Problem 1. Suppose A is a unital Banach algebra and fix $a, b \in A$.

(1) Show that $1 \notin \operatorname{sp}_A(ab)$ if and only if $1 \notin \operatorname{sp}_A(ba)$ using the identity $(1-ba)^{-1} = 1 + b(1-ab)^{-1}a$. Deduce that $\operatorname{sp}_A(ab) \cup \{0\} = \operatorname{sp}_A(ba) \cup \{0\}$.
(2) Show that for any Banach subalgebra $B \subseteq A$ with $1_A \in B$, for every $a \in B$, the spectral radius in B of a is equal to the spectral radius in A of a, i.e., $r_B(a) = r_A(a)$.
(3) Suppose $a, b \in A$ commute. Prove that $r(ab) \leq r(a)r(b)$ and $r(a+b) \leq r(a) + r(b)$.
Hint: By (2), this computation can be performed in the unital commutative Banach subalgebra $B \subseteq A$ generated by a and b. In B, there is a helpful characterization of the spectrum.
(4) Deduce from part (3) that if A is commutative, the spectral radius $r : A \to [0, \infty)$ is continuous.

Problem 2. Let A be a unital Banach algebra. Suppose we have a norm convergent sequence $(a_n) \subset A$ with $a_n \to a$. Prove that for every open neighborhood U of $\operatorname{sp}(a)$, there is an $N > 0$ such that $\operatorname{sp}(a_n) \subset U$ for all $n > N$.

Problem 3. Let $A \in M_n(\mathbb{C})$.

(1) As best as you can, describe $f(A)$ where $f \in \mathcal{O}(\operatorname{sp}(A))$.
Hint: First consider the case that A is a single Jordan block.
(2) Determine as best you can which matrices $A \in M_n(\mathbb{C})$ have square roots, i.e., when there is a $B \in M_n(\mathbb{C})$ such that $B^2 = A$.
Note: Such a B is not necessarily unique.

Problem 4. Suppose A is a C*-algebra and $a \in A$ is normal.

(1) Show a is self-adjoint if and only if $\operatorname{sp}(a) \subset \mathbb{R}$.
(2) Show a is unitary if and only if $\operatorname{sp}(a) \subset \mathbb{T}$.
(3) Show a is a projection if and only if $\operatorname{sp}(a) \subset \{0, 1\}$.

Problem 5. Let A be a C*-algebra.

(1) Show that the following are equivalent for a self-adjoint $a \in A$:
   (a) $\operatorname{sp}(a) \subset [0, \infty)$,
   (b) For all $\lambda \geq \|a\|$, $\|a - \lambda\| \leq \lambda$, and
   (c) There is a $\lambda \geq \|a\|$ such that $\|a - \lambda\| \leq \lambda$.
For now, we will call such elements *spectrally positive.*
Note: It is implicit here that a spectrally positive element is self-adjoint.
(2) Deduce that the spectrally positive elements in a C*-algebra form a closed cone, i.e., $A_+ = \{a \in A | a \geq 0\}$ is closed, and for all $\lambda \in [0, \infty)$ and $a, b \in A_+$, we have $\lambda a + b \in A_+$.
(3) Show a is positive ($a = b^*b$ for some b) if and only if a is spectrally positive ($a = a^*$ and $\operatorname{sp}(a) \subset [0, \infty)$).
Hint: First, if $\operatorname{sp}(a) \subset [0, \infty)$, we can define $a^{1/2}$ via the continuous functional calculus. Now suppose $a = b^*b$ for some $b \in B$. Use the continuous functions $r \mapsto \max\{0, z\}$ and $r \mapsto -\min\{0, z\}$ on $\operatorname{sp}(a)$ to write $a = a_+ - a_-$ where $\operatorname{sp}(a_\pm) \subset [0, \infty)$ and $a_+a_- = a_-a_+ = 0$. Now look at $c = ba_-$. Prove that $\operatorname{sp}(c^*c) \subset (-\infty, 0]$ and $\operatorname{sp}(cc^*) \subset [0, \infty)$ using part (1) of this problem. Use part (1) of Problem 1 to deduce that $c^*c = 0$. Finally, deduce $a_- = 0$, and thus $a = a_+$.

Problem 6. For $a, b \in A$, we say $a \leq b$ if $b - a \geq 0$.

(1) Show that $\leq$ is a partial order.
(2) Show that if $a \leq b$, then for all $c \in A$, $c^*ac \leq c^*bc$.

(3) Suppose $0 \le a \le b$. Prove that $\|a\| \le \|b\|$.

Problem 7. Let A be a C*-algebra. By the hint to part (4) of Problem 4 that for $a \ge 0$, we can define an $a^{1/2} \ge 0$ such that $(a^{1/2})^2 = a$.

(1) Show that if $b \ge 0$ such that $b^2 = a$, then $b = a^{1/2}$.
(2) Prove that if $0 \le a \le b$, then $a^{1/2} \le b^{1/2}$.
(3) Prove that if $0 < a$ ($0 \le a$ and a is invertible), then $0 < a^{-1}$.
(4) Prove that if $0 < a \le b$, then $0 < b$ and $0 < b^{-1} \le a^{-1}$.

Problem 8 (Rieffel, "Preventative Medicine")**.** Consider $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ for $s, t \ge 0$.

(1) Determine for which $s, t \ge 0$ we have $b \ge a$.
(2) Determine for which $s, t \ge 0$ we have $b \ge a_+$.
Note: Since $a = a^*$, a_+ is the positive part defined as in the hint to part (4) of Problem 4.
(3) Find values of $s, t \ge 0$ for which $b \ge a$, $b \ge 0$, and yet $b \not\ge a_+$.
(4) Find values of $s, t \ge 0$ such that $b \ge a_+ \ge 0$, and yet $b^2 \not\ge a_+^2$.
(5) Can you find $s, t \ge 0$ such that $b \ge a_+$ and yet $b^{1/2} \not\ge a_+^{1/2}$?
Note: $a_+^{1/2}$ is the unique positive square root of a_+ from part (1) Problem 7.
(6) Suppose $c, p \in M_2(\mathbb{C})$ such that $c \ge 0$ and $p^2 = p^* = p$ is a projection. Is it always true that $pcp \le c$?

Problem 9. Let $L^2(\mathbb{T})$ denote the space of complex-valued square-integrable 1-periodic functions on $\mathbb{R}$, and let $C(\mathbb{T}) \subset L^2(\mathbb{T})$ denote the subspace of continuous 1-periodic functions.

(a) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
(b) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(\mathbb{T})} = \int_0^1 f(x) \exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

Problem 10. Recall that each $T \in B(H, K)$ induces a bounded sesquilinear form $K \times H \to \mathbb{C}$ given by $B_T(\xi, \eta) = \langle \xi, T\eta \rangle$.

(1) Prove that $T \mapsto B_T$ is an isometric bijective correspondence between operators in $B(H, K)$ and bounded sesquilinear forms $K \times H \to \mathbb{C}$.
Hint: Adapt the proof Lemma 3.2.2 in Analysis Now (see also Exercise 3.2.15 therein).
(2) For $T \in B(H, K)$ corresponding to $B_T : K \times H \to \mathbb{C}$, we define $T^* \in B(K, H)$ to be the unique operator corresponding to the adjoint sesquilinear form $B_T^* : H \times K \to \mathbb{C}$ defined by

$$B_T^*(\eta, \xi) := \overline{B_T(\xi, \eta)} \qquad \Longleftrightarrow \qquad \langle \eta, T^*\xi \rangle = \langle T\eta, \xi \rangle \qquad \eta \in H, \xi \in K.$$

Show that $T \mapsto T^*$ is a conjugate linear isometry of $B(H, K)$ onto $B(K, H)$, and that $\|T^*T\| = \|T\|^2 = \|TT^*\|$.
(3) In the case that $H = K$, deduce the following:
  (a) $B(H)$ with involution $T \mapsto T^*$ is a C*-algebra.
  (b) $T = T^*$ if and only if B_T is self-adjoint. That is, show $T = T^*$ if and only if $\langle T\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in H$.
  (c) $T \ge 0$ if and only if B_T is positive. That is, show $T \ge 0$ if and only if $\langle T\xi, \xi \rangle \ge 0$ for all $\xi \in H$.
  Hint: Use that for $T = T^*$, we have $\inf\{\langle T\xi, \xi \rangle | \xi \in H, \|\xi\| = 1\} = \min\{\lambda | \lambda \in \mathrm{sp}(T)\}$.
  (d) (optional) $T \ge 0$ and T injective if and only if B_T is positive definite.
  Hint: For $S \in B(H)$, $\ker(S) = \ker(S^*S)$, so $T \ge 0$ is injective if and only if $T^{1/2}$ is injective.

(e) (optional) $T > 0$ ($T \geq 0$ and T is invertible) if and only if B_T is positive definite, and H is complete in the norm $\|\xi\|_T := B_T(\xi,\xi)^{1/2}$.
Hint: When B_T is positive definite and H is complete for $\|\cdot\|_T$, apply part (d) and look at the isometry $(H, \|\cdot\|_T) \to (H, \|\cdot\|)$ by $\xi \mapsto T^{1/2}\xi$.

Problem 11. For a Hilbert space H, we can define the *conjugate* Hilbert space $\overline{H} = \{\overline{\xi} | \xi \in H\}$ which has the conjugate vector space structure $\lambda\overline{\xi} + \overline{\eta} = \overline{\overline{\lambda}\xi + \eta}$ and the conjugate inner product $\langle \overline{\eta}, \overline{\xi} \rangle_{\overline{H}} = \langle \xi, \eta \rangle_H$.

(1) Prove that $\overline{H}$ is a Hilbert space.
(2) For $T \in B(H,K)$, define $\overline{T} : \overline{H} \to \overline{K}$ by $\overline{T}\overline{\xi} = \overline{T\xi}$. Prove that $\overline{T} \in B(\overline{H}, \overline{K})$, and $\|T\| = \|\overline{T}\|$.
(3) Prove that $\overline{\cdot}$ is an endofunctor on the the category Hilb of Hilbert spaces with bounded operators ($\overline{\cdot}$ is a functor $\mathsf{Hilb} \to \mathsf{Hilb}$).
(4) For each $H \in \mathsf{Hilb}$, construct a linear isometry u_H of H^* onto $\overline{H}$ satisfying $u_H T^t = \overline{T} u_H$ for all $T \in B(H,K)$ where $T^t \in B(K^*, H^*)$ is the Banach adjoint of T.

Problem 12. For $T \in B(H)$, we define its *numerical radius* as

$$R(T) := \sup_{\|\xi\| \leq 1} |\langle T\xi, \xi \rangle|.$$

Prove that $r(T) \leq R(T) \leq \|T\| \leq 2R(T)$. Deduce that if T is normal, then $\|T\| = R(T)$.

Problem 13. Let A be a C*-algebra. An element $u \in A$ is called a *partial isometry* if u^*u is a projection.

(1) Show that the following are equivalent:
 (a) u is a partial isometry.
 (b) $u = uu^*u$.
 (c) $u^* = u^*uu^*$.
 (d) u^* is a partial isometry.
 Hint: For (a) $\Rightarrow$ (b), apply the C-axiom to $u - uu^*u$.*
(2) We say two projections $p, q \in A$ are *(Murray-von Neumann) equivalent*, denoted $p \approx q$, if there is a partial isometry $u \in A$ such that $uu^* = p$ and $u^*u = q$. Prove that $\approx$ is an equivalence relation on $P(A)$, the set of projections of A.
(3) Describe the set of equivalence classes $P(A)/\approx$ for $A = B(\ell^2)$.

Problem 14 (MO:325725)**.** Suppose A is a unital C*-algebra and $I \leq A$ is an ideal. Let $q : A \to A/I$ be the canonical surjection.

(1) Show that unital $*$-homomorphisms $C[0,1] \to A$ are in canonical bijection with positive elements of A with norm at most 1.
(2) Show that if $a + I \in A/I$ is positive with norm at most 1, there is a positive $\widetilde{a} \in A$ with norm at most 1 such that $\widetilde{a} + I = a + I$.
Hint: Since $\operatorname{sp}_{A/I}(a+I) \subseteq \operatorname{sp}_A(a)$, $f(q(a)) = q(f(a))$ and thus $f(a+I) = f(a) + I$ for all $f \in C(\operatorname{sp}_A(a))$. Now pick f carefully.
(3) Deduce that for every unital $*$-homomorphism $\phi : C[0,1] \to A/I$, there is a unital $*$-homomorphism $\widetilde{\varphi} : C[0,1] \to A$ with $\phi = q \circ \widetilde{\phi}$.
(4) Discuss the connection between the above statement and the Tietze Extension Theorem when A is commutative.

Problem 15. Let H be a Hilbert space. Compute the extreme points of the unit balls of

(1) $\mathcal{K}(H)$,
(2) $\mathcal{L}^1(H)$, and

(3) $B(H)$.

Problem 16. Let H be a Hilbert space. Prove that the trace Tr induces isometric isomorphims:

(1) $\mathcal{K}(H)^* \cong \mathcal{L}^1(H)$, and
(2) $\mathcal{L}^1(H)^* \cong B(H)$.

Problem 17. Suppose H is a Hilbert space and $K \subseteq H$ is a closed subspace. Let $p_K \in B(H)$ be associated orthogonal projection onto K.

(1) Suppose $x \in B(H)$. Prove that:
  (a) $xK \subseteq K$ if and only if $xp_K = p_K x p_K$.
  (b) $x^*K \subseteq K$ if and only if $p_K x = p_K x p_K$.
  (c) $xK \subseteq K$ and $x^*K \subseteq K$ if and only if $[x, p_K] = 0$.
(2) Prove that if $M \subseteq B(H)$ is a $*$-closed subalgebra, then $MK \subseteq K$ if and only if $p_K \in M'$.

Problem 18. Suppose H is a Hilbert space.

(1) Suppose K is another Hilbert space. Define the tensor product Hilbert space $H \overline{\otimes} K$ by completing the algebraic tensor product vector space $H \otimes K$ in the 2-norm associated to the sesquilinear form $\langle \eta \otimes \xi, \eta' \otimes \xi' \rangle := \langle \eta, \eta' \rangle \langle \xi, \xi' \rangle$. Find a unitary isomorphism $H \overline{\otimes} K \cong \bigoplus_{i=1}^{\dim K} H$.
(2) Find a unital $*$-isomorphism $B(\bigoplus_{i=1}^n H) \cong M_n(B(H))$.
Hint: use orthogonal projections.
(3) Suppose $S \subseteq B(H)$, and let $\alpha : B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

Prove that:
  (a) $\alpha(S)' = M_n(S')$, and
  (b) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.
  (c) Deduce that when $0, 1 \in S$, $\alpha(S)'' = \alpha(S'')$.

Problem 19. Let (X, μ) be a σ-finite measure space, and consider the map $M : L^\infty(X, \mu) \to B(L^2(X, \mu))$ given by $(M_f \xi)(x) = f(x)\xi(x)$ for $\xi \in L^2(X, \mu)$.

(1) Prove that M is an isometric unital $*$-homomorphism.
(2) Let $A \subset B(L^2(X, \mu))$ be the image of the map M. Prove that $A = A'$.
Hint: If you're stuck with (2), try the case $X = \mathbb{N}$ with counting measure.

Problem 20. Let H be a Hilbert space. The *weak operator topology (WOT)* on $B(H)$ is the topology induced by the separating family of seminorms $T \mapsto |\langle T\eta, \xi \rangle|$ for $\eta, \xi \in H$. The *strong operator topology (SOT)* on $B(H)$ is induced by the separating family of seminorms $x \mapsto \|T\xi\|_H$ for $\xi \in H$.

(1) Prove that every WOT open set is SOT open. Equivalently, prove that if a net $(T_\lambda)_{\lambda \in \Lambda} \subset B(H)$ converges to $T \in B(H)$ SOT, then $T_\lambda \to T$ WOT.
(2) Prove that the WOT is equal to the SOT on $B(H)$ if and only if H is finite dimensional.
(3) Show that the following are equivalent for a linear functional φ on $B(H)$:
  (a) There are $\eta_1, \dots, \eta_n, \xi_1, \dots, \xi_n \in H$ such that $\varphi(T) = \sum_{i=1}^n \langle T\eta_i, \xi_i \rangle$.
  (b) φ is WOT-continuous.
  (c) φ is SOT-continuous.

Problem 21. Suppose $M \subset B(H)$ is a unital $*$-subalgebra. A vector $\xi \in H$ is called:

- *cyclic* for M if $M\xi$ is dense in H.

- *separating* for M if for every $x, y \in M$, $x\xi = y\xi$ implies $x = y$.

(1) Prove that ξ is cyclic for M if and only if ξ is separating for M'.
(2) Prove that H can be orthogonally decomposed into M-invariant subspaces $H = \bigoplus_{i\in I} K_i$, such that each K_i is cyclic for M (has a cyclic vector). Prove that if H is separable, this decomposition is countable.
(3) Prove that if M is abelian and H is separable, then there is a separating vector in H for M.

Problem 22. Suppose H is a Hilbert space, and (x_λ) is an increasing net of positive operators in $B(H)$ which is bounded above by the positive operator $x \in B(H)$, i.e., $\lambda \leq \mu$ implies $x_\lambda \leq x_\mu$, and $0 \leq x_\lambda \leq x$ for all λ. Prove that the following are equivalent.

(1) $x_\lambda \to x$ SOT.
(2) $x_\lambda \to x$ WOT.
(3) For every $\xi \in H$, $\omega_\xi(x_\lambda) = \langle x_\lambda \xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.
(4) There exists a dense subspace $D \subset H$ such that for every $\xi \in D$, $\omega_\xi(x_\lambda) = \langle x_\lambda\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.

We say an increasing net of positive operators (x_λ) *increases to* $x \in B(H)_+$, denoted $x_\lambda \nearrow x$, if any of the above equivalent conditions hold.
Hint: Show it suffices to prove (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (3). *Try proving these implications.*

Problem 23. Let H be a Hilbert space and let $T \in B(H)$. Prove that the following are equivalent. (You may use any results from last semester that you'd like without proof.)

(1) T is compact and normal.
(2) T has an orthonormal basis of eigenvectors $(e_i)_{i\in I}$ such that the corresponding eigenvalues $\lambda_i \to 0$, with at most countably many of the $\lambda_i \neq 0$.
(3) There is a countable orthonormal subset $(\xi_n)_{n\in\mathbb{N}} \subset H$ and a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and $T = \sum_{n\in\mathbb{N}} \lambda_n |\xi_n\rangle\langle\xi_n|$, which converges in operator norm.
(4) There is a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and a countable family of finite rank projections $E_n \subset B(H)$ such that $T = \sum_{n\in\mathbb{N}} \lambda_n E_n$, which converges in operator norm.
(5) There is a discrete set X equipped with counting measure ν, a function $f \in c_0(X)$, and a unitary $U \in B(\ell^2 X, H)$ such that $T = UM_fU^*$ where $M_f\xi = f\xi$ for $\xi \in \ell^2 X$.
Note: $U \in B(K, H)$ *is unitary if* $UU^* = \mathrm{id}_H$ *and* $U^*U = \mathrm{id}_K$.

Problem 24. Suppose A is a unital C*-algebra. A linear map $\Phi : A \to B(H)$ is called *completely positive* if for every $a = (a_{i,j}) \geq 0$ in $M_n(A)$, $(\Phi(a_{i,j})) \geq 0$ in $M_n(B(H)) \cong B(H^n)$. Such a map is *unital* if $\Phi(1) = 1$.

(1) Show that $\langle x\otimes\eta, y\otimes\xi\rangle := \langle \Phi(y^*x)\eta, \xi\rangle_H$ on $A\otimes H$ linearly extends to a well-defined positive sesquilinear form.
(2) Show that for V a vector space with positive sesquilinear form $B(\,\cdot\,, \cdot\,)$, $N_B = \{v \in V | B(v, v) = 0\}$ is a subspace of V, and B descends to an inner product on V/N_B.
(3) Define K to be completion of $(A \otimes H)/N_{\langle\,\cdot\,,\cdot\,\rangle}$ in $\|\cdot\|_2$. Find a unital $*$-homormophism $\Psi : A \to B(K)$, and an isometry $v \in B(H, K)$ such that $\Phi(m) = v^*\Psi(m)v$.

Problem 25. Suppose $A \subseteq B(H)$ is a unital C*-subalgebra and $\xi \in H$ is a cyclic vector for A. Consider the vector state $\omega_\xi = \langle\,\cdot\,\xi, \xi\rangle$. Prove there is a bijective correspondence between:

(1) positive linear functionals φ on A such that $0 \leq \varphi \leq \omega_\xi$ ($\omega_\xi - \varphi \geq 0$), and
(2) operators $0 \leq x \leq 1$ in A'.

Hint: For $0 \leq x \leq 1$ *in* A', *define* $\varphi_x(a) := \langle ax\xi, \xi\rangle$ *for* $a \in A$. *(Why is* $0 \leq \varphi_x \leq \omega_\xi$*?) For the reverse direction, use the bijective correspondence between sesquilinear forms and operators.*

Problem 26.

(1) Prove that a unital $*$-subalgebra $M \subseteq B(H)$ is a von Neumann algebra if and only if its unit ball is σ-WOT compact.
(2) Let $M \subset B(H)$ be a von Neumann algebra and $\Phi : M \to B(K)$ a unital $*$-homomorphism. Deduce that if Φ is σ-WOT continuous and injective, then $\Phi(M)$ is a von Neumann subalgebra of $B(K)$.

Problem 27. Suppose X is a compact Hausdorff topological space and $E : (X, \mathcal{M}) \to B(H)$ is a Borel spectral measure. Prove that the following conditions are equivalent.

(1) E is regular, i.e., for all $\xi \in H$, $\mu_{\xi,\xi}(S) = \langle E(S)\xi, \xi\rangle$ is a finite regular Borel measure.
(2) For all $S \in \mathcal{M}$, $E(S) = \sup\{E(K) | K \text{ is compact and } K \subseteq S\}$.
(3) For all $S \in \mathcal{M}$, $E(S) = \inf\{E(U) | U \text{ is open and } S \subseteq U\}$

Problem 28. Let H be a separable Hilbert space and $A \subseteq B(H)$ an abelian von Neumann algebra. Prove that the following are equivalent.

(1) A is maximal abelian, i.e., $A = A'$.
(2) A has a cyclic vector $\xi \in H$.
(3) For every norm separable SOT-dense C*-subalgebra $A_0 \subset A$, A_0 has a cyclic vector.
(4) There is a norm separable SOT-dense C*-subalgebra $A_0 \subset A$ such that A_0 has a cyclic vector.
(5) There is a finite regular Borel measure μ on a compact Hausdorff second countable space X and a unitary $u \in B(L^2(X, \mu), H)$ such that $f \mapsto uM_f u^*$ is an isometric $*$-isomorphism $L^\infty(X, \mu) \to A$.

Hints:
For $(1) \Rightarrow (2)$*, use Problem 21.*
For $(3) \Rightarrow (4)$ *it suffices to construct a norm separable SOT-dense C*-algebra. First show that* $A_* = \mathcal{L}^1(H)/A_\perp$ *is a separable Banach space. Then show that* A *is* σ*-WOT separable, which implies SOT-separable. Take* A_0 *to be the unital C*-algebra generated by an SOT-dense sequence.*
For $(4) \Rightarrow (5)$ *show that* A_0 *separable implies* $X = \widehat{A_0}$ *is second countable. Define* $\mu = \mu_{\xi,\xi}$ *on* X*, and show that the map* $C(X) \to H$ *by* $f \mapsto \Gamma^{-1}(f)\xi$ *is a* $\|\cdot\|_2 - \|\cdot\|_H$ *isometry with dense range.*

Problem 29. Suppose $E : (X, \mathcal{M}) \to P(H)$ is a spectral measure with H separable, and let $A \subset B(H)$ be the unital C*-algebra which is the image of $L^\infty(E)$ under $\int \cdot \, dE$. Suppose there is a cyclic unit vector $\xi \in H$ for A.

(1) Show that $\omega_\xi(f) = \langle (\int f dE)\xi, \xi\rangle$ is a faithful state on $L^\infty(E)$ ($\omega_\xi(|f|^2) = 0 \Longrightarrow f = 0$).
(2) Consider the finite non-negative measure $\mu = \mu_{\xi,\xi}$ on $(X, \mathcal{M})$. Show that a measurable function f on $(X, \mathcal{M})$ is essentially bounded with respect to E if and only if f is essentially bounded with respect to μ.
(3) Deduce that for essentially bounded measurable f on $(X, \mathcal{M})$, $\|f\|_E = \|f\|_{L^\infty(X,\mathcal{M},\mu)}$.
(4) Construct a unitary $u \in B(L^2(X, \mathcal{M}, \mu), H)$ such that for all $f \in L^\infty(E) = L^\infty(X, \mathcal{M}, \mu)$, $(\int f dE)u = uM_f$.
(5) Deduce that $A \subset B(H)$ is a maximal abelian von Neumann algebra.

Problem 30. Suppose H is a separable infinite dimensional Hilbert space. Prove that $K(H) \subset B(H)$ is the unique norm closed 2-sided proper ideal.

Problem 31. Classify all abelian von Neumann algebras $A \subset B(H)$ when H is separable.
Hint: Use a maximality argument to show you can write $1 = p + q$ *with* $p, q \in P(A)$ *such that* q *is diffuse and* $p = \sum p_i$ *(SOT) with all* p_i *minimal. Then analyze* Aq *and* Ap*.*

Problem 32. Suppose $M \subseteq B(H)$ is a von Neumann algebra and $p, q \in P(M)$. Define $p \wedge q \in B(H)$ to be the orthogonal projection onto $pH \cap qH$. Prove that $p \wedge q \in M$ two separate ways:

(1) Show that $pH \cap qH$ is M'-invariant, and deduce $p \wedge q \in M$.
(2) Show that $p \wedge q$ is the SOT-limit of $(pq)^n$ as $n \to \infty$.
Hint: You could proceed as follows, but a quicker proof would be much appreciated!
(a) Use (2) of Problem 6 to show $(pq)^n p$ is a decreasing sequence of positive operators.
(b) Show $(pq)^n p$ converges SOT to a positive operator $x \in M$.
(c) Show that $x^2 = x$, and deduce $x \leq p$ is an orthogonal projection.
(d) Show that $xqp = x$, and deduce $xqx = x$.
(e) Show that $x \leq q$, and deduce $x \leq p \wedge q$.
(f) Show that $(p \wedge q)(pq)^n$ converges SOT to both $p \wedge q$ and x, and deduce $x = p \wedge q$.
(g) Finally, show $(pq)^n$ converges SOT to $xq = p \wedge q$.

Define $p \vee q$ as the projection onto $\overline{pH + qH}$. Show that $p \vee q \in M$ in two separate ways:
(1) Prove that $\overline{pH + qH}$ is M'-invariant, and deduce $p \vee q \in M$.
(2) Show that $p \vee q = 1 - (1 - p) \wedge (1 - q)$ and use that $p \wedge q \in M$.

Problem 33. Suppose $N \subseteq M \subset B(H)$ is a unital inclusion of von Neumann algebra and $p \in P(N)$.
(1) Prove that $(N'p) \cap pMp = (N' \cap M)p$.
(2) Deduce that if $p \in P(M)$, $Z(pMp) = Z(M)p$.
(3) Deduce that if $p \in P(M)$ and M is a factor, then pMp is a factor.
(4) Prove that when M is a factor and $p \in P(M)$, the map $M' \to M'p$ by $x \mapsto xp$ is a unital $*$-algebra isomorphism.

Problem 34. Prove that the following conditions are equivalent for a von Neumann algebra $M \subseteq B(H)$:
(1) Every non-zero $q \in P(M)$ majorizes an abelian projection $p \in P(M)$.
(2) M is type I (every non-zero $z \in P(Z(M))$ majorizes an abelian $p \in P(M)$).
(3) There is an abelian projection $p \in P(M)$ whose central support $z(p) = \bigvee_{u \in U(M)} u^*pu \in Z(M)$ is 1_M.

Hints:
For (2) $\Rightarrow$ (3), if $p \in P(M)$ is abelian with $z(p) \neq 1$, then there is an abelian projection $q \in P(M)$ such that $z(q) \leq 1 - z(p)$. Show that $pMq = 0$ and $p + q$ is an abelian projection. Now use Zorn's Lemma.
For (3) $\Rightarrow$ (1), suppose $p \in P(M)$ is abelian with $z(p) = 1$ and $q \in P(M)$ is non-zero. Show there is a non-zero partial isometry $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. Deduce that uu^* is abelian, and then prove u^*u is abelian.

Problem 35. Show that for every von Neumann algebra M, there are unique central projections z_{I}, z_{II_1}, z_{II_∞}, and z_{III} (some of which may be zero) such that

- Mz_{I} is type I, Mz_{II_1} is type II_1, Mz_{II_∞} is type II_∞, and Mz_{III} is type III, and
- $z_{\mathrm{I}} + z_{\mathrm{II}_1} + z_{\mathrm{II}_\infty} + z_{\mathrm{III}} = 1$

Hint: You could proceed as follows:
(1) First, show that if M has an abelian projection p, then $z(p)$ is type I. *Then use a maximality argument to construct z_{I}. For this, you could adapt the hint for (2) $\Rightarrow$ (3) in Problem 34.*
(2) Replacing M, H with $M(1 - z_{\mathrm{I}}), (1 - z_{\mathrm{I}})H$, we may assume M has no abelian projections. Show that if M has a finite central projection z, then Mz is type II_1. *Now use a maximality argument to construct z_{II_1}. This hinges on proving the sum of two orthogonal finite central projections is finite. (Proving this is much easier than proving the sup of two finite projections is finite!)*

(3) By compression, we may now assume that M has no abelian projections and no finite central projections. Show that if M has a nonzero finite projection p, then its central support $z(p)$ satisfies $Mz(p)$ is type II_∞. Use a maximality argument to construct z_{II_∞}.
(4) Compressing one more time, we may assume M has no finite projections, and thus M is purely infinite and type III.

Problem 36. Let $M \subseteq B(H)$ be a finite dimensional von Neumann algebra.

(1) Prove M has a minimal projection.
(2) Deduce that $Z(M)$ has a minimal projection.
(3) Prove that for any minimal projection $p \in Z(M)$, Mp is a type I factor.
(4) Prove that M is a direct sum of matrix algebras.

Problem 37. Suppose $M \subseteq B(H)$ and $N \subseteq B(K)$ are von Neumann algebras, and let $H\overline{\otimes}K$ be the tensor product of Hilbert spaces as in Problem 18.

(1) Show that for every $m \in M$ and $n \in N$, the formula $(m \otimes n)(\eta \otimes \xi) := m\eta \otimes n\xi$ gives a unique well-defined operator $m \otimes n \in B(H\overline{\otimes}K)$.
(2) Let $M\overline{\otimes}N = \{m \otimes n | m \in M, n \in N\}'' \subset B(H\overline{\otimes}K)$. Show that the linear extension of the map from the algebraic tensor product $M \otimes N$ to $M\overline{\otimes}N$ given by $m \otimes n \mapsto m \otimes n$ is a well-defined injective unital $*$-algebra map onto an SOT-dense unital $*$-subalgebra.
Hint for injectivity: Suppose $x = \sum_{i=1}^{k} m_i \otimes n_i$ is not zero in $M \otimes N$. Reduce to the case $\{n_1, \ldots, n_k\}$ is linearly independent and all $m_i \neq 0$. Show that for each $i = 1, \ldots, k$, there exists a $k_i > 0$ and $\{\eta_j^i, \xi_j^i\}_{j=1}^{k_i}$ such that $\sum_{j=1}^{k_i} \langle n_{i'}\eta_j^i, \xi_j^i\rangle = \delta_{i=i'}$. (Sub-hint: Consider $F = \operatorname{span}_{\mathbb{C}}\{n_1, \ldots, n_k\} \subset N$, a closed normed space, and look at $\Phi : H \times \overline{H} \to F^*$ by $(\eta, \xi) \mapsto \langle \cdot\, \eta, \xi\rangle$. Show that $\operatorname{span}_{\mathbb{C}}(\Phi(H)) = F^*$.) Now pick $\kappa, \zeta \in H$ such that $\langle m_1\kappa, \zeta\rangle \neq 0$, and deduce $\sum_{j=1}^{k_1} \langle x(\kappa \otimes \eta_j^1), \zeta \otimes \xi_j^1\rangle_{H\overline{\otimes}K} \neq 0$.
(3) We denote by $B(H) \otimes 1$ the image of $B(H)$ under the map $x \mapsto x \otimes 1 \in B(H\overline{\otimes}K)$. Prove that $B(H) \otimes 1$ is a von Neumann algebra.
Hint: Show that $(B(H) \otimes 1)' = 1 \otimes B(K)$. Then by symmetry, $(1 \otimes B(K))' = B(H) \otimes 1$ is a von Neumann algebra.
(4) Prove that $B(H\overline{\otimes}K) = B(H)\overline{\otimes}B(K)$.
Hint: Calculate the commutant of the image of the algebraic tensor product $(B(H) \otimes B(K))' = \mathbb{C}1$ and use (2).

Problem 38. Let S_∞ be the group of finite permutations of $\mathbb{N}$.

(1) Show that S_∞ is ICC. Deduce that LS_∞ is a II_1 factor.
(2) Give an explicit description of a projection with trace k^{-n} for arbitrary $n, k \in \mathbb{N}$.
Hint: Find such a projection in $\mathbb{C}S_\infty \subset LS_\infty$.
(3) Find an increasing sequence $F_n \subset LS_\infty$ of finite dimensional von Neumann subalgebras such that $LS_\infty = (\bigcup_{n=1}^{\infty} F_n)''$.

Note: A II_1 factor which is generated by an increasing sequence of finite dimensional von Neumann subalgebras as in (3) above is called hyperfinite.

Problem 39. Let M be a von Neuman algebra. Suppose $a, b \in M$ with $0 \leq a \leq b$. Prove there is a $c \in M$ such that $a = c^*bc$. Deduce that a 2-sided ideal in a von Neumann algebra is *hereditary*: $0 \leq a \leq b \in M$ implies $a \in M$.

Problem 40. Let M be a factor. Prove that if M is finite or purely infinite, then M is algebraically simple, i.e., M has no 2-sided ideals.
Note: You may use that a II_1 factor has a (faithful σ-WOT continuous) tracial state.

Problem 41. A positive linear functional $\varphi \in M^*$ is called *completely additive* if for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Suppose $\varphi, \psi \in M^*$ are completely additive and $p \in P(M)$ such that $\varphi(p) < \psi(p)$. Then there is a non-zero projection $q \leq p$ such that $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.
Hint: Choose a maximal family of mutually orthogonal projections $e_i \leq p$ for which $\psi(e_i) \leq \varphi(e_i)$. Consider $e = \bigvee e_i$, and show that $\psi(e) \leq \varphi(e)$. Set $q = p - e$, and show that for all projections $r \leq q$, $\varphi(r) < \psi(r)$. Then show $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.

Problem 42. Show that the following conditions are equivalent for a positive linear functional $\varphi \in M^*$ for a von Neumann algebra M:

(1) φ is σ-WOT continuous,
(2) φ is *normal*: $x_\lambda \nearrow x$ implies $\varphi(x_\lambda) \nearrow \varphi(x)$, and
(3) φ is *completely additive*: for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Hint: For (3) $\Rightarrow$ (1), show if $p \in P(M)$ is non-zero, then pick $\xi \in H$ such that $\varphi(p) < \langle p\xi, \xi \rangle$. Use Problem 41 to find a non-zero $q \leq p$ such that $\varphi(qxq) < \langle xq\xi, q\xi \rangle$ for all $x \in M$. Use the Cauchy-Schwarz inequality to show $x \mapsto \varphi(xq)$ is SOT-continuous, and thus σ-WOT continuous. Now use Zorn's Lemma to consider a maximal family of mutually orthogonal projections $(q_i)_{i \in I}$ for which $x \mapsto \varphi(xq_i)$ is σ-WOT continuous. Show $\sum q_i = 1$. For finite $F \subseteq I$, define $\varphi_F(x) = \sum_{i \in F} \varphi(xq_i)$. Ordering finite subsets by inclusion, we get a net $(\varphi_F) \subset M_*$. Show that $\varphi_F \to \varphi$ in norm in M^*. Deduce that $\varphi \in M_*$ since $M_* \subset M^*$ is norm-closed.

Problem 43. Let $\Phi : M \to N$ be a unital $*$-homomorphism between von Neumann algebras.

(1) Prove that the following two conditions are equivalent:
  (a) Φ is *normal*: $x_\lambda \nearrow x$ implies $\Phi(x_\lambda) \nearrow \Phi(x)$.
  (b) Φ is σ-WOT continuous.
(2) Prove that if Φ is normal, then $\Phi(M) \subset N$ is a von Neumann subalgebra.
  Hint: $\ker(\Phi) \subset M$ *is a σ-WOT closed 2-sided ideal.*
(3) Let φ be a normal state on a a von Neumann algebra M, and let $(H_\varphi, \Omega_\varphi, \pi_\varphi)$ be the cyclic GNS representation of M associated to φ, i.e., $H_\varphi = L^2(M, \varphi)$, $\Omega_\varphi \in H_\varphi$ is the image of $1 \in M$ in H_φ, and $\pi_\varphi(x)m\Omega_\varphi = xm\Omega_\varphi$ for all $x, m \in M$.
  (a) Show that π_φ is normal.
  (b) Deduce that if φ is faithful, then $M \cong \pi_\varphi(M) \subset B(H_\varphi)$ is a von Neumann algebra acting on H_φ.

Problem 44. Suppose $\Phi : M \to N$ is a unital $*$-algebra homomorphism between von Neumann algebras.

(1) Prove that the following conditions imply Φ is normal:
  (a) Φ is SOT-continuous on the unit ball of M.
  (b) Φ is WOT-continuous on the unit ball of M.
  (c) Suppose $N = N'' \subseteq B(H)$. For a dense subspace $D \subseteq H$, $m \mapsto \langle \Phi(m)\eta, \xi \rangle$ is WOT-continuous on M for any $\eta, \xi \in D$.
(2) (optional) Which of the conditions above are equivalent to normality of Φ?

Problem 45. Let M be a finite von Neumann algebra with a faithful σ-WOT continuous tracial state. Let $L^2M = L^2(M, \mathrm{tr})$ where Ω is the image of 1_M in L^2M. Identify M with its image in $B(L^2M)$ by part (3) of Problem 43.

(1) Show that $J : M\Omega \to M\Omega$ by $a\Omega \mapsto a^*\Omega$ is a conjugate-linear isometry with dense range.
(2) Deduce J has a unique extension to L^2M, still denoted J, which is a conjugate-linear unitary, i.e, $J^2 = 1$ and $\langle J\eta, J\xi \rangle = \langle \xi, \eta \rangle$ for all $\eta, \xi \in L^2M$.
  Hint: Look at η, ξ in $M\Omega$.

(3) Calculate $Ja^*Jb\Omega$ for $a, b \in M$. Deduce that $JMJ \subseteq M'$.
(4) Show $\langle Ja^*Jb\Omega, c\Omega\rangle = \langle b\Omega, JaJc\Omega\rangle$ for all $a, b, c \in M$. Deduce $(JaJ)^* = Ja^*J$.
(5) Show $\langle Jy\Omega, a\Omega\rangle = \langle y^*\Omega, a\Omega\rangle$ for all $a \in M$ and $y \in M'$. Deduce $Jy\Omega = y^*\Omega$.
(6) Prove that for $y \in M'$, $(JyJ)^* = Jy^*J$.
Hint: Try the same technique as in (4).
(7) Show for all $a, b \in M$ and $x, y \in M'$, $\langle xJyJa\Omega, b\Omega\rangle = \langle JyJxa\Omega, b\Omega\rangle$.
(8) Deduce that $M' \subseteq (JM'J)' = JMJ$, and thus $M' = JMJ$.

Problem 46. Let Γ be a discrete group, and let $L\Gamma = \{\lambda_g\}'' \subset B(\ell^2\Gamma)$. Consider the faithful σ-WOT continuous tracial state $\operatorname{tr}(x) = \langle x\delta_e, \delta_e\rangle$ on $L\Gamma$.

(1) Show that $u\delta_g = \lambda_g$ uniquely extends to a unitary $u \in B(\ell^2\Gamma, L^2L\Gamma)$ such that for all $x \in L\Gamma$ and $\xi \in \ell^2\Gamma$, $L_x u\xi = ux\xi$ where $L_x \in B(L^2L\Gamma)$ is left multiplication by x, i.e., $L_x(y\Omega) = xy\Omega$.
(2) Deduce from Problem 45 that $L\Gamma' = R\Gamma$.

Problem 47. Use Problem 46 above to give the following alternative characterization of $L\Gamma$. Let

$$\ell\Gamma = \left\{x = (x_g) \in \ell^2\Gamma \middle| x * y \in \ell^2\Gamma \text{ for all } y \in \ell^2\Gamma\right\}$$

where $(x * y)_g = \sum_h x_h y_{h^{-1}g}$. Define a unital $*$-algebra structure on $\ell\Gamma$ by multiplication is convolution, the unit is δ_e, the the indicator function at $e \in \Gamma$ ($\delta_e(g) = \delta_{g=e}$), and the involution $*$ on $\ell\Gamma$ is given on $x \in \ell\Gamma$ by $(x^*)_g := \overline{x_{g^{-1}}}$.

(1) Show that $\ell\Gamma$ is a well-defined unital $*$-algebra under the above operations.
(2) For $x \in \ell\Gamma$ define $T_x : \ell^2\Gamma \to \ell^2\Gamma$ by $T_x y = x * y$. Prove $T_x \in B(\ell^2\Gamma)$.
Hint: Show that for all $x \in \ell\Gamma$ and $y, z \in \ell^2\Gamma$, $\langle T_x y, z\rangle = \langle y, T_{x^*}z\rangle$. Then use the Closed Graph Theorem.
(3) Prove that for all $x \in \ell\Gamma$, $T_x \in L\Gamma$.
Hint: Prove $T_x \in R\Gamma'$ and apply Problem 46.
(4) Deduce that $x \mapsto T_x$ is a unital $*$-algebra isomorphism $\ell\Gamma \to L\Gamma$.

Problem 48. Repeat Problem 47 for the crossed product von Neumann algebra $M \rtimes_\alpha \Gamma$ acting on $L^2M \otimes \ell^2\Gamma \cong L^2(\Gamma, L^2M)$ where M is a finite von Neumann algebra with faithful normal tracial state tr, Γ is a discrete group, and $\alpha : \Gamma \to \operatorname{Aut}(M)$ is an action. Here, we define

$$\ell^2(\Gamma, M) = \left\{x : \Gamma \to M \middle| \sum_g \|x_g\Omega\|^2_{L^2M} < \infty\right\}$$
$$\ell^2(\Gamma, L^2M) = \left\{\xi : \Gamma \to L^2M \middle| \sum_g \|\xi_g\|^2 < \infty\right\} \text{ and}$$
$$M \rtimes_\alpha \Gamma = \left\{x = (x_g) \in \ell^2(\Gamma, M) \middle| x * \xi \in \ell^2(\Gamma, L^2M) \text{ for all } \xi \in \ell^2(\Gamma, L^2M)\right\}.$$

Here, the convolution action is given by $(x * \xi)_g = \sum_h x_h v_h \xi_{h^{-1}g}$ where $v_h \in U(L^2M)$ is the unitary implementing $\alpha_u \in \operatorname{Aut}(M)$. Define an analogous unital $*$-algebra structure on $M\Gamma$ and find a unital $*$-algebra isomorphism $M \rtimes_\alpha \Gamma \to M \rtimes_\alpha \Gamma$.
Hint: Similar to $L\Gamma$, some people write elements of $M \rtimes_\alpha \Gamma$ as formal sums $\sum_g x_g u_g$ which does not converge in any operator topology. Rather, $\sum_g x_g u_g(\Omega \otimes \delta_e)$ converges in $L^2M \otimes \ell^2\Gamma$. These formal sums can be algebraically manipulated to obtain a unital $*$-algebra structure using the covariance condition $u_g m u_g^* = \alpha_g(m)$ for all $g \in \Gamma$ and $m \in M$. Thus

$$\left(\sum_g x_g u_g\right)^* = \sum_g u_g x_g^* = \sum_g u_g x_g^* u_g^* u_g = \sum_g \alpha_g(x_g^*) u_g.$$

Thus for $x = (x_g) \in M \rtimes_\alpha \Gamma$, we define $(x^*)_g = \alpha_g(x_g^*)$. A similar algebraic manipulation gives the formula for multiplication, which is similar to convolution, but involves the action.

Problem 49. Prove that a $*$-isomorphism between von Neumann algebras is automatically normal.

Problem 50. Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group on 2 generators.

(1) Show that $\mathbb{F}_2$ is ICC. Deduce $L\mathbb{F}_2$ is a II_1 factor.
(2) Show that the swap $a \leftrightarrow b$ extends to an automorphism σ of $L\mathbb{F}_2$.
(3) Show that σ is outer.

Problem 51. Prove that irrational rotation on the circle (with Lebesgue/Haar measure) is free and ergodic.

Problem 52. Let M be a finite von Neumann algebra with a faithful normal tracial state.

(1) Show for all $x, y \in M$, $|\operatorname{tr}(xy)| \leq \|y\| \operatorname{tr}(|x|)$.
(2) Show for all $x \in M$, $\operatorname{tr}(|x|) = \sup\{|\operatorname{tr}(xy)||y \in M \text{ with } \|y\| = 1\}$.
(3) Define $\|x\|_1 = \operatorname{tr}(|x|)$ on M. Show that $\|\cdot\|_1$ is a norm on M.
(4) Define a map $\varphi : M \to M_*$ by $x \mapsto \varphi_x$ where $\varphi_x(y) = \operatorname{tr}(xy)$. Show that φ is a well-defined isometry from $(M, \|\cdot\|_1) \to M_*$ with dense range.
(5) Deduce that $L^1(M, \operatorname{tr}) := \overline{M}^{\|\cdot\|_1}$ is isometrically isomorphic to the predual M_*.

Problem 53. Continue the notation of Problem 52. Let $N \subseteq M$ be a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $i : L^1(N, \operatorname{tr}) \to L^1(M, \operatorname{tr})$.
(2) Let $E : M \to N$ be the Banach adjoint of i under the identification $M_* = L^1(M, \operatorname{tr})$ and $N_* = L^1(N, \operatorname{tr})$. Show that E is uniquely characterized by the equation

$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$.

Problem 54. Suppose M is a finite von Neumann algebra with normal faithful tracial state tr and $N \subseteq M$ is a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $L^2(N, \operatorname{tr}) \to L^2(M, \operatorname{tr})$.
(2) Define $e_N \in B(L^2M, L^2N)$ be the orthogonal projection with range $L^2(N, \operatorname{tr}) = \overline{N\Omega}^{\|\cdot\|_2} \subset L^2(M, \operatorname{tr})$. Show that for all $x \in M$, $e_N x e_N^* \subset B(L^2N)$ commutes with the right action of N, and thus defines an element in N by Problem 45.
Hint: Show the inclusion $e_N^* : L^2N \to L^2M$ commutes with the right N action, and deduce e_N commutes with the right N action.
(3) For $x \in M$, define $E(x) = e_N x e_N^*$. Show that $E(x)$ is uniquely characterized by the equation

$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$. *Part (3) implies this definition agrees with that from Problem 53.*

Problem 55. Continue the notation of Problem 54.

(1) Deduce that E is normal.
(2) Deduce $E(1) = 1$ and E is $N{-}N$ bilinear, i.e., for all $x \in M$ and $y, z \in N$, $E(yxz) = yE(x)z$.
(3) Deduce that $E(x^*) = E(x)^*$.
(4) Show that E is completely positive, which was defined in Problem 24.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 54.

(5) Show that $E(x)^*E(x) \le E(x^*x)$ for all $x \in M$.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 54. Show that $e_N^* e_N$ is an orthogonal projection.
(6) Show that E is faithful: $E(x^*x) = 0$ implies $x^*x = 0$.
Hint: Prove this by looking at the vector states $\omega_{n\Omega}$ for $n \in N$.

Problem 56. Suppose M is a finite von Neumann algebra with faithful normal tracial state tr. Suppose further that there is an increasing sequence of von Neumann subalgebras $M_1 \subset M_2 \subset \cdots M$ such that $(\bigcup M_n)'' = M$ (considered as acting on L^2M). Let $E_n : M \to M_n$ be the canonical trace-preserving conditional expectation from Problem 54.

(1) Prove that the $\|\cdot\|_2$-topology agrees with the SOT on the unit ball of M. That is, prove that $x_n \to x$ SOT if and only if $\|x_n\Omega - x\Omega\|_2 \to 0$.
(2) Prove that for all $x \in M$, $\|E_n(x)\Omega - x\Omega\|_2 \to 0$ as $n \to \infty$.
(3) Deduce that $E_n(x) \to x$ SOT as $n \to \infty$.

Problem 57. Suppose Γ is a countable group, and let $\operatorname{Prob}(\Gamma) = \left\{\mu \in \ell^1\Gamma \middle| \mu \ge 0 \text{ and } \sum_g \mu(g) = 1\right\}$.

(1) Prove that $\operatorname{Prob}(\Gamma)$ is weak* dense in the state space of $\ell^\infty\Gamma$.
(2) Let $F \subset \Gamma$ be finite, and consider $\bigoplus_{g\in F} \ell^1\Gamma$ with the (product) weak topology. Let K be the weak closure of $\left\{\bigoplus_{g\in F} g\cdot\mu - \mu \middle| \mu \in \operatorname{Prob}(\Gamma)\right\} \subset \bigoplus_{g\in F} \ell^1\Gamma$. Prove K is convex and norm closed in $\bigoplus_{g\in F} \ell^1\Gamma$.
(3) Now assume Γ is amenable, i.e., there is a left Γ-invariant state on $\ell^\infty\Gamma$. Prove that $0 \in K$. Deduce that Γ has an approximately invariant mean.

Problem 58. Suppose Γ is a countable group, and let $\operatorname{Prob}(\Gamma)$ be as in Problem 57.

(1) Prove that if $a, b \in [0,1]$, then
$$|a-b| = \int_0^1 |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)|\, dr.$$
(2) Deduce that for $\mu \in \operatorname{Prob}(\Gamma)$ and $h \in \Gamma$,
$$\|h\cdot\mu - \mu\|_{\ell^1\Gamma} = \int_0^1 \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|\, dr.$$
(3) For $r \in [0,1]$ and $\mu \in \operatorname{Prob}(\Gamma)$, let $E(\mu, r) = \{g \in \Gamma | \mu(g) > r\}$. Show that for all $h \in \Gamma$, $hE(\mu,r) = \{g \in \Gamma | (h\cdot\mu)(g) > r\}$.
(4) Calculate $\int_0^1 |E(\mu,r)|\, dr$.
(5) Show that for $r \in [0,1]$, $\mu \in \operatorname{Prob}(\Gamma)$, and $h \in \Gamma$,
$$|hE(\mu,r) \triangle E(\mu,r)| = \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|.$$
Deduce that $\|h\cdot\mu - \mu\|_1 = \int_0^1 |hE(\mu,r)\triangle E(\mu,r)|\, dr$.
(6) Suppose now that Γ has an approximate invariant mean, so that for every finite subset $F \subset \Gamma$ and $\varepsilon > 0$, there is a $\mu \in \operatorname{Prob}(\Gamma)$ such that
$$\sum_{h\in F} \|h\cdot\mu - \mu\|_1 < \varepsilon.$$
Show that for the μ corresponding to this F and ε,
$$\int_0^1 \sum_{h\in F} |hE(\mu,r)\triangle E(\mu,r)|\, dr < \varepsilon \int_0^1 |E(\mu,r)|\, dr.$$

Deduce there is an $r \in [0,1]$ such that $|hE(\mu,r) \triangle E(\mu,r)| < \varepsilon |E(\mu,r)|$ for all $h \in F$.
(7) Use (6) above to construct a Følner sequence for Γ.

Problem 59. Recall that an *ultrafilter* ω on a set X is a nonempty collection of subsets of X such that:

- $\emptyset \notin \omega$,
- If $A \subseteq B \subseteq X$ and $A \in \omega$, then $B \in \omega$,
- If $A, B \in \omega$, then $A \cap B \in \omega$, and
- For all $A \subset X$, either $A \in \omega$ or $X \setminus A \in \omega$ (but not both!).

(1) Find a bijection from the set of ultrafilters on $\mathbb{N}$ to $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$.
(2) Let ω be an ultrafilter on $\mathbb{N}$. Let X be a compact Hausdorff space and $f : \mathbb{N} \to X$. We say
   - $x = \lim_{n\to\omega} f(n)$ if for every open neighborhood U of x, $f^{-1}(U) \in \omega$.

   Prove that $\lim_{n\to\omega} f(n)$ always exists for any function $f : \mathbb{N} \to X$.
(3) An ultrafilter on $\mathbb{N}$ is called *principal* if it contains a finite set. Show that every principal ultrafilter on $\mathbb{N}$ contains a unique singleton set, and that any two principal ultrafilters containing the same singleton set are necessarily equal. Thus we may identify the set of principal ultrafilters on $\mathbb{N}$ with $\mathbb{N} \subset \beta\mathbb{N}$.
(4) Determine $\lim_{n\to\omega} f(n)$ for $f : \mathbb{N} \to X$ as in (2) when ω is principal.
(5) An ultrafilter on $\mathbb{N}$ is called *free* or *non-principal* if it does not contain a finite set. Let ω be a free ultrafilter on $\mathbb{N}$. Suppose $\Gamma = \bigcup \Gamma_n$ is a locally finite group and m_n is the uniform probability (Haar) measure on Γ_n. Define $m : 2^\Gamma \to [0,1]$ by $m(A) = \lim_{n\to\omega} m_n(A \cap \Gamma_n)$. Prove that m is a left Γ-invariant finitely additive probability measure on Γ, i.e., Γ is amenable.

Problem 60. Let X be a uniformly convex Banach space and $B \subset X$ a bounded set. Prove that the function $f : X \to [0,\infty)$ given by $f(x) = \sup_{b\in B} \|b - x\|_X$ achieves its minimum at a unique point of X.

Problem 61. Let Γ be a countable discrete group. Show that an affine action $\alpha = (\pi, \beta) : \Gamma \to \mathrm{Aff}(H)$ ($\alpha_g \xi := \pi_g \xi + \beta(g)$ for $\pi_g \in U(H)$ and $\beta(g) \in H$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in \Gamma$) is proper if and only if the cocycle part $\beta : \Gamma \to H$ is proper ($g \mapsto \|\beta(g)\|$ is a proper map).

Problem 62. Recall that the *Schur product* of two matrices $a, b \in M_n(\mathbb{C})$ is given by the entry-wise product: $(a * b)_{i,j} := a_{i,j} b_{i,j}$.

(1) Prove that if $a, b \geq 0$, then $a * b \geq 0$.
(2) Suppose that $p \in \mathbb{R}[z]$ is a polynomial whose coefficients are all non-negative. Prove that if $a \geq 0$, then $p[a] \geq 0$, where $p[a]_{i,j} := p(a_{i,j})$ for $a \in M_n(\mathbb{C})$.
   Note: Here we use the notation $p[a]$ to not overload the functional calculus notation.
(3) Suppose that f is an entire function whose Taylor expansion at 0 has only non-negative real coefficients. Prove that is $a \geq 0$, then $f[a] \geq 0$, where again $f[a]_{i,j} := f(a_{i,j})$ for $a \in M_n(\mathbb{C})$.

Problem 63. Let A be a unital C*-algebra.

(1) Prove that a map $\Phi : A \to M_n(\mathbb{C})$ is completely positive if and only if the map $\varphi : M_n(A) \to \mathbb{C}$ given by $(a_{i,j}) \mapsto \sum_{i,j}^n \Phi(a_{i,j})_{i,j}$ is positive.
   Hint: for one direction, note that $\varphi(a) = \vec{1}^* \Phi(a) \vec{1}$ where $\vec{1} \in \mathbb{C}^n$ is the column vector with all 1s. For the other direction, use GNS with respect to φ, and consider $V : \mathbb{C}^n \to L^2(A, \varphi)$ given by $Ve_i = \pi_\varphi(E_{ij})\Omega_\varphi$ where (E_{ij}) is a system of matrix units in $M_n(\mathbb{C}) \subseteq M_n(A)$. Then use Stinespring.

(2) Let $S \subset A$ be an operator subsystem, and let $\psi : S \to \mathbb{C}$ be a positive linear functional. Prove $\|\psi\| = \psi(1)$. Deduce that any norm-preserving (Hahn-Banach) extension of ψ to A is also positive.
(3) Let $S \subset A$ be an operator subsystem, and let $\Phi : S \to M_n(\mathbb{C})$ be a (unital) completely positive map. Show that Φ extends to a (unital) completely positive map $A \to M_n(\mathbb{C})$.

Problem 64. Suppose Γ is a countable discrete group, and suppose $\varphi : L\Gamma \to L\Gamma$ is a normal completely positive map. Prove that $f : \Gamma \to \mathbb{C}$ given by $f(g) := \operatorname{tr}_{L\Gamma}(\varphi(\lambda_g)\lambda_g^*)$ is a positive definite function.

Problem 65. Prove that the following are equivalent for a finite von Neumann algebra $(M, \operatorname{tr}) \subset B(H)$ with faithful normalized tracial state.
(1) M is amenable, i.e., there is a conditional expectation $E : B(H) \to M$.
(2) There is a sequence $(\varphi_n : M \to M)$ of (normal) trace-preserving completely positive maps such that $\varphi_n \to \mathrm{id}$ pointwise in $\|\cdot\|_M$, and for all $n \in \mathbb{N}$, the induced map $\widehat{\varphi}_n \in B(L^2M)$ given by $m\Omega \mapsto \varphi_n(m)\Omega$ is finite rank.

Problem 66. Suppose that Γ is a countable discrete group such that every cocycle is inner. Suppose (H, π) is a unitary representation and $(\xi_n) \subset H$ is a sequence of unit vectors such that $\|\pi_g\xi_n - \xi_n\| \to 0$ as $n \to \infty$ for all $g \in \Gamma$. Follow the steps below to find a non-zero Γ-invariant vector in H. (We may assume that no ξ_n is fixed by Γ.)
(1) Enumerate $\Gamma = \{g_1, g_2, \dots\}$. Explain why you can pass to a subsequence of (ξ_n) to assume that for all $n \in \mathbb{N}$, $\|\pi_{g_i}\xi_n - \xi_n\| < 4^{-n}$ for all $1 \le i \le n$.
(2) For $n \in \mathbb{N}$, consider the inner cocycles $\beta_n(g) := \xi_n - \pi_g\xi_n$. Let $(K, \sigma) = \bigoplus_{n\in\mathbb{N}}(H, \pi)$. Define $\beta : \Gamma \to K$ by $\beta(g)_n := 2^n\beta_n(g)$. Prove that $\beta(g) \in H$ is well-defined for every $g \in \Gamma$. Then show that β is a cocycle for (K, σ).
(3) Deduce β is inner and thus bounded. Thus there is a $\kappa \in K \setminus \{0\}$ such that $\beta(g) = \kappa - \sigma_g\kappa$ for all $g \in \Gamma$.
(4) Prove that $\|\beta_n(g)\| \to 0$ uniformly for $g \in \Gamma$. That is, show that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $\|\beta_n(g)\| < \varepsilon$ for all $g \in \Gamma$.
(5) Fix $N \in \mathbb{N}$ such that $\|\beta_N(g)\| = \|\xi_N - \pi_g\xi_N\| < 1$ for all $g \in \Gamma$. Show there is a $\xi_0 \in H \setminus \{0\}$ such that $\pi_g\xi_0 = \xi_0$ for all $g \in \Gamma$.
Hint: Look at $\{\pi_g\xi_N | g \in \Gamma\} \subset (H)_1$ *and apply Problem 60.*
(6) (optional) Use a similar trick to finish the proof of (1) $\Rightarrow$ (2) from the same theorem from class.

Problem 67 (optional)**.** As best as you can, edit the equivalent definitions I gave in class for property (T) for a countable discrete group Γ to be relative to a subgroup $\Lambda \le \Gamma$. Then prove all the equivalences.

Problem 68 (Fell's Absorption Principle)**.** Suppose Γ is a countable group and (H, π) is a unitary representation on a separable Hilbert space. Find a unitary $u \in B(\ell^2\Gamma \overline{\otimes} H)$ intertwining $\lambda \otimes \pi$ and $\lambda \otimes 1$, i.e., $u(\lambda_g \otimes \pi_g) = (\lambda_g \otimes 1)u$ for all $g \in \Gamma$.

Problem 69. Suppose $\Gamma \curvearrowright (X, \mu)$ is a free p.m.p. action and $\mathcal{R} = \{(x, gx) | x \in X, g \in \Gamma\}$ is the corresponding countable p.m.p. equivalence relation. Follow the steps below to show $L^\infty(X, \mu) \rtimes \Gamma \cong L\mathcal{R}$.
(1) Prove that $\theta : (x, g) \mapsto (x, g^{-1}x)$ induces a unitary operator $v \in B(L^2\mathcal{R}, L^2(X \times \Gamma, \mu \times \gamma))$ where γ is counting measure on Γ.
(2) Deduce that θ is a p.m.p. isomorphism $(X \times \Gamma, \mu \times \gamma) \to (\mathcal{R}, \nu)$.
(3) Show that $v^*M_fv = \lambda(f)$ for all $f \in L^\infty(X, \mu)$. Here, $(M_f\xi)(x, g) = f(x)\xi(x, g)$ for $\xi \in L^2(X \times \Gamma, \mu \times \gamma)$.

(4) Show that $v^*u_gv = L_{\varphi_g}$ where $\varphi_g \in [\mathcal{R}]$ is the isomorphism $x \mapsto g \cdot x$. Here, $(u_g\xi)(x,h) = \xi(g^{-1}x, g^{-1}h)$ for all $\xi \in L^2(X \times \Gamma, \mu \times \gamma) \cong L^2(X,\mu) \otimes \ell^2\Gamma$.
(5) Deduce that $v^*(L^\infty(X,\mu) \rtimes \Gamma)v \subset L\mathcal{R}$.
(6) Show that conjugation by v takes the commutant of $L^\infty(X,\mu) \rtimes \Gamma$ into $R\mathcal{R}$.
Hint: Show that right multiplication by $L^\infty(X,\mu)$ and the right action of u_g are both taken into $R\mathcal{R}$.
(7) Deduce that $v^*(L^\infty(X,\mu) \rtimes \Gamma)v = L\mathcal{R}$.

Problem 70. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X,μ). Let $A = L^\infty(X,\mu) \subset L\mathcal{R}$. Prove that the von Neumann subalgebra of $B(L^2(\mathcal{R},\nu))$ generated by $A \cup JAJ$ is the von Neumann algebra of multiplication operators by elements of $L^\infty(\mathcal{R},\nu)$.

Problem 71. Let M be a von Neumann algebra. A *weight* on M is a function $\varphi : M_+ \to [0,\infty]$ such that for all $r \in [0,\infty)$ and $x, y \in B(H)_+$, $\varphi(rx + y) = r\varphi(x) + \varphi(y)$, with the convention that for $s \in [0,\infty)$,

$$\infty \cdot s = \begin{cases} \infty & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases}$$

Define

$$\begin{aligned}
\mathfrak{p}_\varphi &= \{x \in M | \varphi(x) < \infty\} \\
\mathfrak{n}_\varphi &= \{x \in M | x^*x \in \mathfrak{p}_\varphi\} \\
\mathfrak{m}_\varphi &= \mathfrak{n}_\varphi^*\mathfrak{n}_\varphi = \left\{ \sum_{i=1}^n x_i^* y_i \middle| x_i, y_i \in \mathfrak{n}_\varphi \text{ for all } i = 1, \ldots, n \right\}.
\end{aligned}$$

(1) Prove that
  (a) $\mathfrak{p}_\varphi$ is a hereditary subcone of M_+, i.e.,
    - (subcone) $r \geq 0$ and $x, y \in \mathfrak{p}_\varphi$ implies $rx + y \in \mathfrak{p}_\varphi$
    - (hereditary) $0 \leq x \leq y$ and $y \in \mathfrak{p}_\varphi$ implies $x \in \mathfrak{p}_\varphi$.
  (b) $\mathfrak{n}_\varphi$ is a left ideal of M.
  Hint: Prove that for all $x, y \in M$, $(x \pm y)^*(x \pm y) \leq 2(x^*x + y^*y)$.
  (c) $\mathfrak{m}_\varphi$ is algebraically spanned by $\mathfrak{p}_\varphi$.
  Hint: Use polarization.
  (d) $\mathfrak{m}_\varphi \cap M_+ = \mathfrak{p}_\varphi$.
  (e) $\mathfrak{m}_\varphi$ is a hereditary $*$-subalgebra of M (hereditary is defined the same way as above).
(2) When $M = B(H)$ and $\varphi = \mathrm{Tr}$, show $\mathfrak{m}_{\mathrm{Tr}} = \mathcal{L}^1(H)$ and $\mathfrak{n}_{\mathrm{Tr}} = \mathcal{L}^2(H)$.