Normed Spaces + Lin maps

X v.sp. over $\mathbb{F}$ ($=\mathbb{R}$ or $\mathbb{C}$).

A <u>norm</u> on X is a fct $\|\cdot\|: X \to [0,\infty)$ s.t.

① (sub-add) $\|x+y\| \le \|x\| + \|y\|$ $\forall x,y \in X$
② (homog.) $\|\lambda x\| = |\lambda| \cdot \|x\|$ $\forall \lambda \in \mathbb{F}$, $x \in X$ — seminorm.
③ (definite) $\|x\| = 0 \iff x = 0$.

- metric induced by norm is $d_{\|\cdot\|}(x,y) = \|x-y\|$.
- if $(X, d_{\|\cdot\|})$ complete, X is a Banach space.

<u>Exs:</u> $L^p(X,\mu)$, ℓ^p, $C_0(X)$, c_0, $C(X)$, c, H,
$B(H)$, $B(X, Y)$ $\quad$ $H^2(\Omega)$
normed ↗ ↖ Banach,

<u>Exercises:</u>
- $\|\cdot\|$, $d(x,\cdot)$ cts, $d(\cdot,\cdot)$
- $B_r(x)$ form sub-basis for topology on X.
  $U \subseteq X$ open $\iff$ $U = \bigcup_{i\in I} B_{r_i}(x_i)$.
- Hausdorff, v.sp. ops are <u>cts</u>.

<u>Prop:</u> $T: X \to Y$ lin. map btwn normed spaces. TFAE
① T cts
② T cts at $x_0 \in X$
③ T <u>bdd</u>: $\exists \alpha > 0$ s.t. $\|Tx\| \le \alpha \|x\|$ $\forall x \in X$.

<u>Pf:</u> ②$\Rightarrow$③: T cts at x_0 $\Rightarrow$ $\exists \delta > 0$ s.t. $\|x - x_0\| \le \delta \Rightarrow \|Tx - Tx_0\| \le 1$.

$\forall y \ne 0$, $\left\| \delta \frac{y}{\|y\|} + x_0 - x_0 \right\| \le \delta$ $\Rightarrow$ $\|Ty\| \le \delta^{-1} \|y\|$.

③$\Rightarrow$①: $\|Tx - Ty\| = \|T(x-y)\| \le \alpha \|x-y\|$.

②

Def: $B(\mathfrak{X}, Y) = \{T : \mathfrak{X} \to Y \mid T \text{ bdd}\}$. v.sp ✓

$\|T\| := \sup\{\|Tx\| \mid \underset{x \in \mathfrak{X}}{\|x\| = 1}\}$. op. norm.

exercises:
- it's a well-defined norm.
- If $T \in B(\mathfrak{X}, Y)$, $S \in B(Y, Z)$ $\|S \circ T\| \leq \|S\| \cdot \|T\|$,

  i.e. op norm is submultiplicative.

  Cor: $B(\mathfrak{X})$ is a normed algebra. (A Banach Alg is a complete normed algebra)

Prop: $\mathfrak{X}, Y$ normed, Y cplete $\Rightarrow$ $B(\mathfrak{X}, Y)$ cplete.

[[Cor: $B(\mathfrak{X})$ a Banach alg when $\mathfrak{X}$ Banach]]

Pf: Let (T_n) be Cauchy in $B(\mathfrak{X}, Y)$. $\forall x \in \mathfrak{X}$, $(T_n x)$ Cauchy in Y. Define $T : \mathfrak{X} \to Y$ by $Tx = \lim_n T_n x$.

exercise: T is linear. (use that $+, \cdot$ are cts.)

$$(*) \quad \|Tx - T_n x\| \underset{\substack{\uparrow \\ \|\cdot\| \text{ cts},\ \circ-\circ \text{ cts}}}{=} \lim_{k \to \infty} \underbrace{\|T_k x - T_n x\|}_{\substack{\leq \|T_k - T_n\| \cdot \|x\| \\ \leq \sup_{l \geq k} \|T_l - T_n\| \cdot \|x\|}} \leq \underbrace{\lim_{k \to \infty} \sup_{l \geq k} \|T_l - T_n\|}_{< \varepsilon \text{ for large } n} \cdot \|x\|$$

$$\|Tx\| = \|Tx - T_n x + T_n x\| \leq \|Tx - T_n x\| + \|T_n x\| \leq \varepsilon \|x\| + \|T_n\| \cdot \|x\| \quad \Rightarrow T \text{ bdd}.$$

Finally, $(*)$ shows $T_n \to T$ in $\|\cdot\|$.

Def: $\mathfrak{X}$ normed $Y \subseteq \mathfrak{X}$ subspace, $Q : \mathfrak{X} \to \mathfrak{X}/Y$ quotient map $x \mapsto x + Y$. Define

$$\|Qx\|_{\mathfrak{X}/Y} = \|x + Y\|_{\mathfrak{X}/Y} = \inf\{\|x - y\| \mid y \in Y\}.$$

HW1: seminorm, Q cts, Y closed $\Rightarrow$ norm, $\mathfrak{X}$ Banach Y closed $\Rightarrow$ $\mathfrak{X}/Y$ Banach

Fact: Q is open, $Q\, B_1^{\mathbb{X}}(0) = B_1^{\mathbb{X}/Y}(0)$ (open balls), not nec. true for closed balls!

Prop: $T \in B(\mathbb{X}, Y)$, $\mathbb{X}, Y$ normed, $Z \subseteq \mathbb{X}$ closed w/ $\ker T \subseteq Z$.
Then T factors uniquely through $\mathbb{X}/Z$.

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{T} & Y \\ {\scriptstyle Q}\searrow & & \nearrow {\scriptstyle \exists! \tilde{T}} \\ & \mathbb{X}/Z & \end{array}$$

* $\tilde{T} Q x := Tx$ well-defined, linear, bdd.
* $\|\tilde{T}\| = \|T\|$.

Pf: $\tilde{T}$ well-def: $x - y \in Z \Rightarrow \tilde{T} Q x = Qx = Qy = \tilde{T} Q y$.
$\tilde{T}$ linear: $\tilde{T}(Qx + Qy) = \tilde{T}(Q(x+y)) = T(x+y) = Tx + Ty = \tilde{T}Qx + \tilde{T}Qy$

norms: ① $\|\tilde{T} Q x\| = \|Tx\| = \|T(x-z)\| \leq \|T\| \cdot \|x - z\| \quad \forall z \in Z$

$\Rightarrow \|\tilde{T} Q x\| \leq \|T\| \inf_{z \in Z} \|x - z\| = \|T\| \cdot \|Qx\|$.

$\Rightarrow \|\tilde{T}\| \leq \|T\|$.

② $\|Q\| \leq 1$, so $\|T\| = \|\tilde{T} Q\| \leq \|\tilde{T}\| \|Q\| \leq \|\tilde{T}\|$.

Prop: $\mathbb{X}$ normed, $Y \subseteq \mathbb{X}$ closed subspace s.t. $\mathbb{X}/Y$, Y Banach.
Then $\mathbb{X}$ Banach.

Pf: Take (x_n) Cauchy in $\mathbb{X}$. Then (Qx_n) Cauchy, so $\exists x \in \mathbb{X}$ s.t. $Qx_n \to Qx$. By def. of $\|\cdot\|_{\mathbb{X}/Y}$ $\exists y_n \in Y$ s.t. $\|x_n - x - y_n\| < \frac{1}{n} + \|Q(x_n - x)\|$

$$\|y_m - y_n\| = \|y_m + x - x_m - (y_n + x - x_n) + (x_m - x_n)\|$$
$$\leq \frac{1}{m} + \|Q(x_m - x)\| + \frac{1}{n} + \|Q(x_n - x)\| + \|x_m - x_n\| \to 0$$

$\Rightarrow (y_n)$ Cauchy, $\exists y \in Y$ s.t. $y_n \to y$.
Finally, $\|x_n - (x + y)\| \leq \|x_n - x - y_n\| + \|y_n - y\| < \frac{1}{n} + \|Q(x - x_n)\| + \|y_n - y\| \to 0$.

<u>Prop:</u> $\mathfrak{X}, \mathcal{Y}$ Banach, $\mathfrak{X}_0 \subseteq \mathfrak{X}$ dense subspace. Every $T_0 \in \mathcal{B}(\mathfrak{X}_0, \mathcal{Y})$ has a ! extension to $T \in \mathcal{B}(\mathfrak{X}, \mathcal{Y})$ w/ $\|T\| = \|T_0\|$.

Pf: If $x \in \mathfrak{X}$, $\exists (x_n) \subset \mathfrak{X}_0$ w/ $x_n \to x$. Define

$Tx := \lim T_0 x_n$.

<u>well-defined:</u> If $(y_n) \subset \mathfrak{X}_0$ w/ $y_n \to x$, show $T_0 y_n \to Tx$

$$\| T_0 y_k - T_0 x_n \| = \| T_0 (y_k - x_n) \| \leq \underbrace{\|T_0\| \cdot \|y_k - x_n\|}_{\text{small}}$$ eventually $< \varepsilon$ ✓

<u>linear:</u> $\left.\begin{array}{l} x_n \to x \\ y_n \to y \end{array}\right\} x_n + y_n \to x+y$. $T(x+y) = \lim T_0(x_n + y_n)$ $= Tx + Ty$ ✓

$$\|Tx\| = \lim_n \|T_0 x_n\| \leq \|T_0\| \lim_n \|x_n\| = \|T_0\| \cdot \|x\|.$$

Clearly $\|T_0\| \leq \|T\|$ (sup over larger set.)

<u>Later:</u> Every normed space $\mathfrak{X}$ embeds densely into a Banach space $\tilde{\mathfrak{X}}$, unique up to <u>isometric iso.</u>

<u>Product spaces:</u> $\mathfrak{X} \times \mathcal{Y}$, can endow w/

$$\|(x,y)\|_1 = \|x\|_{\mathfrak{X}} + \|y\|_{\mathcal{Y}} \qquad \|\cdot\|_p \ldots$$
$$\|(x,y)\|_2 = \left(\|x\|_{\mathfrak{X}}^2 + \|y\|_{\mathcal{Y}}^2\right)^{1/2} \qquad \|(x,y)\|_\infty = \sup\{\|x\|, \|y\|\}.$$

<u>exer:</u> $(\mathfrak{X}_j)_{j \in J}$ ① $\{(x_j) \in \prod \mathfrak{X}_j \mid \sum \|x_j\|^p < \infty\}$ is a Banach space.

direct sum: w/ p-norm $\|(x_j)\|_p = \left(\sum \|x_j\|^p\right)^{1/p}$.

② $\{(x_j) \in \prod \mathfrak{X}_j \mid j \mapsto \|x_j\| \text{ is in } c_0(J)$ [illegible] $\}$

is a ~~are both~~ Banach w/ ∞-norm $\|(x_j)\|_\infty = \sup \|x_j\|$.

direct product: ③ $\{(x_j) \in \prod \mathfrak{X}_j \mid \sup \|x_j\| < \infty\}$ is Banach under ∞-norm.

Baire Category: Suppose X is either:

1. A locally cpt Hausdorff space
2. A complete metric space.

Let $\mathcal{U}$ be a countable collection of dense open subsets of X.
Then $\bigcap_{i=1}^{\infty} V_i$ is dense in X. $(\mathcal{U} = \{V_i\}_{i=1}^{\infty})$

Pf: Let $B_0 \subseteq X$ be nonempty + open. Inductively construct
open $\emptyset \neq B_1 \subseteq V_1 \cap B_0$ and $\emptyset \neq B_n \subset \overline{B_n} \subset V_n \cap B_{n-1}$.

For ①, take B_n so that $\overline{B_n}$ cpt, so $\{\overline{B_n}\}_{n=1}^{\infty}$ are nested cpt sets.

For ②, take B_n to be a ball of radius $\frac{1}{n}$.

Setting $K = \bigcap \overline{B_n}$, we see $\emptyset \neq K \subset \bigcap_{i=1}^{\infty} V_i$.

For ①, $\overline{B_n}$'s nest cpt sets.
For ②, centers of balls are Cauchy seq. $\Rightarrow$ convergent.

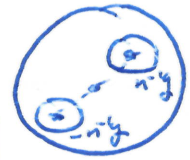
Open Mapping Thm: See ②

Lemma: Let X, Y be Banach spaces and $T \in B(X,Y)$. Suppose $\overline{TB_1^X(0)}$ is dense in $\overline{B_r^Y(0)}$ for some $r>0$. Then $\forall\, 0<\varepsilon<1$, $\overline{B^Y_{(1-\varepsilon)r}(0)} \subset T B_1^X(0)$.

Pf: Let $y \in \overline{B_r^Y(0)}$ and $0<\varepsilon<1$. Then $\exists\, y_0 \in \overline{TB_1^X(0)}$ s.t. $\|y-y_0\| < \varepsilon r$. Inductively find
Pick $y_1 \in \varepsilon \overline{T B_1^X(0)}$ s.t. $\|y-y_0-y_1\| < \varepsilon^2 r$.
$(y_n) \in \varepsilon^n T B_1^X(0)$ s.t. $\|y - \sum_{j=0}^{n} y_j\| < \varepsilon^{n+1} r$. Let $x_n \in X$ s.t.
$\|x_n\| \leq \varepsilon^n$ and $Tx_n = y_n$. Then $\sum x_n = x \in X$ and $Tx = y$. Also
$\|x\| \leq \sum_{n \geq 0} \varepsilon^n = \frac{1}{1-\varepsilon}$. Hence $(1-\varepsilon)^{-1} T \overline{B_1^X(0)} \supset \overline{B_r^Y(0)}$.

OMT: Suppose X, Y Banach and $T \in B(X,Y)$ w/ $TX = Y$. Then T is open.

Pf: Note $Y = \bigcup_n \overline{TB_n^X(0)}$. By BCT, $\exists n > 0$ s.t. $\overline{TB_n^X(0)}$ has nonempty interior. $\exists y \in Y$, $\varepsilon > 0$ s.t. $B_\varepsilon^Y(0)$ is contained. Hence $TB_1(0)$ is dense in $\overline{B^Y_{n^{-1}\varepsilon}(n^{-1}y)}$, and also dense in $\overline{B^Y_{n^{-1}\varepsilon}(0)}$!



$$\left[\ 2B^Y_{n^{-1}\varepsilon}(0) \subseteq \overline{B^Y_{n^{-1}\varepsilon}(n^{-1}y)} - \overline{B^Y_{n^{-1}\varepsilon}(n^{-1}y)}\ \right]$$

[Lemma: If $B \subseteq X$ balanced + convex and $B_r(y) \subseteq TB$, then $B_r(0) \subseteq TB$.]

By the Lemma, $\overline{B^Y_\delta(0)} \subseteq \overline{TB_1^X(0)}$ $\forall\ \delta < n^{-1}\varepsilon$.

(it's enough to show $TB_1(0)$ contains a nbhd of 0.)

Cor: Every bdd bijection $T \in B(X,Y)$ has bdd inverse.

Cor: If X is Banach under $\|\cdot\|_1$ and $\|\cdot\|_2$, $\exists \alpha > 0$ s.t. $\frac{1}{\alpha}\|\cdot\|_1 \le \|\cdot\|_2 \le \alpha\|\cdot\|_1$.

Closed Graph Thm: Suppose $T: X \to Y$ is linear between Banach spaces X and Y s.t. $\underbrace{\text{graph}(T)}_{G(T)} = \{(x,y) \in X \times Y \mid Tx = y\}$ is closed. Then T is bdd.

Pf: Note prod. top on $X \times Y$ is induced by $\|(x,y)\|_\infty = \max\{\|x\|, \|y\|\}$. $G(T) = \text{graph}(T)$ is a closed subspace. $P_X: G(T) \to X$, $(x, Tx) \mapsto x$ is norm decreasing + bijective, so has bdd inverse. $P_Y: G(T) \to Y$, $(x,Tx) \mapsto Tx$ is bdd, and $T = P_Y \circ P_X^{-1}$.

②

OMT: $\mathfrak{X}, Y$ Banach

Lemma: Let $U \subset \mathfrak{X}$ be an open ball w/ center $0_{\mathfrak{X}}$ and $V \subset Y$ open ball w/ center 0_Y. If $T \in B(\mathfrak{X}, Y)$ and $V \subset \overline{TU}$, then $V \subset TU$.

Pf: Let $y \in V$. Take $r \in (0,1)$ s.t. $y \in rV$. Let $\varepsilon \in (0,1)$ to be decided later. Since $y \in \overline{T(rU)}$, $\exists x_0 \in rU$ s.t. $y - Tx_0 \in \varepsilon r V$. Since $y - Tx_0 \in \overline{T\varepsilon r U}$, $\exists x_1 \in \varepsilon r U$ s.t. $y - Tx_0 - Tx_1 \in \varepsilon^2 r V$. Continue to get seq. $(x_n)_{n=0}^{\infty}$ s.t. $x_n \in \varepsilon^n r U$ and $y - \sum_{j=0}^{n} Tx_j \in \varepsilon^{n+1} r U$ $\forall n \in \mathbb{N}$. Then $\sum x_n =: x \in \mathfrak{X}$ conv, and $Tx = y$. Moreover, $x \in \frac{r}{1-\varepsilon} U$, so take ε s.t. $1 - \varepsilon > r$.

OMT: $T \in B(\mathfrak{X}, Y)$ w/ $T\mathfrak{X} = Y$ is open.

Pf: Enough to prove T maps a nbhd of $0_{\mathfrak{X}}$ to a nbhd of 0_Y. Note $Y = \bigcup_{n=1}^{\infty} \overline{TB_n(0)}$. By BCT, $\exists n \in \mathbb{N}$ s.t. $\overline{TB_n(0)}$ contains a nonempty open set, say $Tx_0 + V$ where $x_0 \in B_n(0)$ and V is an open ball in Y w/ center 0_Y. Then $V \subset \overline{TB_n(0)} - Tx_0 \subset \overline{TB_{2n}(0)}$. By the lemma, $V \subset TB_{2n}(0)$.

Lemma: Let X, Y be Banach, U the open unit ball of X,
V the open unit ball of Y.

Suppose $T \in B(X,Y)$ and $\delta > 0$. Consider

1. $\|T^* y^*\| \geq \delta \|y^*\| \quad \forall y^* \in Y^*$
2. $\overline{TU} \supseteq \delta V$
3. $TU \supseteq \delta V$
4. $TX = Y$

Then 1 $\Rightarrow$ 2 $\Rightarrow$ 3 $\Rightarrow$ 4.

Pf: 1 $\Rightarrow$ 2: ~~Note~~ Suppose $y_0 \notin \overline{TU}$. Since $\overline{TU}$ convex, closed, balanced, Use CGT by showing if $(x_n) \subseteq \mathfrak{X}$ w/ $x_n \to x$ and $Tx_n \to y$, then $x = y$. Get control over Tx_n!

Y need not be Banach!

Thm (Banach-Steinhaus / Uniform bddness principle) Consider $\mathfrak{X}, Y$ Banach and a family $\{T_\lambda\}_{\lambda \in \Lambda} \subset B(\mathfrak{X}, Y)$. If $\{T_\lambda x \mid \lambda \in \Lambda\} \subset Y$ is bdd $\forall x \in \mathfrak{X}$, $\{\|T_\lambda\| \mid \lambda \in \Lambda\} \subset [0,\infty)$ is bdd.

Pf: Define $Y_\lambda = Y$ for $\lambda \in \Lambda$, and define $T : \mathfrak{X} \to \prod_{\lambda \in \Lambda} Y_\lambda$ ((y_λ) s.t. $\sup_{\lambda \in \Lambda} \|y_\lambda\| < \infty$ w/ $\|\cdot\|_\infty$) by $Tx = (T_\lambda x)_{\lambda \in \Lambda}$.

Note T is well-defined.

Claim: graph of T is closed.

Pf: If $(x_n) \subseteq \mathfrak{X}$ w/ $x_n \to x$ and $Tx_n \to y$, if $P_\lambda : \prod Y_\lambda \to Y_\lambda$ is proj, $T_\lambda x_n \to P_\lambda y$. Since T_λ bdd, $T_\lambda x_n \to T_\lambda x$, so $T_\lambda x = P_\lambda y$ $\forall \lambda$. Thus $y = Tx$.

Thus by CGT, T is bdd. Since $T_\lambda = P_\lambda T$, $\|T_\lambda\| \le \|T\|$ $\forall \lambda$.

Cor: Suppose (T_λ) is a net in $B(\mathfrak{X}, Y)$ s.t. $(T_\lambda x) \subset \mathfrak{X}$ is bdd and convergent in Y $\forall x \in \mathfrak{X}$. Then $\exists T \in B(\mathfrak{X}, Y)$ s.t. $T_\lambda x \to Tx$ $\forall x \in \mathfrak{X}$.

↳ only need bdd $\forall x \in \mathfrak{X}$, conv. for dense $\mathfrak{X}_0 \subseteq \mathfrak{X}$.

Pf: Define $T : \mathfrak{X} \to Y$ by $Tx = \lim T_\lambda x$. By UBP, $\exists M > 0$ s.t. $\|T_\lambda\| < M$ $\forall \lambda$ and thus $\|Tx\| \le M\|x\|$ $\forall x \in \mathfrak{X}$.

(Exercise) **Cor:** Any wkly conv. seq. in $\mathfrak{X}$ is ~~norm~~ bdd.

~~Recall: $T : \mathfrak{X} \to Y$, $T^* : Y^* \to \mathfrak{X}^*$ by $T^*\varphi = \varphi \circ T$.~~

~~Prop: If $T \in B(\mathfrak{X}, Y)$, $\mathfrak{X}, Y$ normed, then $\|T^*\| \in B(Y^*, \mathfrak{X}^*)$ w/ $\|T\| = \|T^*\|$.~~

Pf: $\|T^*\varphi\| = \sup_{\|x\|=1} |\varphi(Tx)| \leq \sup_{\|x\|=1} \|\varphi\| \cdot \|T\| \cdot \|x\| = \|\varphi\| \cdot \|T\|.$ (4)

$\Rightarrow \|T^*\| \leq \|T\|.$

For $\varepsilon > 0$, $\exists x \in X$ w/ $\|x\|=1$ and $\|Tx\| \geq \|T\| - \varepsilon$. Let $\varphi \in Y^*$ s.t. $\varphi(Tx) = \|Tx\|$ and $\|\varphi\| = 1$. Then

$$\|T\| \leq \varepsilon + \|Tx\| = \varepsilon + \varphi(Tx) = \varepsilon + |\varphi(Tx)| = \varepsilon + |(T^*\varphi)(x)| \leq \varepsilon + \|T^*\|.$$

Since $\varepsilon > 0$ arbitrary, $\|T\| \leq \|T^*\|$.

Exercise: X, Y Banach, $T: X \to Y$, $S: Y^* \to X^*$ lin, $\varphi(Tx) = (S\varphi)x$ $\forall x \in X$, $\varphi \in Y^*$.

Then T, S bdd and $S = T^*$.

Topological vector spaces

Def: A top. $\vec{v}$.sp. (TVS) is a pair $(\mathbb{X}, \tau)$ w/ $\mathbb{X}$ a $\vec{v}$.sp. and τ a topology on $\mathbb{X}$ s.t.

1. τ is Hausdorff
2. $\vec{v}$.sp. operations $+ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $\cdot : \mathbb{F} \times \mathbb{X} \to \mathbb{X}$ are cts.

Exercises:

1. Translation/Dilation are cts, homeomorphisms ($\lambda \neq 0$).
   $T_a x = a + x$ $\quad M_\lambda x = \lambda x$
2. Any linear map $\mathbb{F}^n \to \mathbb{X}$ is cts.
3. If $A \subseteq \mathbb{X}$, $\overline{A} = \bigcap \{ A + \mathcal{U} \mid \mathcal{U} \text{ open nbd of } 0_{\mathbb{X}} \}$
4. $A, B \subseteq \mathbb{X}$, $\overline{A} + \overline{B} \subseteq \overline{A+B}$
5. $Y \subseteq \mathbb{X}$ subspace, so is $\overline{Y}$.

Pf of 3: $x \in \overline{A} \iff (x + \mathcal{U}) \cap A \neq \emptyset$ $\forall$ open nbhd $\mathcal{U}$ of 0
$\iff x \in A - \mathcal{U}$ $\forall \mathcal{U}$
$\iff x \in A + \mathcal{U}$ $\forall \mathcal{U}$ $\quad$ ($\mathcal{U}$ nbhd $\iff -\mathcal{U}$ nbhd)

Convexity: A subset $S \subseteq \mathbb{X}$ ($\vec{v}$sp.) is called convex if
$tS + (1-t) S \subseteq S$, $\forall t \in [0,1]$.

Exercises:

1. If $S \subseteq \mathbb{X}$ convex, $x_1, \dots, x_n \in S$, $\alpha_1, \dots, \alpha_n \in [0,1]$ s.t. $\sum \alpha_i = 1$, then $\sum \alpha_i x_i \in S$.
2. If $\mathbb{X}$ is a TVS and $S \subseteq \mathbb{X}$ convex, then so are $\overline{S}$ and S°.
3. Convex hull: $\operatorname{conv}(S) = \{ \sum_{i=1}^n \alpha_i x_i \mid \alpha_i \in [0,1], \sum \alpha_i = 1, x_1, \dots, x_n \in S, n \in \mathbb{N} \}$.
   Exer: (a) Smallest conv. set containing S. (b) $\mathcal{U}$ open $\Rightarrow \operatorname{conv}(\mathcal{U})$ open. (TVS)

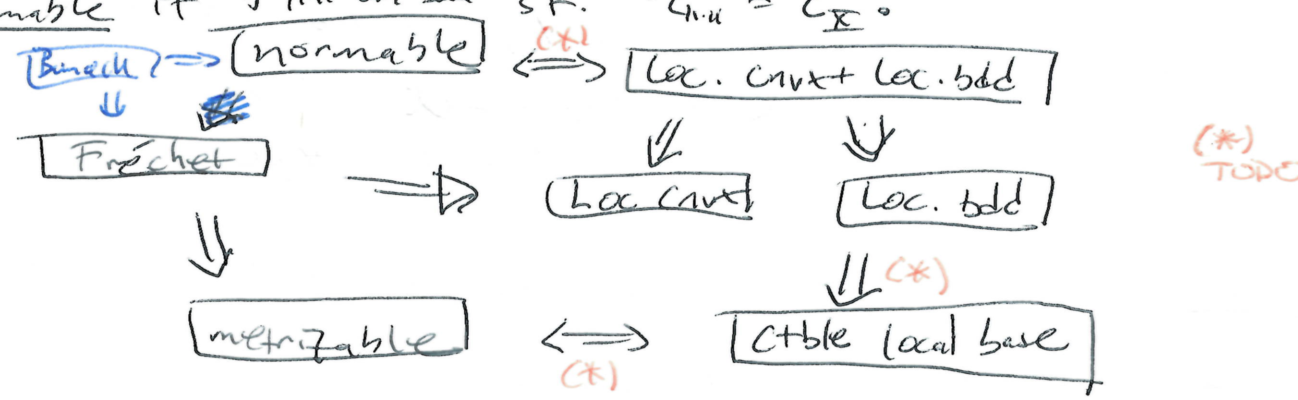
Bdd subsets: A subset $S \subseteq X$ (TVS) is called **bdd** if $\forall$ open nbhd $\mathcal{U}$ of 0_X, $\exists r>0$ s.t. $\forall t>r$, $S \subseteq t\mathcal{U}$.

Exercise: If $S \subseteq X$ bdd, so is $\bar{S}$.

A TVS X is called:

- **locally convex** if there is a local base $\mathcal{B}$ of 0_X consisting of convex sets. (every open $0_X \in \mathcal{U}$ contains a $B \in \mathcal{B}$.)
- **locally bdd** if 0_X has a bdd open nbhd.
- **locally cpt** if 0_X has an open nbhd w/ cpt closure.
- **metrizable** if τ_X is compatible w/ a metric d on X [$\exists d$ s.t. τ_X is induced by d]
- a **Fréchet space** if τ is induced by a complete translation invariant metric and is locally convex.
- **normable** if $\exists \|\cdot\|$ on X s.t. $\tau_{\|\cdot\|} = \tau_X$.

Helpful Table:



Linear maps btw TVS's:

Prop: $T: X \to Y$ linear, X, Y TVS's. TFAE:
1. T cts
2. T cts at 0_X.

$\Rightarrow$ 3. T bdd.

Moreover, T is uniformly cts in the sense that $\forall$ nbhd $\mathcal{U}$ of 0_Y, $\exists$ nbhd V of 0_X s.t. $y - x \in V \Rightarrow Ty - Tx \in \mathcal{U}$.

Pf: ② ⇒ ①: Pick nbhd $\mathcal{U}$ of 0_Y. Since T cts at 0_X,

$\exists$ open V of 0_X s.t. $TV \subseteq \mathcal{U}$. If $y-x \in V$,

$Ty - Tx = T(y-x) \in \mathcal{U}$. Thus $T(x+V) \subseteq Tx + \mathcal{U}$.

① ⇒ ③: Let $S \subseteq X$ bdd. Let $V \subseteq Y$ open nbhd of 0_Y. $\exists$ open nbhd $\mathcal{U}$ of 0_X s.t. $T\mathcal{U} \subseteq V$. S bdd $\Rightarrow$ $S \subseteq t\mathcal{U}$ large t, so $TS \subseteq T(t\mathcal{U}) = tT\mathcal{U} \subset tV$. So TS bdd in Y.

Prop: Let X be a TVS and $\varphi: X \to \mathbb{F}$ a linear ftl.

Suppose $\varphi \neq 0$ $[\exists x \in X$ s.t. $\varphi x \neq 0]$ TFAE:

① φ cts

② $\ker\varphi$ closed

③ $\ker\varphi$ is not dense in X

④ $\exists$ open nbhd $\mathcal{U}$ of 0_X s.t. $\varphi\mathcal{U}$ is bdd in $\mathbb{F}$.

Pf: ① ⇒ ②: $\ker\varphi = \varphi^{-1}(\{0\})$ closed ✓

② ⇒ ③: $\ker\varphi \neq X$ by assumption, so $\ker\varphi = \overline{\ker\varphi} \neq X$.

③ ⇒ ④: Suppose $\overline{\ker\varphi} \neq X$. Then ~~$\exists$ open~~ $(\overline{\ker\varphi})^c$ is

open + nonempty. Let $x \in (\overline{\ker\varphi})^c$ $(\varphi x \neq 0)$

and let $\mathcal{U}$ be an open nbhd of 0_X s.t. $x + \mathcal{U} \cap \ker\varphi = \emptyset$.

$(x + \mathcal{U} \subseteq (\overline{\ker\varphi})^c)$

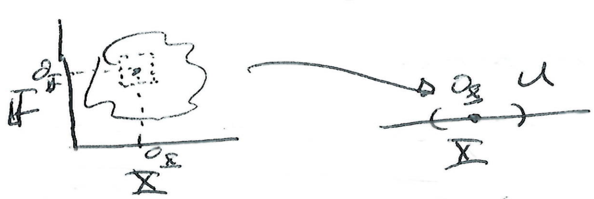
Claim: Every open nbhd of 0_X contains a balanced open nbhd of 0_X.

↳ V balanced if $\alpha V \subseteq V$ $\forall |\alpha| \leq 1$.

Ex: If B balanced, so is $\overline{B}$. If $0 \in B$, so is B°.

Pf: Since scalar mult. is cts, $\cdot : \mathbb{F} \times X \to X$, $\exists \delta > 0$ and

$W \subseteq X$ open nbhd of 0_X s.t. $\alpha W \subseteq \mathcal{U}$ $\forall |\alpha| < \delta$.



Set $V = \bigcup_{|\alpha| < \delta} \alpha W$, open nbhd of 0_X, balanced, $V \subseteq \mathcal{U}$.

Back to pf: WLOG, $\mathcal{U}$ is balanced.

Claim: $\varphi\mathcal{U} \subseteq \mathbb{F}$ is balanced.

Pf: if $\varphi u \in \varphi\mathcal{U}$, $|\alpha| \le 1$, $\alpha\varphi u = \varphi(\alpha u) \in \varphi\mathcal{U}$.

Q: What are the balanced subsets of $\mathbb{F}$?

- $\mathbb{F}$, $\{0_{\mathbb{F}}\}$, $B_r(0_{\mathbb{F}})$, $\overline{B_r}(0_{\mathbb{F}})$

So either $\varphi\mathcal{U}$ is bdd, or $\varphi\mathcal{U} = \mathbb{F}$. If the latter holds,

$\exists y \in \mathcal{U}$ s.t. $\varphi y = -\varphi x$, so $\underset{0_{\mathbb{F}}}{\underset{\shortparallel}{\varphi(x+y)}} \in \varphi(x+\mathcal{U})$

But $x+\mathcal{U} \cap \ker\varphi = \emptyset$, a contradiction!

(4) $\Rightarrow$ (1): Suppose $\exists$ open nbhd $\mathcal{U}$ of 0_X s.t. $|\varphi x| < M$ $\forall x \in \mathcal{U}$.

Show φ cts at 0_X. Let $\varepsilon > 0$, and consider $V = \frac{\varepsilon}{M}\mathcal{U}$.

Then $x \in V \Rightarrow |\varphi x| < \varepsilon$, and we are finished.

Prop: $n \in \mathbb{N}$ and F an n-dim'l subspace of a TVS X. Then:

1. Every iso $\mathbb{F}^n \to F$ is a homeom.
2. F is closed.

Pf: (1) Let $S^{n-1} \subseteq \mathbb{F}^n$ be n-1 sphere $\{x \in \mathbb{F}^n \mid |x| = 1\}$.
Suppose $T: \mathbb{F}^n \to F$ is an iso, (linear, inj), $T\mathbb{F}^n = F$. Let $K = TS^{n-1}$, cpt.
Since $T0 = 0$ and T inj, $0 \notin K$. So $\exists$ balanced nbhd $\mathcal{U}$ of 0_X

s.t. $\mathcal{U} \cap K = \emptyset$. Then $T^{-1}\mathcal{U} = T^{-1}(\mathcal{U} \cap F)$ disj from S^{n-1}!

Since $0 \in T^{-1}\mathcal{U}$, $T^{-1}\mathcal{U}$ is an open nbhd of $0_{\mathbb{F}^n}$ contained in $B_1(0)$.

But $T^{-1}|_F$ is an n-tuple of linear fnls, by prev. prop., T^{-1} cts.

Thus T a homeom.

(2) Let $x \in \overline{F}$, let T, $\mathcal{U}$ be as above. $\exists t > 0$ s.t.

$x \in t\mathcal{U}$, so $x \in \overline{F \cap t\mathcal{U}} \subseteq \overline{T(tB_1(0))} \subseteq T(\underbrace{\overline{tB_1(0)}}_{\text{cpt!}}) = \underset{F.}{\underset{\cap\shortmid}{T(\overline{tB_1(0)})}}$

Cor: $\exists!$ TVS structure on a f. dim'l v.sp. F over $\mathbb{F}$.

Nets:

An index set is a partially ordered set $(\Lambda, \leq)$ s.t. $\forall \lambda_1, \lambda_2 \in \Lambda$, $\exists \lambda_3 \geq \lambda_1, \lambda_2$.

A net in a set X is a fct $\Lambda \xrightarrow{x} X$ where Λ is some index set.

we write $x_\lambda = x(\lambda)$.

A subnet of $(x_\lambda)_{\lambda\in\Lambda}$ is a net $\Gamma \xrightarrow{y} X$ together w/ a map $\Gamma \xrightarrow{z} \Lambda$ s.t. (a) $y = x \circ z$ and (b) $\forall \lambda \in \Lambda$, $\exists \gamma_\lambda \in \Gamma$ s.t. $\lambda \leq z(\gamma)$ $\forall \gamma \geq \gamma_\lambda$. [some include (c) order preserving]

we say a net (x_λ) is

① eventually in $Y \subseteq X$ if $\exists \lambda_Y$ s.t. $\lambda \geq \lambda_Y \Rightarrow x_\lambda \in Y$

② frequently in $Y \subseteq X$ if $\forall \lambda \in \Lambda$, $\exists \lambda' \geq \lambda$ s.t. $x_{\lambda'} \in Y$.

(X, τ) a top. space. Write $\mathcal{O}(x)$ for the set of all open nbhds of $x \in X$. (neighborhood filter)

Say $x_\lambda \to x$ if $\forall \mathcal{U} \in \mathcal{O}(x)$, (x_λ) is eventually in $\mathcal{U}$.

Say x is an acc. pt. of (x_λ) if $\forall \mathcal{U} \in \mathcal{O}(x)$, (x_λ) is frequently in $\mathcal{U}$.

Facts:

① $\forall$ acc. pt. of a net (x_λ), $\exists$ subnet $(y_\gamma) \to x$.

② $x \in \overline{Y} \iff \exists$ net $(x_\lambda) \subseteq Y$ w/ $x_\lambda \to x$.

③ $K \subseteq X$ cpt $\iff$ any net $(x_\lambda) \subseteq K$ has an acc. pt.

④ $f: X \to Y$ cts $\iff [x_\lambda \to x \Rightarrow f(x_\lambda) \to f(x)]$

⑤ Conv. nets in a locally bdd TVS are eventually bdd.
[if $x_\lambda \to 0$, $\exists$ bdd nbhd $\mathcal{U}$ of 0_X. Eventually, $x_\lambda \in \mathcal{U}$.]

Seminormed TVS's:

Let $\mathbb{X}$ be a v.sp. and $\mathcal{M}$ a sep. family of seminorms.

[• $\forall x \neq y$ in $\mathbb{X}$, $\exists m \in \mathcal{M}$ s.t. $m(x-y) \neq 0$. Happens iff:
• the fcts $x \longmapsto m(x-y)$ indexed by $m \in \mathcal{M}$, $y \in \mathbb{X}$ separate pts.]

Def: The initial/seminorm topology $\tau_{\mathcal{M}}$ is the weakest top. s.t.
$\forall m \in \mathcal{M}$, $\forall y \in \mathbb{X}$, $x \longmapsto m(x-y)$ is cts.

Lem: $(\mathbb{X}, \tau_{\mathcal{M}})$ is Hausdorff.

Pf: For $y \neq z$ in $\mathbb{X}$, pick $m \in \mathcal{M}$ s.t. $x \overset{f}{\longmapsto} m(x-y)$ does not vanish at z.
Since f is cts, can pull back disj. open subsets of 0 and $m(z-y)$.

~~Fact~~: A basis for $\mathcal{O}(x)$ is given by the sets
(exercise)

$$\{y \in \mathbb{X} \mid m_i(x-y) < \varepsilon,\ i=1,\dots,n,\ m_1,\dots,m_n \in \mathcal{M}\}$$

sets where finitely many $m \in \mathcal{M}$ are ε-small for $x-y$ $\forall y \in \mathbb{X}$.

These sets are convex: if y_1, y_2 in above set and $t \in [0,1]$, then

$$m_i(x-(ty_1+(1-t)y_2)) = m_i(t(x-y_1)+(1-t)(x-y_2)) \leq t\, m_i(x-y_1)+(1-t)\, m_i(x-y_2) < \varepsilon \quad \forall i=1,\dots,n.$$

Cor: A net (x_λ) conv. to x in $(\mathbb{X}, \tau_{\mathcal{M}}) \Leftrightarrow m(x-x_\lambda) \to 0$ $\forall m \in \mathcal{M}$.

Fact: If Y a top. space and $f: Y \to (\mathbb{X}, \tau_{\mathcal{M}})$, then f is cts.
iff $y_\lambda \to y \Rightarrow m(f(y_\lambda) - f(y)) \to 0$ $\forall m \in \mathcal{M}$.

Prop: $(\mathbb{X}, \tau_{\mathcal{M}})$ is a loc. cnvx TVS.

Pf: Only need to show $+$, $\cdot$ are cts.

$+$: If $x_\lambda \to x$, $y_\lambda \to y$, $m(x_\lambda + y_\lambda - x - y) \leq m(x_\lambda - x) + m(y_\lambda - y) \to 0$.

(*) why can we take (x_λ), (y_λ) same λ?

$\cdot$: If $\alpha_\lambda \to \alpha$ in $\mathbb{F}$ and $x_\lambda \to x$, $m(\alpha_\lambda x_\lambda - \alpha x) =$
$= m(\alpha_\lambda x_\lambda - \alpha_\lambda x + \alpha_\lambda x - \alpha x) \leq m(\alpha_\lambda(x_\lambda - x)) + m((\alpha_\lambda - \alpha)x)$
$\leq |\alpha_\lambda|\, m(x_\lambda - x) + |\alpha_\lambda - \alpha|\, m(x) \to 0.$

Eventually bdd!

Typically, we use a sep. family of fctl's F on $\mathbb{X}$, define $\mathcal{M}_F = \{x \mapsto |\varphi(x)| \mid \varphi \in F\}$.
Then $\tau_{\mathcal{M}_F}$ called the weak topology induced by F.

~~Semi~~ Normed spaces ctd.

If $\mathcal{M}$ is a separating family of seminorms on a v.sp. $\mathfrak{X}$, give $\mathfrak{X}$ initial topology s.t. $\{x \to m(x-y) \mid m \in \mathcal{M}, y \in \mathfrak{X}\}$ are cts.

Call this $\tau_{\mathcal{M}}$. $(\mathfrak{X}, \tau_{\mathcal{M}})$ is a TVS.

$x_\lambda \to x \iff m(x_\lambda - x) \to 0 \quad \forall m \in \mathcal{M}$.

Basis for $\mathcal{O}(x)$ is $\left\{ N(x; m_1, \dots, m_n; \varepsilon) \,\middle|\, \begin{matrix} m_1, \dots, m_n \in \mathcal{M} \\ n \in \mathbb{N}, \varepsilon > 0 \end{matrix} \right\}$ where

$$N(x; m_1, \dots, m_n; \varepsilon) = \{y \in \mathfrak{X} \mid m_i(x-y) < \varepsilon \ \forall i = 1, \dots, n\}.$$

Typical example: let Y be a separating linear space of ftl's on $\mathfrak{X}$, and let $\mathcal{M}_Y = \{x \mapsto |\varphi(x)| \mid \varphi \in Y\}$.

Call $\tau_{\mathcal{M}_Y}$ the weak topology induced by Y.

Exercise: $(\mathfrak{X}, \tau_{\mathcal{M}_Y})^* = Y$.

Symmetric version: Suppose $\mathfrak{X}, Y$ are v.sp in algebraic duality via a bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{X} \times Y \to \mathbb{F}$. ~~IE~~ $\{\mathfrak{X}$ sep. pts of Y and ~~IE~~ Y sep. pts of $\mathfrak{X}\}$ = alg. duality. If we endow $\mathfrak{X}$ + Y w/ respective weak topologies, then $\mathfrak{X}^* = Y$ and $Y^* = \mathfrak{X}$.

Warning: Can have 2 distinct topologies τ_1, τ_2 on $\mathfrak{X}$ s.t. $(\mathfrak{X}, \tau_1)^* = (\mathfrak{X}, \tau_2)$, where both τ_1, τ_2 come from a sep. family of seminorms! (will be accurate later.)

Reverse direction: let's show a locally convex TVS is induced by a sep. family of seminorms.

Def: A <u>Minkowski ftnl</u> on a v.sp. X is a fct
$\mu: X \to \mathbb{R}$ s.t.

① (Subadditive) $\mu(x+y) \leq \mu(x) + \mu(y)$

② (pos. homog.) $\mu(\lambda x) = \lambda \mu(x)$ $\forall \lambda \geq 0$.

<u>Key Lemma</u>: Let $V \subseteq (X, \tau)$ be a convex open nbhd of 0_X.

Define $\mu_V(x) = \inf\{r > 0 \mid r^{-1}x \in V\}$ (V absorbing ⇒ well defined!)

① μ_V is a Minkowski ftnl.

② $rV = \{x \in X \mid \mu_V(x) < r\}$ $\forall r > 0$.

③ μ_V is a <u>seminorm</u> if V is <u>balanced</u>.

④ μ_V is a <u>norm</u> if V is <u>balanced</u> + <u>bdd</u>.

Pf: ① Note if $S \subset \mathbb{R}$ and $r > 0$, $\inf(rS) = r \cdot \inf(S)$.
Hence μ_V is pos. homog. Let $x, y \in X$, and let $r, s > 0$ s.t. $r^{-1}x \in V$ and $s^{-1}y \in V$. Since V convex,

$$(r+s)^{-1}(x+y) = \frac{r}{(r+s)}(r^{-1}x) + \frac{s}{(r+s)}(s^{-1}y) \in V,$$ and thus

$\mu_V(x+y) \leq r+s$. This is true $\forall r > \mu_V(x)$ and $s > \mu_V(y)$,

so $\mu_V(x+y) \leq \mu_V(x) + \mu_V(y)$.

② Since μ_V pos. homog., it suffices to assume $r = 1$.

[$x \in X$ w/ $\mu_V(x) < r \iff r^{-1}\mu_V(x) < 1$
$\iff \mu_V(r^{-1}x) < 1$
$\iff r^{-1}x \in V$
$\iff x \in rV$]

If $x \in V$, then $\exists \varepsilon > 0$ s.t. $(1+\varepsilon)x \in V$ as V is open and $\cdot : \mathbb{F} \times X \to X$ co.
Then $\mu_V(x) \leq \frac{1}{1+\varepsilon} < 1$. Conversely, if $\mu_V(x) < 1$, $\exists r < 1$ s.t. $r^{-1}x \in V$. Since $0 \in V$, V convex, $x = (1-r)0 + r(r^{-1}x) \in V$.

③ immediate from ①: If V balanced, $\lambda \in \mathbb{F}$ ~~then~~ w/ $|\lambda| = 1$, then $\mu_V(\lambda x) = \mu_V(x)$ $\forall x \in X$. [$\lambda x \in V \iff x \in V$ if $|\lambda| = 1$] If $\alpha \in \mathbb{F}$, $\alpha \neq 0$

$\mu_V(\alpha x) = \mu_V(|\alpha| \frac{\alpha}{|\alpha|} x) = |\alpha| \cdot \mu_V(\frac{\alpha}{|\alpha|} x) = |\alpha| \mu_V(x)$.

(4) Suppose $x \neq 0$. Since $\mathfrak{X}$ is Hausdorff, $\exists$ open nbhd $\mathcal{U} \in \mathcal{O}(0_{\mathfrak{X}})$ s.t. $x \notin \mathcal{U}$. Since V bdd, $\exists r > 0$ s.t. $V \subseteq r\mathcal{U}$, iff $r^{-1}V \subseteq \mathcal{U}$. Hence $x \notin r^{-1}V$. By (2), $\mu_V(x) \geq r^{-1}$, and thus μ_V is definite. $[\mu_V(x) < r^{-1} \Leftrightarrow x \in r^{-1}V]$ (3)

Thm: Let $(\mathfrak{X}, \tau)$ be a loc. cnvx TVS. Then $\exists$ a sep. family $\mathcal{M}$ of seminorms on $\mathfrak{X}$ s.t. $\tau = \tau_{\mathcal{M}}$.

Pf: $(\mathfrak{X}, \tau)$ loc. cnvx $\Rightarrow$ $\exists$ a convex balanced local base $\mathcal{B}$ for $\mathcal{O}(0_{\mathfrak{X}})$. (use HW!)

Then $\mathcal{M} = \{\mu_V \mid V \in \mathcal{B}\}$ is a family of seminorms on $\mathfrak{X}$.

Separating: If $x \in \mathfrak{X}$ w/ $x \neq 0$, then $\exists V \in \mathcal{B}$ s.t. $x \notin V$. For this V, $\mu_V(x) \geq 1$.

$\tau_{\mathcal{M}} \subseteq \tau$: It suffices to show $\forall m \in \mathcal{M}$, $\forall y \in \mathfrak{X}$, $x \mapsto m(x-y)$ cts. Let $V \in \mathcal{B}$. By subadditivity, $\forall x, z \in \mathfrak{X}$, μ_V $V \in \mathcal{B}$. y

$$|\mu_V(x-y) - \mu_V(z-y)| \leq \mu_V(x-z) < r$$

if $x - z \in rV$.

$\tau \subseteq \tau_{\mathcal{M}}$: Show $\forall V \in \mathcal{B}$, $V \in \tau_{\mathcal{M}}$. Note $V = \{x \in \mathfrak{X} \mid \mu_V(x) < 1\}$, so $V = \mu_V^{-1}([0,1))$, which is open in $\tau_{\mathcal{M}}$.

Thm: $(\mathfrak{X}, \tau)$ normable $\Leftrightarrow$ $\mathfrak{X}$ loc. cnvx + loc. bdd.
$\Leftrightarrow$ $\exists$ cnvx bdd $V \in \mathcal{O}(0_{\mathfrak{X}})$.

easy: $\Rightarrow$ trivial
$\Leftarrow$ bdd sets are absorbed by V.

Pf:

$\Rightarrow$: If $\mathfrak{X}$ normable, set $V = \{x \in \mathfrak{X} \mid \|x\| < 1\}$.

$\Leftarrow$: Suppose $\exists$ cnvx bdd $V \in \mathcal{O}(0_{\mathfrak{X}})$. WLOG, may assume V balanced.

Claim: $\mathcal{B} = \{rV \mid r > 0\}$ is a cnvx + balanced bdd local base.

Pf: Clear each rV is cnvx, balanced + bdd. If $\mathcal{U} \in \mathcal{O}(0_{\mathfrak{X}})$, $\exists r > 0$ s.t. $V \subseteq r\mathcal{U} \Leftrightarrow r^{-1}V \subseteq \mathcal{U}$, so $\mathcal{B}$ a local base.

Now μ_V is a norm, and $\mu_{rV} = r\mu_V$ $\forall r > 0$. Thus $\tau_{\mathcal{M}_{\mathcal{B}}} = \tau_{\|\cdot\|} = \tau$.

Hahn-Banach Thm: If X is a $\mathbb{R}$-v.sp. and μ a Minkowski ftl on X, and ~~Y~~ $Y \subseteq X$ a subspace, and $\varphi: Y \to \mathbb{R}$ a linear ftl dominated by μ [$\varphi(y) \le \mu(y) \ \forall y \in Y$], then $\exists$ a ftl $\tilde{\varphi}$ on X dominated by μ s.t. $\tilde{\varphi}|_Y = \varphi$.

Pf: Pf (1): If $x \in X \setminus Y$, any extension of φ to $Y + \mathbb{R}x$ is determined by

$\tilde{\varphi}(y + rx) = \varphi(y) + r\alpha \quad r \in \mathbb{R}, y \in Y$. Want to choose $\alpha \in \mathbb{R}$ s.t.

$\varphi(y) + r\alpha \le \mu(y + rx)$. By pos. homog. of μ & φ, we only need to look at $r = \pm 1$. These 2 conditions are:

$$\varphi(y) - m(y - x) \le \alpha \le -\varphi(z) + m(z + x) \quad \forall y, z \in Y.$$

Now: $-\varphi(z) + m(z+x) - \varphi(y) + m(y-x) \ge m(y-x) + m(z+x) - \varphi(y+z)$

$$\ge m(y+z) - \varphi(y+z) \ge 0.$$

So $\exists \alpha \in \left[\sup_y \{ \varphi(y) - m(y-x) \}, \ \inf_z \{ -\varphi(z) + m(z+x) \} \right]$

Pf (2): Use Zorn to extend all the way.

$\Lambda = \{ (Z, \psi) \mid Y \subseteq Z \subseteq X$ subspace, ψ ftl on Z s.t. $\psi|_Y = \varphi$ + $\psi \le \mu \}$

Def: $(Z_1, \psi_1) \le (Z_2, \psi_2) \iff Z_1 \subseteq Z_2$ and $\psi_2|_{Z_1} = \psi_1$.

Now Λ a partially ordered set.

Claim: Every ascending chain in Λ has an upper bd. in Λ.

Pf: If $(Z_i, \psi_i)_{i \in I}$ an ascending chain, let $Z = \bigcup_{i \in I} Z_i$ and define ψ on Z by $\psi(z) = \psi_i(z)$ if $z \in Z_i$.

By Zorn $\exists$ maximal elt $(\tilde{Y}, \tilde{\varphi})$. By part (1), if $\tilde{Y} \ne X$, can extend to $\tilde{Y} + \mathbb{R}x$ for $x \in X \setminus \tilde{Y}$, contradicting maximality.

Hence $\tilde{Y} = X$ and $\tilde{\varphi}$ is as claimed.

All about Hahn-Banach

HB1: $\varphi \le \mu$ can be extended on a $\mathbb{R}$-v.sp....

HB2: If m is a seminorm on a v.sp. $\mathfrak{X}$ and φ a ftl on $Y \subseteq \mathfrak{X}$ subspace s.t. $|\varphi| \le m$ on Y, then $\exists\, \tilde{\varphi}$ on $\mathfrak{X}$ s.t. $|\tilde{\varphi}| \le m$ and $\tilde{\varphi}|_Y = \varphi$.

Pf: If $\mathbb{F} = \mathbb{R}$, finished by HB1, since $\varphi(x) \le m(x) \Leftrightarrow |\varphi(x)| \le m(x)$

[I look at $\frac{\overline{\varphi(x)}}{|\varphi(x)|}x$ for $\varphi(x) \ne 0$.]

Assume $\mathbb{F} = \mathbb{C}$. Regard $\mathfrak{X}_{\mathbb{R}}$ as an $\mathbb{R}$ v.sp. and consider $Re(\varphi): \mathfrak{X}_{\mathbb{R}} \to \mathbb{R}$.

By HB1, $\exists\, \psi: \mathfrak{X}_{\mathbb{R}} \to \mathbb{R}$ s.t. $\psi|_{Y_{\mathbb{R}}} = Re(\varphi)$ and $\psi(x) \le m(x)\ \forall x$.

Define $\tilde{\varphi}: \mathfrak{X} \to \mathbb{C}$ by $\tilde{\varphi}(x) = \psi(x) - i\psi(ix)$. By HW3,

$\tilde{\varphi}$ is a $\mathbb{C}$-linear ftl, and $Re(\tilde{\varphi}) = \psi$. $\Rightarrow Re\,\tilde{\varphi}|_Y = \psi|_Y = Re(\varphi|_Y)$.

Then $\tilde{\varphi}|_Y = \varphi$ again by HW3.

Now, if $x \in \mathfrak{X}$, let $|\alpha| = 1$ so $\tilde{\varphi}(\alpha x) = |\tilde{\varphi}(x)| \in [0,\infty)$. Then

$$|\tilde{\varphi}(x)| = \tilde{\varphi}(\alpha x) = \psi(\alpha x) \le m(\alpha x) = m(x).$$

Cor: If $\mathfrak{X}$ normed, $x \ne 0$, $\exists\, \varphi \in \mathfrak{X}^*$ w/ $\|\varphi\| = 1$ s.t. $\varphi(x) = \|x\|$.

Pf: Define φ on $\mathbb{F}x$ by $\varphi(\alpha x) = |\alpha| \cdot \|x\|$, and note $\|\varphi\| = 1$. Now extend via HB2 w/ $m = \|\cdot\|$.

HB3 (Separating hyperplane Thm) Suppose $A, B \subseteq (\mathfrak{X}, \tau)$ ← TVS are disj., nonempty cvx sets.

① If A is open, $\exists\, \varphi \in \mathfrak{X}^*$ and $r \in \mathbb{R}$ s.t.

$$Re\,\varphi(a) < r \le Re\,\varphi(b) \qquad \forall a \in A,\ b \in B.$$

② If A cpt., B closed, and $(\mathfrak{X}, \tau)$ loc. cvx, $\exists\, \varphi \in \mathfrak{X}^*$ and $r_1, r_2 \in \mathbb{R}$ s.t. $Re\,\varphi(a) < r_1 < r_2 < Re\,\varphi(b)\ \forall a \in A, b \in B$.

Pf: Case 1: $\mathbb{F} = \mathbb{R}$. Choose $a_0 \in A$, $b_0 \in B$, set $x = b_0 - a_0$ and $V = A - B + x$. Then V an open cvx nbhd of $0_{\mathfrak{X}}$, and $x \notin V$. (disj)

Consider the Mink. fnl μ_V of V, and not $\mu_V(x) \geq 1$ (2)
$(x \notin V)$

Define φ_0 on $\mathbb{R}x$ by $\varphi_0(\alpha x) = \alpha$. For $\alpha \geq 0$,

$$\varphi_0(\alpha x) = \alpha \leq \alpha \mu_V(x) = \mu_V(\alpha x), \text{ so } \varphi_0 \leq \mu_V.$$

Extend φ_0 to φ dominated by μ_V by HB1.

<u>Claim</u>: $\varphi \in X^*$.

<u>Pf</u>: Let $\varepsilon > 0$. When $x \in V$, $\varphi(x) \leq \mu_V(x) < 1$, so

$\forall x \in \varepsilon V \cap (-\varepsilon V)$, $|\varphi(x)| < \varepsilon$.

Finally if $a \in A, b \in B$, $a - b + x \in V \Rightarrow \varphi(a-b+x) \leq \mu_V(a-b+x) < 1 = \varphi(x)$.

$\Rightarrow \varphi(a) < \varphi(b)$ $\forall a \in A, b \in B$.

Now A, B cnvx, disj $\Rightarrow$ A, B conn., disj $\Rightarrow$ $\varphi A, \varphi B$ disj intervals w/ φA to the left of φB. A open $\Rightarrow \varphi A$ open (cont of scalar mult.)

So take $\gamma = \inf\{\varphi(b) \mid b \in B\}$.

<u>Step 2</u>: $\mathbb{F} = \mathbb{C}$. Consider $X_{\mathbb{R}}$ as an $\mathbb{R}$-v.sp., and find $\psi : X_{\mathbb{R}} \to \mathbb{R}$ s.t. $\psi(A) < \gamma \leq \psi(B)$. Define $\varphi(x) = \psi(x) - i\psi(ix)$ $\forall x \in X$.

Since $\operatorname{Re}\varphi = \psi$, we're finished, since ψ cts $\Rightarrow \varphi$ cts.

(2) HW4! (optional!)

<u>Cor</u>: If X is loc. cnvx TVS, X^* separates pts of X.

<u>Cor</u>: If $Y \subseteq X$ subspace of loc cnvx TVS and $x_0 \notin \overline{Y}$, then $\exists \varphi \in X^*$ s.t. $\varphi(x_0) = 1$ and $\varphi(y) = 0$ $\forall y \in Y$.

<u>Pf</u>: HB3 w/ $A = \{x_0\}$ and $B = \overline{Y}$. Then $\exists \varphi$ s.t. $\{\varphi(x_0)\}$ and φY are disjoint. But $\varphi Y \subseteq \mathbb{F}$ is a subspace $\Rightarrow \varphi Y = \{0\}$, $\varphi(x_0) \neq 0$.

Now scale by $\varphi(x_0)^{-1}$!

Cor: If $Y \subseteq X$ subspace of a loc. cvx TVS, and $\varphi \in Y^*$, $\exists \tilde{\varphi} \in X^*$ s.t. $\tilde{\varphi}|_Y = \varphi$.

Pf: We may assume $\varphi \neq 0$. Let $Y_0 = \ker\varphi = \{x \in Y \mid \varphi(x) = 0\}$. Pick $y \in Y \setminus Y_0$ s.t. $\varphi(y) = 1$. Since φ cts on Y, $y \notin$ closure of Y_0 in Y. Since Y has relative top., $y \notin$ closure of Y_0 in X. By prev. cor., $\exists \tilde{\varphi} \in X^*$ s.t. $\tilde{\varphi}(y) = 1$ and $\tilde{\varphi}|_{Y_0} = 0$. Then if $x \in Y$, $x - \varphi(x)y \in Y_0$ since $\varphi(y) = 1$. Thus:

$$\tilde{\varphi}(x) - \varphi(x) = \tilde{\varphi}(x) - \varphi(x)\tilde{\varphi}(y) = \tilde{\varphi}(x - \varphi(x)y) = 0,$$

and $\tilde{\varphi}|_Y = \varphi$ ■

Exercise: If X locally cvx TVS, B cvx balanced closed, $x_0 \in X \setminus B$. $\exists \varphi \in X^*$ s.t. $|\varphi(x)| \le 1\ \forall x \in B$, $\varphi(x_0) > 1$.

Last time: $\varphi \in X^*$, (X, τ) TVS, $\varphi \neq 0 \Rightarrow \varphi$ open map.

easy:
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "Q"'] & & \mathbb{F} \\ & X/\ker\varphi \arrow[ur, "\cong"'] & \end{tikzcd}$$
Q open, onto, and $X/\ker\varphi \cong \mathbb{F}$ is a homeom.

$X/\ker\varphi$ ← quotient topology. is a TVS structure.

Harder: If X, Y TVS's, Y f. dim'l, $T: X \to Y$ linear and onto, then T is open. If moreover $\ker(T)$ is closed, then T cts.

when $Y = \mathbb{F}$: Show $\forall$ open balanced $\mathcal{U} \in \mathcal{O}(0_X)$, $\exists V \in \mathcal{O}(0_{\mathbb{F}})$ s.t. $V \subseteq T\mathcal{U}$. (Claim 1: This is sufficient to show T open.)

(Claim 2: $T\mathcal{U} \neq \{0\}$. T onto $\Rightarrow \exists x \in X$ s.t. $Tx \neq 0$, so $T(\alpha x) \neq 0$ $\forall \alpha \neq 0$ in $\mathbb{F}$. But $\mathcal{U}$ abs. $\Rightarrow \exists r > 0$ s.t. $r^{-1}x \in \mathcal{U}$.)

Now let $u \in \mathcal{U}$ st. $Tu = \alpha \neq 0$. Since $\mathcal{U}$ balanced, $\beta u \in \mathcal{U}$ $\forall |\beta| \le 1 \Rightarrow \beta T u = \beta\alpha \in T\mathcal{U}$ $\forall |\beta| \le 1$, and thus contains an open nbhd of $0_{\mathbb{F}}$. ($T\mathcal{U}$ balanced in $\mathbb{F}$)

Because $T\mathcal{U} \neq 0$ and $T\mathcal{U}$ balanced.

Application: Banach limits.

Consider $X = \ell^\infty$, $Y = c \subseteq X$. Define $\varphi = \lim$ on Y,

and $\mu = \limsup$ on X.

Claim 1: μ a Minkowski fn on X.

Pf: $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$.

$\limsup (r x_n) = r \limsup x_n \quad \forall r > 0$.

Claim 2: $\varphi \leq \mu$ on Y (they're equal).

By HB1, $\exists$ extension $\tilde{\varphi}$ to X s.t. $\tilde{\varphi}|_Y = \varphi$ and $\tilde{\varphi} \leq \mu$.

Q: what on earth is $\tilde{\varphi}$?!

$\leadsto$ HB is horribly nonconstructive!

Even better: On $X = \ell^\infty$, define shift op S by

$[\underbrace{S(x_n)}_{Sx}]_n = x_{n+1}$, $\quad m_n(x) = \frac{1}{n}(x_1 + \dots + x_n)$,

$Y \subseteq X$ is subspace s.t. $\lim_{n\to\infty} m_n(x)$ exists.

Define φ on Y by $\varphi(x) = \lim_{n\to\infty} m_n(x)$ and

μ on ℓ^∞ by $\mu(x) = \limsup x_n$.

By HB1, get a $\tilde{\varphi} \in (\ell^\infty)^*$ s.t.

(1) $\tilde{\varphi}(Sx) = \tilde{\varphi}(x) \quad \forall x \in \ell^\infty$

(2) $\liminf x_n \leq \tilde{\varphi}(x) \leq \limsup x_n$