Let $X, Y$ be Banach spaces.

**Definition 1.** A bounded linear map $T \in \mathcal{B}(X,Y)$ is called *Fredholm* if $\dim(\ker T) < \infty$ and $\operatorname{codim}(TX) < \infty$. In this case, we define the *(Fredholm) index* of $T$ to be $\operatorname{ind}(T) := \dim(\ker T) - \operatorname{codim}(TX)$.

**Exercise 2.** As an exercise, verify the following claims:

(1) Invertible bounded maps have index zero.

(2) If $K \in \mathcal{B}(X)$ is compact, then for all $\lambda \in \mathbb{C} \setminus \{0\}$, $K - \lambda 1$ is Fredholm with index zero.

(3) The shift operator $S_\pm \in \mathcal{B}(\ell^p)$ for $1 \leq p \leq \infty$ defined by $(S_\pm x)_n = x_{n\pm 1}$ is Fredholm with index $\pm 1$.

(4) If $X, Y$ are finite dimensional and $T \in \mathcal{B}(X,Y)$, then by the Rank-Nullity Theorem, $\operatorname{ind}(T) = \dim(X) - \dim(Y)$.

**Lemma 3.** *Suppose $E, F \subseteq X$ are closed subspaces with $F$ finite dimensional.*

*(1) The subspace $E + F \subseteq X$ is closed.*

*(2) If in addition $E$ is complemented, then $E + F \subseteq X$ is complemented.*

*Proof.* Consider the canonical surjection $Q : X \to X/E$. Then $QF \subseteq X/E$ is finite dimensional and thus closed, and $E + F = Q^{-1}(QF)$ is closed since $Q$ is continuous.

Suppose now that $E$ is complemented, and let $P \in \mathcal{B}(X)$ be an idempotent with $PX = E$. Then $(1-P)F \subseteq (1-P)X$ is finite dimensional, and thus complemented. Let $Q_0 \in \mathcal{B}((1-P)X)$ be an idempotent with $Q_0(1-P)X = (1-P)F$. Extend $Q_0$ to $Q \in \mathcal{B}(X)$ by $Q = 0$ on $PX$. Then $QP = PQ = 0$, so $P + Q \in \mathcal{B}(X)$ is an idempotent with $(P+Q)X = E + F$. □

**Proposition 4.** *Suppose $T \in \mathcal{B}(X,Y)$ such that* $\operatorname{codim}(TX) < \infty$. *Then $TX$ is closed.*

*Proof.* Pick $y_1, \dots, y_n \in Y$ such that $\{y_1 + TX, \dots, y_n + TX\}$ are a basis for the vector space $Y/TX$, and define $F = \operatorname{span}\{y_1, \dots, y_n\} \subset Y$. Now consider the Banach space $Z = X/\ker T \oplus F$ with the $\ell^1$ norm $\|(x + \ker T) + f\|_Z := \|x + \ker T\|_{X/\ker T} + \|f\|_F$. We define a linear map $S : Z \to Y$ by $S((x + \ker T) + f) = Tx + f$. It is straightforward to verify $S$ is bounded and bijective, and thus invertible by the Open Mapping Theorem. Finally, we have $S(X/\ker T) = TX$ is closed as $S$ is a closed map. □

**Corollary 5.** *If $T \in \mathcal{B}(X,Y)$ is Fredholm, then for any closed subspace $E \subseteq X$, $TE$ is closed.*

*Proof.* Define $S : Z = X/\ker T \oplus F \to Y$ as in the proof of Proposition 4. Since $\ker(T)$ is finite dimensional, by Lemma 3.(1), $E + \ker T$ is closed in $X$, and thus $(E + \ker T)/\ker T$ is closed in $X/\ker T$ and also in $Z$. Finally, we have $S((E + \ker T)/\ker T) = TE$ is closed as $S$ is a closed map. □

**Theorem 6** (Atkinson). *For $T \in \mathcal{B}(X,Y)$, the following are equivalent:*

*(1) $T$ is Fredholm.*

*(2) There is an $S \in \mathcal{B}(Y,X)$ such that $ST - 1$ and $TS - 1$ are both finite rank.*

*(3) There is an $S \in \mathcal{B}(Y,X)$ such that $ST-1$ and $TS-1$ are both compact.*

*(4) There are $S_1, S_2 \in \mathcal{B}(Y,X)$ such that $S_1T-1$ and $TS_2-1$ are both compact.*

*Proof.*

(1) ⇒ (2) : Let $T$ be Fredholm. Let $P \in \mathcal{B}(X)$ be an idempotent such that $\ker(T) = \ker(P)$, and let $Q \in \mathcal{B}(Y)$ be an idempotent such that $TX = QY$. (Note here that $TX$ is closed by Proposition 4 and has finite codimension, and is thus complemented.) We may decompose $T$ as a matrix of operators

$$\begin{pmatrix} QTP & QT(1-P) \\ (1-Q)TP & (1-Q)T(1-P) \end{pmatrix} = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} PX \\ (1-P)X = \ker(P) = \ker(T) \end{pmatrix} \to \begin{pmatrix} QY = TX \\ (1-Q)Y \end{pmatrix}.$$

Now $T_0 \in \mathcal{B}(PX, QY)$ is invertible by the Open Mapping Theorem, so let $S_0 \in \mathcal{B}(QY, PX)$ be its inverse. We may extend $S_0$ to an operator in $\mathcal{B}(Y,X)$ by defining the matrix of operators

$$\begin{pmatrix} S_0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} QY = TX \\ (1-Q)Y \end{pmatrix} \to \begin{pmatrix} PX \\ (1-P)X = \ker(P) = \ker(T) \end{pmatrix}.$$

It is straightforward to verify that

$$ST = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P = 1 - (1-P) \tag{6.a}$$

$$TS = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. = Q = 1 - (1-Q) \tag{6.b}$$

Now $1-P$ has range $\ker(T)$ which is finite dimensional, and $QY = TX$ has finite codimension, so $1-Q$ has finite rank.

(2) ⇒ (3) ⇒ (4) : Trivial.

(4) ⇒ (1) : Note that $S_1T = 1 + K_1$ for some compact $K_1 \in \mathcal{B}(X)$. Then $\ker(T) \subseteq \ker(S_1T) = \ker(1+K_1)$, which is finite dimensional. Now $TS_2 = 1 + K_2$ for some compact $K_2 \in \mathcal{B}(Y)$. Then $(1+K_2)Y \subseteq TX$, and $(1+K_2)Y$ has finite codimension in $Y$. □

**Definition 7.** For $T \in \mathcal{B}(X,Y)$, an $S$ as in Theorem 6.(3) is called *a Fredholm inverse* for $T$.

**Exercise 8.** As an exercise, verify the following properties of Fredholm operators:

(1) $T \in \mathcal{B}(X)$ is Fredholm if and only if $T + \mathcal{K}(X)$ is invertible in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$.

(2) If $T \in \mathcal{B}(X,Y)$ is Fredholm, then so is $T+K$ for all compact operators $K \in \mathcal{K}(X,Y)$.

(3) A Fredholm inverse of a Freholm operator is also Fredholm.

(4) If $S \in \mathcal{B}(Y,X)$ is a Fredholm inverse for $T \in \mathcal{B}(X,Y)$, then for any compact $K \in K(Y)$, $S+K$ is also a Fredholm inverse for $T$.

(5) For any two Fredholm inverses $S_1, S_2 \in \mathcal{B}(Y,X)$ for $T \in \mathcal{B}(X,Y)$, $S_1 - S_2 \in K(Y,X)$.

(6) The product of two Freholm operators is Fredholm.

(7) The adjoint of a Fredholm operator $T \in \mathcal{B}(X,Y)$ is Fredolm, and $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

**Lemma 9.** *Suppose $P, Q \in \mathcal{B}(X)$ are idempotents. Then $PQ = Q$ if and only if $QX \subseteq PX$. In this case, $QP$ is an idempotent with range $QX$ which commutes with $P$.*

*Proof.* Suppose $PQ = Q$, and let $q \in QX$. Then $Pq = PQq = Qq = q$, and thus $q \in PX$. Conversely, suppose $QX \subseteq PX$. We can write $x \in X$ uniquely as $q + y$ with $q \in QX$ and $y \in (1-Q)X$. Then $Qx = q = Pq = PQx$.

Note now that $P$ commutes with $QP$, since $PQP = (PQ)P = QP = QP^2$. Hence $QPQP = Q^2P = QP$, and $QP$ is an idempotent. Clearly $QPX \subseteq QX$. Conversely, if $q \in QX \subseteq PX$, we have $QPq = q$, so $QX \subseteq QPX$. □

**Remarks 10.** Suppose $(E, F)$ are complementary closed subspaces of $X$. Let $P \in \mathcal{B}(X)$ be the idempotent such that $PX = E$ and $(1-P)X = F$.

(1) Suppose $Q \in \mathcal{B}(X)$ is an idempotent with $QX \subseteq PX$. Since the idempotent $QP$ from Lemma 9 commutes with $P$, we see $P - QP$ is an idempotent. Thus we can decompose $E$ into two complementary subspaces $E = E_1 \oplus E_2$ with $E_1 = QX = QPX$ and $E_2 = (P - QP)X$. Thus $X = E_1 \oplus E_2 \oplus F$.

(2) Suppose $Q \in \mathcal{B}(X)$ is an idempotent with $PX \subseteq QX$. Since the idempotent $PQ$ from Lemma 9 commutes with $Q$, $Q - PQ$ is an idempotent, $F_1 = (Q-PQ)X = (1-P)QX \subseteq (1-P)X = F$, and $QX = E \oplus F_1$. Now we claim $F_1$ is complemented in $F$. Indeed, by another application of Lemma 9 to the inclusion $(Q - PQ)X \subseteq (1 - P)X$, we have $R = (1 - P)Q(1 - P)$ is an idempotent with range $(Q-PQ)X$, and $1-P-R$ is an idempotent. Setting $F_2 = (1-P-R)X$, we have $F = F_1 \oplus F_2$ and $X = E \oplus F_1 \oplus F_2$.

**Definition 11.** We call $T \in \mathcal{B}(X, Y)$

- *left Fredholm* if $TX$ is closed and complemented and $\ker T$ is finite dimensional.
- *right Fredholm* if $\ker T$ is complemented and $\operatorname{codim}(TX) < \infty$.

Note that $T$ right Fredholm implies $TX$ is closed and complemented by Proposition 4.

**Theorem 12** (Atkinson)**.** *For $T \in \mathcal{B}(X, Y)$, (1) - (3) below are equivalent, as are (1')- (3'):*

*(1) $T$ is left Fredholm*

*(2) There is an $S \in \mathcal{B}(Y, X)$ such that $ST - 1$ has finite rank.*

*(3) There is an $S \in \mathcal{B}(Y, X)$ such that $ST - 1$ is compact.*

*(1') $T$ is right Fredholm*

*(2') There is an $S \in \mathcal{B}(Y, X)$ such that $TS - 1$ is finite rank.*

*(3') There is an $S \in \mathcal{B}(Y, X)$ such that $TS - 1$ is compact.*

*Proof.*

(1) ⇒ (2): This follows directly from the proof of (1) ⇒ (2) from Theorem 6. Since $TX$ is closed and complemented and $\ker T$ is finite dimensional, we can still define the idempotents $P \in \mathcal{B}(X)$ and $Q \in \mathcal{B}(Y)$. We still get $ST = P = 1 - (1 - P)$ as in (6.a) and $1 - P$ has finite rank. However, it is not necessarily the case that $1 - Q$ has finite rank, since we do not know if $TX$ has finite codimension.

$\underline{(1') \Rightarrow (2')}$ Again, this follows directly from the proof of (1) $\Rightarrow$ (2) from Theorem 6. In this case, we have that $1-Q$ has finite rank since $\operatorname{codim}(TX) < \infty$, but since we do not know if $\ker T$ is finite dimensional, we cannot conclude that $1-P$ has finite rank.

$\underline{(2) \Rightarrow (3) \text{ and } (2') \Rightarrow (3'):}$ Trivial.

$\underline{(3) \Rightarrow (1)}$: Suppose there is an $S \in \mathcal{B}(Y, X)$ such that $ST - 1 = K \in \mathcal{K}(X)$ is compact. As in the proof of Theorem 6, $\ker T \subseteq \ker ST = \ker(1+K)$ is finite dimensional. Now by the spectral analysis of compact operators, there is a pair of closed complementary $K$-invariant subspaces $(E, F)$ of $X$ such that $X = E \oplus F$, $E$ is finite dimensional, $(1+K)|_F$ is invertible, and $(1+K)|_E$ is nilpotent. Thus $T|_F$ has left inverse $(ST)|_F^{-1}S$, and thus $TF$ is closed and complemented. But $TX = TE + TF$ with $TE$ finite dimensional, and thus $X$ is closed and complemented by Lemma 3.(2).

$\underline{(3') \Rightarrow (1')}$: Suppose $TS - 1 = K \in K(Y)$ is compact. As in the proof of Theorem 6, $(1+K)Y \subseteq TX$, and $(1+K)Y$ has finite codimension, as does $TX$. (Again, this implies $TX$ is closed and complemented by Proposition 4.) By the spectral analysis of compact operators, there is a pair of closed complementary $K$-invariant subspaces $(M, N)$ of $Y$ such that $Y = M \oplus N$, $N$ is finite dimensional, $(1+K)|_M$ is invertible, and $(1+K)|_N$ is nilpotent. Since $TS = 1 + K$, we have $M \subseteq TX \subseteq Y$. By Lemma 9 and Remarks 10, we can further decompose $N = N_1 \oplus N_2$ such that $TX = N_2 \oplus M$ and $Y = N_1 \oplus N_2 \oplus M$.

Now define $E = T^{-1}N_2$ and $F = T^{-1}M$. Since $TX = M \oplus N_2$, $(E, F)$ are closed complementary subspaces of $X$. We claim that $\ker(T) = \ker(T|_E) \oplus \ker(T|_F)$. Indeed, if $x \in \ker T$, we can uniquely write $x = e + f$ with $e \in E$ and $f \in F$. If $Tx = Te + Tf = 0$, then since $Te \in N_2$ and $Tf \in M$, we have $Te = -Tf \in N_2 \cap M = (0)$, and thus $Te = Tf = 0$.

It remains to prove that $\ker(T|_E)$ and $\ker(T|_F)$ are complemented in $E$ and $F$ respectively. We can then add idempotents similar to Remarks 10 to see that $\ker(T)$ is complemented. First, $T|_F$ has right inverse $S|_M(TS)|_M^{-1}$, so $\ker(T|_F)$ is complemented. Second, $T|_E \in \mathcal{B}(E, N_2)$ is onto a finite dimensional space. It is easily seen that such a map has a bounded right inverse (any right inverse will do). Hence $\ker(T|_E)$ is complemented. □

**Remark 13.** We see that $T \in \mathcal{B}(X)$ is left/right Fredholm if and only if $T + K(X)$ is left/right invertible in the Calkin algebra $\mathcal{B}(X)/\mathcal{K}(X)$ respectively.

**Exercise 14.** For an exact sequence of finite dimensional vector spaces

$$0 \to V_0 \to V_1 \to \cdots \to V_n \to 0,$$

we have $\sum_{j=0}^{n} (-1)^j \dim(V_j) = 0$.

**Theorem 15** (Multiplication)**.** *If $T \in \mathcal{B}(X, Y)$ and $S \in \mathcal{B}(Y, Z)$ are Fredholm, then $ST$ is Fredholm with* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$.

*Proof.* That $ST$ is Fredholm is contained in Exercise 8. Consider the sequence of finite dimensional vector spaces

$$0 \to \ker T \xrightarrow{i} \ker ST \xrightarrow{T|_{\ker ST}} \ker S \xrightarrow{Q|_{\ker S}} Y/TX \xrightarrow{\widetilde{S}} Z/STX \xrightarrow{q} Z/SY \to 0,$$

where $i$ is inclusion, $Q : Y \to TX$ is the canonical quotient map, $\widetilde{S}(y + TX) = Sy + STX$, and $q(z + STX) = z + SY$. The reader can verify that the above sequence is exact, so by Exercise 14, we have

$$\begin{aligned} 0 &= \dim(\ker T) - \dim(\ker ST) + \dim(\ker S) - \operatorname{codim}(TX) + \operatorname{codim}(STX) - \operatorname{codim}(SY) \\ &= \operatorname{ind}(T) - \operatorname{ind}(ST) + \operatorname{ind}(S). \end{aligned}$$

The proof is complete. □

**Corollary 16.** *If* $S \in \mathcal{B}(Y, X)$ *is a Fredholm inverse for* $T \in \mathcal{B}(X, Y)$*, then* $\operatorname{ind}(S) = -\operatorname{ind}(T)$*.*

*Proof.* Recall that by the Fredholm Alternative for compact operators, $1 + K$ has Fredholm index zero for all compact $K \in \mathcal{B}(X)$. Thus $0 = \operatorname{ind}(1 + K) = \operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$. □

**Theorem 17** (Stability)**.** *For* $T \in \mathcal{B}(X, Y)$ *Fredholm, there is an* $\varepsilon > 0$ *such that for all* $R \in \mathcal{B}(X, Y)$ *with* $\|R\| < \varepsilon$*,* $T + R$ *is Fredholm and* $\operatorname{ind}(T + R) = \operatorname{ind}(T)$*.*

*Proof.* Let $S \in \mathcal{B}(Y, X)$ be a Fredholm inverse of $T$, and set $\varepsilon = \|S\|^{-1}$. Then there are compact operators $K_1 \in \mathcal{B}(X)$ and $K_2 \in \mathcal{B}(Y)$ such that $ST = 1 + K_1$ and $TS = 1 + K_2$. If $\|R\| < \varepsilon$, then $S(T + R) = 1 + K_1 + SR$, and $\|SR\| < 1$. Thus $1 + SR$ and $1 + RS$ are invertible, and

$$(1 + SR)^{-1} S(T + R) = 1 + (1 + SR)^{-1} K_1 \in 1 + \mathcal{K}(X)$$
$$S(T + R)(1 + RS)^{-1} = 1 + K_2(1 + RS)^{-1} \in 1 + \mathcal{K}(Y).$$

Hence $T + R$ is Fredholm. Now using Theorem 15 applied to the above equations, we have

$$0 = \operatorname{ind}((1 + SR)^{-1}) + \operatorname{ind}(S) + \operatorname{ind}(T + R) = 0 - \operatorname{ind}(T) + \operatorname{ind}(T + R).$$

This concludes the proof. □

**Corollary 18.** *The set of Fredholm operators in* $\mathcal{B}(X, Y)$ *is open, and the index is a continuous map from the set of Fredholm operators to* $\mathbb{Z}$*.*

**Corollary 19.** *If* $T \in \mathcal{B}(X, Y)$ *is Fredholm and* $K \in \mathcal{B}(X, Y)$ *is compact, then* $T + K$ *is Fredholm and* $\operatorname{ind}(T + K) = \operatorname{ind}(T)$*.*

*Proof.* That $T + K$ is Fredholm is contained in Exercise 8. By Theorem 17, the function $[0, 1] \to \mathbb{Z}$ by $t \mapsto \operatorname{ind}(T + tK)$ is continuous, and thus constant. □