi \in \mathbb{X}^* \mid |\varphi(x)| \leq 1 \quad \forall x \in \mathcal{U}\} = \underbrace{\bigcap_{x \in \mathcal{U}} \{\varphi \in \mathbb{X}^* \mid |\varphi(x)| \leq 1\}}_{\varphi^{-1} B_{\leq 1}(0_{\mathbb{F}})}$$

Says K is cvx + balanced

Then K is wk* cpt.

Pf: Since $\mathcal{U}$ is absorbing, $\forall x \in \mathbb{X}$, $\exists R_x > 0$ s.t. $x \in R_x \mathcal{U} \Leftrightarrow R_x^{-1} x \in \mathcal{U}$.

Thus $|\varphi(x)| \leq R_x \quad \forall x \in \mathbb{X}$ and $\varphi \in K$. Set

$$P = \prod_{x \in \mathbb{X}} B_{\leq R_x}(0_{\mathbb{F}}) \quad , \text{cpt by Tychonoff's Thm.}$$

Can identify P w/ $\{$fcts $f: \mathbb{X} \to \mathbb{F} \mid |f(x)| \leq R_x\}$. Under this identification, $K \subset \mathbb{X}^* \cap P$. The result follows from the following 2 claims:

Claim 1: Relative wk*-top on K = Relative P-top on K

Claim 2: $K \subseteq P$ is closed.

$\hookrightarrow$ K closed subset of cpt Hausd. space $\Rightarrow$ K cpt.

Pf of Claim 1: We'll construct bases for $\mathcal{O}(\varphi_0)$ for $\varphi_0 \in K$ wrt wk* top and P-top, and we'll show they agree intersected w/ K. For $\varphi_0 \in K$, $x_1, \dots, x_n \in \mathbb{X}$, $\varepsilon > 0$, we define

$$N_1 = N_1(\varphi_0; x_1, \dots, x_n; \varepsilon) = \{\varphi \in \mathbb{X}^* \mid |\varphi(x_i) - \varphi_0(x_i)| < \varepsilon \quad \forall i = 1..n\}$$

$$N_2 = N_2(\varphi_0; x_1, \dots, x_n; \varepsilon) = \{f \in P \mid |f(x_i) - \varphi_0(x_i)| < \varepsilon \quad \forall i = 1, \dots, n\}$$

Then $\{N_1 \mid \overset{n \in \mathbb{N}}{x_1, \dots, x_n} \in \mathbb{X}, \varepsilon > 0\}$ local base for wk* top at φ_0

$\{N_2 \mid \overset{n \in \mathbb{N}}{x_1, \dots, x_n} \in \mathbb{X}, \varepsilon > 0\}$ local base for P-top at φ_0.

Since $K \subset \mathbb{X}^* \cap P$, $N_1 \cap K = N_2 \cap K$, which proves the result.

Pf of Claim 2: Suppose $\varphi_0 \in \overline{K} \subseteq P$. We must show:

(a) φ_0 is linear

(b) $|\varphi_0(x)| \le 1 \quad \forall x \in \mathcal{U}$ $\quad (\Rightarrow \varphi_0 \in X^*$ too.)

$\exists\, \mathcal{U} \in \mathfrak{U}(0_X)$ s.t. $\varphi_0\mathcal{U}$ is bdd in $\mathbb{F}$

Pf of (a): Let $\alpha, \beta \in \mathbb{F}$, $x, y \in X$, $\varepsilon > 0$. Then

$V = \{ f \in P \mid |f(z) - \varphi_0(z)| < \varepsilon$ for $z \in \{x, y\} \}$ is an open nbhd of $\varphi_0 \in P$, so $\exists\, \varphi \in V \cap K \neq \emptyset$. Then

$$|\varphi_0(\alpha x + \beta y) - \alpha\varphi_0(x) - \beta\varphi_0(y)| = |(\varphi_0 - \varphi)(\alpha x + \beta y) + \alpha(\varphi - \varphi_0)(x) + \beta(\varphi - \varphi_0)(y)|$$
$$< (1 + |\alpha| + |\beta|)\,\varepsilon.$$

Since ε was arbitrary, φ_0 is linear.

Pf of (b): If $x \in \mathcal{U}$ and $\varepsilon > 0$, $\exists\, \varphi \in K$ s.t. $|\varphi(x) - \varphi_0(x)| < \varepsilon$.

Thus $|\varphi_0(x)| \le |\varphi(x)| + \varepsilon \le 1 + \varepsilon$. Since ε arbitrary, done.

Cor: If X normed v.sp., the unit ball B^* of X^* is wk* cpt

Pf: Let $\mathcal{U} = B_1(0_X) \subseteq X$. We claim that B^* equals

$$K := \{\varphi \in X^* \mid |\varphi(x)| \le 1 \ \forall x \in \mathcal{U}\}, \text{ wk* cpt by BA Thm.}$$

$\subseteq$: If $\varphi \in B^*$, then $|\varphi(x)| \le \|\varphi\| \cdot \|x\| \underset{x \in \mathcal{U}}{\le} 1 \Rightarrow \varphi \in K$.

$\supseteq$: If $\varphi \in K$, $\|x\| = 1$, and $0 < r < 1$, then $\|rx\| < 1$, so $rx \in \mathcal{U}$.

Then $|\varphi(rx)| = r|\varphi(x)| \le 1$, and $|\varphi(x)| \le r^{-1}$.

Since $0 < r < 1$ was arbitrary, $|\varphi(x)| \le 1$, and

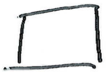
thus $\sup_{\|x\| = 1} |\varphi(x)| \le 1$.

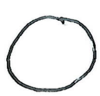
Krein–Milman convexity + cptness. ①

Def: A *face* of a cnvx $S \subseteq \mathfrak{X}$ (a v.sp.) is a nonempty, cnvx subset $F \subseteq S$ s.t. $\lambda x + (1-\lambda) y \in F \Rightarrow x, y \in F$ $\forall x,y \in S$ and $\lambda \in (0,1)$.

An *extreme pt* of S is a one pt face, i.e. it's an $x \in S$ s.t. $x = \lambda y + (1-\lambda) z$ for $\lambda \in (0,1)$, $y, z \in S \Rightarrow y = z = x$.

Examples:

①  $\subset \mathbb{R}^2$. Faces = ? Extreme pts = ?

②  $\subseteq \mathbb{R}^2$. Faces = ? Extreme pts = ?

You'll compute lots of examples for HW!

Thm (Krein–Milman): Suppose $\mathfrak{X}$ is a TVS w/ weak top induced from a sep. family (space) $\mathfrak{X}^*$ of ftl's. For each cnvx cpt $K \subseteq \mathfrak{X}$, the convex hull of the extremal boundary ∂K (set of extreme pts) of K is dense in K.

[K is closed cnvx hull of its extreme pts.]

Pf: Let $K_0 \subseteq K$ be a closed face of K. (Note K is a face of K, so $\exists$.)

① Let $\Lambda = \{$closed faces of $K_0\}$ (note faces of K_0 are faces of K too.) Order Λ by reverse inclusion. We claim every chain has a maximal elt. If $\{F_j\}_{j \in J}$ is a chain of closed faces of K_0, then $\bigcap F_j \neq \emptyset$ as all F_j are nested cpt sets. Now $\bigcap F_j$ is a face "majorizing" all F_j.

By Zorn's Lemma, $\exists$ a minimal closed face $F \subseteq K_0$.

② Let $\varphi \in \mathfrak{X}^*$, set $s = \inf\{\operatorname{Re}\varphi(x) \mid x \in F\}$. Since $x \mapsto \operatorname{Re}\varphi(x)$ is weakly cts, it attains its min s on F. Set $F_\varphi = \{x \in F \mid \operatorname{Re}\varphi(x) = s\}$.

Now F_φ is a closed face of F, and thus of K_0, and since F is minimal, we must have $F_\varphi = F$. Since $\mathfrak{X}^*$ separates pts of $\mathfrak{X}$, F must be a one pt set! Thus $K_0 \cap \partial K \neq \emptyset$ $\forall$ closed faces $K_0 \subseteq K$. ②

③ Consider $\operatorname{conv}(\partial K) \subseteq K$, which is a convex subset. Thus $\overline{\operatorname{conv}(\partial K)}$ is also convex $\subseteq K$ (exercise!). Suppose for contradiction that $\exists x \in K \setminus \overline{\operatorname{conv}(\partial K)}$. Then $\exists$ cnvx $\mathcal{U} \in \mathcal{O}(x)$ disjoint from $\overline{\operatorname{conv}(\partial K)}$. By HB3 (SHT), $\exists\, \varphi \in \mathfrak{X}^*$ and $r \in \mathbb{R}$ s.t.

$$\operatorname{Re} \varphi(x) \in \operatorname{Re} \varphi(\mathcal{U}) < r \leq \operatorname{Re} \varphi(\overline{\operatorname{conv} \partial K}).$$

$\uparrow$ could call A $\qquad$ $\uparrow$ could call B.

Thus $\min\{\operatorname{Re}(\varphi(x)) \mid x \in K\} =: s < t$.

Setting $F_\varphi = \{x \in K \mid \operatorname{Re}\varphi(x) = s\}$, we get a closed face of K, and $F_\varphi \cap \overline{\operatorname{conv} \partial K} = \emptyset$. But, by ①+②, F_φ is a closed face and thus contains an extreme pt., a contradiction!

Exercises: Compute the extreme pts of the unit balls of the follow ~~TVS's~~ Banach spaces. Determine which unit balls are weakly cpt.

① $C(\mathfrak{X})$ $\mathfrak{X}$ infinite cpt Hausdorff space.

② $L^1[0,1]$

③ $L^p[0,1]$ $\quad 1<p<\infty$

④ $L^\infty[0,1]$

⑤ $B(H)$, H infinite dim'l Hilb. space.

⑥ $\mathcal{L}^1(H)$, trace class operators.

⑦ $B(H)_{s.a.}$, self adjoint ops.

⑧ $B(H)_+$, positive ops.

(⑤–⑧) later, after spectral theory.

Applications of Krein-Milman:

Prop: Let $\mathfrak{X}$ be a cpt. Hausd. space. Consider $C(\mathfrak{X})$, and recall $C(\mathfrak{X})^*$ is the space of Radon (regular Borel) measures on $\mathfrak{X}$. Denote it by $M(\mathfrak{X})$.

Let $P(\mathfrak{X}) = \{\mu \in C(\mathfrak{X})^* = M(\mathfrak{X}) \mid \|\mu\| \leq 1 \text{ and } \mu(1) = 1\}$ (prob. measures)

Then $P(\mathfrak{X})$ is a cnvx wk* cpt set whose extreme pts are Dirac measures: $\delta_x(f) = f(x)$ for $f \in C(\mathfrak{X})$.

Pf: Unit ball of $C(\mathfrak{X})^*$ wk* cpt by B.A. $P(\mathfrak{X})$ is wk* closed face of $C(\mathfrak{X})^*$, since $\mu \mapsto \mu(1)$ is a wk* cts fct. So $P(\mathfrak{X})$ is a cnvx wk* cpt set.

Separate Lemma: $\mu \geq 0 \iff \mu(1) = \|\mu\|$ (in both)

Claim 1: If $\mu \in P(\mathfrak{X})$ and $f = \bar{f} \in C(\mathfrak{X})$, $\mu(f) \in \mathbb{R}$.

Pf: if $\mu(f) = a + ib$, then $\forall n$,

$\forall x \in \mathfrak{X}$, $|f(x)|^2 + b^2 n^2 = |f(x) + ibn|^2$

$$a^2 + b^2(1+n)^2 = |\mu(f + ibn)|^2 \leq \|f + ibn\|^2 = \|f\|^2 + b^2 n^2$$

$$\leadsto \quad a^2 + b^2(1+2n) \leq \|f\|^2 \quad \forall n$$

$$\Rightarrow \quad b = 0.$$

Claim 2: If $\mu \in P(\mathfrak{X})$ and $f \geq 0$, then $\mu(f) \geq 0$. "μ a state"

Pf: Since $0 \leq \|f\| - f \leq \|f\|$, $\|f\| - \underbrace{\mu(f)}_{\in \mathbb{R} \text{ by claim } ①} = \mu(\|f\| - f) \leq \|\mu\| \cdot \| \|f\| - f \| \leq \|f\|$.

Hence $\mu(f) \geq 0$.

[Note: Can use these claims to show: If A is a unital C^*-alg. and $\varphi \in A^*$, then $\|\varphi\| = \varphi(1) \Rightarrow \varphi \geq 0$, i.e. $\varphi(a) \geq 0$ $\forall a \geq 0$ in A.]

Cor: $\forall f \in C(\mathfrak{X})$, $\operatorname{Re} \mu(f) = \mu(\operatorname{Re} f) \leq \mu(|f|)$, and $|\mu(f)| \leq \mu(|f|)$.

Claim 3: If $\mu \in P(\mathfrak{X})$ is extreme, then $\mu(fg) = \mu(f)\mu(g)$.

Pf: Step 1: take $f \in C(\mathfrak{X})$ w/ $0 \leq f \leq 1$. Set $\alpha = \mu(f)$, so $0 \leq \alpha \leq 1$.

If $0 < \alpha < 1$, define $\varphi, \psi : C(\mathfrak{X}) \to \mathbb{F}$ by

$$\varphi(g) = \frac{1}{\alpha} \mu(fg) \qquad \psi(g) = \frac{1}{1-\alpha} \mu((1-f)g).$$

Note that $\varphi(1) = \psi(1) = 1$, and

$$|\varphi(g)| = \frac{1}{\alpha}|\mu(fg)| \leq \frac{1}{\alpha}\mu(|fg|) = \frac{1}{\alpha}\mu(f|g|) \leq \|g\|_\infty \frac{\mu(f)}{\alpha} = \|g\|_\infty \quad (\alpha)$$

and thus $\|\varphi\| \leq 1$. Similarly, $\|\psi\| \leq 1$. Thus $\varphi, \psi \in P(X)$.

Now $\mu = \alpha\varphi + (1-\alpha)\psi$ and μ extreme $\Rightarrow$ $\mu = \varphi = \psi$.

Thus $\forall g \in C(X)$, $\frac{1}{\underset{= \mu(f)}{\alpha}}\mu(fg) = \varphi(g) = \mu(g)$ $\Rightarrow$ $\mu(fg) = \mu(f)\mu(g)$.

Clearly this is true when $\mu(f) = 0$ ~~[illegible]~~ $\}_f$ $(*)$.

Similarly, using ψ, it is true when $\mu(f) = 1$ too.

(or replace f w/ $1-f$.)

Now any $h \in C(X)$ is in the linear span of $\{f \in C(X) \mid 0 \leq f \leq 1\}$.

By linearity, $\mu(hg) = \mu(h)\mu(g)$ $\forall g, h \in C(X)$.

<u>Claim 4:</u> $\exists x \in X$ s.t. $\mu = \delta_x = ev_x$.

Suppose for contradiction that $\forall x \in X$, $\exists f \in \ker\mu$ s.t. $f(x) \neq 0$.
Then $\exists U \in \mathcal{O}(x)$ s.t. $f|_U \neq 0$. Since X cpt, $\exists f_1, \ldots, f_n \in \ker\mu$
s.t. $\{x \in X \mid f_i(x) \neq 0$ ~~[illegible]~~ $\}_{i=1}^n$ is an open cover of X.
By Claim 3, $f = \sum f_i \bar{f_i} \in \ker\mu$, but $f(x) > 0$ $\forall x \in X$, so
$1/f \in C(X)$. But $1 = \mu(1) = \mu(f \cdot \frac{1}{f}) = \mu(f)\mu(\frac{1}{f}) = 0$, $\Rightarrow\Leftarrow$.
Thus $\exists x \in X$ s.t. $\ker\mu \subset \ker(ev_x)$. But then
$f - \mu(f)1 \in \ker\mu$ $\forall f \in C(X)$, and thus $f(x) = \mu(f)$ $\forall f$.

<u>Claim 5:</u> ev_x is extreme in $P(X)$ $\forall x \in X$.
Suppose $ev_x = \alpha\varphi + (1-\alpha)\psi$, $\alpha \in (0,1)$, and $\varphi, \psi \in P(X)$. Then
$\alpha|\varphi(f)| \leq \alpha\varphi(|f|) \leq ev_x(|f|) = |f|(x) = |f(x)|$, and so
$\ker ev_x \subset \ker\varphi$. This implies $\varphi = ev_x$, so $\psi = ev_x$ too.

Facts about $C(X)^*$:

Let X be a locally cpt Hausd. space

Let $A = C_?(X, \mathbb{C})$ where $? = c, 0, b$ (if X cpt all are $C(X)$)

Defn for $f \in A$ $\bar{f} \in A$ by $\bar{f}(x) = \overline{f(x)}$ ($x \in X$).

Then $\bar{\ }$ is an involution on A:

- $\bar{\bar{f}} = f$
- $\overline{\lambda f + g} = \bar{\lambda}\bar{f} + \bar{g}$
- $\overline{fg} = \bar{g}\bar{f}$ ($= \bar{f}\bar{g}$ as A commutative, e.g.)

Given an involution, can define real + imag parts:

$$Re(f) = \frac{f + \bar{f}}{2} \qquad Im(f) = \frac{f - \bar{f}}{2i}.$$

Then $f = Re(f) + i\, Im(f)$.

Say $f \geq 0$ if $f(x) \geq 0$ ~~$\forall x \in X$~~.

<u>Claim</u>: $f \geq 0 \iff \exists g \in A$ s.t. $f = \bar{g} g$. (obvious!)

When f is real, can write f uniquely as $f = f_+ - f_-$ w/ $f_+, f_- \geq 0$ and $f_+ f_- = 0$.

<u>Cor</u>: Every $f \in A$ is a lin comb. of 4 pos. fcts.

<u>Defn</u>: A linear fnl $\varphi : A \to \mathbb{C}$ is positive ($\varphi \geq 0$) if $\varphi(f) \geq 0$ $\forall f \geq 0$.

Defn $A_{\mathbb{R}} = C_?(X, \mathbb{R})$. (warning: not same as $X_{\mathbb{R}}$ earlier!)

$\varphi_{\mathbb{R}} = \varphi|_{A_{\mathbb{R}}}$ as an $\mathbb{R}$-linear fnl.

Note $A = A_{\mathbb{R}} + i A_{\mathbb{R}}$.

Exercises: ① $\varphi \geq 0 \iff \varphi_{\mathbb{R}} \geq 0$ ②

② φ completely determined by $\varphi_{\mathbb{R}}$. ($\varphi \mapsto \varphi_{\mathbb{R}}$ inj.)

Def: φ is self-adj or Hermitian if $\varphi_{\mathbb{R}}$ takes values in $\mathbb{R}$.

Def: $\varphi: A \to \mathbb{C}$, define $\varphi^*: A \to \mathbb{C}$ by $\varphi^*(f) = \overline{\varphi(\bar{f})}$.

Define $\varphi^{Re} = \frac{\varphi + \varphi^*}{2}$ $\varphi^{Im} = \frac{\varphi - \varphi^*}{2i}$.

Exercises: ① $\varphi^{Re}, \varphi^{Im}$ Hermitian.

② φ Hermitian $\iff \varphi = \varphi^* \iff \varphi = \varphi^{Re} \iff \varphi^{Im} = 0$.

③ Compare $Re\, \varphi = A \to \mathbb{R}$ and $\varphi^{Re}: A \to \mathbb{C}$.

④ $\forall \psi: A_{\mathbb{R}} \to \mathbb{R}$, $\exists!$ hermitian $\varphi: A \to \mathbb{C}$ s.t. $\varphi_{\mathbb{R}} = \psi$.

A^*: we'll focus on $A = C_0(X, \mathbb{C})$.

Exercises: ① φ bdd $\iff \varphi^*$ bdd $\iff \varphi^{Re}, \varphi^{Im}$ bdd.

② φ hermitian is bdd $\iff \varphi_{\mathbb{R}}$ bdd.

So to study A^*, suffices to study $A^*_{\mathbb{R}}$.

$A = C(X, \mathbb{C})$, X cpt here! $1 \notin C_0(X, \mathbb{C})$!

Lemma: If $\psi \geq 0$ for $\psi: A_{\mathbb{R}} \to \mathbb{R}$, then ψ bdd w/ $\|\psi\| = \psi(1)$.

Pf: $\forall f \in A_{\mathbb{R}, \geq 0}$, $0 \leq f \leq \|f\|_\infty 1$, so $\|f\|_\infty 1 - f \geq 0$.

Thus $\psi(\|f\|_\infty 1 - f) \geq 0 \iff \psi(f) \leq \|f\|_\infty \psi(1)$.

Now $\forall f \in A_{\mathbb{R}}$, $f = f_+ - f_-$, $\psi(f) = \underbrace{\psi(f_+)}_{\geq 0} - \underbrace{\psi(f_-)}_{\geq 0} \leq \psi(f_+) + \psi(f_-)$
$= \psi(|f|)$.

Similarly, $\psi(-f) \leq \psi(|f|)$. So $|\psi(f)| \leq \psi(|f|) \leq \|f\|_\infty \psi(1)$

Facts: ① If $\psi \geq 0$ in $A^*_{\mathbb{R}}$, $\exists!$ positive Radon / reg. Borel meas. μ on X s.t. $\psi = \int \cdot \, d\mu$.

② If $\psi: A_{\mathbb{R}} \to \mathbb{R}$ bdd, $\exists$ s.t. $\psi = \psi_+ - \psi_-$ for ψ_+ and $\psi_- \geq 0$. } $\psi_{\pm}$ not necc.?

③ If $\psi \in A^*_{\mathbb{R}}$, $\exists!$ signed Radon / reg. Borel meas μ on X s.t. $\psi = \int \cdot \, d\mu$.

Let $\mathfrak{X}$ be a ~~Banach~~ normed space and $Y \subseteq \mathfrak{X}$ a subspace.
the annihilator $Y^\perp = \{\varphi \in \mathfrak{X}^* \mid \varphi|_Y = 0\}$.
Similarly, if $Z \subseteq \mathfrak{X}^*$, $Z^\perp = \{x \in \mathfrak{X} \mid \varphi(x) = 0 \ \forall \varphi \in Z\}$.

Exercises: ① $Y \subseteq (Y^\perp)^\perp$ $\forall Y \subseteq \mathfrak{X}$.

② If Y closed, $Y = (Y^\perp)^\perp$.

③ What about Z vs. $(Z^\perp)^\perp$?

$C(\mathfrak{X}, \mathbb{R}) \supseteq A$ unital
$C(\mathfrak{X}, \mathbb{C}) \supseteq A = \bar{A} \ni 1$

Stone-Weierstrass Thm: If $A \subseteq C(\mathfrak{X})$ is a unital alg which separates pts of $\mathfrak{X}$ (cpt. Hausd.), then A is dense in $C(\mathfrak{X})$.

Pf: It suffices to prove if $\mu \in C(\mathfrak{X})^* = M(\mathfrak{X})$ w/ $\mu|_A = 0$, then $\mu = 0$.

Consider $K = A^\perp \cap B^*$, where $A^\perp = \{\varphi \in C(\mathfrak{X})^* = M(\mathfrak{X}) \mid \varphi|_A = 0\}$ and B^* is closed unit ball of $C(\mathfrak{X})^*$. K is closed convex subset of B^* and thus wk* cpt. If $K \neq \emptyset$, by Krein-Milman, $\exists \mu \in \partial_{Ext} K \subset K$.

Assume for contradiction.

Now if $\mu \in \partial_{Ext} K$, clearly $\|\mu\| = 1$. Since A a algebra, $\forall g \in A$, $\mu(g\cdot) \in A^\perp$. If $0 \leq g \leq 1$ $\forall x$, $\varphi = \frac{\mu(g\cdot)}{t} \in K$, $t = \|\mu(g\cdot)\|$

as is $\psi = \frac{\mu((1-g)\cdot)}{\|\mu((1-g)\cdot)\| = (1-t)}$.

Note that $\|\mu(g\cdot)\| + \|\mu((1-g)\cdot)\| = \int g \, d|\mu| + \int (1-g)\, d|\mu| = \int d|\mu| = 1$.

$\Rightarrow$ Setting $t = \|\mu(g\cdot)\|$, $t\varphi + (1-t)\psi = \mu \Rightarrow \varphi = \psi = \mu$.

Now if $\operatorname{supp}(\mu) = \{x \in \mathfrak{X} \mid \int_U d|\mu| > 0 \ \forall U \in \mathcal{O}(x)\}$, since $\mu = \frac{\mu(g\cdot)}{t}$, g must have same value on all pts of $\operatorname{supp}(\mu)$.
But this means $\operatorname{supp}(\mu)$ is a single pt as A separates pts of $\mathfrak{X}$.
$\star$ [! $\exists g$ w/ $0 < g(x) < 1$ $\forall x$ s.t. $g(y) \neq g(z)$!]
Hence $\mu = ev_x$ for some $x \in \mathfrak{X}$. But $1 \in A$, so ~~$\int d\mu \neq 0$~~ $\mu|_A \neq 0$, $\Rightarrow\Leftarrow$.

Lemma: Suppose either

① $1 \in A \subseteq C(X,\mathbb{R})$ or ② $1 \in A = \bar{A} \subseteq C(X,\mathbb{C})$.

Then $\forall y,z \in X$ $\exists$ $0<g<1$ $g \in A$ s.t. $g(y) \neq g(z)$.

Pf: ① A separates pts, so can shift + squish.

② $A = \bar{A}$ separates pts, so $\exists g \in A$ s.t. $\operatorname{Re} g(y) \neq \operatorname{Re} g(z)$.

Then apply ① to $g + \bar{g}$.

Note: The lemma fails w/o $A = \bar{A}$ in ②.

Counterexample: Consider $A = \{\text{poly's}\} \subseteq C(\bar{\mathbb{D}}, \mathbb{C})$ (closed disk.)

Then $\bar{A}^{\|\cdot\|_\infty}$ is the fcts on $\bar{\mathbb{D}}$ which are holo. on $\mathbb{D}$ and cts on $\bar{\mathbb{D}}$.

$\forall$ holo non const f on $\mathbb{D}$, f is open, so $\nexists$ $0<g<1$ to separate pts in interior of $\mathbb{D}$.

Some Complex Analysis:

Let D be a (simply conn.) domain in $\mathbb{C}$.
Let $C(D)$ be cts $\mathbb{C}$-valued fcts on D, and $H(D)$ the holo ($\mathbb{C}$-diff.) fcts on D.

Claim 1: $H(D)$ is a closed subspace of $C(D)$ w/ a Fréchet TVS struc.

Pf: Pick cpt nested sets $(K_n)_{n\in\mathbb{N}}$ s.t. $K_n \subset K_{n+1}^{\circ}$ and $D = \bigcup K_n$.
For $n \in \mathbb{N}$, define $m_n(f) = \|f\|_{C(K_n)}$, seminorm on $C(D)$.
Then $\mathcal{M} = \{m_n \mid n \in \mathbb{N}\}$ is a separating family of seminorms on $C(D)$. Give $C(D)$ the wk topology induced by $\mathcal{M}$.

Exercise: The wk top. induced by $\mathcal{M}$ is compatible w/ the metric $d(f,g) = \sum \frac{1}{2^n} \frac{m_n(f-g)}{1+m_n(f-g)}$. Moreover, this metric is translation invariant, and $C(D)$ is complete in this metric. $\rightsquigarrow$ Fréchet space.

Exercise: $f_n \to f$ in $(C(D), \mathcal{M}\text{-top}) \iff \forall$ cpt $K \subset D$, $f_n|_K \to f|_K$ in $(C(K), \|\cdot\|_\infty)$.

Pf of Claim 1: Suppose $(f_n) \subset H(D)$ w/ $f_n \to f \in C(D)$. Then $\forall$ cpt $K \subset D$, $f_n \to f$ unif. in $C(K)$, so $\oint_\gamma f = 0$ for every simple closed contour $\gamma \subset D$. Thus f is holo. by Morera's thm.

Σ_γ ms$(\gamma) \subset D$.

Runge's Thm: Let D be a simply conn. domain in $\mathbb{C}$ which is bdd. Every holo. fct in D can be approx. uniformly on cpt subsets $K \subset D$ by poly's.

Pf: Every cpt subset of D is contained in a simply conn cpt $K \subset D$. By Jordan's curve thm, there is a closed smooth curve $\gamma \subset D \setminus K$ w/ winding # 1 about K.

Can express $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\,dw \quad \forall z \in K$ by C.I.F.

Now we can approximate this integral by a finite sum, which is a linear comb. of fcts of the form $z \mapsto \frac{1}{w-z}$ w/ $w \in \gamma$.

Claim: $\forall w \notin K$, $z \mapsto \frac{1}{w-z}$ can be uniformly approximated by poly's on K.

Step (1): If $|w| > R = \sup_{z\in K} |z|$, $\frac{1}{w-z} = \sum_0^\infty \frac{z^n}{w^{n+1}}$ $\left(\left|\frac{z}{w}\right|<1\right)$,

and this convergence is uniform in z.

Step (2): Let $\varphi \in C(K)^*$ s.t. $\varphi|_{\text{Polys}} = 0$. We claim that $\varphi(z \mapsto \frac{1}{w-z}) = 0$, so by cor to HB, the claim follows.

Define $g(w) = \varphi(z \mapsto \frac{1}{w-z})$ for $w \notin K$.

Claim: g is analytic (holo.) on K^c.

Step (a): Let $f_w(z) = \frac{1}{w-z}$. Then $\forall h \neq 0$, we have

$\frac{1}{h}(f_{w+h} - f_w)(z) = \frac{-1}{(w+h-z)(w-z)}$. This converges uniformly for

$z \in K$ to $\frac{-1}{(w-z)^2}$ as $h \to 0$.

Step (b): we compute

$$\lim_{h\to 0} \frac{g(w+h) - g(w)}{h} = \lim_{h\to 0} \frac{\varphi(f_{w+h}) - \varphi(f_w)}{h} = \lim_{h\to 0} \varphi\left(\frac{f_{w+h} - f_w}{h}\right)$$

$$= \varphi\left(\lim_{h\to 0} \frac{f_{w+h} - f_w}{h}\right) = \varphi\left(z \mapsto \frac{-1}{(w-z)^2}\right) \in \mathbb{C}.$$

So $g'(w)$ exists $\forall w \in K^c$, and g is holo.

Now for $|w|>R$, $f_w \in \overline{P}^{C(K)}$. Since $\varphi(p)=0$ $\forall p\in P$, by continuity of $\varphi \in C(K)^*$, $\varphi(f_w)=0$ $\forall |w|>R$. Thus $g(w)=0$ $\forall |w|>R$. But g holo and K^c conn. $\Rightarrow g\equiv 0$.

Cor: Poly's dense in $H(D)$. [Pick poly p_n s.t. $\|p_n-f\|_{K_n}<\frac{1}{n}$]

Hardy space: $H^\infty(D)=\{\text{holo } f:D\to\mathbb{C} \mid \sup_{z\in D}|f(z)|<\infty\}$.

Claim: $H^\infty(D)$ a Banach alg. under $\|\cdot\|_\infty$.

Pf: Let $(f_n)\subset H^\infty(D)$ be unif. Cauchy. Then $\forall z\in D$, $(f_n(z))\subset\mathbb{C}$ is Cauchy, so define $f(z)=\lim_n f_n(z)$. Then $\forall$ cpt $K\subset D$, $f_n\to f$ uniformly, so $f\in H(D)$.

Finally, to show $\|f\|_\infty<\infty$, we pick $N>0$ s.t. $\|f_n-f_m\|_\infty\le 1$ $\forall n,m\ge N$. Then $\forall z\in D$, we have $|f(z)|=\lim_n |f_n(z)|\le 1+\|f_N\|_\infty$.

Disk algebra: $A(D):=H^\infty(D)\cap C(\overline{D})$
$=\{f:\overline{D}\to\mathbb{C} \text{ cts} \mid f|_D \text{ holo.}\}$.

Cor: $A(D)$ is a Banach algebra.

~~Lemma~~ Cor: Let $P=\{\text{complex polys}\}$, $P(\overline{D})=\{\text{complex polys}|_{\overline{D}}\}$.

$\overline{P(\overline{D})}^{\|\cdot\|_{C(\overline{D})}}\subseteq A(D)$.

~~Pf~~: [Each $p\in P(\overline{D})$ belongs to $A(D)$.]

Q: Do we get equality?

<u>Mergelyan's Thm</u>: Let K be a ~~simply conn~~ cpt subset of $\mathbb{C}$ s.t. $\mathbb{C}\setminus K$ is conn. Then any $f: K \to \mathbb{C}$ s.t. $f|_{\mathrm{int}(K)}$ (K°) is holo. can be approx. <u>uniformly</u> on K by poly's.

$\leadsto$ $A(D) = \overline{P(\overline{D})}^{\|\cdot\|_{C(\overline{D})}}$ ✓

<u>Pf when D is simply conn., convex that $A(D) = \overline{P(\overline{D})}^{\|\cdot\|_\infty}$:</u>

By translation, we may assume $0 \in D$. Let $f \in A(D)$, $\varepsilon > 0$. Since $\overline{D}$ is cpt, f is <u>unif cts</u>, so $\exists\, r \in (0,1)$ s.t. $\forall z \in \overline{D}$, $|f(z) - f(rz)| < \frac{\varepsilon}{2}$. Define $g(z) = f(rz)$, so that $g: r^{-1}\overline{D} \to \mathbb{C}$ is cts on $r^{-1}\overline{D}$ and holo on $r^{-1}D$. Since D was <u>convex</u> and contains 0, $\boxed{\overline{D} \subset r^{-1}D \subset r^{-1}\overline{D}.}$ By Runge's thm, we can approximate $g(z)$ uniformly by poly's on $\overline{D}$. So $\exists$ poly $p \in P(\overline{D})$ s.t. $\|p - g\|_{C(\overline{D})} < \frac{\varepsilon}{2}$. Then $\forall z \in \overline{D}$,

$$|f(z) - p(z)| \le |f(z) - \underbrace{f(rz)}_{g(z)}| + |g(z) - p(z)| < \varepsilon,$$

and thus $\|f - p\|_\infty < \varepsilon$.