A C^*-algebra A is an involutive Banach algebra s.t.

- $\|a^*a\| = \|a\|^2 \quad \forall a \in A$. ($C^*$-axiom)

- $(\lambda a + bc)^* = \bar{\lambda} a^* + c^* b^*$
- $a^{**} = a \quad \forall a$

Every unital commutative C^*-alg is of the form $C(X)$ for some cpt Hausdorff space X. Moreover, there is a contravariant equivalence of categories

$$\left\{ \begin{array}{c} \text{Cpt Hausd Top Spaces w/} \\ \text{Cts maps} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{unital comm. } C^*\text{-alg's w/} \\ \text{unital } *\text{-homomorphisms.} \end{array} \right\}$$

There is an analogous statement for nonunital C^*-alg's, but must use locally cpt Hausd. spaces and proper maps

Spectral Thm 1: Let A be a unital C^*-alg and $a \in A$ normal ($aa^* = a^*a$). There is a canonical (isometric) unital $*$-iso $C(sp(a)) \cong C^*(a) \subseteq A$. $[sp(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \text{ not inv.}\}]$

Continuous functional calculus:

for any $f \in C(sp(a))$, get $f(a) \in A$ corresponding to f.

Properties:

① CFC extends the holomorphic FC defined for a unital Banach Alg

② (Spectral mapping) $f(sp(a)) = sp(f(a))$.

③ If $g \in C(sp(f(a)))$, $g(f(a)) = (g \circ f)(a)$.

④ If $a_n \to a$ are normal, $\mathcal{U} \supset sp(a)$ is a nbhd, then eventually $sp(a_n) \subset \mathcal{U}$. Moreover, $\forall f \in C(\mathcal{U})$, $f(a_n) \to f(a)$ in A.

Using the CFC in a crucial way, we saw a (unital) C^*-alg has sufficiently many states (pos. linear ftl's of norm 1), and we used this to prove:

Gelfand-Naimark Thm: Every (unital) C^*-alg is (isometrically) $*$-iso to a norm-closed $*$-subalg of $B(H)$ for some Hilbert space H.

Our goal is to extend the CFC to the L^∞-Functional Calculus.

First, we'll need to discuss Operator topologies on $B(H)$: [dualities]

① The weak operator topology is induced by the separating family of seminorms $\{x \mapsto |\langle x\eta,\xi\rangle| \mid \eta,\xi\in H\}$.

- $x_\lambda \to x$ WOT $\iff \langle x_\lambda \eta,\xi\rangle \to \langle x\eta,\xi\rangle \ \forall \eta,\xi\in H$.

② The strong operator topology is induced by the sep. family of seminorms $\{x\mapsto \|x\xi\| \mid \xi\in H\}$.

- $x_\lambda\to x$ SOT $\iff x_\lambda\xi \to x\xi \ \forall \xi\in H$ [pointwise convergence]

More to come!

WOT open $\Rightarrow$ SOT open $\Rightarrow$ norm open.

Since $\|\langle x\eta,\xi\rangle\| \le \|x\eta\|\cdot\|\xi\| \le \|x\|\cdot\|\eta\|\cdot\|\xi\|$, WOT $\le$ SOT $\le$ Norm

Since $\langle x\eta,\xi\rangle = \langle \eta, x^*\xi\rangle$, $*$ is WOT-cts.

Note: $*$ is not SOT cts. $(|e_n\rangle\langle e_1|)_{n\in\mathbb{N}}$ does not converge SOT, but $(|e_1\rangle\langle e_n|)_{n\in\mathbb{N}}$ converges to 0 SOT.

$*$ is SOT cts when restricted to normal ops (not a subspace), since x normal $\iff \|x\xi\| = \|x^*\xi\| \ \forall\xi\in H$.

Note: Multiplication is separately WOT/SOT cts in each variable, but not jointly cts WOT/SOT.

Lemma: $N=\{x\in B(H)\mid x^2=0\}$ is SOT dense in $B(H)$.

Pf: The sets $\{x\in B(H) \mid \|(x-x_0)\xi_i\|<\varepsilon, i=1,\dots,n\}$ where $x_0\in B(H)$ and ξ_i's are lin. indep form a base for SOT. We'll show each of these contains an elt. of N. Choose $\eta_1,\dots,\eta_n\in H$ s.t. $K=\{\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n\}$ lin. indep and $\|x_0\xi_i-\eta_i\|<\varepsilon \ \forall i$. Defining $x\xi_i=\eta_i$, $x\eta_i=0$, $x=0$ on $K^\perp$ works.

Lemma: Mult. is jointly SOT cts on $B_r(0)\times B(H)$ $\forall r>0$. In particular, mult. is jointly SOT cts on bdd sets.

Pf:

$$\|(xy-x_0y_0)\xi\| = \|(xy - xy_0 + xy_0 - x_0y_0)\xi\|$$
$$\le \underbrace{\|x\|}_{\text{bdd}}\cdot\underbrace{\|(y-y_0)\xi\|}_{y\to y_0 \text{ SOT}} + \underbrace{\|(x-x_0)\overbrace{y_0\xi}^{\text{one }\vec{\ }\text{ in }H}\|}_{x\to x_0 \text{ SOT}}$$

Prop: For $\varphi \in B(H)^*$, TFAE:

1. $\exists\, \xi_1, \dots, \xi_n ; \eta_1, \dots, \eta_n \in H$ s.t. $\varphi(x) = \sum_{i=1}^n \langle x\eta_i, \xi_i\rangle$ $\forall x \in B(H)$.
2. φ WOT-cts.
3. φ SOT-cts.

Pf: $1 \Rightarrow 2 \Rightarrow 3$: clear.

$3 \Rightarrow 1$: Suppose φ SOT cts. Then $\varphi^{-1}(B_1^c(0))$ is SOT open, and thus $\exists\, \eta_1, \dots, \eta_n \in H$ s.t. $\max_{i=1,\dots,n} \|x\eta_i\| < 1 \Rightarrow |\varphi(x)| < 1$.

Consider the three seminorms $p(x) := |\varphi(x)|$, $m(x) = \left[\sum_{i=1}^n \|x\eta_i\|^2\right]^{1/2}$, and $M(x) = \max_{i=1,\dots,n} \|x\eta_i\|$. We know

1. $[M(x) < 1 \Rightarrow p(x) < 1] \iff p(x) \leq M(x)$!!
2. $M(x) \leq m(x)$.

Hence $\forall x \in B(H)$, $|\varphi(x)|^2 \leq \sum_{i=1}^n \|x\eta_i\|^2$.

Consider $\eta = (\eta_i)_{i=1}^n \in H^n$, and let $H_0 = \overline{\{x\eta := (x\eta_i)_{i=1}^n \mid x \in B(H)\}} \subseteq H^n$.

amplified rep'n of $B(H)$ on H^n.

Define ψ on H_0 by $\psi(x\eta) = \varphi(x)$. By above,
$|\psi(x\eta)| \leq \left[\sum_{i=1}^n \|x\eta_i\|^2\right]^{1/2} = \|x\eta\|_{H_0}$, so $\psi \in H_0^*$. Using HB, can extend ψ to all of H^n, so by Riesz-Rep thm, $\exists\, \xi_1, \dots, \xi_n \in H$ s.t. $\psi = \langle \cdot, \xi\rangle$ w/ $\xi = (\xi_i)_{i=1}^n$. Thus $\forall x \in B(H)$,
$$\varphi(x) = \psi(x\eta) = \langle x\eta, \xi\rangle = \langle (x\eta_i)_{i=1}^n, (\xi_i)_{i=1}^n\rangle = \sum_{i=1}^n \langle x\eta_i, \xi_i\rangle.$$

Note: In the above pf, we used the amplified rep'n of $B(H)$ on H^n given by $x(\eta_i)_{i=1}^n := (x\eta_i)_{i=1}^n$.

Cor: Every SOT closed cvx set is WOT closed.

Def: For $S \subseteq B(H)$, define $S' = \{x \in B(H) \mid xs = sx \ \forall s \in S\}$. a commutant

Exercises:
1. $S \subseteq T \Rightarrow T' \subseteq S'$.
2. $S \subseteq S''$.
3. $S' = S'''$.

Exercises:
1. Find an (isometric) $*$-iso $B(\bigoplus_{i=1}^n H) \cong M_n(B(H))$.
2. Show $M_n(S)' = \{x \in M_n(S') \mid x \text{ is constant along the diagonal}\}$.
3. Show $\{y \in M_n(S) \mid y \text{ constant along the diagonal}\}' = M_n(S')$.

Warmup Thm: If $M \subseteq M_n(\mathbb{C})$ is a unital $*$-closed subalg, then $M = M''$.

Pf: It suffices to prove $y \in M'' \Rightarrow y \in M$. Fix $y \in M''$. Consider the amplified action of M on $\mathbb{C}^n \otimes \mathbb{C}^n$ by $x(\eta \otimes \xi) = x\eta \otimes \xi$. Letting (e_i) be the std ONB of $\mathbb{C}^n$, get a unitary iso $\mathbb{C}^n \otimes \mathbb{C}^n \cong \bigoplus_{i=1}^n \mathbb{C}^n$ by $\eta \otimes \xi \xleftrightarrow{\text{unitary}} \begin{bmatrix} \xi_1 \eta \\ \vdots \\ \xi_n \eta \end{bmatrix}$ where $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{C}^n$.

[Better: $H \otimes K \cong \bigoplus_{i=1}^{\dim K} H$ as Hilb. spaces]

It is straightforward to calculate that $B(\bigoplus_{i=1}^n \mathbb{C}^n) \cong M_n[M_n(\mathbb{C})]$. Now under this unitary iso, the amplified action of M corresponds to the diagonal matrices which are a single elt of M along the diagonal.

$$\begin{tikzcd}
\eta \otimes \xi \arrow[r, leftrightarrow] \arrow[d, mapsto, "x \otimes 1"'] & \begin{bmatrix} \xi_1 \eta \\ \vdots \\ \xi_n \eta \end{bmatrix} \arrow[d, mapsto] \\
x\eta \otimes \xi \arrow[r, leftrightarrow] & \begin{bmatrix} \xi_1 x \eta \\ \vdots \\ \xi_n x \eta \end{bmatrix} = \underline{\begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix} \begin{bmatrix} \xi_1 \eta \\ \vdots \\ \xi_n \eta \end{bmatrix}}
\end{tikzcd}$$

Now by the exercise, $(\text{amplified } M)' \cap M_n(M_n(\mathbb{C})) = M_n(M')$ and $(\text{amplified } M)'' = M_n(M')' = \{x \in M_n(M'') \mid x \text{ constant along diagonal}\}$.

Consider the vector $\xi := (e_i)_{i=1}^n \in \bigoplus_{i=1}^n \mathbb{C}^n$ [std basis elts in each coord.]

Let $V = M\xi \subseteq \bigoplus_{i=1}^n \mathbb{C}^n$. Then $MV \subseteq V$ as M is an algebra. Since $M = M^*$, P_V commutes w/ M, i.e., $P_V \in (\text{amplified } M)' = M_n(M')$. So if $y \in M''$, $\operatorname{diag}(y, \ldots, y) \in M_n(M')'$ commutes w/ P_V, and thus $\boxed{yM\xi = M\xi!}$ Since $1 \in M$, $\exists x \in M$ s.t. $y(1\xi) = x\xi$. In particular, $ye_i = xe_i$ $\forall i = 1, \ldots, n$, so $y = x \in M$.

Thm (von Neumann): For $M \subseteq B(H)$ a unital $*$-subalgebra, $M'' = \overline{M}^{SOT}$.

Pf: Commutants are WOT-closed, since if $(x_\lambda) \subset M$ and $x_\lambda \to x$ WOT, $\forall y \in M'$, $\forall \eta, \xi \in H$, $\langle xy\eta, \xi\rangle \leftarrow \langle x_\lambda y \eta, \xi \rangle = \langle y x_\lambda \eta, \xi \rangle \to \langle yx\eta, \xi\rangle$.

Now since M'' cvx, it is SOT-closed, and $\overline{M}^{SOT} \subseteq M''$.

Now suppose $y \in M''$, and consider $\{x \in B(H) \mid \|(x-y)\xi_i\| < \varepsilon \quad i=1,\dots,n\}$, a basic SOT-open nbhd of y. We want to show M intersects this nbhd nontrivially. Let $\xi = (\xi_i)_{i=1}^n \in \bigoplus_{i=1}^n H$, and consider the amplified rep'n of $B(H)$ on $\bigoplus_{i=1}^n H$. Let $V = \overline{M\xi} \subseteq H^n$, which is M-invariant. Hence $P_V \in (\text{amplified } M)' = M_n(M')$.

Then $\operatorname{diag}(y,\dots,y) \in M_n(M')'$ commutes w/ P_V. Since $1 \in M$, $y\xi = (y\xi_i)_{i=1}^n \in V = \overline{M\xi}$, so $\exists x \in M$ s.t. $\|x\xi_i - y\xi_i\| < \varepsilon \quad \forall i=1,\dots,n$.

<u>Cor (von Neumann bicommutant thm)</u>: If $M \subseteq B(H)$ is a unital $*$-subalg, TFAE:

1. $M = M''$
2. M is WOT-closed
3. M is SOT-closed

Such a unital $*$-subalg of $B(H)$ is called a <u>von Neumann</u> algebra.

<u>Examples</u>:

1. $M_n(\mathbb{C}) \cong B(H)$ for $\dim H = n$.
2. Any finite dim'l unital $*$-subalg. of $B(H)$.
3. $B(H)$ itself
4. $L^\infty(X,\mu)$ for a σ-finite measure space (X,μ). (Hw 1!)
5. If $S \subseteq B(H)$, $(S \cup S^*)''$ is a vNa.
6. Let Γ be a discrete gp. Let $H = \ell^2\Gamma = \{\xi : \Gamma \to \mathbb{C} \mid \sum_g |\xi(g)|^2 < \infty\}$ w/ inner product $\langle \eta, \xi \rangle = \sum_g \eta(g)\overline{\xi(g)}$. An ONB is given by $\{\delta_g : h \mapsto \delta_{g=h} \mid g \in \Gamma\}$. $\forall g \in \Gamma$, define $\lambda_g \in \mathcal{U}(\ell^2\Gamma)$ by $(\lambda_g \xi)(h) = \xi(g^{-1}h)$. Then $\lambda_g \lambda_h = \lambda_{gh}$. Map $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ is called the <u>left regular rep'n</u>. $\operatorname{Span}_{\mathbb{C}} \lambda\Gamma \cong \mathbb{C}\Gamma$, gp alg. $\overline{\lambda\Gamma}^{\|\cdot\|} =: C^*_r\Gamma$, reduced gp C^*-alg. $L\Gamma = (\lambda\Gamma)''$ <u>gp vNa</u>.

<u>Open Problem</u>: Is $L\mathbb{F}_2 \cong L\mathbb{F}_3$?

Kaplansky Density Thm: Let $M \subseteq B(H)$ be a $*$-subalg.

(1) The unit ball of $M_{s.a.}$ is SOT-dense in the unit ball of $(\overline{M}^{SOT})_{s.a.}$

(2) The unit ball of M_+ is SOT-dense in the unit ball of $(\overline{M}^{SOT})_+$.

(3) The unit ball of M is SOT-dense in the unit ball of $\overline{M}^{SOT}$.

Note: Also true for $\mathcal{U}(M)$ and $\mathcal{U}(M'')$ when M a unital C^*-alg, but this uses the $L^\infty FC$, so we'll skip it for now.

Step 1: If $p \in \mathbb{C}[z, \bar{z}]$ is a poly in z and $\bar{z}$, $x \mapsto p(x)$ is SOT-cts on bdd sets of normal ops in $B(H)$.

Pf: Mult is SOT-cts on bdd sets, and $*$ is SOT-cts on normal ops.

Note: Step 1 holds on bdd sets of non-normal ops for noncommutative poly's $p \in \mathbb{C}\langle z, \bar{z}\rangle$ in z and $\bar{z}$.

Step 2: If $f \in C(\mathbb{C})$, then $x \mapsto f(x)$ is SOT-cts on bdd sets of normal ops in $B(H)$.

Pf: Suppose (x_λ) is a bdd net of normal ops w/ $x_\lambda \to x$ SOT. Then x is normal, and $\exists R > 0$ s.t. $sp(x), sp(x_\lambda) \subseteq B_R(0)$. Then $f|_{B_R(0)}$ can be uniformly approx by poly's in $z, \bar{z}$. The result now follows from Step 1.

[Let $\xi \in H$. Let $\varepsilon > 0$. Pick $p \in \mathbb{C}[z, \bar{z}]$ unif. approx f w/i $\frac{\varepsilon}{3\|\xi\|}$. Pick λ_0 so that $\lambda \geq \lambda_0 \Rightarrow \|(p(x) - p(x_\lambda))\xi\| < \frac{\varepsilon}{3}$. Then $\forall \lambda \geq \lambda_0$,

$$\|(f(x) - f(x_\lambda))\xi\| \leq \|(f(x) - p(x))\xi\| + \|(p(x) - p(x_\lambda))\xi\| + \|(p(x_\lambda) - f(x_\lambda))\xi\|$$
$$\leq \frac{\varepsilon}{3\|\xi\|}\cdot\|\xi\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\|\xi\|}\cdot\|\xi\| = \varepsilon. \,\rfloor$$

Step 3: (Cayley Transform) The map $x \mapsto (x-i)(x+i)^{-1}$ is SOT-cts from $B(H)_{s.a.} \to \mathcal{U}(H)$.

Pf: Suppose (x_λ) is a net of s.a. ops w/ $x_\lambda \to x$ SOT (so x is s.a.). By the spectral mapping thm, $sp[(x_\lambda + i)^{-1}] = [\underbrace{sp(x_\lambda)}_{\subseteq \mathbb{R}} + i]^{-1} \subseteq \overline{B_1(0)}^{c}$. Since $(x_\lambda + i)^{-1}$ is normal, $\|(x_\lambda + i)^{-1}\| = r[(x_\lambda+i)^{-1}] \leq 1$. Hence $\forall \xi \in H$,

$$\|(x-i)(x+i)^{-1}\xi - (x_\lambda - i)(x_\lambda+i)^{-1}\xi\| = \|(x_\lambda+i)^{-1}[\underbrace{(x_\lambda+i)(x-i) - (x_\lambda - i)(x+i)}_{2i(x-x_\lambda)}](x+i)^{-1}\xi\|$$
$$\leq 2\|\underbrace{(x - x_\lambda)}_{\to 0 \text{ SOT}}\underbrace{(x+i)^{-1}\xi}_{\text{one } \vec{v}}\| \to 0$$

Note: The Cayley transform is a Möbius transformation which sends $\mathbb{R} \to S^1$, since $\frac{t-i}{t+i}\cdot\frac{t-i}{t-i} = \frac{(t-i)^2}{t^2+1} = \frac{t^2-1}{t^2+1} - i\frac{2t}{t^2+1}$,

and $(t^2-1)^2+(2t)^2 = (t^2+1)^2$. Alternatively, a Möbius transformation must map $\mathbb{R}$ to a line or circle in $\mathbb{C}$, and we calculate:

$$\left.\begin{aligned} 0 &\mapsto \frac{-i}{i} = -1, \\ 1 &\mapsto \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{-2i}{2} = -i \\ -1 &\mapsto \frac{-1-i}{-1+i} = \frac{(-1-i)^2}{2} = \frac{2i}{2} = i \end{aligned}\right\} \text{ in } S^1.$$

By Spectral mapping thm, $(x-i)(x+i)^{-1}$ has spectrum in S^1 and is normal, and thus it is a unitary.

The inverse of the Cayley transform is $t \mapsto i\frac{1+t}{1-t}$.

[The inverse of the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ $ad-bc\neq 0$ is given by $z \mapsto \frac{dz-b}{-cz+a}$. Cayley is $a=c=1$, $b=-i$, $d=i$.]

Step 4: $\forall f \in C_0(\mathbb{R})$, $x \mapsto f(x)$ is SOT-cts on s.a. ops.

Pf: Let $f \in C_0(\mathbb{R})$. Define $g: S^1 \to \mathbb{C}$ by $g(t) = \begin{cases} f(i\frac{1+t}{1-t}) & t\neq 1 \\ 0 & t=1 \end{cases}$

Then $g = f\circ C^{-1}$ where C^{-1} is inverse of Cayley Trans.

By Step 2, g is SOT-cts on $\mathcal{U}(H)$. Now $f = g\circ C$, and C is SOT-cts $B(H)_{s.a.} \to \mathcal{U}(H)$ by Step 3. Hence f is SOT-cts as a composite of 2 SOT-cts maps.

Step 5: For Kaplansky Density Thm, we may assume M is a C^*-alg. ($\|\cdot\|$-closed.)

Pf: We prove:

(3') unit ball of M is $\|\cdot\|$-dense in unit ball of $\overline{M}^{\|\cdot\|}$.

[If $x \in \overline{M}^{\|\cdot\|}$ w/ $\|x\|\leq 1$, pick $(x_n)\subset M$ w/ $x_n \to x$ in $\|\cdot\|$. Then $\|x_n\| \to \|x\| \leq 1$, so passing to a subseq., we may assume $\|x_n\| \leq 1+\frac{1}{n}$. Then $\frac{n}{n+1}x_n \to x$, and $\|\frac{n}{n+1}x_n\| \leq \frac{n}{n+1}(\frac{n+1}{n}) = 1$.]

(1') unit ball of $M_{s.a.}$ is $\|\cdot\|$-dense in unit ball of $(\overline{M}^{\|\cdot\|})_{s.a.}$.

[If $x \in (\overline{M}^{\|\cdot\|})_{s.a.}$ w/ $\|x\|\leq 1$, pick $(x_n)\subset M$ w/ $\|x_n\|\leq 1$ and $x_n \to x$ as in (3'). Then $\frac{x_n+x_n^*}{2} \to x$ in $\|\cdot\|$, and this seq. is in unit ball of $M_{s.a.}$.]

(2') unit ball of M_+ is $\|\cdot\|$-dense in unit ball of $(\overline{M}^{\|\cdot\|})_+$

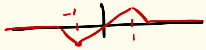
[If $x \in (\overline{M}^{\|\cdot\|})_+$ w/ $\|x\| = 1$, write $x = y^*y$, so $\|y\|\leq 1$. Pick a seq. $(y_n) \subseteq M$ w/ $\|y_n\|\leq 1$ as in (3'), w/ $y_n \to y$. Then $y_n^*y_n \to y^*y = x$ and $\|y_n^*y_n\|\leq 1$.]

Finally, we note that SOT-closed $\Rightarrow$ $\|\cdot\|$-closed, so $\overline{M}^{\|\cdot\|} \subseteq \overline{M}^{SOT}$, and $x_n \to x$ in $\|\cdot\|$ $\Rightarrow$ $x_n \to x$ SOT. Hence if R is $\|\cdot\|$-dense in S and S is SOT-dense in T, then R is SOT-dense in T.

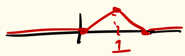
[Suppose $t \in T$, and consider $V = \{x \in T \mid \|(x-t)\xi_i\| < \varepsilon,\ i=1,\dots,n\}$. Since S is SOT-dense in T, $\exists\, s \in S \cap V$. Now $S \cap V$ is $\|\cdot\|$-open in S, so $\exists\, r \in R \cap S \cap V$. Hence $\exists\, r \in R \cap V$.]

Pf of KDT: By Step 5, we assume M is a C^*-alg.

(1) First, we'll show $M_{s.a.}$ is dense in $(\overline{M}^{SOT})_{s.a.}$. Suppose $x \in (\overline{M}^{SOT})_{s.a.}$. Let $(x_\lambda) \subset M$ s.t. $x_\lambda \to x$ SOT. Then $x_\lambda \to x$ WOT. Since $*$ is WOT-cts, $x_\lambda^* \to x^* = x$ WOT, so $\frac{x_\lambda + x_\lambda^*}{2} \to x$ WOT. Hence $M_{s.a.}$ is WOT-dense in $(\overline{M}^{SOT})_{s.a.}$, but $M_{s.a.}$ is cvx, so $\overline{M_{s.a.}}^{SOT} = \overline{M}^{WOT} = (\overline{M}^{SOT})_{s.a.}$

Now suppose $\|x\| \le 1$ in addition. By above, $\exists\, (x_\lambda) \subset M_{s.a.}$ w/ $x_\lambda \to x$ SOT. Consider $f \in C_0(\mathbb{R})$ s.t. $f(t) = t$ $\forall t \in [-1,1]$, and $\|f\|_\infty = 1$, e.g. 

. By Step 4, $f(x_\lambda) \to f(x) = x$ SOT. But by spectral mapping, $\|f(x_\lambda)\| \le 1$ $\forall \lambda$.

(2) Suppose $x \in (\overline{M}^{SOT})_+$ w/ $\|x\| \le 1$. By (1), $\exists\, (x_\lambda) \subset (\overline{M}^{SOT})_{s.a.}$ w/ $\|x_\lambda\| \le 1$ $\forall \lambda$ and $x_\lambda \to x$ SOT. Let $f \in C_0(\mathbb{R})$ s.t. $f = 0$ on $(-\infty, 0]$, $f(t) = t$ $\forall t \in [0,1]$, and $\|f\|_\infty = 1$, e.g., 

. By Step 4, $f(x_\lambda) \to f(x) = x$ SOT, and by spectral mapping, $\|f(x_\lambda)\| \le 1$ and $f(x_\lambda) \ge 0$.

(3) First, we prove $M_2(M)$ is SOT-dense in $M_2(\overline{M}^{SOT})$ on $H \oplus H$. Suppose $(x_{ij}) \in M_2(\overline{M}^{SOT})$, and let $(x_{ij}^\lambda) \subset M$ s.t. $x_{ij}^\lambda \to x_{ij}$ SOT. Then

$$\forall \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \in H^2,\ \begin{bmatrix} x_{11}^\lambda & x_{12}^\lambda \\ x_{21}^\lambda & x_{22}^\lambda \end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = \begin{bmatrix} x_{11}^\lambda \xi_1 + x_{12}^\lambda \xi_2 \\ x_{21}^\lambda \xi_1 + x_{22}^\lambda \xi_2 \end{bmatrix} \longrightarrow \begin{bmatrix} x_{11}\xi_1 + x_{12}\xi_2 \\ x_{21}\xi_1 + x_{22}\xi_2 \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}$$

Now suppose $x \in (\overline{M}^{SOT})$ w/ $\|x\| \le 1$. Then $\begin{bmatrix} 0 & x \\ x^* & 0\end{bmatrix} \in M_2(\overline{M}^{SOT})$ is s.a., so $\exists \left(\begin{bmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{bmatrix}\right) \in M_2(M)$ w/ $\left\|\begin{bmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{bmatrix}\right\| \le 1$ s.t. $\begin{bmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{bmatrix} \to \begin{bmatrix} 0 & x \\ x^* & 0\end{bmatrix}$ SOT.

Then $\|b_\lambda\| \le 1$, and $b_\lambda \to x$ SOT.

[$\forall \eta, \xi \in H$ w/ $\|\eta\| = \|\xi\| = 1$, $\left\|\begin{bmatrix}0\\ \eta\end{bmatrix}\right\| = \left\|\begin{bmatrix}\xi\\ 0\end{bmatrix}\right\| = 1$, so $\left|\left\langle \begin{bmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{bmatrix}\begin{bmatrix}0\\ \eta\end{bmatrix}, \begin{bmatrix}\xi\\ 0\end{bmatrix}\right\rangle\right| =$

$= |\langle b_\lambda \eta, \xi\rangle| \le 1$ by C.S. on H^2. We conclude $\|b_\lambda\| \le 1$.

Finally, $\forall \xi \in H$, $\begin{bmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{bmatrix}\begin{bmatrix}0\\ \xi\end{bmatrix} = \begin{bmatrix} b_\lambda \xi \\ d_\lambda \xi\end{bmatrix} \to \begin{bmatrix} 0 & x \\ x^* & 0\end{bmatrix}\begin{bmatrix}0\\ \xi\end{bmatrix} = \begin{bmatrix} x\xi \\ 0\end{bmatrix}$, so $b_\lambda \xi \to x\xi$.]

Predual: By your HW, $B(H) = \mathcal{L}^1(H)^*$, implemented by Tr.

Def: The σ-wot is the weak* topology on $B(H) = \mathcal{L}^1(H)^*$.

Prop: For a functional φ on $B(H)$, TFAE:

1. $\exists (\eta_n), (\xi_n) \subseteq H$ w/ $\sum \|\eta_n\|^2, \sum \|\xi_n\|^2 < \infty$ s.t. $\varphi(x) = \sum \langle x\eta_n, \xi_n \rangle$
2. Same as 1 w/ $(\eta_n), (\xi_n)$ seq. of pairwise $\perp$ $\vec{v}$'s.
3. $\exists t \in \mathcal{L}^1(H)$ trace-class s.t. $\varphi(x) = \mathrm{Tr}(xt)$ $\forall x$
4. φ is σ-wot cts.

Pf: By a HW problem last semester, $3 \Leftrightarrow 4$. It is clear $2 \Rightarrow 1$.

$1 \Rightarrow 3$: Let $H_0 = \mathrm{span}\{\eta_n, \xi_k\} \subseteq H$, and let (e_n) be an ONB for H_0.

If $\dim H_0 < \infty$, we may express each η_n, ξ_n as a finite linear comb. of the e_n's to obtain scalars λ_{ij} s.t.

$$\varphi(x) = \sum \lambda_{ij} \langle x e_i, e_j \rangle = \sum \langle x t e_j, e_j \rangle = \mathrm{Tr}(xt)$$

where $t = \sum \lambda_{ij} |e_i\rangle\langle e_j|$ is finite rank.

If $\dim H_0 = \infty$, define $t_1, t_2 \in B(H)$ by $t_i|_{H_0^\perp} = 0$ and

$t_1 = \sum |\eta_n\rangle\langle e_n|$ and $t_2 = \sum |\xi_n\rangle\langle e_n|$, which are bdd.

Then $\mathrm{Tr}(t_1^* t_1) = \sum \langle t_1^* t_1 e_n, e_n \rangle = \sum \|\eta_n\|^2 < \infty$, and similarly $\mathrm{Tr}(t_2^* t_2) = \sum \|\xi_n\|^2 < \infty$, so $t_1, t_2 \in \mathcal{L}^2(H)$.

Hence $t = t_1 t_2^* \in \mathcal{L}^1(H)$ $\boxed{\text{HW!}}$, and

$$\mathrm{Tr}(xt) = \mathrm{Tr}(x t_1 t_2^*) = \mathrm{Tr}(t_2^* x t_1) = \sum \langle x t_1 e_n, t_2 e_n \rangle$$
$$= \sum \langle x \eta_n, \xi_n \rangle = \varphi(x).$$

$3 \Rightarrow 2$: Let $t = u|t|$ be polar decomposition. Then $|t|$ is positive cpt op., so $\exists$ ON seq. (e_n) and $(\lambda_n) \subset [0,\infty)$ w/ $\lambda_n \to 0$ s.t. $|t| = \sum \lambda_n |e_n\rangle\langle e_n|$.

Since $|t| \in \mathcal{L}^1(H)$, $\sum \lambda_n < \infty$. Define $\eta_n = \lambda_n^{1/2} u e_n$, $\xi_n = \lambda_n^{1/2} e_n$. Then (η_n) are pairwise $\perp$ as u is a p.i. s.t. $u^* u e_n = e_n$. Obviously (ξ_n) are pairwise $\perp$. Finally, $\forall x \in B(H)$,

$$\varphi(x) = \mathrm{Tr}(xt) = \sum \langle x t e_n, e_n \rangle = \sum \langle x \lambda_n u e_n, e_n \rangle$$
$$= \sum \langle x \eta_n, \xi_n \rangle.$$

Cor: If φ is a σwot cts ftl on $B(H)$ and $\varphi \geq 0$, $\varphi = \sum \langle \cdot\, \xi_n, \xi_n \rangle$ for some ON sequence (ξ_n) w/ $\sum \|\xi_n\|^2 < \infty$.

Pf: By the Prop., $\varphi = \mathrm{Tr}(\cdot\, t)$ for some $t \in \mathcal{L}^1(H)$. $\forall \xi \in H$, we have

$\langle t\xi, \xi\rangle = \mathrm{Tr}(|\xi\rangle\langle\xi|\, t) = \varphi(|\xi\rangle\langle\xi|) \geq 0$, so $t \geq 0$.

Hence $\exists$ ON seq. (e_n) s.t. $t = \sum \lambda_n |e_n\rangle\langle e_n|$ w/ $(\lambda_n) \subset [0,\infty)$, and $\sum \lambda_n < \infty$. Setting $\xi_n = \lambda_n^{1/2} e_n$ works.

Cor: The unit ball of $B(H)$ is σ-WOT cpt.

Pf: Immediate from Banach-Alaoglu and σ-WOT = wk* top.

Prop: On bdd subsets of $B(H)$, σ-WOT = WOT. In particular, the unit ball of $B(H)$ is WOT-cpt.

Pf: The identity map $(B(H), \sigma\text{-WOT}) \longrightarrow (B(H), \text{WOT})$ is cts and injective. Restricting to the unit ball of $B(H)$, we get a cts bij. from a cpt Hausd. space to a Hausd. space, which must be a homeomorphism.

Remark: The σ-WOT is the same as the pullback of the WOT of $\alpha(B(H))$ where $\alpha : B(H) \longrightarrow B(\bigoplus_{n=1}^{\infty} H)$ is the countably infinite amplification.

Def: The σ-SOT is the pull back of the SOT of $\alpha(B(H))$ where α is as in the Remark. That is, we have $x_\lambda \longrightarrow x$ σ-SOT $\iff \forall (\xi_n) \subset H$ w/ $\sum \|\xi_n\|^2 < \infty$,

$$\sum \|(x - x_\lambda)\xi_n\|^2 \longrightarrow 0.$$

Remark: we have

$$\begin{array}{ccccc} \sigma\text{-WOT} & \leq & \sigma\text{-SOT} & \leq & \text{Norm}. \\ \vee\!\shortmid & & \vee\!\shortmid & & \\ \text{WOT} & \leq & \text{SOT} & & \end{array}$$

Exercise: A linear ftl φ on $B(H)$ is σ-WOT cts $\iff$ σ-SOT cts.

Exercise: For a unital $*$-subalg $M \subseteq B(H)$, TFAE:

1. $M = M''$
2. M is closed σ-WOT
3. M is closed σ-SOT.

Exercise: On bdd subsets of $B(H)$, σ-SOT = SOT.

Thm: Let $M \subseteq B(\mathcal{H})$ be a vNa. There is a Banach space M_* s.t. M is isometrically iso to $(M_*)^*$. Moreover, the σ-WOT topology on M is the wk* top.

Pf: Identify $B(\mathcal{H}) = \mathcal{L}^1(\mathcal{H})^*$. Consider the preannihilator

$$M_\perp = \{ t \in \mathcal{L}^1(\mathcal{H}) \mid \operatorname{Tr}(xt) = 0 \ \forall x \in M \}.$$

Then $M_\perp \subseteq \mathcal{L}^1(\mathcal{H})$ is a $\|\cdot\|_1$-closed subspace, and we define $M_* = \mathcal{L}^1(\mathcal{H})/M_\perp$. Since $M \subseteq B(\mathcal{H})$ is $\underset{= \text{wk}*}{\sigma\text{-WOT}}$ closed, it follows that $M = (M_\perp)^\perp$. Considering the canonical surjection $Q: \mathcal{L}^1(\mathcal{H}) \to M_*$, we may identify $Q^*: (M_*)^* \hookrightarrow \mathcal{L}^1(\mathcal{H})^* = B(\mathcal{H})$ as an isometric embedding onto M.

Remark: M_* is called the predual, and is ! up to isometric iso. (Thm of Sakai). He also showed a C^*-alg is a vNa $\Leftrightarrow$ it has a predual.

On to the Spectral Thm + L^∞FC:

Start w/ an example: cpt normal ops.

Let $T \in B(\mathcal{H})$ be cpt and normal. Recall that $\exists$ an ONB of eigenvectors (e_i) for T, and that the corresp. e.val's $\underbrace{\lambda_i \to 0}_{\text{countably many nonzero}}$.

We can think of this in several equivalent ways:

① $\underbrace{T = \sum \lambda_i |e_i\rangle\langle e_i|}_{\text{converges in } \|\cdot\|_{op}}$ where $|\eta\rangle\langle\xi|$ is a rank one op $\zeta \mapsto \langle \zeta, \xi \rangle \eta$.

② $T = \underbrace{\sum_{\lambda \in sp(T)\setminus\{0\}} \lambda E_\lambda}_{\text{converges in } \|\cdot\|_{op}}$ where E_λ is the (finite rank) projection onto the eigenspace assoc. to λ.

③ $T = u M_f u^*$ where $u \in B(\ell^2 X, \mathcal{H})$ for some set X w/ counting measure, u is unitary $[uu^* = \mathrm{id}_{\mathcal{H}},\ u^*u = \mathrm{id}_{\ell^2 X}]$ $f \in c_0(X)$, and $M_f \in B(\ell^2 X)$ by $M_f \xi = f\xi$.

Lemma: For any increasing net (x_λ) of s.a. ops which is $\|\cdot\|$-bdd, $\exists$ a s.a. op $x = \operatorname{lub} x_\lambda$ s.t. $x_\lambda \le x$ $\forall \lambda$, and x minimal with this property, and $x_\lambda \longrightarrow x$ SOT.

Pf: Since $\|\cdot\|$-closed ball of radius R is WOT-cpt, $\exists$ a WOT-cluster pt x of (x_λ). Then $\forall \xi \in H$, $\langle x_\lambda \xi, \xi \rangle \nearrow \langle x\xi, \xi \rangle$, so $x \ge x_\lambda$ $\forall \lambda$. Finally, $\sqrt{x - x_\lambda} \longrightarrow 0$ SOT, and mult. is jointly SOT-cts on bdd sets, so $x_\lambda \longrightarrow x$ SOT.

Cor: If $(p_i)_{i \in I}$ is a family of mutually $\perp$ proj's, then $\sum p_i$ converges as an SOT-limit of finite sums.

Pf: Note that $\Lambda = \{$finite subsets $\lambda \subseteq I\}$ is directed under inclusion, and $(P_\lambda := \sum_{i \in \lambda} p_i)$ is an increasing net which is bdd above. Apply the lem.

Spectral Measures: Let X be a set and $\mathcal{Q}$ a σ-alg of subsets of X. A spectral measure on $(X, \mathcal{Q})$ is a Hilb. space H and a projection-valued fct $E : \mathcal{Q} \longrightarrow P(H)$ s.t.

- $E(\emptyset) = 0$ and $E(X) = 1$
- If (S_n) is a seq. of disjoint subsets in $\mathcal{Q}$, $S = \bigcup_{n=1}^{\infty} S_n$, then $\sum_n E(S_n)$ converges to $E(S)$ SOT.

Observe that $\forall \eta, \xi \in H$, $\mu_{\eta,\xi}(S) := \langle E(S)\eta, \xi \rangle$ is a finite $\mathbb{C}$-meas. on X. If X is a loc. cpt space and $\mathcal{Q}$ the Borel σ-alg, we call E regular if $\mu_{\eta,\xi}$ is a regular Borel measure $\forall \eta, \xi \in H$.

Facts about spectral measures:

① If $S_1 \cap S_2 = \emptyset$, $E(S_1) \perp E(S_2)$.

Pf: we know $E(S_1 \cup S_2) = E(S_1) + E(S_2)$ is a proj. The result follows from:

Exercise: If $p, q \in P(H)$ s.t. $p + q \in P(H)$, then $p \perp q$.

Pf: Let $\xi \in pH$. Then $\|\xi\|^2 \ge \|(p+q)\xi\|^2 = \|\xi\|^2 + \|q\xi\|^2 + 2\langle q\xi, \xi \rangle = \|\xi\|^2 + 3\|q\xi\|^2$. Hence $q\xi = 0$, and $qp = 0$.

② $E(S_1 \cap S_2) = E(S_1)E(S_2)$

Pf: $E(S_1)E(S_2) = [E(S_1 \setminus S_2) + E(S_1 \cap S_2)][E(S_2 \setminus S_1) + E(S_1 \cap S_2)] = E(S_1 \cap S_2)$ by ①.

③ If $S_1 \subset S_2$, $E(S_1)E(S_2) = E(S_1)$ [immediate from ②]

Let $E: (X, \mathcal{Q}) \longrightarrow P(H)$ be a spectral measure. We say that measurable fct f on X is essentially bdd w.r.t. E if $\exists c > 0$ s.t. $E(\{|f| > c\}) = 0$. For such f, we define $\|f\|_E$ to be the inf of all such $c > 0$, so $E(\{|f| > \|f\|_E\}) = 0$. Let

$L^\infty(E)$ = set of (eq. classes) of fcts ess. bdd wrt E.

Fact: $L^\infty(E)$ is a commutative unital C^*-alg under $\|\cdot\|_E$.

Def of $\int f\, dE$:

Step ①: If $f = \sum_{i=1}^{n} c_i \chi_{S_i}$ is a simple fct in $L^\infty(E)$, define

$$\int f\, dE = \sum_{i=1}^{n} c_i E(S_i).$$

well-defined: Show if $\sum_{i=1}^{n} c_i \chi_{S_i} = 0$, then $\sum_{i=1}^{n} c_i E(S_i) = 0$. For $\lambda \subset \{1, 2, \ldots, n\}$, let $S_\lambda = \left[\bigcap_{i \in \lambda} S_i\right] \setminus \left[\bigcup_{i \notin \lambda} S_i\right]$. Then the sets $\{S_\lambda \mid \lambda \subset \{1, 2, \ldots, n\}\}$ are mutually disjoint, and $S_i = \bigcup_{i \in \lambda} S_\lambda$. Then

$$0 = \sum c_i \chi_{S_i} = \sum c_i \Big(\sum_{i \in \lambda} \chi_{S_\lambda}\Big) = \sum_\lambda \Big(\sum_{i \in \lambda} c_i\Big) \chi_\lambda \Rightarrow \sum_{i \in \lambda} c_i = 0 \ \forall \lambda.$$

Thus $\sum c_i E(S_i) = \sum_i c_i E\Big(\bigcup_{i \in \lambda} S_\lambda\Big) = \sum c_i \Big(\sum_{i \in \lambda} E(S_\lambda)\Big) = \sum_\lambda \underbrace{\Big(\sum_{i \in \lambda} c_i\Big)}_{0} E(S_\lambda) = 0$

Step ②: $\left\| \int f\, dE \right\| = \|f\|_E$.

Pf: As in Step ①, $\sum c_i \chi_{S_i} = \sum_\lambda \Big(\sum_{i \in \lambda} c_i\Big) \chi_{S_\lambda}$ and $\int f\, dE = \sum_\lambda \Big(\sum_{i \in \lambda} c_i\Big) E(S_\lambda)$, where all the S_λ are disjoint. Both norms are $\max_\lambda \left|\sum_{i \in \lambda} c_i\right|$ s.t. $E(S_\lambda) \neq 0$.

Step ③: Get a linear isometry $f \mapsto \int f\, dE$ from simple fcts in $L^\infty(E)$ to $B(H)$. Since simple fcts are dense in $L^\infty(E)$, this map extends !ly by continuity to an isometry $L^\infty(E) \to B(H)$.

Remarks:

① $\forall f \in L^\infty(E)$, $\int \bar{f}\, dE = \left(\int f\, dE\right)^*$.

② $\forall f, g \in L^\infty(E)$, $\left(\int f\, dE\right)\left(\int g\, dE\right) = \int fg\, dE$.

For ①+②, look at simple fcts.

③ $\forall f \in L^\infty(E)$, $\forall \eta, \xi \in H$, $\left\langle \left(\int f\, dE\right)\eta, \xi \right\rangle = \int f\, d\mu_{\eta,\xi}$.

④ If $(f_\lambda) \subset L^\infty(E)$ is a bdd increasing net of non-neg fcts, and $f = \sup f_\lambda \in L^\infty(E)$, then $\int f_\lambda\, dE$ increases to $\int f\, dE$, converging in SOT.

Pf: For $\xi \in H$, define $\mu_{\xi,\xi}(S) = \langle E(S)\xi, \xi \rangle$, a non-negative finite measure on $(X, \mathcal{A})$. Since $f_\lambda \nearrow f \in L^\infty(E)$ and ν_ξ finite, $f \in L^1(\nu_\xi)$. By the monotone convergence thm,

$$\left\langle \left(\int f_\lambda \, dE\right)\xi, \xi \right\rangle = \int f_\lambda \mu_{\xi,\xi} \nearrow \int f \, d\mu_{\xi,\xi} = \left\langle \left(\int f \, dE\right)\xi, \xi \right\rangle.$$

Hence $\int f_\lambda dE \nearrow \int f dE$.

(5) [Spectral Mapping] $sp\left(\int f \, dE\right) = \text{ess. range}(f)$ in $L^\infty(E)$

$$= \{\lambda \in \mathbb{C} \mid \forall \varepsilon > 0,\ E(\{|f - \lambda| < \varepsilon\}) \neq 0\}$$

Pf: First, suppose $\exists \varepsilon > 0$ s.t. $E(S := \{|f-\lambda| < \varepsilon\}) = 0$. Define $g \in L^\infty(E)$ by

$$g(z) = \begin{cases} (f(z) - \lambda)^{-1} & \text{if } |f(z) - \lambda| \geq \varepsilon \iff z \notin S \\ 0 & \text{if } |f(z) - \lambda| < \varepsilon \iff z \in S \end{cases} \qquad \left[\|g\|_E \leq \varepsilon^{-1}\right]$$

Then $\int g \, dE \left(\int f dE - \lambda\right) = \int g(f-\lambda) \, dE = \underbrace{\int_S g(f-\lambda) \, dE}_{=0} + \int_{X \setminus S} \underbrace{g(f-\lambda)}_{=1} \, dE = E(X \setminus S) = 1.$

Conversely, if $E(S_\varepsilon := \{|f-\lambda| < \varepsilon\}) \neq 0$ $\forall \varepsilon > 0$, then for unit v. $\xi \in E(S_\varepsilon)H$,

$$\left\|\left(\int f dE - \lambda\right)\xi\right\| = \left\|\left(\int f dE - \lambda\right)E(S_\varepsilon)\xi\right\| \leq \left\|\int_{S_\varepsilon} (f - \lambda) \, dE\right\| \leq \varepsilon.$$

Hence λ is an approximate eval for $\int f dE$, and thus $\lambda \in sp\left(\int f dE\right)$.

Spectral Thm: Let $A \subseteq B(H)$ be a unital commutative C^*-alg and $\Gamma: A \xrightarrow{\sim} C(\hat{A})$ the Gelfand transform. There is a unique regular Borel spectral measure E on $\hat{A}$ s.t. $\forall f \in C(X)$, $\Gamma^{-1}(f) = \int f \, dE$. Moreover, $\int \cdot \, dE$ is an isometric $*$-hom. $L^\infty(E) \longrightarrow A'' \subset B(H)$.

Pf: Let $X = \hat{A}$. Observe that $\forall \eta, \xi \in H$, $\exists!$ finite regular Borel measure $\mu_{\eta,\xi}$ on X s.t. $\langle \Gamma^{-1}(f)\eta, \xi \rangle = \int f \, d\mu_{\eta,\xi}$ $\forall f \in C(X)$.

First, $\forall \xi \in H$, $\mu_{\xi,\xi}$ is a non-negative finite regular Borel measure. Since X cpt Hausdorff, X is normal, so by regularity of $\mu_{\xi,\xi}$, $\forall$ open $S \subseteq X$, $\mu_{\xi,\xi}(S) \overset{(*)}{=} \sup\{\int f \, dE \mid f \in C(X), 0 \leq f \leq 1, \text{supp}(f) \subseteq S\}$.

Since $(\eta, \xi) \longmapsto \mu_{\eta,\xi}$ is linear in η, conj. linear in ξ, by polarization,

$4\mu_{\eta,\xi} = \sum_{k=0}^{3} i^k \mu_{\eta+i^k\xi,\,\eta+i^k\xi}$, and thus by (*), $\overline{\mu_{\eta,\xi}(S)} = \mu_{\xi,\eta}(S)$ $\forall \eta,\xi\in H$.

Now notice for Borel $S\subseteq X$, the map $(\eta,\xi)\longmapsto \mu_{\eta,\xi}(S)$ is sesq. lin. and bdd. Hence $\exists!$ op $E(S)\in B(H)$ s.t. $\mu_{\eta,\xi}(S)=\langle E(S)\eta,\xi\rangle$.

Claim 1: $E:(X, \text{Borel } \sigma\text{-alg}) \longrightarrow B(H)$ is a regular spectral measure which takes values in $P(A'')$.

Step ①: $E(\emptyset)=0$ and $E(X)=1$.

Pf: $\forall \eta,\xi$, $\langle E(\emptyset)\eta,\xi\rangle = \mu_{\eta,\xi}(\emptyset)=0$ and $\langle E(X)\eta,\xi\rangle = \mu_{\eta,\xi}(X)=\langle\eta,\xi\rangle$.

Step ②: $E(S)^* = E(S)$.

Pf: $\forall \eta,\xi$, $\langle E(S)^*\eta,\xi\rangle = \langle \eta, E(S)\xi\rangle = \overline{\mu_{\xi,\eta}(S)} = \mu_{\eta,\xi}(S) = \langle E(S)\eta,\xi\rangle$.

Step ③: If $S_1\cap S_2=\emptyset$, $E(S_1\cup S_2) = E(S_1)+E(S_2)$.

Pf: $\forall \eta,\xi$, $\langle E(S_1\cup S_2)\eta,\xi\rangle = \mu_{\eta,\xi}(S_1\cup S_2) = \mu_{\eta,\xi}(S_1)+\mu_{\eta,\xi}(S_2) = \langle (E(S_1)+E(S_2))\eta,\xi\rangle$.

Step ④: $\forall \eta,\xi\in H$, $\forall f\in C(X)$, $d\mu_{\Gamma^{-1}(f)\eta,\xi} = f\,d\mu_{\eta,\xi}$.

Pf: $\forall g\in C(X)$, $\int g\,d\mu_{\Gamma^{-1}(f)\eta,\xi} = \langle \Gamma^{-1}(g)\Gamma^{-1}(f)\eta,\xi\rangle = \langle \Gamma^{-1}(gf)\eta,\xi\rangle = \int gf\,d\mu_{\eta,\xi}$.

Step ⑤: $\forall \eta,\xi\in H$, $\forall f\in C(X)$, $\forall S\subseteq X$ Borel, $d\mu_{E(S)\eta,\xi} = \chi_S\,d\mu_{\eta,\xi}$.

Pf: $\int f\,d\mu_{E(S)\eta,\xi} = \langle \Gamma^{-1}(f)E(S)\eta,\xi\rangle = \langle E(S)\eta, \Gamma^{-1}(\bar f)\xi\rangle = \mu_{\eta,\Gamma^{-1}(\bar f)\xi}(S)$

$= \overline{\mu_{\Gamma^{-1}(\bar f)\xi,\eta}(S)} = \overline{\int \chi_S \bar f\,d\mu_{\xi,\eta}} = \int \chi_S f\,d\mu_{\eta,\xi}$.

Step ⑥: $E(S_1\cap S_2) = E(S_1)E(S_2)$. In particular, $E(S)=E(S)^2\ [=E(S)^*]$

Pf: $\forall \eta,\xi$, $\langle E(S_1)E(S_2)\eta,\xi\rangle = \mu_{E(S_2)\eta,\xi}(S_1) = \int \underbrace{\chi_{S_1}\chi_{S_2}}_{\chi_{S_1\cap S_2}}\,d\mu_{\eta,\xi} = \langle E(S_1\cap S_2)\eta,\xi\rangle$.

Step ⑦: If (S_n) a seq. of disj Borel sets, $E(\bigcup_n S_n) = \sum_n E(S_n)$ conv. in SOT.

Pf: Let $\xi\in H$ and $N\in\mathbb{N}$. Then $E(S)-\sum_{i=1}^N E(S_i) = E(S) - E(\bigcup_{i=1}^N S_i) = E(S\setminus \bigcup_{i=1}^N S_i)$.

Then $\|E(S)\xi - \sum_{i=1}^N E(S_i)\xi\|^2 = \|E(S\setminus\bigcup_{i=1}^N S_i)\xi\|^2 = \langle E(S\setminus \bigcup_{i=1}^N S_i)\xi,\xi\rangle$

$= \mu_{\xi,\xi}(S\setminus\bigcup_{i=1}^N S_i) \longrightarrow 0$ as $N\to\infty$.

Step ⑧: $\forall$ Borel $S\subseteq X$, $E(S)\in A''$.

Pf: Let $x\in A'$ and $\eta,\xi\in H$. We claim $\mu_{x\eta,\xi} = \mu_{\eta,x^*\xi}$. Indeed, $\forall f\in C(X)$,

$\int f\,d\mu_{x\eta,\xi} = \langle \Gamma^{-1}(f)x\eta,\xi\rangle = \langle x\Gamma^{-1}(f)\eta,\xi\rangle = \langle \Gamma^{-1}(f)\eta, x^*\xi\rangle = \int f\,d\mu_{\eta,x^*\xi}$.

Thus, $\forall S\subseteq X$, we have:

$\langle E(S)x\eta,\xi\rangle = \mu_{x\eta,\xi}(S) = \mu_{\eta,x^*\xi}(S) = \langle E(S)\eta,x^*\xi\rangle = \langle xE(S)\eta,\xi\rangle$.

Summarizing so far, $E:(X, \text{Borel } \sigma\text{-alg}) \longrightarrow P(A'')$ is a regular Borel spectral measure, so $\int \cdot \, dE : L^\infty(E) \longrightarrow A''$ is an isometric $*$-hom.

Claim 2: $\forall f \in C(X)$, $\Gamma^{-1}(f) = \int f \, dE$

Pf: Let $f \in C(X)$. $\forall \eta, \xi \in H$, $\langle \Gamma^{-1}(f)\eta, \xi \rangle = \int f \, d\mu_{\eta,\xi}$.

Now $f \in L^\infty(E)$, so $\langle (\int f \, dE)\eta, \xi \rangle = \int f \, d\mu_{\eta,\xi}$ $\forall \eta, \xi$.

Hence $\Gamma^{-1}(f) = \int f \, dE$ $\forall f \in C(X)$.

Remarks: Claim 2 shows $C(X) \subseteq L^\infty(E)$ injectively, and thus $\forall$ nonempty open $U \subseteq X$, $E(U) \neq 0$. [use Urysohn's Lemma.]

Claim 3: E is the unique regular Borel spectral measure s.t. $\forall f \in C(X)$, $\int f \, dE = \Gamma^{-1}(f)$.

Pf: Suppose F is another such regular Borel spectral measure. $\forall \eta, \xi \in H$, define $\nu_{\eta,\xi}(S) := \langle F(S)\eta, \xi \rangle$, regular Borel meas. on X.

Then $\forall \eta, \xi \in H$, $\forall f \in C(X)$,

$$\int f \, d\nu_{\eta,\xi} = \langle (\int f \, dF)\eta, \xi \rangle = \langle (\int f \, dE)\eta, \xi \rangle = \int f \, d\mu_{\eta,\xi}.$$

Hence $\forall \eta, \xi$, $\nu_{\eta,\xi} = \mu_{\eta,\xi}$, so $E(S) = F(S)$ $\forall$ Borel S.

Remark: The id map from the unital C^*-alg $B^\infty(X)$ of bdd Borel fcts on X to $L^\infty(E)$ is a unital $*$-hom, and thus norm decreasing.

Question: When is it injective? Surjective?

Borel/L^∞ fnl calculus: Let $x \in B(H)$ be normal and set $N(x) = \overbrace{C^*(x)''}^{\text{Abelian vNa!}}$.

There is a unique regular Borel spectral measure E on $sp(x)$ s.t.

$\int \mathrm{id} \, dE = x$ where $\mathrm{id}(z) = z$ $\forall z \in sp(x)$.

Pf: If $\int \mathrm{id} \, dE = x$, then $\forall$ poly's $p \in \mathbb{C}[z, \bar{z}]$, $\int p \, dE = p(x)$, so by continuity, $\int f \, dE = f(x)$ $\forall f \in C(X)$. We are now finished by the uniqueness condition in the Spectral Thm.

Note: we may thus unambiguously denote $\int f \, dE = f(x)$ $\forall f \in L^\infty(E)$.

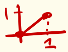
Prop: Suppose $x \in B(H)$ normal and $f \in L^\infty(E)$. Let F be the regular Borel spectral measure for $f(x) \in B(H)$. Then $\forall g \in L^\infty(F)$, $g\circ f \in L^\infty(E)$ and $(g\circ f)(x) = g(f(x))$.

Pf: First, $sp(f(x)) = \text{ess range}(f)$ in $L^\infty(E)$ and g is Borel measble on $sp(f(x))$, so $g\circ f$ is Borel measble on $sp(x)$, and defines an elt of $L^\infty(E)$. It suffices to prove that the assignment $S \xmapsto{G} (\chi_S \circ f)(x) \in P(vN(x))$ is a regular Borel spectral measure on $sp(f(x))$ s.t. $\int id\, dG = f(x)$. Notice that $\chi_S\circ f = \chi_{f^{-1}(S)}$, so $G(S) = E(f^{-1}(S))$, and $\forall \eta,\xi \in H$, $\mu^G_{\eta,\xi}$ on ess.range(f) is the pushforward of $\mu^E_{\eta,\xi}$ via $f: sp(x) \to sp(f(x))$. Hence $\mu^G_{\eta,\xi}$ is regular Borel. [finite Borel meas. on 2nd ctble loc. cpt space is regular, and ess.range(f) is cpt $\subseteq \mathbb{C}$.] Finally, if we approx $id \in B^\infty(sp(f(x)))$ by simple fcts $g_n \to id$ in $\|\cdot\|_\infty$, then $g_n\circ f \to f$ in $\|\cdot\|_\infty$ in $B^\infty(sp(x))$, so $g_n \circ f\to f$ in $L^\infty(E)$, and

$$\int id\, dG = \lim \int g_n\, dG = \lim (g_n\circ f)(x) = \lim\int g_n\circ f\, dE = \int f\, dE = f(x).$$

Immediate easy applications:

① Let $M \subseteq B(H)$ be a vNa. Then M is the $\|\cdot\|$-closure of span $P(M)$.

Pf: It suffices to approximate any positive op. in the unit ball of M by a linear comb. of proj's. Just approximate  uniformly by a simple fct.

② Let $L \subseteq M$ be a nonzero left ideal. Then L contains a nonzero proj.

Pf: Let $x\in L$, $x\neq 0$. Then $x^*x \in L\setminus\{0\}$. WLOG, $\|x^*x\|=1$. Let $0<\varepsilon<1$.
Consider $f(t) = \frac{1}{t} \chi_{[\varepsilon,1]}(t)$. Then $f(x^*x)x^*x = \chi_{[\varepsilon,1]}(x^*x)\in L$.

Note: $\chi_{[\varepsilon,1]}(x^*x) \neq 0$, since $\int\cdot\, dE$ extends CFC, and $\exists$ cts fcts on $sp(x^*x)$ whose support lies in $[\varepsilon,1]$.

③ $\mathcal{U}(M)$ is path conn in $\|\cdot\|$-top.

Pf: Let $u\in \mathcal{U}(M)$ and let log be a branch of the logarithm. Then
$u = e^{f(u)} = e^{i\underbrace{(-if(u))}_{x}}$, where $x=-if(u)$ is s.a. by spectral mapping.
Now $u_t := e^{itx}$ is a $\|\cdot\|$-cts path of unitaries from u to 1 in $\mathcal{U}(M)$.

④ If $A \subseteq B(H)$ a unital C^*-alg, then $\mathcal{U}(A)$ is SOT-dense in $\mathcal{U}(A'')$.

Pf: Suppose $u \in \mathcal{U}(A'')$. Let $x \in A''_{s.a.}$ s.t. $u = e^{ix}$. By Kaplansky Density, $\exists$ a net $(x_\lambda) \subset A_{s.a.}$ s.t. $\|x_\lambda\| \leq \|x\|$ $\forall \lambda$ s.t. $x_\lambda \to x$ SOT. Let $f \in C_0(\mathbb{R})$ s.t. $f(t) = e^{it}$ on $[-\|x\|, \|x\|]$. Then f is SOT-cts, and $\underset{\mathcal{U}(A)}{f(x_\lambda)} \xrightarrow{SOT} \underset{u}{f(x)}$.

Abelian vNa's + Multiplicity Theory

HW: Let $A \subseteq B(H)$ be an abelian vNa and H separable. TFAE:

① A has a cyclic $\vec{v}$.

② $\forall$ $\|\cdot\|$-separable SOT-dense C^*-subalg $A_0 \subseteq A$, A_0 has a cyclic $\vec{v}$.

③ $\exists$ $\|\cdot\|$-sep. SOT-dense C^*-subalg $A_0 \subseteq A$ w/ a cyclic $\vec{v}$.

④ $\exists$ cpt Hausd. 2nd ctble space X and a finite regular Borel meas μ on X and a unitary $u \in B(L^2(X,\mu), H)$ s.t. $f \mapsto u M_f u^*$ is an isometric $*$-iso $A \cong L^\infty(X,\mu)$

⑤ $A = A'$ (A is maximal abelian).

Def: $x \in B(H)$ normal is called multiplicity free if $C^*(x) \subseteq B(H)$ has a cyclic $\vec{v}$, which is equivalent to $vN(x) = vN(x)'$ $[= C^*(x)']$.

HW: Let $E: (X, \mathcal{A}) \to B(H)$ be a spectral measure and $A \subseteq B(H)$ the abelian unital C^*-alg which is the image of $L^\infty(E)$ under $\int \cdot \, dE$. If A has a cyclic $\vec{v}$,

① $\exists$ a non-negative finite measure μ on $(X, \mathcal{A})$ s.t. $\underbrace{L^\infty(E) = L^\infty(X, \mathcal{A}, \mu)}_{\text{as eq. classes of msble fcts.}}$ w/ $\|f\|_E = \|f\|_\infty$

② $\exists$ a unitary $u \in B(L^2(X, \mathcal{A}, \mu), H)$ s.t. $\forall f \in L^\infty(X, \mathcal{A}, \mu)$, $u M_f u^* = \int f \, dE$, and

③ $A = A'$

If you do the above HW problems as expected, you can put the proofs together to get the following:

Cor: Suppose $x \in B(H)$ is normal + mult. free. There exist:

- a finite non-negative regular Borel measure μ on $X = sp(x)$
- a unitary $u \in B(L^2(X, \mu), H)$ s.t.

① $L^\infty(X,\mu) = L^\infty(E_x)$ ← regular Borel spectral meas.

② $\forall f \in L^\infty(X,\mu)$, $u M_f u^* = \int f \, dE_x = f(x)$.

③ The map $L^\infty(X,\mu) \ni f \mapsto f(x)$ is an isometric $*$-iso onto $vN(x)$.

Thm: Suppose $\mathcal{H}$ is separable and $x \in B(\mathcal{H})$ is normal.

① $\exists$ a seq. (p_n) of pairwise $\perp$ proj's in $P(C^*(x)')$ s.t. Previous HW.

- $\sum p_n = 1$ SOT and
- $p_n \mathcal{H}$ is a cyclic subspace for $C^*(x)$ $\forall n$.

② $\forall n$, $\exists$ a non-neg. finite regular Borel meas. μ_n on $X = sp(x)$ and a unitary $u_n \in B(L^2(X, \mu_n), p_n\mathcal{H})$ s.t.

- $\forall f \in B^\infty(X)$, $u_n M_f u_n^* = f(x) p_n = f(x p_n)$, and
- the map $L^\infty(X, \mu_n) \ni f \longmapsto f(x) p_n \in \underbrace{[vN(x)]p_n}_{\text{compression of vNa by a proj. in the commutant.}}$ is an isometric $*$-iso.

③ Setting $\mu = \sum \frac{1}{2^n} \mu_n$,

- $\forall f \in L^\infty(X, \mu)$, $\sum u_n M_f u_n^* = \sum f(x) p_n = f(x)$ conv. SOT, and
- the map $L^\infty(X, \mu) \ni f \longmapsto f(x)$ is a (σ-WOT cts) isometric $*$-iso onto $vN(x)$.

To prove this, we'll need a few facts.

Prop: Suppose $M \subseteq B(\mathcal{H})$ is a vNa and $p \in P(M')$ is a projection. Then $pMp \subseteq B(p\mathcal{H})$ is a vNa w/ $(pMp)' = M'p$.

Pf: If $xp \in M'p$, certainly $xp(pmp) = pxpmp = pmpx = (pmp)(xp)$ $\forall m \in M$. Hence $M'p \subseteq (pMp)'$. The converse requires a clever trick.

Claim: Every elt in a vNa is a linear comb. of 4 unitaries.

Pf: First, any elt is a lin. comb. of 2 s.a. ops. If a is s.a. w/ $\|a\| \le 1$, $u = a + i\sqrt{1-a^2}$ is unitary and $a = \frac{1}{2}(u + u^*)$. Scaling does the general case.

Back to pf: Suppose $u \in (pMp)'$ unitary, set $K = \overline{Mp\mathcal{H}}$ and let $q = P_K$. Since K is M and M'-invariant, $q \in Z(M)$. Extend u to K by

$$\tilde{u} \sum x_i p \xi_i := \sum x_i u p \xi_i \quad \text{for } x_i \in M,\ \xi_i \in \mathcal{H}.$$

Claim: $\tilde{u}$ is a well-defined isometry in $B(K)$.

Pf: $\|\tilde{u} \sum x_i p \xi_i\|^2 = \sum_{i,j} \langle x_i u p \xi_i, x_j u p \xi_j \rangle \overset{[u,p]=0}{=} \sum_{i,j} \langle p x_j^* x_i p u \xi_i, u \xi_j \rangle \overset{u \in (pMp)'}{=}$

$= \sum_{i,j} \langle u p x_j^* x_i p \xi_i, u \xi_j \rangle = \sum_{i,j} \langle p x_j^* x_i p \xi_i, \xi_j \rangle = \cdots = \|\sum x_i p \xi_i\|^2.$

Now, by construction, $\tilde{u}$ commutes w/ action of M on $K = \overline{MpH}$.

Thus $\tilde{u}q \in M' \subset B(H)$, and $u = (\tilde{u}q)p \in M'p$.

[Notice $\forall m \in M$, $\forall \xi \in H$, $\tilde{u}qm\xi = \tilde{u}m\underbrace{q\xi}_{\in K} = m\tilde{u}\underbrace{q\xi}_{\in K}$. Hence $[m, \tilde{u}q] = 0$.

Next, $\tilde{u}qp\xi = \tilde{u}1p\xi = 1up\xi = uq\xi = u\xi$ $\forall \xi \in pH$, so $\tilde{u}qp = u$.]

Defi: For $p \in M$, pMp and $M'p$ are compressions/reductions of M, M'.