Def: • A <u>probability space</u> is a complete measure space $(X, \underbrace{\mathcal{A}}_{\sigma\text{-algebra, suppress.}}, \mu)$ s.t. $\mu(X) = 1$.

• A probability measure preserving (pmp) <u>isomorphism</u> b/t<u>wn</u> prob. spaces (X_i, μ_i), $i=1,2$, is an invertible fct $\theta: Y_1 \to Y_2$ s.t.:

- $Y_i \subseteq X_i$ is <u>conull</u> $[\mu_i(X_i \setminus Y_i) = 0]$
- θ, θ^{-1} are both <u>measurable</u> + <u>measure preserving</u>

• $\mathrm{Aut}(X,\mu) = \{\text{pmp iso's } (X,\mu) \to (X,\mu)\} / \text{equality a.e.}$

• A <u>pmp action</u> of Γ on (X,μ) is a gp. hom $\Gamma \to \mathrm{Aut}(X,\mu)$.

Def: A prob space is called <u>standard</u> if it is isomorphic to:

- $[0,1]$ w/ Lebesgue measure
- a countable set w/ (weighted) counting measure, or
- a disjoint union of the above 2 types.

Def: A <u>countable/discrete equivalence rel'n</u> is an eq. rel'n $\mathcal{R} \subset X \times X$ on a standard prob. space s.t.

- $\mathcal{R}$ is Borel m'ble
- the equivalence classes of $\mathcal{R}$ are countable.

<u>Induced measure on $\mathcal{R}$:</u> Let $\pi_i : X \times X \to X$ be proj. on i^{th} coord.

For $\mathcal{R} \subset X \times X$, define σ-finite ν_i, $i=1,2$, by:

$$\nu_i(C) := \int_X |\pi_i^{-1}(x) \cap C| \, d\mu(x).$$

Define $[[\mathcal{R}]] := \{\text{Borel iso's } \underset{\cap}{\overset{X}{}}A \xrightarrow[\sim]{\varphi} \underset{\cap}{\overset{X}{}}B \mid \mathrm{graph}(\varphi) \subseteq \mathcal{R}\}$. $\boxed{\text{Full gpoid}}$

Def/Thm: [AP, Lem 1.5.2] For $\mathcal{R} \subset X \times X$, TFAE, and we call $\mathcal{R}$ <u>pmp</u>:

① $\nu_1 = \nu_2$ [$\mathcal{R}$ preserves μ]

② $\forall [\varphi: A \to B] \in [[\mathcal{R}]]$, φ is measure preserving: $\mu(\varphi(A')) = \mu(A')$ $\forall A' \subset A$.

Example: Let $\Gamma \curvearrowright (X,\mu)$ be a pmp action of a countable discrete Γ.

Define $\underset{\text{is pmp.}}{\mathcal{R}} = \{(x, gx) \mid x \in X\}$, the <u>orbit equivalence rel'n</u>.

Def: Let $\mathcal{R} \subset X \times X$ be a pmp countable eq. rel'n. For $\varphi \in [[\mathcal{R}]]$ s.t. $\varphi : A \xrightarrow{\sim} B$, define $L_\varphi, R_\varphi \in B(L^2\mathcal{R})$ by

$$(L_\varphi \xi)(x,y) := \chi_B(x)\, \xi(\varphi^{-1}(x), y)$$
$$(R_\varphi \xi)(x,y) := \chi_B(y)\, \xi(x, \varphi^{-1}(y))$$

Define $L\mathcal{R} := \{ L_\varphi \mid \varphi \in [[\mathcal{R}]] \}'' \subset B(L^2\mathcal{R})$

$R\mathcal{R} := \{ R_\varphi \mid \varphi \in [[\mathcal{R}]] \}'' \subset B(L^2\mathcal{R})$.

Note: $[L_\varphi, R_\psi] = 0 \quad \forall \varphi, \psi \in [[\mathcal{R}]]$.

Exercises: [some will be HW.]

1. $\forall \varphi \in [[\mathcal{R}]]$, $L_\varphi^* = L_{\varphi^{-1}}$ and $L_\varphi L_\psi = L_{\varphi\psi}$ — suitable notion of composition!
2. L_φ is a p.i. $\forall \varphi$.
3. L_{id_A} is a projection $\forall A \subset X$.
4. $\lambda : L^\infty(X, \mu) \to B(L^2\mathcal{R})$ by $(\lambda(f)\xi)(x,y) := f(x)\xi(x,y)$ defines a faithful, normal unital $*$-alg. hom.
5. $\underbrace{L^\infty(X,\mu) \cong \text{im}(\lambda)}_{\text{identify w/ image.}} \subset L\mathcal{R}$. [note $\forall A \subset X$, $\lambda(\chi_A) = L_{id_A} \in L\mathcal{R}$.]
6. $\forall f \in L^\infty(X,\mu)$, $\forall \varphi \in [[\mathcal{R}]]$, $L_\varphi \lambda_f L_\varphi^* = \lambda_{f \circ \varphi^{-1}}$.
7. The diagonal $\Delta \subset \mathcal{R}$ satisfies $\nu(\Delta) = \int_X |\pi_l^{-1}(x) \cap \Delta| \, d\mu(x) = \mu(X) = 1$. Defining $\text{tr} : L\mathcal{R} \to \mathbb{C}$ by $\omega_{\chi_\Delta} = \langle \cdot\, \chi_\Delta, \chi_\Delta \rangle$ defines a faithful normal tracial state on $L\mathcal{R}$.
8. χ_Δ is cyclic + separating for $L\mathcal{R}$, so $L^2\mathcal{R} \cong L^2(L\mathcal{R}, \text{tr})$.
9. $L\mathcal{R}' = R\mathcal{R}$.

Def: Let (M, tr) be a finite vNa w/ faithful normal tracial state. A von Neumann subalgebra $A \subseteq M$ is called Cartan if:

1. $A \subseteq M$ is maximal abelian
2. $\underbrace{N_M(A)}_{\text{normalizer of } A \text{ in } M} = \{ u \in \mathcal{U}(M) \mid uAu^* = A \}$ generates M as a vNa.

[3. (for free when (M,tr) tracial: $\exists$ normal cond. exp. $E : M \to A$.]

Def: The gpoid normalizer of $A \subset M$ is
$$GN_M(A) := \{ \text{p.i. } v \in M \mid vAv^* \subset A \text{ and } v^*Av \subset A \}.$$

Note: $v \in GN_M(A) \Rightarrow vv^*, v^*v \in P(A)$, and $vAv^* = Avv^*$ and $v^*Av = Av^*v$.

The map $x \mapsto vxv^*$ is a $*$-iso $Av^*v \xrightarrow{\sim} Avv^*$.

Note $GN_M(A)$ is closed under product and adjoint.

Lemma: Let $A \subset (M, tr) \subset B(H)$. Then
$$GN_M(A) = \{ uq \mid u \in N_M(A) \text{ and } q \in P(A) \}.$$
Hence $N_M(A)'' = GN_M(A)'' \subset B(H)$.

Pf: HW!

Thm: $L^\infty(X, \mu) \subset L\mathcal{R}$ is Cartan.

Pf: Identify $L^2\mathcal{R} = L^2(L\mathcal{R}, tr)$. For $x \in L\mathcal{R}$, write $\hat{x} := x\Omega \in L^2\mathcal{R}$.

Claim 1: $\forall a \in L^\infty(X, \mu)$ $\operatorname{supp}(\hat{a}) \subset \Delta$.

Pf: $\forall b \in L^\infty(X, \mu)$, $ab = ba \Rightarrow ab\Omega = ba\Omega \Rightarrow b(x)a(x,y) = b(y)a(x,y)$ for a.e. $(x,y) \in \mathcal{R}$. Varying $b \in L^\infty(X, \mu)$ yields the result.

① Suppose $u \in \mathcal{U}(L\mathcal{R})$ s.t. $[u, a] = 0$ $\forall a \in L^\infty(X, \mu)$.

Claim 2: $\operatorname{supp}(\hat{u}) \subset \Delta$.

Pf: Similar to pf of Claim 1.

Now since we may identify $L^2(X, \mu) = L^2\Delta \subset L^2\mathcal{R}$, we define $f: X \to \mathbb{C}$ by $f(x) := \hat{u}(x, x)$.

Claim 3: $f \in L^\infty(X, \mu)$.

Pf: Since $[u, a] = 0$ $\forall a \in L^\infty(X, \mu)$, $\underset{\text{bdd}}{[u\hat{a}]}(x, x) = a(x)\hat{u}(x,x) = \underset{\text{mult by } f \text{ bdd} \Rightarrow L^\infty}{f(x)\hat{a}(x, x)}$.

Finally, as χ_Δ separating, $u\chi_\Delta = f\chi_\Delta \Rightarrow u = f \in L^\infty(X, \mu)$.

② Recall that $L\mathcal{R} = \{ L_\varphi \mid \varphi \in [[\mathcal{R}]] \}'' \subset B(L^2\mathcal{R})$.

Claim 4: Every $L_\varphi \in GN_{L\mathcal{R}}(L^\infty(X, \mu))$.

Pf: Recall $L_\varphi \lambda_f L_\varphi^* = \lambda_{f \circ \varphi^{-1}}$.

Thm: Let $\Gamma \curvearrowright (X,\mu)$ be a free pmp action. Let $\mathcal{R} = \{(x,gx) \mid x\in X, g\in\Gamma\}$ be the associated countable pmp eq. rel'n. Then $\exists$ spatial iso $\underbrace{L^\infty(X,\mu)}_{\text{Cartan!}} \rtimes \Gamma \cong L\mathcal{R}$ extending $L^\infty(X,\mu) \ni f \mapsto X(f)$. $\boxed{\text{HW!}}$

Remark: $L\mathbb{F}_n$, $n\geq 2$ is not a gp-measure space vNa.

- (Voiculescu 1996) $L\mathbb{F}_n$ has no Cartan.
- (Ozawa 2004) $L\mathbb{F}_n$ is solid: $\forall$ diffuse $M \subseteq L\mathbb{F}_n$, $M' \cap L\mathbb{F}_n$ is amenable
- (Ozawa–Popa 2007) $L\mathbb{F}_n$ is strongly solid: $\forall$ diffuse amenable vN subalg $M \subseteq L\mathbb{F}_n$, $N_{L\mathbb{F}_n}(M)''$ is amenable. [$\Rightarrow$ non-Gamma, prime, no Cartan]

Strongly solid $\Rightarrow$ solid: Suppose $M \subseteq L\mathbb{F}_n$ is diffuse. Let $Q \subseteq M$ be diffuse + amenable. Note $M' \cap L\mathbb{F}_n \subseteq Q' \cap L\mathbb{F}_n \subseteq \underbrace{N_{L\mathbb{F}_n}(Q)''}_{\text{amenable!}} \subseteq L\mathbb{F}_n$. The result now follows from:

Fact: If (M,tr) is a tracial amenable vNa and $N \subseteq M$ is a vN subalg, then N is amenable.

Pf: Let $E: B(L^2M) \to M$ be a cond. exp. Let $E_N: M \to N$ be the canonical trace-preserving cond. exp. Then $E_N \circ E: B(L^2M) \to N$ is a cond. exp.

Let $\mathcal{R}_i$, $i=1,2$, be two countable pmp eq. rel's on (X_i,μ_i), $i=1,2$. Let $\theta: (X_1,\mu_1) \to (X_2,\mu_2)$ be a pmp iso of prob. spaces.

- We say θ is an iso from $\mathcal{R}_1$ to $\mathcal{R}_2$ if $(\theta\times\theta)(\mathcal{R}_1) = \mathcal{R}_2$ up to null sets.

Suppose now $\mathcal{R}_i$ is the pmp orbit eq. rel'n of the countable gp $\Gamma_i \curvearrowright (X_i,\mu_i)$, i.e., $\mathcal{R}_i = \{(x,gx) \mid g\in\Gamma_i, x\in X_i\}$.

- We say the actions are orbit equivalent if $\exists$ an iso from $\mathcal{R}_1$ to $\mathcal{R}_2$, i.e., $\exists$ iso $\theta: (X_1,\mu_1) \to (X_2,\mu_2)$ s.t. $\theta(\Gamma_1 \cdot x_1) = \Gamma_2 \cdot \theta(x_1)$ a.e. $x_1 \in X_1$.

Thm: Suppose $\mathcal{R}_i$ is a pmp eq. rel'n on (X_i,μ_i) for $i=1,2$ and $\theta: (X_1,\mu_1) \to (X_2,\mu_2)$ is a pmp iso. TFAE:

1. θ is an iso from $\mathcal{R}_1$ to $\mathcal{R}_2$.
2. θ induces an iso $L\mathcal{R}_1 \cong L\mathcal{R}_2$.

In the case R_i is the orbit eq. rel'n of $\Gamma_i \curvearrowright (X_i, \mu_i)$, $i=1,2$:

<u>Cor</u>: For free actions $\Gamma_i \curvearrowright (X_i,\mu_i)$ and a pmp iso $\theta: (X_1,\mu_1) \to (X_2,\mu_2)$, <u>TFAE</u>:

① The actions are orbit equivalent via θ

② The map $f \mapsto f\circ\theta^{-1}$ extends to an iso $L^\infty(X_1,\mu_1)\rtimes\Gamma_1 \cong L^\infty(X_2,\mu_2)\rtimes\Gamma_2$.

<u>Pf of Cor</u>: Identify $L^\infty(X_i,\mu_i)\rtimes\Gamma_i \cong LR_i$ via $\boxed{HW!}$

<u>Pf of Thm</u>: Suppose $\theta:(X_1,\mu_1)\longrightarrow(X_2,\mu_2)$ is a pmp iso.

<u>①⇒②</u>: Define $u: L^2R_1 \longrightarrow L^2R_2$ by $u\xi = \xi\circ(\theta^{-1}\times\theta^{-1})$, which is <u>unitary</u> s.t. $u(LR_1)u^* = LR_2$, as $uL_\varphi u^* = L_{\theta^{-1}\circ\varphi\circ\theta}$ $\forall \varphi\in[[R_1]]$:

$$(uL_\varphi u^*)\xi(x,y) = [L_\varphi u^*]\xi(\theta^{-1}(x),\theta^{-1}(y)) = \chi_B(\theta^{-1}(x))\,[u^*\xi]\,(\varphi^{-1}(\theta^{-1}(x)),\theta^{-1}(y))$$
$$= \chi_{\theta(B)}(x)\,\xi(\underbrace{\theta(\varphi^{-1}(\theta^{-1}(x)))}_{(\theta^{-1}\varphi\circ\theta)^{-1}}, y) = [L_{\theta^{-1}\circ\varphi\circ\theta}\,\xi](x,y).$$

Moreover, if $\varphi = id_A$ for $A\subset(X_1,\mu_1)$, $uL_{id_A}u^* = L_{id_{\theta(A)}}$, so we see conjugation by u takes $L^\infty(X_1,\mu_1)$ to $L^\infty(X_2,\mu_2)$.

<u>②⇒①</u>: Let $A_i = L^\infty(X_i,\mu_i)\subset LR_i$. Let $tr_i = \omega_{\chi_{\Delta_i}}$ be the canonical trace on LR_i, and recall $J_i\xi(x,y) = \overline{\xi(y,x)}$ on $L^2R_i \cong L^2(LR_i,tr_i)$. Let $\alpha: LR_1\to LR_2$ be an iso extending $f\mapsto f\circ\theta^{-1}$. Then $\alpha(\chi_{\Delta_1}) = \chi_{\Delta_2}$, so $tr_2\circ\alpha = tr_1$ on LR_1, and we get a <u>unitary</u> <u>implementation</u> of α on L^2R_1: $\exists u\in B(L^2R_1, L^2R_2)$ <u>unitary</u> s.t.

- $uxu^* = \alpha(x)$ $\forall x\in LR_1$, and
- $uJ_1 = J_2u$.

Recall $L^\infty R_i = (A_i\cup JA_iJ)'' \subset B(L^2R_i)$ by $\boxed{HW!}$. If $a\in A_1$, $\xi\in L^2R_2$,

$$(uau^*\xi)(x,y) = (\alpha(a)\xi)(x,y) = a(\theta^{-1}(x))\,\xi(x,y) \quad\text{and}$$
$$(uJ_1aJ_1u^*\xi)(x,y) = (J_2uau^*J_2\xi)(x,y) = \overline{(uau^*J_2\xi)(y,x)} = a(\theta^{-1}(y))\,\overline{J_2\xi(y,x)}$$
$$= (J_2a(\theta^{-1}(y))J_2\xi)(x,y).$$

Sketch. Thus $u(L^\infty R_1)u^* = L^\infty R_2$, so $\exists$ pmp iso $\phi: R_1\longrightarrow R_2$ s.t. $\phi_*\nu_1 \approx \nu_2$, and $u\lambda(f)u^* = \lambda(f\circ\phi^{-1})$ $\forall f\in L^\infty R_1$. Whenever $f(x,y) = g(x)h(y)$ w/ $g,h\in L^\infty(X_1,\mu_1)$, $(f\circ\phi)(x,y) = f(\theta(x),\theta(y))$. Since such fcts <u>generate</u> $L^\infty R_i$ as a vNa, we have $\phi(x,y) = (\theta(x),\theta(y))$ a.e. Thus θ is an iso R_1 to R_2.

Let $\mathcal{R}$ be a countable pmp eq. rel'n on (X,μ).

Def: The full group $[\mathcal{R}]$ is the subset of $[[\mathcal{R}]]$ of $\varphi: X \xrightarrow{\sim} X$.

For $f \in L^\infty\mathcal{R}$ and $\varphi \in [\mathcal{R}]$, define $(\varphi \cdot f)(x,y) := f(\varphi^{-1}(x), y)$.

Def: A countable pmp eq. rel'n $\mathcal{R}$ on (X,μ) is called amenable if $\exists$ a state Φ on $L^\infty\mathcal{R}$ s.t. $\Phi(\varphi \cdot f) = \Phi(f)$ $\forall f \in L^\infty\mathcal{R}, \forall \varphi \in [\mathcal{R}]$

Def: $\mathcal{R}$ is called hyperfinite if $\exists$ increasing seq. $(\mathcal{R}_n)$ of sub-eq. rel'ns, each w/ finite orbits, s.t. $\cup \mathcal{R}_n = \mathcal{R}$, up to null sets. If moreover $\mathcal{R}$ is ergodic [[$\forall A \subseteq X$ m'ble, $\pi_2[\pi_1^{-1}(A)] = \pi_1[\pi_2^{-1}(A)] = A \Rightarrow \mu(A) \in \{0,1\}$]] [[$\Leftrightarrow L\mathcal{R}$ a factor]] w/ infinite orbits, we call $\mathcal{R}$ a hyperfinite II_1 eq. rel'n. [[$\Rightarrow L\mathcal{R} \cong R$, the hyperfinite II_1 factor.]]

Thm (Dye): ① A ctble pmp eq. rel'n $\mathcal{R}$ is hyperfinite $\Leftrightarrow$ it is iso to the orbit eq. rel'n of $\mathbb{Z}$ acting on (X,μ) for some pmp action.

② Any two pmp $\mathbb{Z}$-actions are orbit equivalent.

$\Rightarrow \exists!$ hyperfinite type II_1 eq. rel'n.

Thm (Ornstein-Weiss): ② above holds for any two countable amenable gps.

Cor: Any two ergodic pmp actions of infinite countable amenable gps are orbit equivalent.

Thm (Zimmer): $\mathcal{R}$ amenable $\Leftrightarrow L\mathcal{R}$ amenable.

Thm (Connes-Feldman-Weiss): $\mathcal{R}$ amenable $\Leftrightarrow \mathcal{R}$ hyperfinite.

Question (vN): Does every nonamenable gp contain $\mathbb{F}_2$?

Answer (Ol'shanskii): No, Tarski monster gps are counterexamples.

– many easier counter-examples since.

Thm (Gaboriau-Lyons): Let Γ be countable + nonamenable, and consider the Bernoulli action $\Gamma \curvearrowright [0,1]^\Gamma$. $\exists$ measurable, ergodic, essentially free action $\mathbb{F}_2 \curvearrowright [0,1]^\Gamma$ s.t. the orbit eq. rel'n of the Γ-action contains the orbit eq. rel'n of the $\mathbb{F}_2$-action.

Cocycle Superrigidity

We now want to find conditions on Γ and $\Gamma \curvearrowright (X,\mu)$ to lift orbit equivalence to conjugacy. Suppose $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are free + ergodic and <u>orbit equivalent</u>, so $\exists$ a measure preserving bij. $\theta: X \to Y$ (up to null sets) s.t. $\theta(\Gamma\cdot x) = \Lambda\cdot\theta(x)$ a.e. $x \in X$.

<u>Claim:</u> $\forall g \in \Gamma$, a.e. $x \in X$, $\exists!\, \ell \in \Lambda$ s.t. $\theta(gx) = \ell\,\theta(x)$.

Pf: Since $\theta(\Gamma\cdot x) = \Lambda\cdot\theta(x)$, $\exists\, \ell \in \Lambda$ s.t. $\theta(gx) = \ell\,\theta(x)$ a.e. $x \in X$.
If $\exists\, \ell_1, \ell_2 \in \Lambda$ s.t. $\theta(gx) = \ell_1\theta(x) = \ell_2\theta(x)$, then $\ell_2^{-1}\ell_1 \in \mathrm{Stab}(\theta(x))$.
But $\mathrm{Stab}(\theta(x)) \leq \Lambda$ which acts freely, so $\mathrm{Stab}(\theta(x)) = \{e\}$ a.e. $x \in X$. Thus $\ell_1 = \ell_2$ a.e. $x \in X$.

<u>Def:</u> A <u>Λ-valued cocycle</u> for $\Gamma \curvearrowright (X,\mu)$ is a m'ble fct $\beta: \Gamma\times X \to \Lambda$ s.t. $\beta(gh,x) = \beta(g,hx)\beta(h,x)$ $\forall g,h \in \Gamma$, a.e. $x \in X$.

<u>Example:</u> Defining $\beta: \Gamma\times X \to \Lambda$ on (g,x) to be the $!\,\ell \in \Lambda$ as in the claim above defines a cocycle.

<u>Observe:</u> $\theta(ghx) = \beta(g,hx)\theta(hx) = \underbrace{\beta(g,hx)\,\beta(h,x)}_{=\beta(gh,x).}\theta(x)$

<u>Def:</u> For $\varphi: X \to \Lambda$, define $d\varphi: \Gamma\times X \to \Lambda$ by $(d\varphi)(g,x) = \varphi(gx)\varphi(x)^{-1}$.

<u>Observe:</u> $(d\varphi)(gh,x) = \varphi(ghx)\varphi(x)^{-1} = \varphi(ghx)\varphi(hx)^{-1}\varphi(hx)\varphi(x)^{-1} = (d\varphi)(g,hx)(d\varphi)(h,x)$.

We say cocycles $\alpha,\beta: \Gamma\times X \to \Lambda$ are cohomologous, $\alpha\sim\beta$, if $\exists$ $\varphi: X \to \Lambda$ s.t. $\alpha(g,x) = \underline{\varphi(gx)\,\beta(g,x)\,\varphi(x)^{-1}}$ $\forall g \in \Gamma$, a.e. $x \in X$.

order necessary for cocycle condition!

<u>Remark:</u> Suppose cocycle $\beta: \Gamma\times X \to \Lambda$ is <u>independent</u> of $x \in X$. Define $\pi: \Gamma \to \Lambda$ by $\pi(g) = \beta(g,x)$ for any $x \in X$. Then π is a gp. hom:
$\pi(g)\pi(h) = \beta(g,hx)\beta(h,x) = \beta(gh,x) = \pi(gh)$ $\forall g,h \in \Gamma$.

Def: Say $\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ are <u>stably orbit equivalent</u> [SOE] if
$\exists\, A \subseteq X$, $B \subseteq Y$ nonnull and $\theta : A \to B$ measure <u>scaling</u> s.t.
$\theta(\Gamma\cdot x \cap A) = [\Lambda \cdot \theta(x)] \cap B$ a.e. $x \in A$.

- we call $\nu(B)/\mu(A)$ the <u>index</u> of the stable orbit equivalence.

→ Can associate a <u>cocycle</u> to SOE as well. [omitted.]

<u>Thm (Zimmer):</u> Suppose Γ has no nontrivial finite subgroup, $\Gamma \curvearrowright (X,\mu) \underset{SOE}{\sim} \Lambda \curvearrowright (Y,\nu)$ via $\theta : A \xrightarrow{\sim} B$, and let α be the associated cocycle. If α is cohomologous to a gp. hom $\pi : \Gamma \to \Lambda$, then

1. $\pi : \Gamma \to \Lambda$ is injective
2. $\pi(\Gamma) \leq \Lambda$ has finite index equal to the index of the SOE
3. $\Lambda \curvearrowright Y \cong \mathrm{Ind}_{\theta(\Gamma)}^{\Lambda}(\Gamma \curvearrowright X)$ <u>induced</u> action.

<u>Note:</u> when $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are <u>OE</u>, then π is a <u>gp iso.</u>

- Gives a strategy to lift <u>OE</u> to conjugacy

<u>Cocycle Superrigidity:</u> Given conditions on Γ, $\Gamma \curvearrowright (X,\mu)$, Λ, and $\beta : \Gamma \times X \to \Lambda$, can conclude $\beta \sim \pi : \Gamma \to \Lambda$, a gp hom.

<u>Example:</u>

<u>Thm (Popa 2005):</u> Suppose Γ <u>has (T)</u> and $\Gamma \curvearrowright X$ is a <u>Bernoulli action</u>. Then <u>any</u> cocycle $\beta : \Gamma \times X \to \Lambda$ into <u>any</u> discrete Λ is cohomologous to a homomorphism $\pi : \Gamma \to \Lambda$.