Elementary Properties of vNa's $\qquad M \subseteq B(H)$ vNa.

Lemma: Every elt in a vNa is a linear comb. of 4 unitaries.

Pf: First, any elt is a lin. comb. of 2 s.a. ops. If a is s.a. w/ $\|a\| \le 1$, $u = a + i\sqrt{1-a^2}$ is unitary and $a = \frac{1}{2}(u + u^*)$. Scaling does the general case.

Cor: Let $x \in M$ s.t. $x = u|x|$ is the polar decomposition. Then $u, |x| \in M$.

Pf: $|x| = (x^*x)^{1/2} \in C^*(x, x^*) \subseteq M$ as M is a C^*-alg. Notice $\forall$ unitary $v \in \mathcal{U}(M')$, $x = vxv^* = vu|x|v^* = vuv^*|x|$. Notice $\forall |x|\xi \in |x|H$, $vuv^*|x|\xi = x\xi$. Moreover, $\ker(vuv^*) = v\ker(u) = v\ker(x) = \ker(x)$ as $[v, x] = 0$. By the uniqueness properties of u, $vuv^* = u$ $\forall v \in \mathcal{U}(M')$. By the lemma, $u \in M'' = M$.

Prop: Suppose $M \subseteq B(H)$ is a vNa and $p \in P(M)$ is a projection. Then $pMp \subseteq B(pH)$ is a vNa w/ $(pMp)' = M'p$.

Pf: If $xp \in M'p$, certainly $xp(pmp) = pxpmp = pmpx = (pmp)(xp)$ $\forall m \in M$. Hence $M'p \subseteq (pMp)'$. The converse requires a clever trick.

Suppose $u \in (pMp)'$ unitary, let $K = \overline{MpH}$ and let $q = P_K$.

Since K is M and M'-invariant, $q \in Z(M)$. Extend u to K by

$$\tilde{u} \sum x_i p \xi_i := \sum x_i u p \xi_i \quad \text{for } x_i \in M,\ \xi_i \in H.$$

Claim: $\tilde{u}$ is a well-defined isometry in $B(K)$.

Pf:
$$\|\tilde{u} \textstyle\sum x_i p \xi_i\|^2 = \displaystyle\sum_{i,j} \langle x_i u p \xi_i, x_j u p \xi_j \rangle \overset{[u,p]=0}{=} \sum_{i,j} \langle p x_j^* x_i p u \xi_i, u \xi_j \rangle \overset{u \in (pMp)'}{=}$$
$$= \sum_{i,j} \langle u p x_j^* x_i p \xi_i, u \xi_j \rangle = \sum_{i,j} \langle p x_j^* x_i p \xi_i, \xi_j \rangle = \cdots = \|\textstyle\sum x_i p \xi_i\|^2.$$

Now, by construction, $\tilde{u}$ commutes w/ action of M on $K = \overline{MpH}$.

Thus $\tilde{u}q \in M' \subset B(H)$, and $u = (\tilde{u}q)p \in M'p$.

[Notice $\forall m \in M$, $\forall \xi \in H$, $\tilde{u}qm\xi = \tilde{u}m\underbrace{q\xi}_{\in K} = m\tilde{u}\underbrace{q\xi}_{\in K}$. Hence $[m, \tilde{u}q] = 0$.
Next, $\tilde{u}qp\xi = \tilde{u}1p\xi = 1up\xi = up\xi = u\xi$ $\forall \xi \in pH$, so $\tilde{u}qp = u$.]

Defn: For $p \in M$, pMp and $M'p$ are compressions/reductions of M, M'.

Cor: $Z(pMp) = Z(M)p$.

Pf: Again, it's clear $Z(M)p \subseteq pMp \cap M'p$. The converse is similar to before, and it's on your HW!

Lattice of projections: Suppose M a vNa and $p, q \in P(M)$.

Define $p \wedge q =$ proj onto $pH \cap qH$ HW: $p\wedge q, p \vee q \in M$

$p \vee q =$ proj onto $\overline{pH + qH}$

Factors: A *factor* is a vNa w/ trivial center, i.e., $M' \cap M = Z(M) = \mathbb{C}1$.

Note: Just as vNa's come in pairs M, M', M is a factor $\Leftrightarrow$ M' is.

Prop (Ergodic property of factors): Suppose M is a factor and $p, q \in P(M) \setminus \{0\}$. Then $\exists x \in M$ s.t. $pxq \neq 0$. Moreover, can choose $x \in \mathcal{U}(M)$.

Pf: Suppose $\forall u \in \mathcal{U}(M)$, $puq = 0$. Then $(u^*pu)q = 0$ $\forall u \in \mathcal{U}(M)$.

Consider $z = \bigvee_{u \in \mathcal{U}(M)} u^*pu \in P(M)$ [this is the lub of the net of proj's $P_F := \bigvee_{u \in F} u^*pu$ w/ $F \subseteq \mathcal{U}(M)$ finite.] Since mult is separately SOT-cts, $[z, u] = 0$ $\forall u \in \mathcal{U}(M)$, and $z \in Z(M) = \mathbb{C}1$.

Since $p \leq z$, $z = 1$. But again as mult is separately SOT-cts,

$q = (\bigvee u^*pu)q = 0$, $\cancel{\Rightarrow\Leftarrow}$. [[Observe $P_1 q = 0 = P_2 q \Leftrightarrow qP_1 = 0 = qP_2$. For such P_1, P_2, q, $\forall \eta, \xi \in H$, $q[P_1\eta + P_2\xi] = 0$. Thus $q(P_1 \vee P_2) = 0 \Leftrightarrow (P_1 \vee P_2)q = 0$. Thus $\forall$ finite $F \subseteq \mathcal{U}(M)$, $P_F q = 0$, so $(\bigvee upu^*)q = \text{SOT-lim } P_F q = 0.$]]

Cor: Let $p, q \in P(M) \setminus \{0\}$, M a factor. Then $\exists$ a nonzero p.i. $u \in M$ s.t. $uu^* \leq p$ and $u^*u \leq q$.

Pf: Pick $x \in M$ s.t. $pxq \neq 0$, and let $pxq = u|pxq|$ be polar decomp.

Recall $\ker(u) = \ker(pxq) \supseteq \ker(q)$, so $u^*u = P_{\ker(u)^\perp} \leq P_{\ker(q)^\perp} = q$.

Moreover, by construction of the p.i. u,

$uH = uu^*H \subseteq \overline{pxqH} \subseteq pH$, so $uu^* \leq p$.

Cor: In previous corollary, can find p.i. $u \in M$ s.t. $uu^* = p$ or $u^*u = q$.

Pf: Order p.i.'s in M w/ $uu^* \le p$ and $u^*u \le q$ by $u \le v$ if $uu^* \le vv^* \le p$ and $u^*u \le v^*v \le q$. If $uu^* \ne p$ and $u^*u \ne q$, $\exists$ nonzero p.i. $v \in M$ s.t. $vv^* \le p - uu^*$ and $v^*v \le q - u^*u$. Thus $u+v$ is a p.i. w/ $u \lneq u+v$. By Zorn, $\exists$ maximal p.i. u, and this u must satisfy $uu^* = p$ or $u^*u = q$.

Prop: Suppose M is a vNa and $p, q \in P(M) \setminus \{0\}$. TFAE:

1. $z(p)z(q) \ne 0$ where $z(p) = \bigvee_{u \in \mathcal{U}(M)} u^*pu$ is the central supp.
2. $\exists u \in \mathcal{U}(M)$ s.t. $puq \ne 0$
3. $\exists$ nonzero p.i. $u \in M$ s.t. $uu^* \le p$ and $u^*u \le q$.

Pf: $\neg 2 \Rightarrow \neg 1$: If $puq = 0$ $\forall u \in \mathcal{U}(M)$, $z(p)q = 0$. But then $\forall u \in \mathcal{U}(M)$, $0 = uz(p)qu^* = z(p)uqu^*$, so $z(p)z(q) = 0$.
$2 \Rightarrow 3$: take polar decomp. of $puq \ne 0$ for $u \in \mathcal{U}(M)$ as before.
$\neg 1 \Rightarrow \neg 3$: Suppose $z(p)z(q) = 0$. If $u \in M$ is a p.i. s.t. $uu^* \le p$ and $u^*u \le q$, then $u = uu^*uu^*u = puq = z(p)uz(q) = uz(q)z(q) = 0$.

Def: A (nonzero) proj p is a vNa M is called:

- minimal if $q \in P(M)$ w/ $q \le p \Rightarrow q \in \{0, p\}$.
- abelian if pMp is abelian.
- diffuse if $\nexists$ an abelian proj $q \le p$.

Obviously minimal $\Rightarrow$ abelian.

Examples:

1. $B(H)$ has minimal proj's. p minimal $\Leftrightarrow$ $\operatorname{rank}(p) = \dim(pH) = \operatorname{Tr}(p) = 1$.
2. $1 \in L^\infty([0,1], \lambda)$ is diffuse.
   Lebesgue

Exercise: Suppose μ is a finite non-negative regular Borel meas. on a cpt Hausd. top. space X. Show that the min. proj's of $L^\infty(X,\mu)$ corresp. to atoms of X, i.e. $x\in X$ s.t. $\mu(\{x\})>0$.

Defi: A vNa M is called type I if $\forall z\in P(Z(M))\setminus\{0\}$, $\exists p\in P(M)$ abelian s.t. $p\le z$.

Exercises:

(1) p minimal $\Leftrightarrow$ $pMp=\mathbb{C}p$.

Pf: Clearly $\mathbb{C}p\subseteq pMp$, so we show p minimal $\Leftrightarrow$ $pMp\subseteq\mathbb{C}p$.

$\Rightarrow$: If $q\in P(pMp)$, then $q\in P(M)$ and $q\le p$, so $q\in\{0,p\}$. The only vNa w/ exactly 2 proj's is $\cong\mathbb{C}$, so $pMp=\mathbb{C}p$.

$\Leftarrow$: Let $q\in P(M)$ w/ $q\le p$. Then $pqp=q=\lambda p$, so $q\in\{0,p\}$.

(2) p abelian + M a factor $\Rightarrow$ p minimal.

Pf: M a factor $\Rightarrow$ pMp is an abelian factor. The only abelian factor is $\mathbb{C}$ (up to unital $*$-iso.)

(3) A factor is type I $\Leftrightarrow$ it has a minimal proj.

Pf: First observe that M a factor implies $P(Z(M))\setminus\{0\}=\{1\}$.

$\Rightarrow$: type I $\Rightarrow$ $\exists p\in P(M)$ abelian (≤ 1). By (2), p is minimal.

$\Leftarrow$: If $\exists p\in P(M)$ minimal, then $p\le 1$ and p is abelian.

Examples of type I vNa's: abelian vNa's, $B(H)$

Classification of type I factors: If M is a type I factor acting on a Hilbert space L, $\exists$ Hilbert spaces H,K and a unitary $u\in B(H\otimes K, L)$ s.t. $u^*Mu=B(H)\otimes 1$.

Pf: Let $\{p_i\}_{i\in I}\subseteq M$ be a maximal family of mutually $\perp$ minimal proj's [which exists by Zorn's Lemma.]

Claim: $\sum p_i = 1$, so $L=\bigoplus p_i L$.

Pf: If $\sum p_i \neq 1$, $\exists$ a nonzero p.i. $u \in M$ s.t. $uu^* \leq p_1$ and $u^*u \leq 1 - \sum p_i$.

By minimality of p_1, $uu^* = p_1$, so u^*u is also minimal!

[(observe $u^*uMu^*u = u^*\underbrace{uu^*}_{p_1}\underbrace{uMu^*}_{\subseteq M}\underbrace{uu^*}_{p_1}u \subseteq u^*p_1\mathbb{C}u = u^*u\mathbb{C}$.)]

But $u^*u \perp p_i$ $\forall i$, contradicting the maximality of $\{p_i\}$.

Now $\forall i$, choose a nonzero p.i. $e_{1,i} \in M$ s.t. $e_{1,i}e_{1,i}^* \leq p_1$ and $e_{1,i}^*e_{1,i} \leq p_i$.
(take $e_{1,1} = p_1 = e_{1,1}^*$.)

Again by minimality, $e_{1,i}e_{1,i}^* = p_1$ and $e_{1,i}^*e_{1,i} = p_i$. Then $\forall x \in M$, we have $x = (\sum p_i)x(\sum p_j) = \sum_{i,j} p_i x p_j$ converges SOT. But note

$$p_i x p_j = e_{1,i}^* e_{1,i} x e_{1,j}^* e_{1,j} = e_{1,i}^* \underbrace{p_1 e_{1,i} x e_{1,j}^* p_1}_{:= \lambda_{i,j} \in \mathbb{C}p_1} e_{1,j} = \lambda_{i,j} e_{1,i}^* e_{1,j}.$$

$$\Rightarrow x = \sum \lambda_{i,j} e_{1,i}^* e_{1,j}, \text{ converges in SOT. Hence } M = \{e_{i,j}\}''.$$

Define $H = \ell^2(I)$ and $K = p_1 L$. If $\{\xi_j\}_{j \in J}$ is an ONB for K, then an ONB for $H \bar{\otimes} K$ is given by $\{\delta_i \otimes \xi_j \mid i \in I, j \in J\}$. Define $u : H \bar{\otimes} K \to L$ by $u(\delta_i \otimes \xi_j) = \sum_i e_{1,i}^* \xi_j$.

Claim: u extends uniquely to a unitary in $B(H \bar{\otimes} K, L)$.

Pf: Since $\{\xi_j\}$ is an ONB of $p_1 L$ and $e_{1,i}^* \in B(p_1 L, p_i L)$ is unitary, $\{e_{1,i}^* \xi_j \mid j \in J\}$ is an ONB of $p_i L$. Since $L = \bigoplus p_i L$, we have $\{e_{1,i}^* \xi_j \mid i \in I, j \in J\}$ is an ONB of L. Next,

$$\sum_{i,j} \lambda_{i,j} \delta_i \otimes \xi_j \in H \bar{\otimes} K \iff \sum |\lambda_{i,j}|^2 < \infty.$$

Hence $\left\| \sum_{i,j} \lambda_{i,j} e_{1,i}^* \xi_j \right\|_L^2 = \sum |\lambda_{i,j}|^2 = \left\| \sum_{i,j} \lambda_{i,j} \delta_i \otimes \xi_j \right\|^2_{H \bar{\otimes} K}$, and u is an isometry with dense image.

Observe $(u^* e_{1,i} u)(\delta_{i'} \otimes \xi_j) = u^* e_{1,i} e_{1,i'}^* \xi_j = u^* e_{1,i} p_i p_{i'} e_{1,i'}^* \xi_j$

$$= \delta_{i=i'} u^* \xi_j = \delta_{i=i'} u^* \underbrace{e_{1,1}^*}_{p_1} \xi_j = \delta_{i=i'} \delta_1 \otimes \xi_j.$$

Hence $u^* e_{1,i} u = E_{1,i} \otimes 1$ where $E_{1,i}$ has a 1 in the $(1,i)$-position and 0s everywhere else. Hence $u^* M u = B(H) \otimes 1$.

[(If $S = S^*$ s.t. $M = S''$, then $u^* M u = (u^* S u)''$.)]

In the previous Thm, we exploited a system of matrix units of size $|I|$, i.e., a family $\{e_{i,j} \mid i,j \in I\}$ s.t.

- $e_{i,j}^* = e_{j,i}$ • $e_{i,j} e_{k,\ell} = \delta_{j=k} e_{i,\ell}$ • $\sum e_{i,i} = 1$ (SOT)

Observe: If $\{p_i\}$ is a family of mutually $\perp$ proj's s.t. $\sum p_i = 1$ (SOT) and $\{e_{1,i}\}$ is a family s.t. $e_{1,1} = p_1$, $e_{1,i} e_{1,i}^* = p_1$, and $e_{1,i}^* e_{1,i} = p_i$ $\forall i$, then setting $e_{i,j} = e_{1,i}^* e_{1,j}$ completes the $\{e_{1,i}\}$ to a SMU $\{e_{i,j}\}$.

Def: We say M is a type I_n factor if M is a type I factor and $M \cong B(H)$ w/ $\dim(H) = n$.

Thm: If M is a type I_m factor, then its type I_n unital subfactors $N \subseteq M$ are uniquely determined up to unitary conjugacy in M by the k s.t.

- $\forall p \in P(N)$ minimal pMp a type I_k factor.

Moreover, $kn = m$.

Pf: Suppose N_1 and N_2 are two type I_n subfactors w/ SMU's $\{e_{ij}\}$ and $\{f_{ij}\}$ respectively. Let k be s.t. $e_{11} M e_{11}$ is a type I_k factor. Then $1 = \sum e_{ii}$, and each e_{ii} is the (SOT) $\perp$ sum of k minimal proj's in $e_{ii} M e_{ii}$, so $kn = m$. The same argument works for N_2.

Claim: If $p \in P(e_{11} M e_{11})$ is minimal, then p is minimal in M.

Pf: Suppose $q \leq p$. Then $q \leq p \leq e_1$, so $q \leq e_1$. Then $q \in \{0, p\}$.

Claim: $\exists$ a p.i. $u \in M$ s.t. $uu^* = e_{11}$ and $u^* u = f_{11}$.

Pf: write $e_{11} = \sum p_i$ and $f_{11} = \sum q_i$ $\perp$ (SOT) sums of k min proj's. By minimality, $\forall i$, $\exists u_i$ s.t. $u_i u_i^* = p_i$, $u_i^* u_i = q_i$. Now $u = \sum u_i$ works.

Finally, $v = \underbrace{\sum e_{i1} u f_{1i}}_{\text{SOT sum.}}$ is a unitary s.t. $v f_{ij} v^* = e_{ij}$ $\forall i,j$, so $v N_2 v^* = N_1$.

Comparison of proj's:

If $p,q \in P(M)$, say $p \precsim q$ [\prec] if $\exists$ a p.i. $u \in M$ s.t. $uu^* = p$ and $u^*u \leq q$. Say $p \approx q$ if $\exists$ a p.i. $u \in M$ s.t. $uu^* = p$ and $u^*u = q$.

Observe:

① $\precsim$ is reflexive and transitive.

Pf: reflexive: obvious as each p is a p.i.

transitive: Suppose $uu^* = p$, $u^*u \leq q = vv^*$, and $v^*v \leq r$. Then

$$\underset{u=uu^*u}{uvv^*u^* = uqu^*} = u\underbrace{u^*uq}_{=u^*u}u^* = uu^*uu^* = p \text{ and}$$

$$v^*u^*uv \leq v^*qv = v^*vv^*v = v^*v \leq r.$$

② $\approx$ is an equivalence rel'n.

Pf: reflexive: obvious as in ①.

symmetric: u a p.i. $\Longleftrightarrow$ u^* a p.i.

transitive: Suppose $uu^* = p$, $u^*u = q = vv^*$, and $v^*v = r$. Then

$$uvv^*u^* = uqu^* = uu^*uu^* = p \text{ and } v^*u^*uv = v^*qv = v^*vv^*v = r.$$

Thm: $\precsim$ is a partial order.

Pf: Suppose $p \precsim q$ and $q \precsim p$. Let $u,v \in M$ be p.i.'s in M s.t.

$$uu^* = p, \quad u^*u \leq q, \quad vv^* = q, \quad v^*v \leq p.$$

Claim: Defining $\begin{bmatrix} p_0 \\ q_0 \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$ and $\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} v^*q_nv \\ u^*p_nu \end{bmatrix}$ inductively defines two decreasing sequences of proj's in M.

Pf: p_0, q_0 are proj's. $p_1 = v^*q_0v = v^*vv^*v = v^*v \leq p = p_0$ and similarly, q_1 is a proj $\leq q_0$. Suppose p_n, q_n are proj's $\leq p_{n-1}, q_{n-1}$ respectively. Then $p_{n+1} = v^*q_nvv^*q_nv = v^*q_nqq_nv = v^*q_nv = p_{n+1}$ is a proj and $p_{n+1} = v^*q_nv \leq v^*q_{n-1}v = p_n$. Similarly q_{n+1} is a proj. $\leq q_n$.

Indeed $e \mapsto v^*ev$ gives order presving map $\{\text{proj's} \leq q\} \to \{\text{proj's} \leq p\}$. Similarly for $e \mapsto u^*eu$.

Define $p_\infty = \bigwedge_{n=0}^{\infty} p_n$ and $q_\infty = \bigwedge_{n=0}^{\infty} q_n$ (proj's onto $\bigcap p_nH$ and $\bigcap q_nH$)

Claim: $v^* q_\infty v = p_\infty$ and $q_\infty v v^* q_\infty = q_\infty$. Hence $p_\infty \approx q_\infty$.

Pf: mult is separately SOT-cts, and $q_n \to q_\infty$ SOT.

Now $p = (p_0 - p_1) + (p_1 - p_2) + \cdots + p_\infty$ (SOT) sum of $\perp$ proj's

$q = (q_0 - q_1) + (q_1 - q_2) + \cdots + q_\infty$

$\forall i$, $v^*(q_i - q_{i-1})v = p_{i+1} - p_i$ and $u^*(p_i - p_{i-1})u = q_{i+1} - q_i$.

Hence $(q_i - q_{i-1})v$ is a p.i. witnessing $p_{i+1} - p_i \approx q_i - q_{i-1}$ and similarly for $(p_i - p_{i-1})u$. Hence we construct a p.i. witnessing $p \approx q$ via:

$$p = (p_0 - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_4) + \cdots + p_\infty$$

$$q = (q_0 - q_1) + (q_1 - q_2) + (q_2 - q_3) + (q_3 - q_4) + \cdots + q_\infty$$

and adding up these p.i.'s (SOT).

Cor: If M is a factor, $\precsim$ is a total order on proj's.

Pf: If $p, q \in M \setminus \{0\}$, $\exists$ p.i. $u \in M$ s.t. $uu^* = p$ or $u^*u = q$, so $p \precsim q$ or $q \precsim p$.

Def: A proj $p \in P(M)$ is called:

① finite if $\forall q \le p$, $q \approx p \Rightarrow q = p$.

② infinite if $\exists q \le p$ w/ $q \ne p$ s.t. $q \approx p$.

An infinite projection p is called:

- purely infinite if $\nexists$ a nonzero finite $q \le p$
- properly infinite if $\forall z \in P(Z(M))$ s.t. $zp \ne 0$, zp is infinite.

A vNa M is called finite/(purely/properly) infinite if 1_M is respectively.

Exercise: Abelian vNa's are finite, so p abelian $\Rightarrow p$ finite.

Def: M is called type III if M is purely infinite.

M is called type II if M has no abelian projections and every $z \in P(Z(M)) \setminus \{0\}$ majorizes a nonzero finite projection.

- type II_1: M type II and M finite.
- type II_∞: M type II and $\nexists\, z \in P(Z(M)) \setminus \{0\}$ finite.

II_1 factors exist: Let Γ be a countable group.

Consider $L\Gamma = \{\lambda_g \mid g\in\Gamma\}'' \subseteq B(\ell^2\Gamma)$ where $(\lambda_g\xi)(h) = \xi(g^{-1}h)$.

Claim 1: $\forall x\in L\Gamma$, $\exists\, (c_g)_{g\in\Gamma}\in\ell^2\Gamma$ s.t. $x\delta_e = \sum c_g\delta_g$. Moreover:

- $x^*\delta_e = \sum \overline{c_{g^{-1}}}\,\delta_g$
- if $x\delta_e = \sum c_g\delta_g$ and $y\delta_e = \sum d_h\delta_h$, $xy\delta_e = \sum_g\left(\sum_h c_h d_{h^{-1}g}\right)\delta_g$

$\underline{(c_g)*(d_h)}\in\ell^2\Gamma$!

Pf: Clearly $x\delta_e\in\ell^2\Gamma$, so (c_g) exists and is uniquely defined.

$\forall h\in\Gamma$, $\langle x^*\delta_e,\delta_h\rangle = \langle\delta_e, x\rho_{h^{-1}}\delta_e\rangle = \langle\rho^*_{h^{-1}}\delta_e, x\delta_e\rangle = \sum_g \overline{c_g}\langle\delta_{h^{-1}},\delta_g\rangle = \overline{c_{h^{-1}}}$.

$\forall k\in\Gamma$, $\langle xy\delta_e,\delta_k\rangle = \langle\rho^*_{k^{-1}}y\delta_e, x^*\delta_e\rangle = \sum_{h,g} d_h c_{g^{-1}}\langle\rho^*_{k^{-1}}\delta_h,\delta_g\rangle = \delta_{k^{-1}h=g}\sum_h d_h c_{h^{-1}k}$

Note: Here, ρ_h is right action of Γ on $\ell^2\Gamma$: $\underline{(\rho_h\xi)(g) = \xi(gh)}$.

Notice $\rho_h\in\mathcal{U}(\ell^2\Gamma)\cap L\Gamma'$.

$\rho_h\delta_g = \delta_{gh^{-1}}$!

Claim 2: $\forall x\in L\Gamma$, $h\in\Gamma$, $x\delta_h = \sum_g c_g\delta_{gh} = \sum_g c_{gh^{-1}}\delta_g$

Pf: $x\delta_h = x\rho_{h^{-1}}\delta_e = \rho_{h^{-1}}x\delta_e = \rho_{h^{-1}}\sum c_g\delta_g = \sum{}_g\delta_{gh}$.

Claim 3: δ_e is cyclic and separating for $L\Gamma$ on $\ell^2\Gamma$.

Pf: Clearly $\mathbb{C}[\Gamma]\delta_e\subseteq L\Gamma\delta_e$ is dense in $\ell^2\Gamma$. If $x,y\in L\Gamma$ s.t. $x\delta_e = \sum c_g\delta_g = y\delta_e$, then by claim 2, $x\delta_h = y\delta_h$ $\forall h\Rightarrow x=y$.

Claim 4: $\mathrm{tr} = \langle\,\cdot\,\delta_e,\delta_e\rangle$ is a faithful trace on $L\Gamma$, and $\mathrm{tr}(x) = c_e$.

Pf: $\langle xy\delta_e,\delta_e\rangle = \sum_h c_h d_{h^{-1}} = \sum_g d_g c_{g^{-1}} = \langle yx\delta_e,\delta_e\rangle$, so tr is a trace.

Since δ_e is separating, tr is faithful: $\langle x^*x\delta_e,\delta_e\rangle = \|x\delta_e\|_2^2 = 0 \Leftrightarrow x=0$.

Claim 5: All proj's in $L\Gamma$ are finite.

Pf: Suppose $uu^* = p$ and $u^*u = q\le p$. Then $\mathrm{tr}(p-q) = \mathrm{tr}(uu^*-u^*u) = 0$, so $p-q=0$ since tr is faithful.

Prop: If Γ is ∞ ICC (all nontrivial conj. classes infinite), $L\Gamma$ is a II_1 factor.

Pf: Suppose $x\in Z(L\Gamma)$ w/ $x\delta_e = \sum c_g\delta_g$. Then $\lambda_{h^{-1}}x\lambda_h\delta_e = \sum c_{hgh^{-1}}\delta_g = \sum c_g\delta_g$, so $(c_g)\in\ell^2\Gamma$ is constant along conjugacy classes. We conclude $c_g = 0$ $\forall g\neq e$. Hence $x\in\mathbb{C}1$ by Claim 4. Thus $L\Gamma$ is a factor. Since M is finite by Claim 5 and ∞-dim'l since Γ is infinite, $L\Gamma$ is type II_1.

Ideals: Let M be a vNa and $I \subseteq M$ a left ideal.

Lemma: Let M be a vNa and $I \subseteq M$ a σ-WOT/WOT closed left ideal. Then $M = Mp$ for some $p \in P(M)$.

Pf: I σ-WOT closed $\Rightarrow$ I $\|\cdot\|$-closed. Thus I has a right approx. id. (e_λ) s.t. $\forall \lambda$, $0 \le e_\lambda \le 1$, $\lambda \le \lambda' \Rightarrow e_\lambda \le e_{\lambda'}$, and $\|a - ae_\lambda\| \to 0$ $\forall a \in I$. Now $p := \bigvee_\lambda e_\lambda \in I$ as I σ-WOT/WOT closed. Hence $\|p - pe_\lambda\| \to 0$ $\Rightarrow p = p^2 = p^*$, and $ap = p$ $\forall a \in I$. Thus

$$I = Ip \subseteq Mp \subseteq I, \text{ so equality holds.}$$

Cor: A left ideal $I \subseteq M$ is σ-WOT/σ-SOT closed $\Leftrightarrow$ WOT/SOT closed.

Pf: Both are of the form Mp for $p \in P(M)$.

Cor: A σ-WOT closed 2-sided ideal is of the form Mz, $z \in P(Z(M))$.

Pf: Since I is a σ-WOT closed left ideal, $I = Mp$ for some $p \in P(M)$. Since I is 2-sided, I is $*$-closed, and p is a left identity for I. Hence $\forall x \in M$, $xp = pxp = px$, and $p \in Z(M)$.

Lemma: A σ-WOT cts tracial state tr on a factor M is faithful.

Pf: Let $I = \{x \in M \mid tr(x^*x) = 0\}$. Since $x^*y^*yx \le \|y^*y\| x^*x$, I is a left ideal. Since tr is a trace, I is 2-sided. By C-S, $tr(x^*x) = 0 \Leftrightarrow tr(xy) = 0$ $\forall y \in M \Leftrightarrow x \in \bigcap_{y \in M} \ker(\underbrace{tr(\cdot y)}_{\sigma\text{-WOT cts!}})$ so I is σ-WOT closed. Thus $I = Mz$ for a $z \in P(Z(M))$. Since M is a factor, $z \in \{0,1\}$. Since $tr(1) = 1$, $z = 0$, so $I = (0)$.

Prop: An ∞-dim'l factor M w/ a σ-WOT cts trace is type II_1.

Pf: We must show M has no abelian projections and is finite. By the lemma, tr is faithful. If $1 = uu^*$ and $p = u^*u$, then $tr(1-p) = tr(uu^* - u^*u) = 0$, so $p = 1$, and M is finite. Now $\forall p \in P(M)$, pMp is a factor. Thus p abelian $\Leftrightarrow pMp = p\mathbb{C} \Rightarrow p$ min. But $B(H)$ for $\dim H = \infty$ has no such trace! [HW!]

Suppose that M is a II_1 factor w/ σ-WOT cts (faithful) tracial state tr:

Lemma: $p \precsim q \iff tr(p) \le tr(q)$. Moreover, $p \approx q \iff tr(p) = tr(q)$.

Pf: $\Rightarrow$: $tr(p) = tr(uu^*) = tr(u^*u) \le tr(q)$.

$\Leftarrow$: Since M is a factor, $p \precsim q$ or $q \precsim p$. By $\Rightarrow$ step, $q \precsim p$ implies $tr(q) = tr(uu^*) = tr(u^*u) \le tr(p) \le tr(q)$, so $u^*u = p$ since tr is faithful. Thus $p \approx q$, so $p \precsim q$.

The final statement now follows formally, and we omit the pf.

Prop: $\forall p \in P(M)\setminus\{0\}$ and $\forall 0 < \varepsilon < tr(p)$, $\exists\, q \le p$ s.t. $0 < tr(q) < \varepsilon$.

Pf: Let $\delta = \inf\{tr(q) \mid q \in P(M)\setminus\{0\},\ q \le p\}$. If $0 < \delta \le tr(p)$, $\exists$ $q \in P(M)\setminus\{0\}$ s.t. $q \le p$ and $tr(q) - \delta < \delta$ by the def. of inf. Since q is not minimal as $M \not\cong B(H)$, $\exists\, 0 \ne r \lneq q \le p$. Then $\delta \le tr(r)$, so $tr(q-r) = tr(q) - tr(r) \le tr(q) - \delta < \delta$, a contradiction.

Thm: If M a II_1 factor w/ σ-WOT cts (faithful) tracial state tr, then $tr[P(M)] = [0,1]$.

Pf: For $r \in (0,1)$, let $S = \{p \in P(M) \mid tr(p) \le r\}$. By Zorn, $\exists$ a max elt $q \in S$. If $tr(q) < r$, $1-q \ne 0$. By the proposition, $\exists$ $0 \ne q' \lneq 1-q$ s.t. $0 < tr(q') < r - tr(q)$. Then $q+q' \in P(M)$, $q+q' \gneq q$, and $0 < tr(q+q') = tr(q) + tr(q') \le tr(q) + r - tr(q) = r$, a contradiction.

Exercises: Let M be a II_1 factor w/ σ-WOT cts tracial state tr

① $\forall n \in \mathbb{N}$, $\exists$ a unital subfactor $N \subseteq M$ s.t. $N \cong M_n(\mathbb{C})$.

② M is algebraically simple, i.e., M has no 2-sided ideals.

Cor: Any two σ-WOT cts tracial states on M agree.

Pf: It suffices to show these traces agree on projections. By Ex. ① above, the traces agree on proj's w/ trace $\frac{1}{n}$, $n \in \mathbb{N}$, as $\exists!$ trace on $M_n(\mathbb{C})$. Now for an arbitrary proj. p, can build an $\perp$ seq. of proj's (p_i) s.t. $p = \sum p_i$ and $tr(p_i) = \frac{1}{n_i}$, $n_i \in \mathbb{N}$ $\forall i$, using the Prop. above.

Cor: A finite vNa M w/ a σ-WOT cts faithful trace tr is a II_1 factor $\iff \forall$ σ-WOT cts tracial state φ, $\varphi = tr$.

Pf: $\Rightarrow$: This was the content of the previous corollary.

$\Leftarrow$: Suppose M not a factor. Then let $z \in P(Z(M)) \setminus \{0,1\}$. Then $\varphi(x) := \frac{1}{tr(z)} tr(xz)$ is a σ-WOT cts tracial state w/ $\varphi \neq tr$ as $\varphi(1-z) = 0 \neq tr(1-z)$ as tr is faithful.

The hyperfinite II_1 factor: For $n \in \mathbb{N}$, let $A_n = \bigotimes^n M_2(\mathbb{C})$.
Include $A_n \hookrightarrow A_{n+1}$ by $x \mapsto x \otimes 1$. Let $A_\infty = \varinjlim A_n$. Since $A_n \cong M_{2^n}(\mathbb{C})$ then A_n has a ! normalized (faithful positive) trace tr_n.
Thus $tr_\infty := \varinjlim tr_n$ on A_∞ is the ! trace, and it is faithful + positive.
For $H = L^2(A_\infty, tr_\infty)$, and note A_∞ acts by bdd ops on the left.

$$\|xa\Omega\|^2 = tr(a^* x^* x a) \le \|x^* x\|_{A_n} tr_\infty(a^* a) = \|x\|^2_{A_n} \|a\Omega\|_2^2$$ [[$x^*x \le \|x^*x\|_{A_n}$ in A_n.]]

Since tr_∞ is tracial, the right action is also bdd! [[requires tracial!]]
Let $R = (A_\infty)'' \subseteq B(L^2 A_\infty, tr_\infty)$. We claim R is a II_1 factor.

Claim ①: $tr = \langle \cdot \Omega, \Omega \rangle$ is a σ-WOT cts tracial state on R s.t. $tr|_{A_\infty} = tr_\infty$.

Pf: For $x \in A_\infty$, $tr(x) = \langle x\Omega, \Omega\rangle = tr_\infty(x)$, so $tr|_{A_\infty} = tr_\infty$ on A_∞.
Since tr is a vector state, it is SOT cts. For $x, y \in R$, pick bdd nets $(x_\lambda), (y_\lambda) \subset A_\infty$ w/ $x_\lambda \to x$, $y_\lambda \to y$ SOT by Kaplansky density. Then
$tr(xy) = \lim tr(x_\lambda y_\lambda) = \lim tr_\infty(x_\lambda y_\lambda) = \lim tr_\infty(y_\lambda x_\lambda) = tr(yx)$.

Claim ②: tr is the ! σ-WOT cts tracial state on R.

Pf: Let φ be another σ-WOT cts tracial state on R. Then $\varphi|_{A_\infty} = tr_\infty$, since tr_∞ is the ! trace on A_∞. Then by σ-WOT continuity, if $(x_\lambda) \subset A_\infty$ w/ $x_\lambda \to x \in R$, $\varphi(x) = \lim \varphi(x_\lambda) = \lim tr_\infty(x_\lambda) = tr(x)$.

Claim ③: tr is faithful on R, so R is a II_1 factor.

Pf: Suppose $tr(x^*x) = 0$. Then $\forall a \in A_\infty$, since right mult by a is bdd and commutes w/ the left A_∞ action and thus w/ R,
$$\|xa\Omega\|_2^2 = \|xR_a\Omega\|_2^2 = \|R_a x\Omega\|_2^2 \le \|R_a\|^2 \|x\Omega\|_2^2 = \|R_a\|^2 tr(x^*x) = 0.$$
By density of $A_\infty\Omega \subseteq L^2(A_\infty, tr_\infty)$, $x = 0$.
Thus tr is faithful, and the unique σ-WOT cts tracial state. So R is a II_1 factor.

Note: To construct a II_1-factor, we exploited the trace. Our next step is to show all II_1 factors have a faithful σ-WOT cts tracial state.

Some important results about comparison of proj's.

Unless otherwise stated, below, M is a vNa, $p, q \in P(M)$.

① If $x \in M$, proj onto $\overline{xH}$ and $\ker(x)$ are in M, and $P_{\overline{xH}} \approx P_{\overline{x^*H}}$.

Pf: a) $x = u|x|$ the polar decomp. Then $uH = \overline{xH}$ and $\ker(u) = \ker(x)$. So uu^* is proj onto $\overline{xH}$ and $1 - u^*u$ is proj. onto $\ker(x)$.

b) $\ker(u)^\perp = \ker(x)^\perp = \overline{x^*H}$. Hence u^*u is proj onto $\overline{x^*H}$.

② (Kaplansky's Formula) $p \vee q - p \approx q - p \wedge q$.

Pf: Consider $x = (1-p)q$. Then $\ker(x) = \ker(q) \oplus (p \wedge q)H$, so proj. onto $\ker(x)$ is $(1-q) + p\wedge q$. Then proj onto $\overline{x^*H} = 1 - [(1-q) + p \wedge q] = q - p\wedge q$. Since $x = [(1-(1-q))(1-p)]^*$, the above argument tells us the proj. onto $\overline{xH}$ is $(1-p) - (1-p)\wedge(1-q) = 1 - p - (1 - p\vee q) = p \vee q - p$. These are $\approx$ by ①.

③ If $p_1 \precsim q_1$, $p_2 \precsim q_2$, and $q_1 q_2 = 0$, then $p_1 \vee p_2 \precsim q_1 + q_2$.

Pf: $p_1 \vee p_2 - p_2 \approx p_1 - p_1 \wedge p_2 \precsim q_1$ by ②, and $p_1 \vee p_2 = (p_1 \vee p_2 - p_2) + p_2 \precsim q_1 + q_2$.

④ (Comparison Thm) $\exists\, z \in P(Z(M))$ s.t. $pz \precsim qz$ and $(1-q)z \precsim (1-p)z$.

Pf: By Zorn, $\exists$ max families $\{p_i\}$, $\{q_i\}$ of mutually $\perp$ proj's s.t. $\sum p_i \le p$, $\sum q_i \le q$, and $p_i \approx q_i \ \forall i$. Let $z_1 = z(p - \sum p_i)$ [central supp.] and $z_2 = z(q - \sum q_i)$. By maximality, $z_1 z_2 = 0$, so

$$(p - \textstyle\sum p_i) \le z_1 \le 1 - z_2 \quad \Longrightarrow \quad z_2(p - \textstyle\sum p_i) = 0.$$
$$(q - \textstyle\sum q_i) \le z_2 \quad \Longrightarrow \quad (1-z_2)(q - \textstyle\sum q_i) = 0.$$

Since $\sum p_i \approx \sum q_i$,

$z_2 \in P(Z(M))$

- $z_2 p = z_2 \sum p_i \approx z_2 \sum q_i \le z_2 q$ and
- $(1-z_2) q = (1-z_2)\sum q_i \approx (1-z_2)\sum p_i \le (1-z_2) p$.

$(1-z_2) \in P(Z(M))$.

⑤ If p, q are finite, so is $p \vee q$.

Pf: Omitted. Uses ③, ④, and in a properly infinite vNa, $\exists p$ s.t. $p \approx 1-p \approx 1$.

(6) If p, q finite and $p \approx q$, then $1-p \approx 1-q$. Hence $\exists u \in \mathcal{U}(M)$ s.t. $u^* p u = q$.

Pf: By (5), $p \vee q$ is finite, so replacing M by $(p\vee q)M(p \vee q)$, we may assume M is finite. [I we'll only use this result for finite M, which is why we've omitted the proof of (5). Also, note (5) is much easier when $p, q \in P(Z(M))$.] By (4), $\exists\, z \in P(Z(M))$ s.t. $(1-p)z \precsim (1-q)z$ and $(1-q)(1-z) \precsim (1-p)(1-z)$. Since we can consider Mz and $M(1-z)$ separately, we may assume $(1-p) \approx r \leq 1-q$. Since $1 = (1-p)+p \approx r+q \leq (1-q)+q = 1$ and M is finite, $r+q = 1$. Thus $1-p \approx r = 1-q$.

Now if $vv^* = p$, $v^*v = q$ and $ww^* = 1-p$, $w^*w = 1-q$, $u = v+w$ satisfies $u^* p u = (v+w)^* p (v+w) = v^* p v = v^* v v^* v = q$.

(7) If $p, q \in P(M)$ finite with $p, q \leq r$, then
(a) If $p \approx q$, $r-p \approx r-q$. (b) If $p \precsim q$, $r-q \precsim r-p$.

Pf: $p, q \leq r \Rightarrow p \vee q \leq r$. Passing to $(p\vee q)M(p\vee q)$, by (5), we may assume M is finite and $r = 1$. [Again, we'll only use (7) for M finite.] (a) now follows by (6). For (b), let $s \in P(M)$ s.t. $p \approx s \leq q$. By (6), $1-p \approx 1-s \geq 1-q$.

(8) If (q_n) is an increasing seq. of finite proj's and $p \in P(M)$ s.t. $q_n \precsim p$ $\forall n$, then $\bigvee q_n \precsim p$.

Pf: We inductively construct an $\perp$ seq. of proj's $p_n \leq p$ s.t. $p_0 \approx q_1$ and $\forall n \in \mathbb{N}$, $p_n \approx q_{n+1} - q_n$. Then $\bigvee_{n=1}^{\infty} q_n = q_1 + \sum_{n=1}^{\infty} (q_{n+1} - q_n) \approx \sum_{n=0}^{\infty} p_n \leq p$.

By assumption, $q_1 \precsim p$, so $\exists\, p_0 \leq p$ s.t. $q_1 \approx p_0$. Suppose we have $p_0, p_1, \ldots, p_n$.

<u>Claim:</u> $q_{n+2} - q_{n+1} \precsim p - \sum_0^n p_i \rightsquigarrow \exists\, p_{n+1} \leq p \perp$ to $p_0, \ldots, p_n$ as desired.

Observe $q_{n+2} \precsim p$, so $\exists$ p.i. u s.t. $uu^* = q_{n+2}$, $e_{n+2} := u^*u \leq p$. Since $q_{n+2} \geq q_{n+1}$, $e_{n+1} := u^* q_{n+1} u \leq u^* q_{n+2} u = e_{n+2}$, and $e_{n+1} \approx q_{n+1}$.

Then $u^*(q_{n+2} - q_{n+1})u = e_{n+2} - e_{n+1}$, so $q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1}$.

Also, $q_{n+1} = (q_{n+1} - q_n) + (q_n - q_{n-1}) + \cdots + (q_2 - q_1) + q_1 \approx \sum_0^n p_i \leq p$. Since q_{n+2} is finite, so are $e_{n+2}, e_{n+1}, \sum_0^n p_i$. Finally,

$q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1} = (p - e_{n+1}) - (p - e_{n+2}) \leq p - e_{n+1} \precsim p - \sum_0^n p_i$

by (7), since $e_{n+1} \approx q_{n+1} \approx \sum_0^n p_i$ and $e_{n+1}, \sum_0^n p_i \leq p$ finite.

(9) Suppose M a finite vNa, and (p_n) an infinite sequence of mutually $\perp$ proj's. $\forall n$, suppose $q_n \approx p_n$. Then $q_n \to 0$ $\underline{\text{SOT}}$. thus also σ-WOT.

Pf: By induction using (3), $\forall m \le n$, $\bigvee_m^n q_i \precsim \sum_m^n p_i \le \sum_m^\infty p_i$. Since $\bigvee_m^n q_i$ is increasing in n, $\bigvee_m^\infty q_i \precsim \sum_m^\infty p_i$ by (8). Let $p_0 = 1 - \sum_{i=1}^\infty p_i$.

By (2), $p_0 + \sum_{i=1}^{m-1} p_i = 1 - \sum_m^\infty p_i \precsim 1 - \bigvee_m^\infty q_i \le 1 - \bigwedge_{m=1}^\infty \bigvee_m^\infty q_i$.

Again by (8), $1 = p_0 + \sum_{i=1}^\infty p_i \precsim 1 - \bigwedge_{m=1}^\infty \bigvee_m^\infty q_i$. Since M is finite, $0 = \bigwedge_{m=1}^\infty \underbrace{\bigvee_m^\infty q_i}_{\text{decreasing as } m \to \infty} = \text{SOT-}\lim_{m\to\infty} \underbrace{\bigvee_m^\infty q_i}_{\ge q_m}$. Hence $\forall \xi \in H$,

$\|q_m \xi\|^2 = \langle q_m \xi, \xi \rangle \le \langle \bigvee_m^\infty q_i \xi, \xi \rangle \to 0$ as $m \to \infty$. Thus $q_m \to 0$ SOT.

Recall: σ-WOT on M is wk* top induced by $(M_*)^* = M$. Thus we may identify M_* w/ the σ-WOT cts linear ftls on M.

Def: For a vNa M, let $S(M) = \{\sigma\text{-WOT cts states of } M\} \subset M_*$.
Note that $\mathcal{U}(M) \curvearrowright S(M)$ by $\varphi \longmapsto \varphi(u^* \cdot u)$.

Lemma: Let M be a vNa and $\varphi \in M^*$ a state. TFAE:

1. φ is tracial: $\varphi(xy) = \varphi(yx)$ $\forall x, y \in M$.
2. $\forall x \in M$, $\varphi(xx^*) = \varphi(x^*x)$.
3. $\forall u \in \mathcal{U}(M)$, $\varphi(u^*xu) = \varphi(x)$ $\forall x \in M$.

Pf: (1)$\Rightarrow$(2): obvious

(2)$\Rightarrow$(3): $\forall x \ge 0$, $\varphi(uxu^*) = \varphi(ux^{1/2}x^{1/2}u^*) = \varphi(x^{1/2}u^*u x^{1/2}) = \varphi(x)$. Now every $x \in M$ is a linear combination of 4 pos. elts.

(3)$\Rightarrow$(1): If $u \in \mathcal{U}(M)$, $\varphi(xu) = \varphi(u^*uxu) = \varphi(ux)$. Now every $y \in M$ is a linear comb. of 4 unitaries.

Goal: To construct a trace in $S(M)$ for a finite vNa M, find a fixed pt in $S(M)$ for the action of $\mathcal{U}(M)$. We'll use:

Thm (Ryll-Nardzewski): Let X be a Banach space and $K \subseteq X$ a wkly cpt cvx subset. Suppose $G \subseteq B(X)$ and $GK \subseteq K$. Then $\exists k \in K$ s.t. $gk = k$ $\forall g \in G$.

$\forall u \in \mathcal{U}(M)$, define $\pi_u \in B(M_*)$ via $\pi_u \varphi = \varphi(u^* \cdot u)$.

[Observe: $\forall u \in \mathcal{U}(Z(M))$, $\pi_u \varphi = \varphi$ $\forall \varphi \in M_*$, so $\mathcal{U}(Z(M)) \leq \ker \pi$.]

Hence $G = \pi[\mathcal{U}(M)] \leq B(M_*)$.

Thm: Suppose M is a finite vNa, and fix $\varphi \in S(M)$. Let $K_0 = G\varphi = \{\varphi(u^* \cdot u) \mid u \in \mathcal{U}(M)\} \subseteq S(M)$, and let $K \subseteq S(M)$ be the wkly closed cnvx hull of K_0 in M_*. Then K is weakly compact.

Cor: $\exists$ a σ-WOT cts tracial state on M.

Pf: By Ryll-Nardzewski, $\exists$ a G-fixed pt $\mathrm{tr} \in K \subseteq S(M)$, which is a trace.

We'll first prove the above thm, then Ryll-Nardzewski.

Lemma: For a positive ($\Rightarrow$ bdd) linear fnl $\varphi \in M^*$, TFAE:

① φ is σ-WOT cts.

② φ is normal: $\forall\, 0 \leq x_\lambda \nearrow x$, $\varphi(x_\lambda) \nearrow \varphi(x)$.

③ φ is completely additive: $\forall$ family (p_i) of mutually $\perp$ proj's, $\sum \varphi(p_i) = \varphi(\sum p_i)$.

Pf: HW!

Remark: $\forall\, 0 \leq \varphi \in M^*$, $\forall$ finite subsets $F \subseteq I$, $\sum_{i \in F} \varphi(p_i) = \varphi(\sum_{i \in F} p_i) \leq \varphi(\sum p_i)$, so $\sum \varphi(p_i) \leq \varphi(\sum p_i)$. Hence φ is completely additive iff $\forall (p_i)$ family of $\perp$ proj's, $\forall \varepsilon > 0$, $\exists$ finite $F \subseteq I$ s.t. $\varphi(\sum_{i \notin F} p_i) \leq \varepsilon$.

[Observe: $\sum \varphi(p_i) = \sup_{F \subseteq I} \sum_{i \in F} \varphi(p_i) = \sup_{F \subseteq I} \varphi(\sum_{i \in F} p_i) = \sup_{F \subseteq I} \varphi(\sum p_i) - \varphi(\sum_{i \notin F} p_i)$

$= \varphi(\sum p_i) - \inf_{F \subseteq I} \varphi(\sum_{i \notin F} p_i)$.]

Proof of Thm: Recall that the relative wk* top on $X \subseteq X^{**}$ is the weak topology. So to show $K \subseteq M_*$ is wkly cpt, by the Banach-Alaoglu thm, it suffices to prove $K \subseteq M_*^{**} = M^*$ is wk* closed, as $K \subseteq (M^*)_1$ which is wk* cpt.

Let $\psi \in \overline{K}$, the wk* closure of K in M^*. We'll show ψ is completely additive, and thus $\psi \in M_* \Rightarrow \psi \in K$. Suppose not for contradiction. Then $\exists$ a family $(p_i)_{i\in I}$ of mutually $\perp$ proj's and an $\varepsilon > 0$ s.t. $\forall$ finite $F \subseteq I$, $\psi(\sum_{i\notin F} p_i) > \varepsilon$.

<u>Claim</u>: $\exists$ a seq. of unitaries $(u_j) \subseteq \mathcal{U}(M)$ and a seq. of mutually $\perp$ proj's $(q_j) \subset M$ s.t. $\varphi(u_j^* q_j u_j) > \varepsilon$ $\forall j$.

<u>Observe</u>: This claim gives us our contradiction! Since the (q_j) are mutually $\perp$ and $u_j^* q_j u_j \approx q_j$, by ⓔ, $u_j^* q_j u_j \longrightarrow 0$ <u>SOT</u>, and thus σ-WOT. But $\varphi \in S(M)$ is σ-WOT cts, and $\varphi(u_j^* q_j u_j) > \varepsilon$ $\forall j$, a contradiction.

<u>Pf of Claim</u>: By induction.

<u>$j=1$</u>: Note $\psi(\sum p_i) > \varepsilon$. Thus $\exists \phi_1 \in K_0$ s.t. $\phi_1(\sum p_i) > \varepsilon$.

[Conv(K_0) (convex hull) is wkly dense in K which is wk* dense in $\overline{K}$. Hence Conv(K_0) is wk* dense in $\overline{K}$. Thus $\forall \delta > 0$, $\exists \phi \in$ Conv(K_0) s.t. $|(\psi - \phi)(\sum p_i)| < \delta$ as $\{\phi \in M^* \mid |(\psi-\phi)(\sum p_i)| < \delta\}$ is wk* open nbhd of ψ. Since $\psi(\sum p_i) > \varepsilon$, choosing δ small so $\phi(\sum p_i) > \varepsilon$. Now if $\phi = \sum_{j=1}^n \lambda_j \phi_j$ cvx comb. of $\phi_j \in K_0$, $\exists j$ s.t. $\phi_j(\sum p_i) > \varepsilon$. Else, $\phi(\sum p_i) \leq \varepsilon$ ~~$\Rightarrow\Leftarrow$~~.]

Since ϕ_1 is completely additive, $\exists$ finite set $F_1 \subseteq I$ s.t. $\phi_1(\sum_{i\in F_1} p_i) > \varepsilon$.
Since $\psi(\sum_{i\notin F_1} p_i) > \varepsilon$, $\exists \phi_2 \in K_0$ s.t. $\phi_2(\sum_{i\notin F_1} p_i) > \varepsilon$. Again, since ϕ_2 is completely additive, $\exists$ finite $F_2 \subseteq I \setminus F_1$ s.t. $\phi_2(\sum_{i\in F_2} p_i) > \varepsilon$. Since $\psi(\sum_{i\notin F_2} p_i) > \varepsilon$, $\exists \phi_3 \in K_0$ etc.

Now, $\forall j \in \mathbb{N}$, write $\phi_j = \varphi(u_j^* \cdot u_j)$ for $u_j \in \mathcal{U}(M)$, and define $q_j = \sum_{i\in F_j} p_i$. Since all the F_j are <u>disjoint</u>, the proj's (q_j) are mutually $\perp$. By construction,

$$\phi_j(\sum_{i\in F_j} p_i) = \phi_j(q_j) = \varphi(u_j^* q_j u_j) > \varepsilon \quad \forall j \in \mathbb{N}.$$

Theorem (Ryll-Nardzewski): Let $\mathfrak{X}$ be a Banach space and $K \subseteq \mathfrak{X}$ cnvx nonempty which is weakly cpt. Let G be a gp of isometries of $\mathfrak{X}$ which preserves K. Then $\exists x \in K$ s.t. $g(x) = x$ $\forall g \in G$.

Note: We'll use the proof below due to Jacob Lurie, which assumes $G \subseteq B(\mathfrak{X})$. Note that $\forall u \in \mathcal{U}(M)$, $T_u : M_* \longrightarrow M_*$ by $\varphi \longmapsto \varphi(u^* \cdot u)$ is linear + bdd, so $T[\mathcal{U}(M)] \subseteq B(M_*)$.

Warmup: Suppose $\exists T \in B(\mathfrak{X})$ s.t. $TK \subseteq K$. $\exists x \in K$ s.t. $Tx = x$.

Pf: For $n \in \mathbb{N}$, define $T_n = \frac{1}{n} \sum_{j=0}^{n-1} T^j$. As K cnvx, $T_n K \subseteq K$ $\forall n \in \mathbb{N}$. Let $K_n = T_n K \subseteq K$. We claim $\{K_n\}$ has FIP, so $\bigcap K_n \neq \emptyset$. Indeed, $K_{n_1} \cap \cdots \cap K_{n_j} \supseteq T_{n_1} T_{n_2} \cdots T_{n_j} K$, as $T_n T_m = T_m T_n$ $\forall m, n$. Let $x \in \bigcap K_n$. Then $\forall n \in \mathbb{N}$, $\exists y \in K$ s.t. $x = T_n y$. Hence

$$Tx - x = T\left[\frac{1}{n} \sum_0^{n-1} T^j y\right] - \frac{1}{n} \sum_0^{n-1} T^j y = \frac{1}{n}\left[T^n y - y\right] \in \frac{1}{n}[K - K].$$

Since $K - K$ wkly cpt, it is bdd, so $\forall$ open nbhd $\mathcal{U} \ni 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n}(K-K) \subseteq \mathcal{U}$. Hence $Tx - x \in \mathcal{U}$ $\forall$ open nbhd $\mathcal{U}$ of 0, so $Tx = x$.

Pf of Thm: [pf where $G \subseteq B(\mathfrak{X})$, not necessarily affine isometries.]

Step 1: Can assume G is finitely generated.

Pf: Write $G = \bigcup G_i$, all G_i finitely gen. Then $K^G = \bigcap K^{G_i}$. By cptness of K and FIP, $K^{G_i} \neq \emptyset$ $\forall i$ $\Rightarrow$ $K^G \neq \emptyset$.

Step 2: We may assume for contradiction $G = \langle g_1, \ldots, g_n \rangle$ and $\exists x \in K$ s.t. $\frac{1}{n} \sum g_i(x) = x$, but $g_i(x) \neq x$ $\forall i$.

Pf: By step 1, we may assume $G = \langle g_1, \ldots, g_m \rangle$. Let $T = \frac{1}{m} \sum g_i \in B(\mathfrak{X})$. By the warmup, $\exists x \in K$ s.t. $Tx = x$. If $g_i(x) = x$ $\forall i$, we're finished. Else, relabelling, we have $g_i(x) \neq x$ $\forall 1 \leq i \leq n \leq m$ and $g_i(x) = x$ $\forall n < i \leq m$. Then

$$x = Tx = \frac{1}{m} \sum g_i(x) = \frac{1}{m} \sum_{i=1}^{n} g_i(x) + \frac{m-n}{m} x \Rightarrow \frac{n}{m} x = \frac{1}{m} \sum_{i=1}^{n} g_i(x)$$

$$\Rightarrow x = \frac{1}{n} \sum_{i=1}^{n} g_i(x).$$

Step 3: Replace K w/ the weak closed cnvx hull of $Gx \subseteq K$ (wk cpt) and $\mathfrak{X}$ by closure of $\operatorname{span}\{g_{n_1} g_{n_2} \cdots g_{n_j} x \mid n_1, \dots, n_j \in \{1, \dots, n\}\} \subseteq \mathfrak{X}$, which is a separable Banach space.

Lemma: Let $\varepsilon > 0$ s.t. $\|g_i(x) - x\| > \varepsilon$ $\forall i = 1, \dots, n$. $\exists$ a wkly cpt cnvx subset $K' \subsetneq K$ s.t. $\operatorname{diam}(K \setminus K') \le \varepsilon$.

Proof of Thm assuming the Lemma: Since K is wk closed cnvx hull of Gx and $K' \subsetneq K$ is wkly closed, cpt cnvx, $\exists h \in G$ s.t. $hx \notin K'$. Then $hx = hTx = \frac{1}{n} \sum_{i=1}^{n} hg_i(x) \notin K'$. Since K' cnvx, $\exists i \in \{1, \dots, n\}$ s.t. $hg_i(x) \notin K'$, so $hx, hg_i(x) \in K \setminus K'$. Since $\operatorname{diam}(K \setminus K') \le \varepsilon$, we have $\|hx - hg_i(x)\| \le \varepsilon$. Since $h \in G$ is an isometry, $\|x - g_i(x)\| \le \varepsilon$, $\Rightarrow\Leftarrow$.

Pf of the Lemma: Let $E = \partial_{ext}(K) \subseteq K$, extreme pts. By the Krein-Milman Thm, K is closed cnvx hull of E. Let $\overline{E} \subseteq K$ be the weak closure of E. Let $B = B_{\frac{\varepsilon}{3}}[0]$, closed ball of radius $\varepsilon/3$. Note B is cnvx + closed, so B is closed weakly as well since the $\|\cdot\|$ and wk top have same closed cnvx sets. Since $\mathfrak{X}$ separable, $\exists$ seq. $(y_i) \subseteq \mathfrak{X}$ s.t. $(y_i + B)$ covers $\mathfrak{X}$. Thus $(y_i + B) \cap \overline{E}$ is a cover of the wkly cpt set $\overline{E}$. By Baire category thm, $\exists i$ s.t. $(y_i + B) \cap \overline{E}$ has nonempty interior $\mathcal{U}$ in $\overline{E}$ wrt. the relative wk top. on $\overline{E}$.
Define $K_1 =$ wk closed cnvx hull of $\overline{E} \setminus \mathcal{U}$ and $K_2 =$ wk closed cnvx hull of $(y_i + B) \cap \overline{E}$. Then K_1, K_2 wk closed cnvx subsets of K. Since K is the closed cnvx hull of $E \subseteq (\overline{E} \setminus \mathcal{U}) \cup (y_i + B)$, it is the cnvx join of $K_1 + K_2$, i.e., $K = \operatorname{im} \varphi$ where $\varphi : K_1 \times K_2 \times [0,1] \to \mathfrak{X}$ by $(a,b,t) \mapsto ta + (1-t)b$.
For $\delta > 0$, let $K(\delta) = \operatorname{im} \varphi|_{K_1 \times K_2 \times [\delta, 1]}$. We claim if δ sufficiently small, we can take $K' = K(\delta)$.

(i) Each $K(\delta)$ is wk closed cnvx subset of K:

closed: φ is (wk$\times$wk$\times$std)-wk cts, so $K(\delta)$ is wk cpt, and thus wk closed.

wk top on $\mathfrak{X}$ is a TVS structure.

Cnvx: a) $\forall \delta \in (0,1]$, $\delta K_1 + (1-\delta) K_2$ is convex.

b) $\varphi(K_1 \times K_2 \times [\delta, 1]) = \varphi(K_1 \times (\delta K_1 + (1-\delta) K_2) \times [0,1])$ which is cnvx.

$\subseteq$: If $t \in [\delta, 1]$, $ta + (1-t)b = sa + (1-s)[\delta a + (1-\delta) b]$
for $s \in [0,1]$ s.t. $(1-s)(1-\delta) = (1-t) \iff t = \delta + s(1-\delta)$.

$\supseteq$: If $s \in [0,1]$, $s a_1 + (1-s)[\delta a_2 + (1-\delta) b] = ta + (1-t)b$
for $t = \delta + s - s\delta \in [\delta, 1]$ as before and $a = \frac{s a_1 + (1-s)\delta a_2}{\delta + s - s\delta} \in K_1$.

② For $\delta > 0$ sufficiently small, $\operatorname{diam}(K \setminus K(\delta)) \le \varepsilon$.

Pf: Note $K \subset B_R[0]$ for some $R > 0$. [$K \subseteq X \subseteq X^{**}$ is ptwise bdd as maps on X^* by compactness, thus uniformly bdd.] If $y, y' \in K \setminus K(\delta)$, then $\exists\, 0 \le t, t' < \delta$, $v, v' \in K_1$, and $w, w' \in K_2$ s.t. $y = tv + (1-t)w$ and $y' = t'v' + (1-t')w'$. Then

$$\|y - y'\| \le t[\|v\| + \|w\|] + t'[\|v'\| + \|w'\|] + \underbrace{\|w - w'\|}_{w, w' \in K_2 \subseteq y_i + B\text{, which has diameter } \frac{2}{3}\varepsilon.} \le 4\delta R + \tfrac{2}{3}\varepsilon$$

Now choose $\delta < \frac{\varepsilon}{12R}$.

③ For $\delta > 0$, $K(\delta) \subsetneq K$.

Pf: Since $\mathcal{U} \subseteq \overline{E}$ is a nonempty open subset, $\exists y \in E \cap \mathcal{U}$. [Let $u \in \mathcal{U} \subseteq \overline{E}$. If if $u \notin E$, $u \in \partial E$, so $\exists \varepsilon > 0$ s.t. $B_\varepsilon(u) \subseteq \mathcal{U}$. But $B_\varepsilon(u) \cap E \ne \emptyset$.] We claim that $y \notin K(\delta)$. Since $y \in E$ is an extreme pt. of K, it suffices to show $y \notin K_1$. [If $y \notin K_1$ and $y = tv + (1-t)w$ for $v \in K_1$ and $w \in K_2$, we must have $t = 0$. Thus since y is extreme, $v = w = y$. Since $v \in K_1$, y cannot be written as $tv + (1-t)w$ for $t \in [\delta, 1]$, $v \in K_1$, $w \in K_2$.]
Since $(X, \text{wk top})$ is loc. cnvx, $\exists$ a wkly open cnvx nbhd $V \subseteq X$ s.t. the wk closure $\overline{V}$ satisfies $(y - \overline{V}) \cap \overline{E} \subseteq \mathcal{U}$. [Can use $\overline{E}$ wkly cpt, so $\overline{E}$ is wkly normal.] Since $\overline{E} \setminus \mathcal{U}$ is wkly cpt, it admits a wkly open cover $\{z_i + V\}_{i=1}^k$ where each $z_i \in \overline{E} \setminus \mathcal{U}$. Thus K_1 is contained in the closed cnvx hull of $\bigcup_{i=1}^k (z_i + V) \cap \overline{E} \supseteq \overline{E} \setminus \mathcal{U}$. In turn, $\bigcup_{i=1}^k (z_i + V) \cap \overline{E}$ is contained in the cnvx join of the $(z_i + \overline{V}) \cap K$.
If $y \in K_1$, $y \in (z_i + \overline{V}) \cap K$ for some i. Then $z_i \in (y - \overline{V}) \cap \overline{E} \subseteq \mathcal{U}$, a contradiction to $z_i \in \overline{E} \setminus \mathcal{U}$.