Problem 1. Suppose X is a normed space and $Y \subset X$ is a subspace. Define $Q : X \to X/Y$ by $Qx = x + Y$. Define

$$\|Qx\|_{X/Y} = \inf \{\|x - y\|_X | y \in Y\}.$$

(1) Prove that $\| \cdot \|_{X/Y}$ is a well-defined seminorm.
(2) Show that if Y is closed, then $\| \cdot \|_{X/Y}$ is a norm.
(3) Show that in the case of (2) above, $Q : X \to X/Y$ is continuous and open.
Optional: is Q continuous or open only in the case of (1)?
(4) Show that if X is Banach, so is X/Y.

Problem 2. Suppose F is a finite dimensional vector space.

(1) Show that for any two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on F, there is a $c > 0$ such that $\|f\|_1 \leq c\|f\|_2$ for all $f \in F$. Deduce that all norms on F induce the same vector space topology on F.
(2) Show that for any two finite dimensional normed spaces F_1 and F_2, all linear maps $T : F_1 \to F_2$ are continuous.
Optional: Show that for any two finite dimensional vector spaces F_1 and F_2 endowed with their vector space topologies from part (1), all linear maps $T : F_1 \to F_2$ are continuous.
(3) Let X, F be normed spaces with F finite dimensional, and let $T : X \to F$ be a linear map. Prove that the following are equivalent:
(a) T is bounded, and
(b) $\ker(T)$ is closed.
Hint: One way to do (b) implies (a) uses Problem 1 part (3) and part (2) of this problem.

Problem 3. Consider $L^p[0, 1]$ for $0 < p < 1$.

(1) Show that $d(f, g) = \int_0^1 |f(t) - g(t)|^p \, dt$ is a well-defined translation-invariant metric on $L^p[0, 1]$.
(2) Show that $L^p[0, 1]$ with the metric in (1) above is a complete metric space.
(3) Prove that the only convex open subsets of $L^p[0, 1]$ are $\emptyset$ and $L^p[0, 1]$.
(4) Deduce that if (X, τ) is a locally convex topological vector space and $T : L^p[0, 1] \to X$ is a continuous linear map, then $T = 0$.

Problem 4. Let (X, τ) be a topological vector space. For $x \in X$, let $\mathcal{O}(x)$ denote the collection of open neighborhoods of x. Prove the following assertions.

(1) Every open $U \in \mathcal{O}(0_X)$ is absorbing.
(2) If $U, V \subseteq X$ are open, then so is $U + V$.
(3) If $U \subseteq X$ is open, so is $\operatorname{Conv}(U)$, the convex hull of U.
(4) Every convex $U \in \mathcal{O}(0_X)$ contains a balanced convex $V \in \mathcal{O}(0_X)$.

Problem 5. Suppose $\varphi, \varphi_1, \ldots, \varphi_n$ are linear functionals on a vector space X. Prove that the following are equivalent.

(1) $\varphi \in \sum_{k=1}^n \alpha_k \varphi_k$ where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.
(2) There is an $\alpha > 0$ such that for all $x \in X$, $|\varphi(x)| \leq \alpha \max_{k=1,\ldots,n} |\varphi_k(x)|$.
(3) $\bigcap_{k=1}^n \ker(\varphi_k) \subset \ker(\varphi)$.

Problem 6. Suppose X is a vector space and Y is a separating linear space of functionals on X. Endow X with the weak topology induced by Y. Prove that a linear functional φ on X is weakly continuous if and only if $\varphi \in Y$.

Problem 7. Suppose X is a $\mathbb{C}$-vector space and $\varphi : X \to \mathbb{C}$ is a $\mathbb{C}$-linear functional. Let $X_{\mathbb{R}}$ be X considered as an $\mathbb{R}$-vector space.

(1) Prove $\mathrm{Re}(\varphi) : X_{\mathbb{R}} \to \mathbb{R}$ is an $\mathbb{R}$-linear functional.
(2) Show that for any two $\mathbb{C}$-linear functionals $\varphi_1, \varphi_2 : X \to \mathbb{C}$, $\varphi_1 = \varphi_2$ if and only if $\mathrm{Re}(\varphi_1) = \mathrm{Re}(\varphi_2)$ as $\mathbb{R}$-linear functionals on $X_{\mathbb{R}}$.
(3) Suppose $\psi : X_{\mathbb{R}} \to \mathbb{R}$ is an $\mathbb{R}$-linear functional. Show that $\psi_{\mathbb{C}} : X \to \mathbb{C}$ given by $\psi_{\mathbb{C}}(x) = \psi(x) - i\psi(ix)$ is a $\mathbb{C}$-linear functional.
(4) Let $\psi = \mathrm{Re}(\varphi) : X_{\mathbb{R}} \to \mathbb{R}$ as an $\mathbb{R}$-linear functional. Deduce that $\varphi = \psi_{\mathbb{C}}$.

Problem 8. Suppose (X, τ) is a topological vector space, and $A, B \subset X$ are disjoint nonempty subsets with A compact and B closed.

(1) Show that for every $U \in \mathcal{O}(0_X)$ and $n \in \mathbb{N}$, there is a $V \in \mathcal{O}(0_X)$ and

$$\sum_{i=1}^{n} V = \underbrace{V + \cdots + V}_{n \text{ summands}} \subseteq U.$$

(2) Prove there is a $V \in \mathcal{O}(0_X)$ such that $(A + V) \cap (B + V) = \emptyset$.
This means that all topological vector spaces are T_3.
(3) Suppose in addition that (X, τ) is locally convex and A, B are convex. Show there is a $\varphi \in X^*$ and $r_1, r_2 \in \mathbb{R}$ such that $\varphi(a) < r_1 < r_2 < \varphi(b)$ for all $a \in A$ and $b \in B$.

Problem 9 (Banach Limits)**.** Show that there is a $\varphi \in (\ell^\infty)^*$ satisfying the following two conditions:

(1) Letting $S : \ell^\infty \to \ell^\infty$ be the shift operator $(Sx)_n = x_{n+1}$ for $x = (x_n)_{n\in\mathbb{N}}$, $\varphi = \varphi \circ S$.
(2) For all $x \in \ell^\infty$, $\liminf x_n \leq \varphi(x) \leq \limsup x_n$.

Problem 10 (Goldstine's Theorem)**.** Let X be a normed vector space with closed unit ball B. Let B^{**} be the unit ball in X^{**}, and let $i : X \to X^{**}$ be the canonical inclusion. Show that $i(B)$ is weak* dense in B^{**}.
Note: recall that the weak topology on X^{**} is the weak topology induced by X^*.*

Problem 11. Let X be a compact Hausdorff topological space. For $x \in X$, define $\mathrm{ev}_x : C(X) \to \mathbb{F}$ by $\mathrm{ev}_x(f) = f(x)$.

(1) Prove that $\mathrm{ev}_x \in C(X)^*$, and find $\| \mathrm{ev}_x \|$.
(2) Show that the map $\mathrm{ev} : X \to C(X)^*$ given by $x \mapsto \mathrm{ev}_x$ is a homeomorphism onto its image, where the image has the relative weak* topology.

Problem 12. A Banach space X is called:

- *uniformly convex* if for any two sequences $(x_n), (y_n) \subset X$, if $\|x_n\| = \|y_n\| = 1$ for all n and $\|\frac{1}{2}(x_n + y_n)\| \to 1$, then $\|x_n - y_n\| \to 0$.
- *strictly convex* if the equality $\|x + y\| = \|x\| + \|y\|$ always implies that x and y are proportional.

(1) Prove that every nonempty closed convex subset of a uniformly convex Banach space has a unique element with minimal norm.
(2) Show that if Y is a compact Hausdorff space which is not a singleton, then $C(Y)$ is not uniformly convex.
(3) Show that every uniformly convex Banach space is reflexive.
(4) Compare uniform and strict convexity. Does one imply the other?

You may do the problems in any order you wish!

Problem 13. Compute the extreme points of the unit balls of the following Banach spaces. Then for each Banach space X below, either prove there exists a Banach space Y such that Y^* is isometrically isomorphic to X, or prove that no such Y exists.

(1) c_0.
(2) ℓ^1.

(3) $L^1[0,1]$ with Lebesgue measure.
(4) $L^p(X,\mu)$ where (X,μ) is a complete measure space and $1 < p < \infty$.
(5) $L^\infty(X,\mu)$ where (X,μ) is a complete measure space.

Problem 14 (Adapted from `https://www.math.ksu.edu/~nagy/func-an-2007-2008/bs-3.pdf`). Let X be a locally compact Hausdorff topological space, and let $A = C_?(X,\mathbb{C})$ be one of the following complex algebras: $C_c(X,\mathbb{C}), C_0(X,\mathbb{C}), C(X,\mathbb{C}), C_b(X,\mathbb{C})$. Define the real algebra $A_\mathbb{R} = C_?(X,\mathbb{R})$. Clearly $A = A_\mathbb{R} + iA_\mathbb{R}$. We call $f \in A$ *positive*, denoted $f \geq 0$, if $f(x) \geq 0$ for all $x \in X$. Consider a linear functional $\varphi : A \to \mathbb{C}$, and let $\varphi_\mathbb{R} = \varphi|_{A_\mathbb{R}}$ as an $\mathbb{R}$-linear functional.
Warning: Here, the notation $A_\mathbb{R}$ does not agree with the same notation from Problem 7!

(1) We call φ *positive*, denoted $\varphi \geq 0$, if $\varphi(f) \geq 0$ whenever $f \geq 0$. Prove that $\varphi \geq 0$ if and only if $\varphi_\mathbb{R} \geq 0$.
(2) Show that the map $\varphi \mapsto \varphi_\mathbb{R}$ is injective.
(3) We call φ *self-adjoint* or *hermitian* if $\varphi_\mathbb{R}$ is $\mathbb{R}$-valued. Define $\varphi^* : A \to \mathbb{C}$ by $\varphi^*(f) = \overline{\varphi(\overline{f})}$. Show that $\varphi^{\mathrm{Re}} := \frac{1}{2}(\varphi + \varphi^*)$ and $\varphi^{\mathrm{Im}} = \frac{1}{2i}(\varphi - \varphi^*)$ are hermitian.
(4) Show that φ^{Re} is not the same as $\mathrm{Re}(\varphi)$, but $\mathrm{Re}(\varphi)|_{A_\mathbb{R}} = \varphi^{\mathrm{Re}}|_{A_\mathbb{R}}$.
(5) Show that $\varphi \mapsto \varphi^*$ is an *involution*, i.e., $(\lambda\varphi + \psi)^* = \overline{\lambda}\varphi^* + \psi^*$ and $(\varphi^*)^* = \varphi$ for all $\lambda \in \mathbb{C}$ and all linear functionals $\varphi, \psi : A \to \mathbb{C}$.
(6) Show that the following are equivalent:
  (a) φ is hermitian
  (b) $\varphi = \varphi^*$
  (c) $\varphi = \varphi^{\mathrm{Re}}$
  (d) $\varphi^{\mathrm{Im}} = 0$.
(7) Adapt (3) of Problem 7 to show that for any $\mathbb{R}$-linear functional $\psi : A_\mathbb{R} \to \mathbb{R}$, there is a unique hermitian $\mathbb{C}$-linear functional $\varphi : A \to \mathbb{C}$ such that $\varphi_\mathbb{R} = \psi$.
Warning: It does not make sense to define $\varphi(f) = \psi(f) - i\psi(if)$ as in Problem 7 due to the different definition of $A_\mathbb{R}$!

Problem 15 (adapted from `https://www.math.ksu.edu/~nagy/func-an-2007-2008/bs-3.pdf`). Continue the notation from Problem 14, only now restrict to $A = C_?(X,\mathbb{C})$ for one of the complex algebras $C_c(X,\mathbb{C}), C_0(X,\mathbb{C}), C_b(X,\mathbb{C})$ so that the sup norm $\|\cdot\|_\infty$ is well-defined.
Note that if X is compact, then all three of the above algebras are equal to $C(X,\mathbb{C})$, so we have not lost this example.

(1) Prove that the following are equivalent:
  (a) φ is bounded
  (b) φ^* is bounded
  (c) φ^{Re} and φ^{Im} are bounded.
  In this case, show that $\|\varphi\| = \|\varphi^*\|$.
(2) Show that for a hermitian φ, φ is bounded if and only if $\varphi_\mathbb{R}$ is bounded. In this case, show that $\|\varphi\| = \|\varphi_\mathbb{R}\|$.
(3) Prove that the set $A_h^* = \{\varphi \in A^* | \varphi \text{ is hermitian}\}$ is an $\mathbb{R}$-linear closed subspace of A^*. Deduce that $A_h^* \to A_\mathbb{R}^*$ by $\varphi \mapsto \varphi_\mathbb{R}$ is an isometric isomorphism.
(4) Show that A_h^* is weak* closed in A^*, and that the isomorphism $A_h^* \to A_\mathbb{R}^*$ by $\varphi \mapsto \varphi_\mathbb{R}$ is a weak* homeomorphism.

Problem 16. Let X be a normed space. Prove that every norm closed convex subset $S \subset X$ is weakly closed.

Problem 17.

(1) Suppose X and Y are Banach spaces and $(T_\lambda) \subset B(X,Y)$ is a net such that for all $x \in X$, $(T_\lambda x)$ is bounded and convergent in Y. Show there exists a $T \in B(X,Y)$ such that $T_\lambda x \to$

Tx for all $x \in X$.

Recall that $T_\lambda \to T$ in the strong operator topology if and only if $T_\lambda x \to Tx$ for all $x \in X$. It is induced by the separating family of seminorms $T \mapsto \|Tx\|_Y$ for $x \in X$.

(2) Show that every weakly convergent sequence in a normed space X is norm bounded.

Problem 18. Suppose X, Y are Banach spaces and $T : X \to Y$ and $S : Y^* \to X^*$ are linear maps such that $\varphi(Tx) = (S\varphi)x$ for all $x \in X$ and $\varphi \in Y^*$. Prove that S and T are bounded and $S = T^*$.

Problem 19. Suppose X, Y are Banach spaces and $T \in B(X, Y)$. Prove that the following are equivalent:

(a) T is bounded below, i.e., there is an $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for every $x \in X$.

(b) T is injective and TX is closed in Y.

Problem 20. Provide examples of the following:

(1) Normed spaces X, Y and a discontinuous linear map $T : X \to Y$ with closed graph.

(2) Normed spaces X, Y and a family of linear operators $\{T_\lambda\}_{\lambda\in\Lambda}$ such that $(T_\lambda x)_{\lambda\in\Lambda}$ is bounded for every $x \in X$, but $(\|T_\lambda\|)_{\lambda\in\Lambda}$ is not bounded.

Problem 21 (Sarason)**.** Let K be a compact metric space, F a nonempty closed subset of K, and $I(F)$ the ideal of functions in $C(K)$ that vanish on F. Show $I(F)$ is complemented in $C(K)$ by showing the following.

(1) Suppose X, Y are Banach spaces and $T \in B(X, Y)$. Then T is right invertible if and only if T is surjective and $\ker(T)$ is complemented.

(2) To prove the theorem, it suffices to show that the restriction map $R\colon C(K) \longrightarrow C(F)$ given by $Rf = f|_F$ has a right inverse.

(3) There is a countable partition of unity $\{u_n\}_{n\in\mathbb{N}}$ of continuous functions $K \setminus F \to [0, 1]$ such that for each $n \in \mathbb{N}$, the diameter of $\operatorname{supp}(u_n)$ is less than or equal to the distance of $\operatorname{supp}(u_n)$ from F.

(4) With $\{u_n\}_{n\in\mathbb{N}}$ as above, choose for each $n \in \mathbb{N}$ a point $x_n \in F$ such that

$$\operatorname{dist}(x_n, \operatorname{supp}(u_n)) = \operatorname{dist}(\operatorname{supp}(u_n), F).$$

For $f \in C(F)$, define the function Sf on K by

$$(Sf)(x) = \begin{cases} f(x) & \text{if } x \in F \\ \sum_{n=1}^{\infty} u_n(x)f(x_n) & \text{if } x \in K \setminus F. \end{cases}$$

Then S is a right inverse of R.

Problem 22 (Sarason)**.** Prove c_0 is uncomplemented in ℓ^∞ by establishing the following assertions and then combining them to obtain the desired conclusion.

(1) If E is a complemented subspace of a Banach space X, then X/E is isomorphic to a subspace of X.

(2) The bidual of a separable Banach space is separable in its weak* topology.

(3) If the dual of a Banach space is weak* separable, then the dual of each of its subspaces is weak* separable.

(4) There is a family $\{N_t | 0 \leq t \leq 1\}$ of infinite subsets of $\mathbb{N}$ such that $N_s \cap N_t$ is finite for $s \neq t$.

(5) For $0 \leq t \leq 1$, define the sequence χ_t in ℓ^∞ by

$$\chi_t(n) = \begin{cases} 1 & \text{if } n \in N_t \\ 0 & \text{if } n \notin N_t. \end{cases}$$

Then each functional in $(\ell^\infty/c_0)^*$ annihilates all but finitely many of the cosets $\chi_t + c_0$. It follows that $(\ell^\infty/c_0)^*$ is not weak* separable.

Don't forget to finally prove that c_0 is uncomplemented in ℓ^∞!

Problem 23. Show that the following operators are well-defined, bounded, and compact.

(1) For $F \in C([0,1]^2)$, define $T : C[0,1] \to C[0,1]$ by $(Tf)(x) = \int_0^1 F(x,y)f(y)\,dy$.
(2) For $f \in C^1[0,1]$, define $\|f\|_{C^1[0,1]} = \|f\|_\infty + \|f'\|_\infty$. Let $T : C^1[0,1] \to C[0,1]$ be inclusion.
(3) For $1 < p \leq \infty$, define the Volterra operator $V : L^p[0,1] \to L^p[0,1]$ by $(Vf)(x) = \int_0^x f(y)\,dy$.

Hint: Use the Arzelà-Ascoli Theorem.

Problem 24 (Sarason)**.** Prove that the Volterra operator $V : L^1[0,1] \to L^1[0,1]$ by $(Vf)(x) = \int_0^x f(y)\,dy$ is compact.
Hint: Calculate V^* and show it is compact.

Problem 25. Suppose X, Y are Banach spaces and $T : X \to Y$ is a linear transformation.

(1) Show that if $T \in B(X,Y)$, then T is weak-weak continuous. That is, if $x_\lambda \to x$ in the weak topology on X induced by X^*, then $Tx_\lambda \to Tx$ in the weak topology on Y induced by Y^*.
(2) Show that if T is norm-weak continuous, then $T \in B(X,Y)$.
(3) Show that if T is weak-norm continuous, then T has finite rank.

Problem 26. Suppose X is a Banach space. Prove the following are equivalent:

(1) X is separable.
(2) The relative weak* topology on the closed unit ball of X^* is metrizable.

Deduce that the closed unit ball of X^* is weak* sequentially compact.

Problem 27. Suppose X is a Banach space. Prove the following are equivalent:

(1) X^* is separable.
(2) The relative weak topology on the closed unit ball of X is metrizable.

Prove that in this case, X is also separable.

Problem 28 (Eberlein-Smulian)**.** Suppose X is a Banach space. Prove the following are equivalent:

(1) X is reflexive.
(2) The closed unit ball of X is weakly compact.
(3) The closed unit ball of X is weakly sequentially compact.

Optional: *How do you reconcile Problems 26, 27, and 28? That is, how do you reconcile the fact that there exist separable Banach spaces which are not reflexive?*

Problem 29. Let X, Y be Banach spaces and $K \in B(X,Y)$. Consider the following statements.

(a) K is compact
(b) $K|_B : B \to Y$ is weak-norm continuous, where B is the norm closed unit ball of X, endowed with the relative weak topology.
(c) K maps weakly convergent sequences in X to norm convergent sequences in Y.

Now for the problem:

(1) Prove that $(a) \Rightarrow (b) \Rightarrow (c)$.
(2) Prove $(c) \Rightarrow (a)$ when in addition X is reflexive.
(3) Prove that every weakly convergent sequence in ℓ^1 is norm convergent. Explain why this fact does not imply that the norm and weak topologies on ℓ^1 are the same. Deduce that $(c) \Rightarrow (a)$ fails without the additional assumption that X is reflexive.

Hints: *For $(a) \Rightarrow (b)$, you could prove that every $T \in B(X,Y)$ is weak-weak continuous as in Problem 25.*
*For $(c) \Rightarrow (a)$, you could show that the closed unit ball $B \subset X$ is weakly **sequentially** compact as in Problem 28.*

<u>Optional:</u> How do you reconcile (3) of Problem 25 with (b) above? That is, how do you reconcile the fact that any weak-norm continuous operator has finite rank with the existence of compact operators which are not finite rank?

Problem 30. Suppose X, Y are Banach spaces and $T \in B(X, Y)$ is Fredholm. Prove that T^* is Fredholm with $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$.

Problem 31. Suppose X is a Banach space.

(1) Suppose that $T \in B(X)$ such that T^2 is Fredholm. Prove T is Fredholm.
(2) Show that the shift operator on ℓ^p for $1 \leq p \leq \infty$ does not have a square root.

Problem 32. Suppose X is a Banach space and $P, Q \in B(X)$ are idempotents.

(1) Show that $QX \subseteq PX$ if and only if $PQ = Q$. Assuming this holds:
  (a) Show $QP \in B(X)$ is an idempotent with $QPX = QX$.
  (b) Show QX is complemented in PX.
  (c) Find an idempotent $R \in B(X)$ such that $RX \subseteq (1-Q)X \cap PX$ and $RX \oplus QX = PX$. Then find an idempotent $S \in B(X)$ such that $SX \subseteq (1-Q)X$ and $RX \oplus SX = (1-Q)X$.
(2) Show that $(1-P)X \subseteq (1-Q)X$ if and only if $QP = Q$.

Problem 33. Let A be a unital Banach algebra. Show that the map $a \mapsto a^{-1}$ on $G(A)$, the set of invertible elements of A, is continuous.

Problem 34. Let X be a Banach space, let $GL(X)$ denote the invertible elements in $B(X)$, and let $F(X)$ denote the Fredholm operators on X.

(1) Show that each connected component of $GL(X)$ (respectively $F(X)$) is locally path connected. Deduce that each connected component is path connected.
(2) Let $\pi_0(GL(X))$ and $\pi_0(F(X))$ denote the sets of (path)connected components of $GL(X)$ and $F(X)$ respectively. Show that the multiplication map $\mu : \pi_0(F(X)) \times \pi_0(F(X)) \to \pi_0(F(X))$ by $([S], [T]) \mapsto [ST]$ is well-defined.
(3) Show that $\pi_0(F(X))$ with multiplication μ is a group, where the inverse of $[T]$ is $[S]$ where S is a Fredholm inverse of T.
Hint: Consider the path $[0, 1] \to F(X)$ by $t \mapsto 1 + t(TS - 1)$.
(4) Repeat the analogous versions of (2) and (3) for $\pi_0(GL(X))$.
Hint: Use that $T \mapsto T^{-1}$ is continuous on $GL(X)$.
(5) Prove that $\operatorname{ind} : \pi_0(F(X)) \to \mathbb{Z}$ by $[T] \mapsto \operatorname{ind}(T)$ is a well-defined group homomorphism.
(6) Prove that the inclusion $GL(X) \to F(X)$ induces a well-defined group homomorphism $\phi : \pi_0(GL(X)) \to \pi_0(F(X))$.
(7) Prove that $\ker(\operatorname{ind}) = \phi(\pi_0(GL(X)))$.
Hint: For $T \in F(X)$ with index zero, decompose X into $E_0 \oplus F_0$ and into $E_1 \oplus F_1$ such that $\dim(E_0) = \dim(E_1) < \infty$ and T induces an invertible map $F_0 \to F_1$. Then find a path which connects T to an invertible operator by perturbing by an isomorphism $E_0 \to E_1$.

Problem 35. Let A be a unital Banach algebra, and let X be a Banach space.

(1) An element $b \in A$ is called a *topological zero divisor* (in A) if there is a sequence (a_n) in A such that $\|a_n\| = 1$ for all $n \in \mathbb{N}$ and both $a_n b \to 0$ and $b a_n \to 0$. Prove every element on the boundary of the set of invertible elements of A is a topological zero divisor.
(2) A complex number λ is called an *approximate eigenvalue* of the operator $T \in B(X)$ if there is a sequence (x_n) of unit vectors in X such that $Tx_n - \lambda x_n \to 0$. Prove every point on the boundary of $\operatorname{sp}(T)$ is an approximate eigenvalue of T.

Problem 36. Let A be a unital Banach algebra. Suppose we have a norm convergent sequence $(a_n) \subset A$ with $a_n \to a$. Prove that for every open neighborhood U of $\operatorname{sp}(a)$, there is an $N > 0$ such that $\operatorname{sp}(a_n) \subset U$ for all $n > N$.

Problem 37. Let X be a Banach space, and let $[a,b] \subset \mathbb{R}$ be a compact interval. Let $C([a,b],X)$ be the space of continuous functions $[a,b] \to X$, where X has the norm topology.

(1) Show that every $f \in C([a,b],X)$ is uniformly continuous.
(2) Prove that $C([a,b],X)$ is a Banach space under the norm $\|f\|_\infty := \sup_{t\in[a,b]} \|f(t)\|_X$.

Problem 38. Let X be a Banach space. In this problem, we show that the Riemann integral for continuous paths $\gamma : [a,b] \to X$ is well-defined and is compatible with X^*. Fix a continuous path $\gamma : [a,b] \to X$.

(1) A *partition* of $[a,b]$ is a finite list $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$. We say $P \leq Q$ or Q *refines* P if $P \subseteq Q$ as sets. Clearly $\leq$ is a partial order on partitions. Show that partitions form a directed set under $\leq$.
(2) A *tagged partition* of $[a,b]$ is a pair (P,u) where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition of $[a,b]$ and $u \in [a,b]^n$ such that $t_{i-1} \leq u_i \leq t_i$ for all $i = 1,\dots,n$. Show that the partial order given on partitions in (1) induces a preorder on tagged partitions.
Note: A preorder is reflexive and transitive, but need not be anti-symmetric.
(3) For a tagged partition (P,u), let $x_{(P,u)} = \sum_{i=1}^n \gamma(u_i)(t_i - t_{i-1})$. Show that $(x_{P,u})$ is a *norm* convergent net in X.
Hint: Take a limit as $\|P\| = \max\{\Delta_i := t_i - t_{i-1} | i = 1,\dots n\} \to 0$ and use Problem 37.
Note: Some authors define nets using preorders instead of partial orders. We need only consider a net defined using a preorder for this problem, so let's do so.
(4) Define $\int_0^1 \gamma(t)\,dt = \lim x_{(P,u)}$. Prove that for every $\varphi \in X^*$, $\varphi(\int_a^b \gamma(t)\,dt) = \int_a^b \varphi(\gamma(t))\,dt$, where the right hand side is the Riemann integral of $\varphi \circ \gamma : [a,b] \to \mathbb{C}$.
(5) Show that $\left\|\int_a^b \gamma(t)\,dt\right\| \leq \int_a^b \|\gamma(t)\|\,dt$. Deduce that $\int_a^b : C([a,b],X) \to X$ is a bounded linear transformation.

Problem 39. Let A be a unital Banach algebra. Show that the holomorphic functional calculus satisfies the following properties.

(1) Suppose $a \in A$ and $K \subset \mathbb{C}$ is compact such that $\operatorname{sp}(a) \subset K^\circ$. Show there is an $M_K > 0$ such that for any $f \in H(K^\circ)$ which has a continuous extension to K, $\|f(a)\| \leq M_K \|f\|_{C(K)}$.
(2) Suppose $(a_n) \subset A$ is a norm convergent sequence with $a_n \to a$. Show that for all $f \in \mathcal{O}(\operatorname{sp}(a))$, $f(a_n) \to f(a)$ as $n \to \infty$.
Note: For $f \in \mathcal{O}(\operatorname{sp}(a))$, note that f is holomorphic on some open set U containing K. By Problem 36, we know that eventually $\operatorname{sp}(a_n) \subset U$, so eventually $f(a_n)$ is well-defined.

Problem 40. Let A be a unital Banach algebra, and let $a, p \in A$ such that $ap = pa$.

(1) Show that for every $f \in \mathcal{O}(\operatorname{sp}(a))$, $f(a)p = pf(a)$.
Hint: First prove the result for rational f, and then apply Runge's Theorem.
(2) Suppose from here on p is an idempotent. Show that pAp is a unital Banach algebra.
(3) Prove that $\operatorname{sp}_{pAp}(pa) \subseteq \operatorname{sp}_A(a)$.
(4) Prove that for every $f \in \mathcal{O}(\operatorname{sp}_A(a))$, $f(ap) = pf(a)$ when viewed in the image of the holomorphic functional calculus $\mathcal{O}(\operatorname{sp}_{pAp}(pa)) \ni f \mapsto f(pa) \in pAp$.
Hint: First verify that the proof of the uniqueness of the holomorphic functional calculus $\mathcal{O}(\operatorname{sp}_A(a)) \ni f \mapsto f(a) \in A$ also proves the following fact.

Fact. Suppose that U is an open neighborhood of $\operatorname{sp}_A(a)$, and $\Phi : H(U) \to A$ is a homomorphism such that

- $\Phi(z \mapsto 1) = 1_A$ and $\Phi(z \mapsto z) = a$, and
- If $(f_n) \subset H(U)$ converges locally uniformly to f, then $\Phi(f_n) \to \Phi(f)$.

Then $\Phi(f) = f(a)$ for all $f \in H(U)$, i.e., Φ is the holomorphic functional calculus restricted to $H(U) \subseteq \mathcal{O}(\operatorname{sp}_A(a))$.

Now show that for any open neighborhood U of $\mathrm{sp}_A(a) \supseteq \mathrm{sp}_{pAp}(pa)$, $\Phi : H(U) \to pAp$ given by $\Phi(f) = p(f(a))$ is a homomorphism which satisfies the criteria in the above fact with A and a replaced by pAp and pa respectively.

(5) (optional) Suppose $\mathrm{sp}_A(a) = K_1 \cup K_2$, a disjoint union of two nonempty compact sets. Let U_1, U_2 be disjoint non-empty subsets of $\mathbb{C}$ such that $K_i \subseteq U_i$. Suppose further that the idempotent $p = \chi_{U_1}(a)$ where χ_{U_1} is the indicator function for U_1. See if $\mathrm{sp}_{pAp}(ap) = K_1$.

Problem 41. Let $A \in M_n(\mathbb{C})$. As best as you can, describe $f(A)$ where $f \in \mathcal{O}(\mathrm{sp}(A))$.
Hint: First consider the case that A is a single Jordan block.

Problem 42. Let A be a unital commutative Banach algebra, $G(A)$ the group of invertible elements of A, and G_1 the connected component of $G(A)$ containing 1_A. Show that G_1 is the subgroup generated by $B_1(1_A) \subset G(A)$. Deduce that every element of G_1 has a logarithm.
Hint: For the second part, show that for all $g \in B_1(1_A)$, $\mathrm{sp}_A(g) \subset B_1(1)$. Then show that whenever x, y are commuting elements in a Banach algebra, $e^{x+y} = e^x e^y$.

Problem 43. Suppose A is a unital Banach algebra and fix $a, b \in A$.

(1) Show that $1 \notin \mathrm{sp}_A(ab)$ if and only if $1 \notin \mathrm{sp}_A(ba)$ using the identity $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$. Deduce that $\mathrm{sp}_A(ab) \cup \{0\} = \mathrm{sp}_A(ba) \cup \{0\}$.
(2) Show that for any Banach subalgebra $B \subseteq A$ with $1_A \in B$, for every $a \in B$, the spectral radius in B of a is equal to the spectral radius in A of a, i.e., $r_B(a) = r_A(a)$.
(3) Suppose $a, b \in A$ commute. Prove that $r(ab) \leq r(a)r(b)$ and $r(a + b) \leq r(a) + r(b)$.
Hint: By (2), this computation can be performed in the unital commutative Banach subalgebra $B \subseteq A$ generated by a and b. In B, there is a helpful characterization of the spectrum.
(4) Deduce from part (3) that if A is commutative, the spectral radius $r : A \to [0, \infty)$ is continuous.

Problem 44. Determine as best you can which matrices $A \in M_n(\mathbb{C})$ have square roots, i.e., when there is a $B \in M_n(\mathbb{C})$ such that $B^2 = A$.
Note: Such a B is not necessarily unique.

Problem 45 (Sarason)**.** The Volterra algebra is $L^1[0, 1]$ with truncated convolution multiplication:

$$(f * g)(t) = \int_0^t f(t - s)g(s)\, ds.$$

Prove that every element of the Volterra algebra is quasinilpotent.
Hint: Prove that for every $f \in C[0, 1]$,

$$\|f^n\| = \|\underbrace{f * \cdots * f}_{n\ copies}\| \leq \frac{\|f\|_\infty^n}{n!}.$$

Then deduce that $r(f) = 0$ for all $f \in C[0, 1]$, and apply part (4) of Problem 43.

Problem 46. Prove that for $n \geq 2$, there are no multiplicative linear functionals on $M_n(\mathbb{C})$.

Problem 47.

(1) Suppose A is a unital commutative Banach algebra which is generated by a single $a \in A$, i.e., polynomials in a are dense in A. Find a homeomorphism $\widehat{A} \cong \mathrm{sp}_A(a)$.
(2) Calculate the Gelfand space with its topology together with the Gelfand transform for the following Banach algebras.
    (a) $\ell^1(\mathbb{Z}_{\geq 0})$ with truncated convolution multiplication $(a * b)(n) = \sum_{k=0}^n a(n - k)b(k)$.
    Hint: Use (1) and mimic the calculations from class for $\ell^1(\mathbb{Z})$.

(b) The disk algebra $A(\mathbb{D}) = \left\{f \in C(\overline{\mathbb{D}}) \middle| f|_{\mathbb{D}} \text{ is holomorphic}\right\}$.
Hint: Use (1) together with the relationship between $\mathrm{sp}_A(b)$ *and* $\mathrm{sp}_B(b)$ *when* $1_A \in B \subseteq A$ *is a unital inclusion of Banach algebras with* B *generated by* b.

Problem 48. Consider $L^p[0,1]$ for $p \in \{1, \infty\}$ with respect to Lebesgue measure.

(1) Prove that no multiplicative linear functional $\varphi : L^\infty[0,1] \to \mathbb{C}$ is weak* continuous under the identification $L^\infty[0,1] \cong L^1[0,1]^*$.

(2) Let X be the Gelfand space of $L^\infty[0,1]$. For a measurable $E \subset [0,1]$, denote the characteristic function of E by χ_E, and define $\widehat{E} = \{x \in X | \Gamma(\chi_E)(x) = 1\}$, where $\Gamma : L^\infty[0,1] \to C(X)$ is the Gelfand transform.
   (a) Prove that the sets $\widehat{E} \subset X$ are both closed and open.
   (b) Show that the sets $\widehat{E} \subset X$ form a base for the Gelfand topology.
   (c) Show that the closure of every open subset $U \subset X$ is open, i.e., X is extremally disconnected.

Hint: This is a hint distilled from a solution I gave to the problem back in graduate school, so use with caution! First, the following claim is helpful for the problem. (There are other related helpful statements too.)

Claim. For measurable $E, F \subset [0,1]$, $\widehat{E^c} = \widehat{E}^c$ and $\widehat{E \cap F} = \widehat{E} \cap \widehat{F}$.

For part (c), suppose $U \subseteq X$ *is open. We know that* $U = \bigcup_{i \in I} \widehat{E_i}$ *for measurable subsets* $E_i \subset [0,1]$ *where* I *is some index set. Find a countable subset* $J \subset I$ *such that* $E := \bigcup_{j \in J} E_j$ *satisfies* $m(E_i \setminus E) = 0$ *for all* $i \in I$. *Use the claim to prove* $\widehat{E_i} \subseteq \widehat{E}$ *for all* $i \in I$, *and deduce* $\overline{U} \subseteq \widehat{E}$.
Now suppose for contradiction $\overline{U} \neq \widehat{E}$. *Pick a* $\varphi \in \widehat{E} \setminus \overline{U}$, *which is open. This means there is a measurable* $F \subseteq [0,1]$ *with* $m(F) > 0$ *such that* $\varphi \in \widehat{F} \subseteq \widehat{E} \setminus \overline{U}$ *by part (b). Now look at* φ *applied to* $\chi_E \chi_F = \chi_{E \cap F}$ *and derive a contradiction.*

Problem 49. Suppose A is a C*-algebra and $a \in A$ is normal.

(1) Show a is self-adjoint if and only if $\mathrm{sp}(a) \subset \mathbb{R}$.
(2) Show a is unitary if and only if $\mathrm{sp}(a) \subset \mathbb{T}$.
(3) Show a is a projection if and only if $\mathrm{sp}(a) \subset \{0, 1\}$.

Problem 50. Let A be a C*-algebra.

(1) Show that the following are equivalent for a self-adjoint $a \in A$:
   (a) $\mathrm{sp}(a) \subset [0, \infty)$,
   (b) For all $\lambda \geq \|a\|$, $\|a - \lambda\| \leq \lambda$, and
   (c) There is a $\lambda \geq \|a\|$ such that $\|a - \lambda\| \leq \lambda$.

   For now, we will call such elements *spectrally positive.*
   Note: It is implicit here that a spectrally positive element is self-adjoint.

(2) Deduce that the spectrally positive elements in a C*-algebra form a closed cone, i.e., $A_+ = \{a \in A | a \geq 0\}$ is closed, and for all $\lambda \in [0, \infty)$ and $a, b \in A_+$, we have $\lambda a + b \in A_+$.

(3) Show a is positive ($a = b^*b$ for some b) if and only if a is spectrally positive ($a = a^*$ and $\mathrm{sp}(a) \subset [0, \infty)$).
Hint: First, if $\mathrm{sp}(a) \subset [0, \infty)$, *we can define* $a^{1/2}$ *via the continuous functional calculus. Now suppose* $a = b^*b$ *for some* $b \in B$. *Use the continuous functions* $r \mapsto \max\{0, z\}$ *and* $r \mapsto -\min\{0, z\}$ *on* $\mathrm{sp}(a)$ *to write* $a = a_+ - a_-$ *where* $\mathrm{sp}(a_\pm) \subset [0, \infty)$ *and* $a_+ a_- = a_- a_+ = 0$. *Now look at* $c = ba_-$. *Prove that* $\mathrm{sp}(c^*c) \subset (-\infty, 0]$ *and* $\mathrm{sp}(cc^*) \subset [0, \infty)$ *using part (1) of this problem. Use part (1) of Problem 43 to deduce that* $c^*c = 0$. *Finally, deduce* $a_- = 0$, *and thus* $a = a_+$.

Problem 51. For $a, b \in A$, we say $a \leq b$ if $b - a \geq 0$.

(1) Show that $\leq$ is a partial order.
(2) Show that if $a \leq b$, then for all $c \in A$, $c^*ac \leq c^*bc$.
(3) Suppose $0 \leq a \leq b$. Prove that $\|a\| \leq \|b\|$.

Problem 52. Let A be a C*-algebra. By the hint to part (4) of Problem 49 that for $a \geq 0$, we can define an $a^{1/2} \geq 0$ such that $(a^{1/2})^2 = a$.

(1) Show that if $b \geq 0$ such that $b^2 = a$, then $b = a^{1/2}$.
(2) Prove that if $0 \leq a \leq b$, then $a^{1/2} \leq b^{1/2}$.
(3) Prove that if $0 < a$ ($0 \leq a$ and a is invertible), then $0 < a^{-1}$.
(4) Prove that if $0 < a \leq b$, then $0 < b$ and $0 < b^{-1} \leq a^{-1}$.

Problem 53 (Rieffel, "Preventative Medicine")**.** Consider $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ for $s, t \geq 0$.

(1) Determine for which $s, t \geq 0$ we have $b \geq a$.
(2) Determine for which $s, t \geq 0$ we have $b \geq a_+$.
Note: Since $a = a^*$, a_+ is the positive part defined as in the hint to part (4) of Problem 49.
(3) Find values of $s, t \geq 0$ for which $b \geq a$, $b \geq 0$, and yet $b \not\geq a_+$.
(4) Find values of $s, t \geq 0$ such that $b \geq a_+ \geq 0$, and yet $b^2 \not\geq a_+^2$.
(5) Can you find $s, t \geq 0$ such that $b \geq a_+$ and yet $b^{1/2} \not\geq a_+^{1/2}$?
Note: $a_+^{1/2}$ is the unique positive square root of a_+ from part (1) Problem 52.
(6) Suppose $c, p \in M_2(\mathbb{C})$ such that $c \geq 0$ and $p^2 = p^* = p$ is a projection. Is it always true that $pcp \leq c$?

Problem 54. Let $L^2(\mathbb{T})$ denote the space of complex-valued square-integrable 1-periodic functions on $\mathbb{R}$, and let $C(\mathbb{T}) \subset L^2(\mathbb{T})$ denote the subspace of continuous 1-periodic functions.

(a) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
(b) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(\mathbb{T})} = \int_0^1 f(x) \exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

Problem 55. Recall that each $T \in B(H, K)$ induces a bounded sesquilinear form $K \times H \to \mathbb{C}$ given by $B_T(\xi, \eta) = \langle \xi, T\eta \rangle$.

(1) Prove that $T \mapsto B_T$ is an isometric bijective correspondence between operators in $B(H, K)$ and bounded sesquilinear forms $K \times H \to \mathbb{C}$.
Hint: Adapt the proof Lemma 3.2.2 in Analysis Now (see also Exercise 3.2.15 therein).
(2) For $T \in B(H, K)$ corresponding to $B_T : K \times H \to \mathbb{C}$, we define $T^* \in B(K, H)$ to be the unique operator corresponding to the adjoint sesquilinear form $B_T^* : H \times K \to \mathbb{C}$ defined by

$$B_T^*(\eta, \xi) := \overline{B_T(\xi, \eta)} \qquad \Longleftrightarrow \qquad \langle \eta, T^*\xi \rangle = \langle T\eta, \xi \rangle \qquad \eta \in H, \xi \in K.$$

Show that $T \mapsto T^*$ is a conjugate linear isometry of $B(H, K)$ onto $B(K, H)$, and that $\|T^*T\| = \|T\|^2 = \|TT^*\|$.
(3) In the case that $H = K$, deduce the following:
  (a) $B(H)$ with involution $T \mapsto T^*$ is a C*-algebra.
  (b) $T = T^*$ if and only if B_T is self-adjoint. That is, show $T = T^*$ if and only if $\langle T\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in H$.
  (c) $T \geq 0$ if and only if B_T is positive. That is, show $T \geq 0$ if and only if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.
  Hint: Use that for $T = T^*$, we have $\inf\{\langle T\xi, \xi \rangle | \xi \in H, \|\xi\| = 1\} = \min\{\lambda | \lambda \in \mathrm{sp}(T)\}$.

(d) (optional) $T \geq 0$ and T injective if and only if B_T is positive definite.
Hint: For $S \in B(H)$, $\ker(S) = \ker(S^*S)$, so $T \geq 0$ is injective if and only if $T^{1/2}$ is injective.

(e) (optional) $T > 0$ ($T \geq 0$ and T is invertible) if and only if B_T is positive definite, and H is complete in the norm $\|\xi\|_T := B_T(\xi, \xi)^{1/2}$.
Hint: When B_T is positive definite and H is complete for $\|\cdot\|_T$, apply part (d) and look at the isometry $(H, \|\cdot\|_T) \to (H, \|\cdot\|)$ by $\xi \mapsto T^{1/2}\xi$.

Problem 56. For a Hilbert space H, we can define the *conjugate* Hilbert space $\overline{H} = \{\overline{\xi} | \xi \in H\}$ which has the conjugate vector space structure $\lambda\overline{\xi} + \overline{\eta} = \overline{\overline{\lambda}\xi + \eta}$ and the conjugate inner product $\langle \overline{\eta}, \overline{\xi} \rangle_{\overline{H}} = \langle \xi, \eta \rangle_H$.

(1) Prove that $\overline{H}$ is a Hilbert space.
(2) For $T \in B(H, K)$, define $\overline{T} : \overline{H} \to \overline{K}$ by $\overline{T}\overline{\xi} = \overline{T\xi}$. Prove that $\overline{T} \in B(\overline{H}, \overline{K})$, and $\|T\| = \|\overline{T}\|$.
(3) Prove that $\overline{\cdot}$ is an endofunctor on the the category Hilb of Hilbert spaces with bounded operators ($\overline{\cdot}$ is a functor $\mathsf{Hilb} \to \mathsf{Hilb}$).
(4) For each $H \in \mathsf{Hilb}$, construct a linear isometry u_H of H^* onto $\overline{H}$ satisfying $u_H T^t = \overline{T} u_H$ for all $T \in B(H, K)$ where $T^t \in B(K^*, H^*)$ is the Banach adjoint of T.

Problem 57. For $T \in B(H)$, we define its *numerical radius* as

$$R(T) := \sup_{\|\xi\| \leq 1} |\langle T\xi, \xi \rangle|.$$

Prove that $r(T) \leq R(T) \leq \|T\| \leq 2R(T)$. Deduce that if T is normal, then $\|T\| = R(T)$.

Problem 58. Suppose we have sequence $(\xi_n) \subset H$ converges weakly to $\xi \in H$. Show there exists a subsequence (ξ_{n_k}) such that $\frac{1}{N} \sum_{k=1}^{N} \xi_{n_k} \to \xi$ in norm as $N \to \infty$.

Problem 59 (Sarason)**.** Find the norm of the Volterra operator V on $L^2[0, 1]$ given by $(Vf)(x) = \int_0^x f(y)\, dy$.
Hint: Recall V is compact by part (3) of Problem 23. See also Exercise 3.4.7 in Analysis Now.

Problem 60. Let A be a C*-algebra. An element $u \in A$ is called a *partial isometry* if u^*u is a projection.

(1) Show that the following are equivalent:
  (a) u is a partial isometry.
  (b) $u = uu^*u$.
  (c) $u^* = u^*uu^*$.
  (d) u^* is a partial isometry.

  Hint: For (a) $\Rightarrow$ (b), apply the C-axiom to $u - uu^*u$.*
(2) We say two projections $p, q \in A$ are *(Murray-von Neumann) equivalent*, denoted $p \approx q$, if there is a partial isometry $u \in A$ such that $uu^* = p$ and $u^*u = q$. Prove that $\approx$ is an equivalence relation on $P(A)$, the set of projections of A.
(3) Describe the set of equivalence classes $P(A)/\approx$ for $A = B(\ell^2)$.

Problem 61. Let H be a Hilbert space. Compute the extreme points of the unit balls of

(1) $\mathcal{K}(H)$,
(2) $\mathcal{L}^1(H)$, and
(3) $B(H)$.

Problem 62. Let H be a Hilbert space. Prove that the trace Tr induces isometric isomorphims:

(1) $\mathcal{K}(H)^* \cong \mathcal{L}^1(H)$, and
(2) $\mathcal{L}^1(H)^* \cong B(H)$.

Problem 63. Suppose H is a Hilbert space and $K \subseteq H$ is a closed subspace. Let $p_K \in B(H)$ be associated orthogonal projection onto K.

(1) Suppose $x \in B(H)$. Prove that:
 (a) $xK \subseteq K$ if and only if $xp_K = p_K x p_K$.
 (b) $x^*K \subseteq K$ if and only if $p_K x = p_K x p_K$.
 (c) $xK \subseteq K$ and $x^*K \subseteq K$ if and only if $[x, p_K] = 0$.
(2) Prove that if $M \subseteq B(H)$ is a $*$-closed subalgebra, then $MK \subseteq K$ if and only if $p_K \in M'$.

Problem 64. Suppose H is a Hilbert space.

(1) Suppose K is another Hilbert space. Define the tensor product Hilbert space $H \overline{\otimes} K$ by completing the algebraic tensor product vector space $H \otimes K$ in the 2-norm associated to the sesquilinear form $\langle \eta \otimes \xi, \eta' \otimes \xi' \rangle := \langle \eta, \eta' \rangle \langle \xi, \xi' \rangle$. Find a unitary isomorphism $H \overline{\otimes} K \cong \bigoplus_{i=1}^{\dim K} H$.
(2) Find a unital $*$-isomorphism $B(\bigoplus_{i=1}^{n} H) \cong M_n(B(H))$.
 Hint: use orthogonal projections.
(3) Suppose $S \subseteq B(H)$, and let $\alpha : B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

 Prove that:
 (a) $\alpha(S)' = M_n(S')$, and
 (b) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.
 (c) Deduce that we always have $\alpha(S)'' = \alpha(S'')$.

Problem 65. Let (X, μ) be a σ-finite measure space, and consider the map $M : L^\infty(X, \mu) \to B(L^2(X, \mu))$ given by $(M_f \xi)(x) = f(x)\xi(x)$ for $\xi \in L^2(X, \mu)$.

(1) Prove that M is an isometric unital $*$-homomorphism.
(2) Let $A \subset B(L^2(X, \mu))$ be the image of the map M. Prove that $A = A'$.
 Hint: If you're stuck with (2), try the case $X = \mathbb{N}$ with counting measure.

Problem 66. Let H be a Hilbert space. The *weak operator topology (WOT)* on $B(H)$ is the topology induced by the separating family of seminorms $T \mapsto |\langle T\eta, \xi \rangle|$ for $\eta, \xi \in H$. The *strong operator topology (SOT)* on $B(H)$ is induced by the separating family of seminorms $x \mapsto \|T\xi\|_H$ for $\xi \in H$.

(1) Prove that every WOT open set is SOT open. Equivalently, prove that if a net $(T_\lambda)_{\lambda \in \Lambda} \subset B(H)$ converges to $T \in B(H)$ SOT, then $T_\lambda \to T$ WOT.
(2) Prove that the WOT is equal to the SOT on $B(H)$ if and only if H is finite dimensional.
(3) Show that the following are equivalent for a linear functional φ on $B(H)$:
 (a) There are $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n \in H$ such that $\varphi(T) = \sum_{i=1}^{n} \langle T\eta_i, \xi_i \rangle$.
 (b) φ is WOT-continuous.
 (c) φ is SOT-continuous.

Problem 67. Suppose $M \subset B(H)$ is a unital $*$-subalgebra. A vector $\xi \in H$ is called:

- *cyclic* for M if $M\xi$ is dense in H.
- *separating* for M if for every $x, y \in M$, $x\xi = y\xi$ implies $x = y$.

(1) Prove that ξ is cyclic for M if and only if ξ is separating for M'.

(2) Prove that H can be orthogonally decomposed into M-invariant subspaces $H = \bigoplus_{i\in I} K_i$, such that each K_i is cyclic for M (has a cyclic vector). Prove that if H is separable, this decomposition is countable.
(3) Prove that if M is abelian and H is separable, then there is a separating vector in H for M.

Problem 68. Suppose H is a Hilbert space, and (x_λ) is an increasing net of positive operators in $B(H)$ which is bounded above by the positive operator $x \in B(H)$, i.e., $\lambda \le \mu$ implies $x_\lambda \le x_\mu$, and $0 \le x_\lambda \le x$ for all λ. Prove that the following are equivalent.

(1) $x_\lambda \to x$ SOT.
(2) $x_\lambda \to x$ WOT.
(3) For every $\xi \in H$, $\omega_\xi(x_\lambda) = \langle x_\lambda \xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.
(4) There exists a dense subspace $D \subset H$ such that for every $\xi \in D$, $\omega_\xi(x_\lambda) = \langle x_\lambda \xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.

We say an increasing net of positive operators (x_λ) *increases to* $x \in B(H)_+$, denoted $x_\lambda \nearrow x$, if any of the above equivalent conditions hold.
Hint: Show it suffices to prove (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (3). *Try proving these implications.*

Problem 69. Let H be a Hilbert space and let $T \in B(H)$. Prove that the following are equivalent. (You may use any results from last semester that you'd like without proof.)

(1) T is compact and normal.
(2) T has an orthonormal basis of eigenvectors $(e_i)_{i\in I}$ such that the corresponding eigenvalues $\lambda_i \to 0$, with at most countably many of the $\lambda_i \neq 0$.
(3) There is a countable orthonormal subset $(\xi_n)_{n\in\mathbb{N}} \subset H$ and a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and $T = \sum_{n\in\mathbb{N}} \lambda_n |\xi_n\rangle\langle\xi_n|$, which converges in operator norm.
(4) There is a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and a countable family of finite rank projections $E_n \subset B(H)$ such that $T = \sum_{n\in\mathbb{N}} \lambda_n E_n$, which converges in operator norm.
(5) There is a discrete set X equipped with counting measure ν, a function $f \in c_0(X)$, and a unitary $U \in B(\ell^2 X, H)$ such that $T = U M_f U^*$ where $M_f \xi = f\xi$ for $\xi \in \ell^2 X$.
Note: $U \in B(K, H)$ *is unitary if* $UU^* = \mathrm{id}_H$ *and* $U^*U = \mathrm{id}_K$.

Problem 70. Suppose A is a unital C*-algebra. A linear map $\Phi : A \to B(H)$ is called *completely positive* if for every $a = (a_{i,j}) \ge 0$ in $M_n(A)$, $(\Phi(a_{i,j})) \ge 0$ in $M_n(B(H)) \cong B(H^n)$. Such a map is *unital* if $\Phi(1) = 1$.

(1) Show that $\langle x\otimes\eta, y\otimes\xi\rangle := \langle \Phi(y^*x)\eta, \xi\rangle_H$ on $A\otimes H$ linearly extends to a well-defined positive sesquilinear form.
(2) Show that for V a vector space with positive sesquilinear form $B(\,\cdot\,,\,\cdot\,)$, $N_B = \{v \in V | B(v, v) = 0\}$ is a subspace of V, and B descends to an inner product on V/N_B.
(3) Define K to be completion of $(A \otimes H)/N_{\langle\,\cdot\,,\,\cdot\,\rangle}$ in $\|\cdot\|_2$. Find a unital $*$-homormophism $\Psi : A \to B(K)$, and an isometry $v \in B(H, K)$ such that $\Phi(m) = v^*\Psi(m)v$.

Problem 71. Suppose $A \subseteq B(H)$ is a unital C*-subalgebra and $\xi \in H$ is a cyclic vector for A. Consider the vector state $\omega_\xi = \langle\,\cdot\,\xi, \xi\rangle$. Prove there is a bijective correspondence between:

(1) positive linear functionals φ on A such that $0 \le \varphi \le \omega_\xi$ ($\omega_\xi - \varphi \ge 0$), and
(2) operators $0 \le x \le 1$ in A'.

Hint: Use the bijective correspondence between sesquilinear forms and operators.

Problem 72.

(1) Prove that a unital $*$-subalgebra $M \subseteq B(H)$ is a von Neumann algebra if and only if its unit ball is σ-WOT compact.

(2) Let $M \subset B(H)$ be a von Neumann algebra and $\Phi : M \to B(K)$ a unital $*$-homomorphism. Deduce that if Φ is σ-WOT continuous and injective, then $\Phi(M)$ is a von Neumann subalgebra of $B(K)$.

Problem 73. Suppose X is a compact Hausdorff topological space and $E : (X, \mathcal{B}) \to B(H)$ is a Borel spectral measure. Prove that the following conditions are equivalent.

(1) E is regular, i.e., for all $\xi \in H$, $\mu_{\xi,\xi}(S) = \langle E(S)\xi, \xi\rangle$ is a finite regular Borel measure.
(2) For all $S \in \mathcal{B}$, $E(S) = \sup\{E(K)|K$ is compact and $K \subseteq S\}$.
(3) For all $S \in \mathcal{B}$, $E(S) = \inf\{E(U)|U$ is open and $S \subseteq U\}$

Problem 74. Let H be a separable Hilbert space and $A \subseteq B(H)$ an abelian von Neumann algebra. Prove that the following are equivalent.

(1) A is maximal abelian, i.e., $A = A'$.
(2) A has a cyclic vector $\xi \in H$.
(3) For every norm separable SOT-dense C*-subalgebra $A_0 \subset A$, A_0 has a cyclic vector.
(4) There is a norm separable SOT-dense C*-subalgebra $A_0 \subset A$ such that A_0 has a cyclic vector.
(5) There is a finite regular Borel measure μ on a compact Hausdorff second countable space X and a unitary $u \in B(L^2(X, \mu), H)$ such that $f \mapsto uM_f u^*$ is an isometric $*$-isomorphism $L^\infty(X, \mu) \to A$.

Hints:
For (1) $\Rightarrow$ (2)*, use Problem 67.*
For (3) $\Rightarrow$ (4) *it suffices to construct a norm separable SOT-dense C*-algebra. First show that* $A_* = \mathcal{L}^1(H)/A_\perp$ *is a separable Banach space. Then show that* A *is* σ*-WOT separable, which implies SOT-separable. Take* A_0 *to be the unital C*-algebra generated by an SOT-dense sequence.*
For (4) $\Rightarrow$ (5) *show that* A_0 *separable implies* $X = \widehat{A_0}$ *is second countable. Define* $\mu = \mu_{\xi,\xi}$ *on* X*, and show that the map* $C(X) \to H$ *by* $f \mapsto \Gamma^{-1}(f)\xi$ *is a* $\|\cdot\|_2 - \|\cdot\|_H$ *isometry with dense range.*

Problem 75. Suppose $E : (X, \mathcal{A}) \to P(H)$ is a spectral measure, and let $A \subset B(H)$ be the unital C*-algebra which is the image of $L^\infty(E)$ under $\int \cdot \, dE$. Suppose there is a cyclic unit vector $\xi \in H$ for A.

(1) Show that $\omega_\xi(f) = \langle (\int f dE)\xi, \xi\rangle$ is a faithful state on $L^\infty(E)$ ($\omega_\xi(|f|^2) = 0 \Longrightarrow f = 0$).
(2) Consider the finite non-negative measure $\mu = \mu_{\xi,\xi}$ on $(X, \mathcal{A})$. Show that a measurable function f on $(X, \mathcal{A})$ is essentially bounded with respect to E if and only if f is essentially bounded with respect to μ.
(3) Deduce that for essentially bounded measurable f on $(X, \mathcal{A})$, $\|f\|_E = \|f\|_{L^\infty(X,\mathcal{A},\mu)}$.
(4) Construct a unitary $u \in B(L^2(X, \mathcal{A}, \mu), H)$ such that for all $f \in L^\infty(E) = L^\infty(X, \mathcal{A}, \mu)$, $(\int f dE)u = uM_f$.
(5) Deduce that $A \subset B(H)$ is a maximal abelian von Neumann algebra.

Problem 76. Suppose H is a separable infinite dimensional Hilbert space. Prove that $K(H) \subset B(H)$ is the unique norm closed 2-sided proper ideal.

Problem 77. Classify all abelian von Neumann algebras $A \subset B(H)$ when H is separable.
Hint: Use a maximality argument to show you can write $1 = p + q$ *with* $p, q \in P(A)$ *such that* q *is diffuse and* $p = \sum p_i$ *(SOT) with all* p_i *minimal. Then analyze* Aq *and* Ap*.*

Problem 78. Suppose $M \subseteq B(H)$ is a von Neumann algebra and $p, q \in P(M)$. Define $p \wedge q \in B(H)$ to be the orthogonal projection onto $pH \cap qH$. Prove that $p \wedge q \in M$ two separate ways:

(1) Show that $pH \cap qH$ is M'-invariant, and deduce $p \wedge q \in M$.

(2) Show that $p \wedge q$ is the SOT-limit of $(pq)^n$ as $n \to \infty$.
Hint: You could proceed as follows, but a quicker proof would be much appreciated!
   (a) *Use (2) of Problem 51 to show $(pq)^n p$ is a decreasing sequence of positive operators.*
   (b) *Show $(pq)^n p$ converges SOT to a positive operator $x \in M$.*
   (c) *Show that $x^2 = x$, and deduce $x \leq p$ is an orthogonal projection.*
   (d) *Show that $xqp = x$, and deduce $xqx = x$.*
   (e) *Show that $x \leq q$, and deduce $x \leq p \wedge q$.*
   (f) *Show that $(p \wedge q)(pq)^n$ converges SOT to both $p \wedge q$ and x, and deduce $x = p \wedge q$.*
   (g) *Finally, show $(pq)^n$ converges SOT to $xq = p \wedge q$.*

Define $p \vee q$ as the projection onto $\overline{pH + qH}$. Show that $p \vee q \in M$ in two separate ways:

(1) Prove that $\overline{pH + qH}$ is M'-invariant, and deduce $p \vee q \in M$.
(2) Show that $p \vee q = 1 - (1-p) \wedge (1-q)$ and use that $p \wedge q \in M$.

Problem 79. Suppose $N \subseteq M \subset B(H)$ is a unital inclusion of von Neumann algebra and $p \in P(N)$.

(1) Prove that $(N'p) \cap pMp = (N' \cap M)p$.
(2) Deduce that if $p \in P(M)$, $Z(pMp) = Z(M)p$.
(3) Deduce that if $p \in P(M)$ and M is a factor, then pMp is a factor.
(4) Prove that when M is a factor and $p \in P(M)$, the map $M' \to M'p$ by $x \mapsto xp$ is a unital $*$-algebra isomorphism.

Problem 80. Prove that the following conditions are equivalent for a von Neumann algebra $M \subseteq B(H)$:

(1) Every non-zero $q \in P(M)$ majorizes an abelian projection $p \in P(M)$.
(2) M is type I (every non-zero $z \in P(Z(M))$ majorizes an abelian $p \in P(M)$).
(3) There is an abelian projection $p \in P(M)$ whose central support $z(p) = \bigvee_{u \in U(M)} u^* p u \in Z(M)$ is 1_M.

Hints:
For (2) $\Rightarrow$ (3), if $p \in P(M)$ is abelian with $z(p) \neq 1$, then there is an abelian projection $q \in P(M)$ such that $z(q) \leq 1 - z(p)$. Show that $pMq = 0$ and $p + q$ is an abelian projection. Now use Zorn's Lemma.
For (3) $\Rightarrow$ (1), suppose $p \in P(M)$ is abelian with $z(p) = 1$ and $q \in P(M)$ is non-zero. Show there is a non-zero partial isometry $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. Deduce that uu^* is abelian, and then prove u^*u is abelian.

Problem 81. Show that for every von Neumann algebra M, there are unique central projections z_{I}, z_{II_1}, z_{II_∞}, and z_{III} (some of which may be zero) such that

- Mz_{I} is type I, Mz_{II_1} is type II_1, Mz_{II_∞} is type II_∞, and Mz_{III} is type III, and
- $z_{\mathrm{I}} + z_{\mathrm{II}_1} + z_{\mathrm{II}_\infty} + z_{\mathrm{III}} = 1$

Hint: You could proceed as follows:

(1) First, show that if M has an abelian projection p, then $z(p)$ is type I. *Then use a maximality argument to construct z_{I}. For this, you could adapt the hint for (2) $\Rightarrow$ (3) in Problem 80.*
(2) Replacing M, H with $M(1 - z_{\mathrm{I}}), (1 - z_{\mathrm{I}})H$, we may assume M has no abelian projections. Show that if M has a finite central projection z, then Mz is type II_1. *Now use a maximality argument to construct z_{II_1}. This hinges on proving the sum of two orthogonal finite central projections is finite. (Proving this is much easier than proving the sup of two finite projections is finite!)*

(3) *By compression, we may now assume that M has no abelian projections and no finite central projections. Show that if M has a nonzero finite projection p, then its central support $z(p)$ satisfies $Mz(p)$ is type II_∞. Use a maximality argument to construct z_{II_∞}.*
(4) *Compressing one more time, we may assume M has no finite projections, and thus M is purely infinite and type* III.

Problem 82. Let $M \subseteq B(H)$ be a finite dimensional von Neumann algebra.

(1) Prove M has a minimal projection.
(2) Deduce that $Z(M)$ has a minimal projection.
(3) Prove that for any minimal projection $p \in Z(M)$, Mp is a type I factor.
(4) Prove that M is a direct sum of matrix algebras.

Problem 83. Suppose $M \subseteq B(H)$ and $N \subseteq B(K)$ are von Neumann algebras, and let $H\overline{\otimes}K$ be the tensor product of Hilbert spaces as in Problem 64.

(1) Show that for every $m \in M$ and $n \in N$, the formula $(m \otimes n)(\eta \otimes \xi) := m\eta \otimes n\xi$ gives a unique well-defined operator $m \otimes n \in B(H\overline{\otimes}K)$.
(2) Let $M\overline{\otimes}N = \{m \otimes n | m \in M, n \in N\}'' \subset B(H\overline{\otimes}K)$. Show that the linear extension of the map from the algebraic tensor product $M \otimes N$ to $M\overline{\otimes}N$ given by $m \otimes n \mapsto m \otimes n$ is a well-defined injective unital $*$-algebra map onto an SOT-dense unital $*$-subalgebra.
Hint for injectivity: Suppose $x = \sum_{i=1}^{k} m_i \otimes n_i$ is not zero in $M \otimes N$. Reduce to the case $\{n_1, \ldots, n_k\}$ is linearly independent and all $m_i \neq 0$. Show that for each $i = 1, \ldots, k$, there exists a $k_i > 0$ and $\{\eta_j^i, \xi_j^i\}_{j=1}^{k_i}$ such that $\sum_{j=1}^{k_i} \langle n_{i'}\eta_j^i, \xi_j^i\rangle = \delta_{i=i'}$. (Sub-hint: Consider $F = \operatorname{span}_{\mathbb{C}}\{n_1, \ldots, n_k\} \subset N$, a closed normed space, and look at $\Phi : H \times \overline{H} \to F^*$ by $(\eta, \xi) \mapsto \langle \cdot\, \eta, \xi\rangle$. Show that $\operatorname{span}_{\mathbb{C}}(\Phi(H)) = F^*$.) Now pick $\kappa, \zeta \in H$ such that $\langle m_1\kappa, \zeta\rangle \neq 0$, and deduce $\sum_{j=1}^{k_1} \langle x(\kappa \otimes \eta_j^1), \zeta \otimes \xi_j^1\rangle_{H\overline{\otimes}K} \neq 0$.
(3) We denote by $B(H) \otimes 1$ the image of $B(H)$ under the map $x \mapsto x \otimes 1 \in B(H\overline{\otimes}K)$. Prove that $B(H) \otimes 1$ is a von Neumann algebra.
Hint: Show that $(B(H) \otimes 1)' = 1 \otimes B(K)$. Then by symmetry, $(1 \otimes B(K))' = B(H) \otimes 1$ is a von Neumann algebra.
(4) Prove that $B(H\overline{\otimes}K) = B(H)\overline{\otimes}B(K)$.
Hint: Calculate the commutant of the image of the algebraic tensor product $(B(H) \otimes B(K))' = \mathbb{C}1$ and use (2).

Problem 84. Let S_∞ be the group of finite permutations of $\mathbb{N}$.

(1) Show that S_∞ is ICC. Deduce that LS_∞ is a II_1 factor.
(2) Give an explicit description of a projection with trace k^{-n} for arbitrary $n, k \in \mathbb{N}$.
Hint: Find such a projection in $\mathbb{C}S_\infty \subset LS_\infty$.
(3) Find an increasing sequence $F_n \subset LS_\infty$ of finite dimensional von Neumann subalgebras such that $LS_\infty = (\bigcup_{n=1}^\infty F_n)''$.

Note: A II_1 factor which is generated by an increasing sequence of finite dimensional von Neumann subalgebras as in (3) above is called hyperfinite.

Problem 85. Let M be a von Neuman algebra. Suppose $a, b \in M$ with $0 \leq a \leq b$. Prove there is a $c \in M$ such that $a = c^*bc$. Deduce that a 2-sided ideal in a von Neumann algebra is *hereditary*: $0 \leq a \leq b \in M$ implies $a \in M$.

Problem 86. Let M be a factor. Prove that if M is finite or purely infinite, then M is algebraically simple, i.e., M has no 2-sided ideals.
Note: You may use that a II_1 factor has a (faithful σ-WOT continuous) tracial state.

Problem 87. A positive linear functional $\varphi \in M^*$ is called *completely additive* if for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Suppose $\varphi, \psi \in M^*$ are completely additive and $p \in P(M)$ such that $\varphi(p) < \psi(p)$. Then there is a non-zero projection $q \leq p$ such that $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.
Hint: Choose a maximal family of mutually orthogonal projections $e_i \leq p$ for which $\psi(e_i) \leq \varphi(e_i)$. Consider $e = \bigvee e_i$, and show that $\psi(e) \leq \varphi(e)$. Set $q = p - e$, and show that for all projections $r \leq q$, $\varphi(r) < \psi(r)$. Then show $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.

Problem 88. Show that the following conditions are equivalent for a positive linear functional $\varphi \in M^*$ for a von Neumann algebra M:

(1) φ is σ-WOT continuous,
(2) φ is *normal*: $x_\lambda \nearrow x$ implies $\varphi(x_\lambda) \nearrow \varphi(x)$, and
(3) φ is *completely additive*: for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Hint: For (3) $\Rightarrow$ (1), show if $p \in P(M)$ is non-zero, then pick $\xi \in H$ such that $\varphi(p) < \langle p\xi, \xi\rangle$. Use Problem 87 to find a non-zero $q \leq p$ such that $\varphi(qxq) < \langle xq\xi, q\xi\rangle$ for all $x \in M$. Use the Cauchy-Schwarz inequality to show $x \mapsto \varphi(xq)$ is SOT-continuous, and thus σ-WOT continuous. Now use Zorn's Lemma to consider a maximal family of mutually orthogonal projections $(q_i)_{i\in I}$ for which $x \mapsto \varphi(xq_i)$ is σ-WOT continuous. Show $\sum q_i = 1$. For finite $F \subseteq I$, define $\varphi_F(x) = \sum_{i\in F} \varphi(xq_i)$. Ordering finite subsets by inclusion, we get a net $(\varphi_F) \subset M_*$. Show that $\varphi_F \to \varphi$ in norm in M^*. Deduce that $\varphi \in M_*$ since $M_* \subset M^*$ is norm-closed.

Problem 89. Let $\Phi : M \to N$ be a unital $*$-homomorphism between von Neumann algebras.

(1) Prove that the following two conditions are equivalent:
  (a) Φ is *normal*: $x_\lambda \nearrow x$ implies $\Phi(x_\lambda) \nearrow \Phi(x)$.
  (b) Φ is σ-WOT continuous.
(2) Prove that if Φ is normal, then $\Phi(M) \subset N$ is a von Neumann subalgebra.
  Hint: $\ker(\Phi) \subset M$ *is a σ-WOT closed 2-sided ideal.*
(3) Let φ be a normal state on a a von Neumann algebra M, and let $(H_\varphi, \Omega_\varphi, \pi_\varphi)$ be the cyclic GNS representation of M associated to φ, i.e., $H_\varphi = L^2(M, \varphi)$, $\Omega_\varphi \in H_\varphi$ is the image of $1 \in M$ in H_φ, and $\pi_\varphi(x)m\Omega_\varphi = xm\Omega_\varphi$ for all $x, m \in M$.
  (a) Show that π_φ is normal.
  (b) Deduce that if φ is faithful, then $M \cong \pi_\varphi(M) \subset B(H_\varphi)$ is a von Neumann algebra acting on H_φ.

Problem 90. Suppose $\Phi : M \to N$ is a unital $*$-algebra homomorphism between von Neumann algebras.

(1) Prove that the following conditions imply Φ is normal:
  (a) Φ is SOT-continuous on the unit ball of M.
  (b) Φ is WOT-continuous on the unit ball of M.
  (c) Suppose $N = N'' \subseteq B(H)$. For a dense subspace $D \subseteq H$, $m \mapsto \langle \Phi(m)\eta, \xi\rangle$ is WOT-continuous on M for any $\eta, \xi \in D$.
(2) (optional) Which of the conditions above are equivalent to normality of Φ?

Problem 91. Let M be a finite von Neumann algebra with a faithful σ-WOT continuous tracial state. Let $L^2M = L^2(M, \mathrm{tr})$ where Ω is the image of 1_M in L^2M. Identify M with its image in $B(L^2M)$ by part (3) of Problem 89.

(1) Show that $J : M\Omega \to M\Omega$ by $a\Omega \mapsto a^*\Omega$ is a conjugate-linear isometry with dense range.
(2) Deduce J has a unique extension to L^2M, still denoted J, which is a conjugate-linear unitary, i.e, $J^2 = 1$ and $\langle J\eta, J\xi\rangle = \langle \xi, \eta\rangle$ for all $\eta, \xi \in L^2M$.
  Hint: Look at η, ξ in $M\Omega$.

(3) Calculate $Ja^*Jb\Omega$ for $a, b \in M$. Deduce that $JMJ \subseteq M'$.
(4) Show $\langle Ja^*Jb\Omega, c\Omega\rangle = \langle b\Omega, JaJc\Omega\rangle$ for all $a, b, c \in M$. Deduce $(JaJ)^* = Ja^*J$.
(5) Show $\langle Jy\Omega, a\Omega\rangle = \langle y^*\Omega, a\Omega\rangle$ for all $a \in M$ and $y \in M'$. Deduce $Jy\Omega = y^*\Omega$.
(6) Prove that for $y \in M'$, $(JyJ)^* = Jy^*J$.
Hint: Try the same technique as in (4).
(7) Show for all $a, b \in M$ and $x, y \in M'$, $\langle xJyJa\Omega, b\Omega\rangle = \langle JyJxa\Omega, b\Omega\rangle$.
(8) Deduce that $M' \subseteq (JM'J)' = JMJ$, and thus $M' = JMJ$.

Problem 92. Let Γ be a discrete group, and let $L\Gamma = \{\lambda_g\}'' \subset B(\ell^2\Gamma)$. Consider the faithful σ-WOT continuous tracial state $\mathrm{tr}(x) = \langle x\delta_e, \delta_e\rangle$ on $L\Gamma$.

(1) Show that $u\delta_g = \lambda_g$ uniquely extends to a unitary $u \in B(\ell^2\Gamma, L^2L\Gamma)$ such that for all $x \in L\Gamma$ and $\xi \in \ell^2\Gamma$, $L_xu\xi = ux\xi$ where $L_x \in B(L^2L\Gamma)$ is left multiplication by x, i.e., $L_x(y\Omega) = xy\Omega$.
(2) Deduce from Problem 91 that $L\Gamma' = R\Gamma$.

Problem 93. Use Problem 92 above to give the following alternative characterization of $L\Gamma$. Let

$$\ell\Gamma = \left\{x = (x_g) \in \ell^2\Gamma \middle| x * y \in \ell^2\Gamma \text{ for all } y \in \ell^2\Gamma\right\}$$

where $(x * y)_g = \sum_h x_h y_{h^{-1}g}$. Define a unital $*$-algebra structure on $\ell\Gamma$ by multiplication is convolution, the unit is δ_e, the the indicator function at $e \in \Gamma$ ($\delta_e(g) = \delta_{g=e}$), and the involution $*$ on $\ell\Gamma$ is given on $x \in \ell\Gamma$ by $(x^*)_g := \overline{x_{g^{-1}}}$.

(1) Show that $\ell\Gamma$ is a well-defined unital $*$-algebra under the above operations.
(2) For $x \in \ell\Gamma$ define $T_x : \ell^2\Gamma \to \ell^2\Gamma$ by $T_xy = x * y$. Prove $T_x \in B(\ell^2\Gamma)$.
Hint: Show that for all $x \in \ell\Gamma$ and $y, z \in \ell^2\Gamma$, $\langle T_xy, z\rangle = \langle y, T_{x^*}z\rangle$. Then use the Closed Graph Theorem similar to Problem 18.
(3) Prove that for all $x \in \ell\Gamma$, $T_x \in L\Gamma$.
Hint: Prove $T_x \in R\Gamma'$ and apply Problem 92.
(4) Deduce that $x \mapsto T_x$ is a unital $*$-algebra isomorphism $\ell\Gamma \to L\Gamma$.

Problem 94. Repeat Problem 93 for the crossed product von Neumann algebra $M \rtimes_\alpha \Gamma$ acting on $L^2M \otimes \ell^2\Gamma \cong L^2(\Gamma, L^2M)$ where M is a finite von Neumann algebra with faithful normal tracial state tr, Γ is a discrete group, and $\alpha : \Gamma \to \mathrm{Aut}(M)$ is an action. Here, we define

$$\ell^2(\Gamma, M) = \left\{x : \Gamma \to M \middle| \sum_g \|x_g\Omega\|^2_{L^2M} < \infty\right\}$$

$$\ell^2(\Gamma, L^2M) = \left\{\xi : \Gamma \to L^2M \middle| \sum_g \|\xi_g\|^2 < \infty\right\} \text{ and}$$

$$M \rtimes_\alpha \Gamma = \left\{x = (x_g) \in \ell^2(\Gamma, M) \middle| x * \xi \in \ell^2(\Gamma, L^2M) \text{ for all } \xi \in \ell^2(\Gamma, L^2M)\right\}.$$

Here, the convolution action is given by $(x * \xi)_g = \sum_h x_h v_h \xi_{h^{-1}g}$ where $v_h \in U(L^2M)$ is the unitary implementing $\alpha_u \in \mathrm{Aut}(M)$. Define an analogous unital $*$-algebra structure on $M\Gamma$ and find a unital $*$-algebra isomorphism $M \rtimes_\alpha \Gamma \to M \rtimes_\alpha \Gamma$.
Hint: Similar to $L\Gamma$, some people write elements of $M \rtimes_\alpha \Gamma$ as formal sums $\sum_g x_gu_g$ which does not converge in any operator topology. Rather, $\sum_g x_gu_g(\Omega \otimes \delta_e)$ converges in $L^2M \otimes \ell^2\Gamma$. These formal sums can be algebraically manipulated to obtain a unital $*$-algebra structure using the covariance condition $u_gmu_g^* = \alpha_g(m)$ for all $g \in \Gamma$ and $m \in M$. Thus

$$\left(\sum_g x_gu_g\right)^* = \sum_g u_gx_g^* = \sum_g u_gx_g^*u_g^*u_g = \sum_g \alpha_g(x_g^*)u_g.$$

Thus for $x = (x_g) \in M \rtimes_\alpha \Gamma$, we define $(x^*)_g = \alpha_g(x_g^*)$. A similar algebraic manipulation gives the formula for multiplication, which is similar to convolution, but involves the action.

Problem 95. Prove that a $*$-isomorphism between von Neumann algebras is automatically normal.

Problem 96. Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group on 2 generators.

(1) Show that $\mathbb{F}_2$ is ICC. Deduce $L\mathbb{F}_2$ is a II_1 factor.
(2) Show that the swap $a \leftrightarrow b$ extends to an automorphism σ of $L\mathbb{F}_2$.
(3) Show that σ is outer.

Problem 97. Prove that irrational rotation on the circle (with Lebesgue/Haar measure) is free and ergodic.

Problem 98. Let M be a finite von Neumann algebra with a faithful normal tracial state.

(1) Show for all $x, y \in M$, $|\operatorname{tr}(xy)| \leq \|y\| \operatorname{tr}(|x|)$.
(2) Show for all $x \in M$, $\operatorname{tr}(|x|) = \sup\{|\operatorname{tr}(xy)| | y \in M \text{ with } \|y\| = 1\}$.
(3) Define $\|x\|_1 = \operatorname{tr}(|x|)$ on M. Show that $\|\cdot\|_1$ is a norm on M.
(4) Define a map $\varphi : M \to M_*$ by $x \mapsto \varphi_x$ where $\varphi_x(y) = \operatorname{tr}(xy)$. Show that φ is a well-defined isometry from $(M, \|\cdot\|_1) \to M_*$ with dense range.
(5) Deduce that $L^1(M, \operatorname{tr}) := \overline{M}^{\|\cdot\|_1}$ is isometrically isomorphic to the predual M_*.

Problem 99. Continue the notation of Problem 98. Let $N \subseteq M$ be a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $i : L^1(N, \operatorname{tr}) \to L^1(M, \operatorname{tr})$.
(2) Let $E : M \to N$ be the Banach adjoint of i under the identification $M_* = L^1(M, \operatorname{tr})$ and $N_* = L^1(N, \operatorname{tr})$. Show that E is uniquely characterized by the equation

$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$.

Problem 100. Suppose M is a finite von Neumann algebra with normal faithful tracial state tr and $N \subseteq M$ is a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $L^2(N, \operatorname{tr}) \to L^2(M, \operatorname{tr})$.
(2) Define $e_N \in B(L^2M, L^2N)$ be the orthogonal projection with range $L^2(N, \operatorname{tr}) = \overline{N\Omega}^{\|\cdot\|_2} \subset L^2(M, \operatorname{tr})$. Show that for all $x \in M$, $e_N x e_N^* \subset B(L^2N)$ commutes with the right action of N, and thus defines an element in N by Problem 91.
Hint: Show the inclusion $e_N^* : L^2N \to L^2M$ commutes with the right N action, and deduce e_N commutes with the right N action.
(3) For $x \in M$, define $E(x) = e_N x e_N^*$. Show that $E(x)$ is uniquely characterized by the equation

$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$. *Part (3) implies this definition agrees with that from Problem 99.*

Problem 101. Continue the notation of Problem 100.

(1) Deduce that E is normal.
(2) Deduce $E(1) = 1$ and E is $N{-}N$ bilinear, i.e., for all $x \in M$ and $y, z \in N$, $E(yxz) = yE(x)z$.
(3) Deduce that $E(x^*) = E(x)^*$.
(4) Show that E is completely positive, which was defined in Problem 70.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 100.

(5) Show that $E(x)^*E(x) \leq E(x^*x)$ for all $x \in M$.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 100. Show that $e_N^* e_N$ is an orthogonal projection.
(6) Show that E is faithful: $E(x^*x) = 0$ implies $x^*x = 0$.
Hint: Prove this by looking at the vector states $\omega_{n\Omega}$ for $n \in N$.

Problem 102. Suppose M is a finite von Neumann algebra with faithful normal tracial state tr. Suppose further that there is an increasing sequence of von Neumann subalgebras $M_1 \subset M_2 \subset \cdots M$ such that $(\bigcup M_n)'' = M$ (considered as acting on $L^2 M$). Let $E_n : M \to M_n$ be the canonical trace-preserving conditional expectation from Problem 100.

(1) Prove that the $\|\cdot\|_2$-topology agrees with the SOT on the unit ball of M. That is, prove that $x_n \to x$ SOT if and only if $\|x_n\Omega - x\Omega\|_2 \to 0$.
(2) Prove that for all $x \in M$, $\|E_n(x)\Omega - x\Omega\|_2 \to 0$ as $n \to \infty$.
(3) Deduce that $E_n(x) \to x$ SOT as $n \to \infty$.

Problem 103. Suppose Γ is a countable group, and let $\operatorname{Prob}(\Gamma) = \left\{\mu \in \ell^1\Gamma \middle| \mu \geq 0 \text{ and } \sum_g \mu(g) = 1\right\}$.

(1) Prove that $\operatorname{Prob}(\Gamma)$ is weak* dense in the state space of $\ell^\infty\Gamma$.
(2) Let $F \subset \Gamma$ be finite, and consider $\bigoplus_{g\in F} \ell^1\Gamma$ with the (product) weak topology. Let K be the weak closure of $\left\{\bigoplus_{g\in F} g\cdot\mu - \mu \middle| \mu \in \operatorname{Prob}(\Gamma)\right\} \subset \bigoplus_{g\in F} \ell^1\Gamma$. Prove K is convex and norm closed in $\bigoplus_{g\in F} \ell^1\Gamma$.
(3) Now assume Γ is amenable, i.e., there is a left Γ-invariant state on $\ell^\infty\Gamma$. Prove that $0 \in K$. Deduce that Γ has an approximately invariant mean.

Problem 104. Suppose Γ is a countable group, and let $\operatorname{Prob}(\Gamma)$ be as in Problem 103.

(1) Prove that if $a, b \in [0,1]$, then
$$|a-b| = \int_0^1 |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)|\,dr.$$
(2) Deduce that for $\mu \in \operatorname{Prob}(\Gamma)$ and $h \in \Gamma$,
$$\|h\cdot\mu - \mu\|_{\ell^1\Gamma} = \int_0^1 \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|\,dr.$$
(3) For $r \in [0,1]$ and $\mu \in \operatorname{Prob}(\Gamma)$, let $E(\mu, r) = \{g \in \Gamma | \mu(g) > r\}$. Show that for all $h \in \Gamma$, $hE(\mu, r) = \{g \in \Gamma | (h\cdot\mu)(g) > r\}$.
(4) Calculate $\int_0^1 |E(\mu, r)|\,dr$.
(5) Show that for $r \in [0,1]$, $\mu \in \operatorname{Prob}(\Gamma)$, and $h \in \Gamma$,
$$|hE(\mu, r) \triangle E(\mu, r)| = \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|.$$
Deduce that $\|h\cdot\mu - \mu\|_1 = \int_0^1 |hE(\mu, r)\triangle E(\mu, r)|\,dr$.
(6) Suppose now that Γ has an approximate invariant mean, so that for every finite subset $F \subset \Gamma$ and $\varepsilon > 0$, there is a $\mu \in \operatorname{Prob}(\Gamma)$ such that
$$\sum_{h\in F} \|h\cdot\mu - \mu\|_1 < \varepsilon.$$
Show that for the μ corresponding to this F and ε,
$$\int_0^1 \sum_{h\in F} |hE(\mu, r)\triangle E(\mu, r)|\,dr < \varepsilon \int_0^1 |E(\mu, r)|\,dr.$$

Deduce there is an $r \in [0,1]$ such that $|hE(\mu,r)\triangle E(\mu,r)| < \varepsilon|E(\mu,r)|$ for all $h \in F$.

(7) Use (6) above to construct a Følner sequence for Γ.

Problem 105. Recall that an *ultrafilter* ω on a set X is a nonempty collection of subsets of X such that:

- $\emptyset \notin \omega$,
- If $A \subseteq B \subseteq X$ and $A \in \omega$, then $B \in \omega$,
- If $A, B \in \omega$, then $A \cap B \in \omega$, and
- For all $A \subset X$, either $A \in \omega$ or $X \setminus A \in \omega$ (but not both!).

(1) Find a bijection from the set of ultrafilters on $\mathbb{N}$ to $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$.

(2) Let ω be an ultrafilter on $\mathbb{N}$. Let X be a compact Hausdorff space and $f : \mathbb{N} \to X$. We say
  - $x = \lim_{n\to\omega} f(n)$ if for every open neighborhood U of x, $f^{-1}(U) \in \omega$.

  Prove that $\lim_{n\to\omega} f(n)$ always exists for any function $f : \mathbb{N} \to X$.

(3) An ultrafilter on $\mathbb{N}$ is called *principal* if it contains a finite set. Show that every principal ultrafilter on $\mathbb{N}$ contains a unique singleton set, and that any two principal ultrafilters containing the same singleton set are necessarily equal. Thus we may identify the set of principal ultrafilters on $\mathbb{N}$ with $\mathbb{N} \subset \beta\mathbb{N}$.

(4) Determine $\lim_{n\to\omega} f(n)$ for $f : \mathbb{N} \to X$ as in (2) when ω is principal.

(5) An ultrafilter on $\mathbb{N}$ is called *free* or *non-principal* if it does not contain a finite set. Let ω be a free ultrafilter on $\mathbb{N}$. Suppose $\Gamma = \bigcup \Gamma_n$ is a locally finite group and m_n is the uniform probability (Haar) measure on Γ_n. Define $m : 2^\Gamma \to [0,1]$ by $m(A) = \lim_{n\to\omega} m_n(A \cap \Gamma_n)$. Prove that m is a left Γ-invariant finitely additive probability measure on Γ, i.e., Γ is amenable.

Problem 106. Let X be a uniformly convex Banach space and $B \subset X$ a bounded set. Prove that the function $f : X \to [0,\infty)$ given by $f(x) = \sup_{b\in B} \|b - x\|_X$ achieves its minimum at a unique point of X.

Problem 107. Let Γ be a countable discrete group. Show that an affine action $\alpha = (\pi, \beta) : \Gamma \to \operatorname{Aff}(H)$ ($\alpha_g\xi := \pi_g\xi + \beta(g)$ for $\pi_g \in U(H)$ and $\beta(g) \in H$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in \Gamma$) is proper if and only if the cocycle part $\beta : \Gamma \to H$ is proper ($g \mapsto \|\beta(g)\|$ is a proper map).

Problem 108. Recall that the *Schur product* of two matrices $a, b \in M_n(\mathbb{C})$ is given by the entry-wise product: $(a * b)_{i,j} := a_{i,j}b_{i,j}$.

(1) Prove that if $a, b \geq 0$, then $a * b \geq 0$.

(2) Suppose that $p \in \mathbb{R}[z]$ is a polynomial whose coefficients are all non-negative. Prove that if $a \geq 0$, then $p[a] \geq 0$, where $p[a]_{i,j} := p(a_{i,j})$ for $a \in M_n(\mathbb{C})$.
  Note: Here we use the notation $p[a]$ to not overload the functional calculus notation.

(3) Suppose that f is an entire function whose Taylor expansion at 0 has only non-negative real coefficients. Prove that is $a \geq 0$, then $f[a] \geq 0$, where again $f[a]_{i,j} := f(a_{i,j})$ for $a \in M_n(\mathbb{C})$.

Problem 109. Let A be a unital C*-algebra.

(1) Prove that a map $\Phi : A \to M_n(\mathbb{C})$ is completely positive if and only if the map $\varphi : M_n(A) \to \mathbb{C}$ given by $(a_{i,j}) \mapsto \sum_{i,j}^n \Phi(a_{i,j})_{i,j}$ is positive.
  Hint: for one direction, note that $\varphi(a) = \vec{1}^*\Phi(a)\vec{1}$ where $\vec{1} \in \mathbb{C}^n$ is the column vector with all 1s. For the other direction, use GNS with respect to φ, and consider $V : \mathbb{C}^n \to L^2(A,\varphi)$ given by $Ve_i = \pi_\varphi(E_{ij})\Omega_\varphi$ where (E_{ij}) is a system of matrix units in $M_n(\mathbb{C}) \subseteq M_n(A)$. Then use Stinespring.

(2) Let $S \subset A$ be an operator subsystem, and let $\psi : S \to \mathbb{C}$ be a positive linear functional. Prove $\|\psi\| = \psi(1)$. Deduce that any norm-preserving (Hahn-Banach) extension of ψ to A is also positive.
(3) Let $S \subset A$ be an operator subsystem, and let $\Phi : S \to M_n(\mathbb{C})$ be a (unital) completely positive map. Show that Φ extends to a (unital) completely positive map $A \to M_n(\mathbb{C})$.

Problem 110. Suppose Γ is a countable discrete group, and suppose $\varphi : L\Gamma \to L\Gamma$ is a normal completely positive map. Prove that $f : \Gamma \to \mathbb{C}$ given by $f(g) := \mathrm{tr}_{L\Gamma}(\varphi(\lambda_g)\lambda_g^*)$ is a positive definite function.

Problem 111. Prove that the following are equivalent for a finite von Neumann algebra $(M, \mathrm{tr}) \subset B(H)$ with faithful normalized tracial state.
(1) M is amenable, i.e., there is a conditional expectation $E : B(H) \to M$.
(2) There is a sequence $(\varphi_n : M \to M)$ of (normal) trace-preserving completely positive maps such that $\varphi_n \to \mathrm{id}$ pointwise in $\|\cdot\|_M$, and for all $n \in \mathbb{N}$, the induced map $\widehat{\varphi}_n \in B(L^2M)$ given by $m\Omega \mapsto \varphi_n(m)\Omega$ is finite rank.

Problem 112. Suppose that Γ is a countable discrete group such that every cocycle is inner. Suppose (H, π) is a unitary representation and $(\xi_n) \subset H$ is a sequence of unit vectors such that $\|\pi_g\xi_n - \xi_n\| \to 0$ as $n \to \infty$ for all $g \in \Gamma$. Follow the steps below to find a non-zero Γ-invariant vector in H. (We may assume that no ξ_n is fixed by Γ.)
(1) Enumerate $\Gamma = \{g_1, g_2, \dots\}$. Explain why you can pass to a subsequence of (ξ_n) to assume that for all $n \in \mathbb{N}$, $\|\pi_{g_i}\xi_n - \xi_n\| < 4^{-n}$ for all $1 \le i \le n$.
(2) For $n \in \mathbb{N}$, consider the inner cocycles $\beta_n(g) := \xi_n - \pi_g\xi_n$. Let $(K, \sigma) = \bigoplus_{n\in\mathbb{N}}(H, \pi)$. Define $\beta : \Gamma \to K$ by $\beta(g)_n := 2^n\beta_n(g)$. Prove that $\beta(g) \in H$ is well-defined for every $g \in \Gamma$. Then show that β is a cocycle for (K, σ).
(3) Deduce β is inner and thus bounded. Thus there is a $\kappa \in K \setminus \{0\}$ such that $\beta(g) = \kappa - \sigma_g\kappa$ for all $g \in \Gamma$.
(4) Prove that $\|\beta_n(g)\| \to 0$ uniformly for $g \in \Gamma$. That is, show that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $\|\beta_n(g)\| < \varepsilon$ for all $g \in \Gamma$.
(5) Fix $N \in \mathbb{N}$ such that $\|\beta_N(g)\| = \|\xi_N - \pi_g\xi_N\| < 1$ for all $g \in \Gamma$. Show there is a $\xi_0 \in H \setminus \{0\}$ such that $\pi_g\xi_0 = \xi_0$ for all $g \in \Gamma$.
Hint: Look at $\{\pi_g\xi_N | g \in \Gamma\} \subset (H)_1$ and apply Problem 106.
(6) (optional) Use a similar trick to finish the proof of (1) $\Rightarrow$ (2) from the same theorem from class.

Problem 113 (optional)**.** As best as you can, edit the equivalent definitions I gave in class for property (T) for a countable discrete group Γ to be relative to a subgroup $\Lambda \le \Gamma$. Then prove all the equivalences.

Problem 114 (Fell's Absorption Principle)**.** Suppose Γ is a countable group and (H, π) is a unitary representation on a separable Hilbert space. Find a unitary $u \in B(\ell^2\Gamma\overline{\otimes}H)$ intertwining $\lambda \otimes \pi$ and $\lambda \otimes 1$, i.e., $u(\lambda_g \otimes \pi_g) = (\lambda_g \otimes 1)u$ for all $g \in \Gamma$.

Problem 115. Suppose $\Gamma \curvearrowright (X, \mu)$ is a free p.m.p. action and $\mathcal{R} = \{(x, gx) | x \in X, g \in \Gamma\}$ is the corresponding countable p.m.p. equivalence relation. Follow the steps below to show $L^\infty(X, \mu) \rtimes \Gamma \cong L\mathcal{R}$.
(1) Prove that $\theta : (x, g) \mapsto (x, g^{-1}x)$ induces a unitary operator $v \in B(L^2\mathcal{R}, L^2(X \times \Gamma, \mu \times \gamma))$ where γ is counting measure on Γ.
(2) Deduce that θ is a p.m.p. isomorphism $(X \times \Gamma, \mu \times \gamma) \to (\mathcal{R}, \nu)$.
(3) Show that $v^*M_fv = \lambda(f)$ for all $f \in L^\infty(X, \mu)$. Here, $(M_f\xi)(x, g) = f(x)\xi(x, g)$ for $\xi \in L^2(X \times \Gamma, \mu \times \gamma)$.

(4) Show that $v^*u_gv = L_{\varphi_g}$ where $\varphi_g \in [\mathcal{R}]$ is the isomorphism $x \mapsto g \cdot x$. Here, $(u_g\xi)(x,h) = \xi(g^{-1}x, g^{-1}h)$ for all $\xi \in L^2(X \times \Gamma, \mu \times \gamma) \cong L^2(X,\mu) \otimes \ell^2\Gamma$.
(5) Deduce that $v^*(L^\infty(X,\mu) \rtimes \Gamma)v \subset L\mathcal{R}$.
(6) Show that conjugation by v takes the commutant of $L^\infty(X,\mu) \rtimes \Gamma$ into $R\mathcal{R}$.
Hint: Show that right multiplication by $L^\infty(X,\mu)$ and the right action of u_g are both taken into $R\mathcal{R}$.
(7) Deduce that $v^*(L^\infty(X,\mu) \rtimes \Gamma)v = L\mathcal{R}$.

Problem 116. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). Let $A = L^\infty(X,\mu) \subset L\mathcal{R}$. Prove that the von Neumann subalgebra of $B(L^2(\mathcal{R},\nu))$ generated by $A \cup JAJ$ is the von Neumann algebra of multiplication operators by elements of $L^\infty(\mathcal{R},\nu)$.

Problem 117. Let M be a von Neumann algebra. A *weight* on M is a function $\varphi : M_+ \to [0,\infty]$ such that for all $r \in [0,\infty)$ and $x,y \in B(H)_+$, $\varphi(rx+y) = r\varphi(x) + \varphi(y)$, with the convention that for $s \in [0,\infty)$,

$$\infty \cdot s = \begin{cases} \infty & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases}$$

Define

$$\begin{aligned} \mathfrak{p}_\varphi &= \{x \in M | \varphi(x) < \infty\} \\ \mathfrak{n}_\varphi &= \{x \in M | x^*x \in \mathfrak{p}_\varphi\} \\ \mathfrak{m}_\varphi &= \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi = \left\{ \sum_{i=1}^n x_i^* y_i \middle| x_i, y_i \in \mathfrak{n}_\varphi \text{ for all } i = 1,\dots,n \right\}. \end{aligned}$$

(1) Prove that
  (a) $\mathfrak{p}_\varphi$ is a hereditary subcone of M_+, i.e.,
    - (subcone) $r \geq 0$ and $x,y \in \mathfrak{p}_\varphi$ implies $rx+y \in \mathfrak{p}_\varphi$
    - (hereditary) $0 \leq x \leq y$ and $y \in \mathfrak{p}_\varphi$ implies $x \in \mathfrak{p}_\varphi$.
  (b) $\mathfrak{n}_\varphi$ is a left ideal of M.
  Hint: Prove that for all $x,y \in M$, $(x \pm y)^*(x \pm y) \leq 2(x^*x + y^*y)$.
  (c) $\mathfrak{m}_\varphi$ is algebraically spanned by $\mathfrak{p}_\varphi$.
  Hint: Use polarization.
  (d) $\mathfrak{m}_\varphi \cap M_+ = \mathfrak{p}_\varphi$.
  (e) $\mathfrak{m}_\varphi$ is a hereditary $*$-subalgebra of M (hereditary is defined the same way as above).
(2) When $M = B(H)$ and $\varphi = \mathrm{Tr}$, show $\mathfrak{m}_{\mathrm{Tr}} = \mathcal{L}^1(H)$ and $\mathfrak{n}_{\mathrm{Tr}} = \mathcal{L}^2(H)$.