1. Banach algebras

Let A be a unital Banach algebra. The spectrum of $a \in A$ is

$$\mathrm{sp}(a) = \left\{\lambda \in \mathbb{C} \middle| a - \lambda 1 \notin A^\times\right\},$$

which is a non-empty compact subset of $B_{r(a)}(0)$. Here, $r(a)$ is the spectral radius:

$$r(a) = \lim \|a^n\|^{1/n}.$$

For each $a \in A$, the holomorphic functional calculus (HFC) gives a unital algebra homomorphism $\mathcal{O}(\mathrm{sp}(a)) \to A$ given by

$$f \longmapsto f(a) := \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{a - z}\,dz$$ **TODO: picture**

such that

- If $\mathrm{sp}(a) \subset U$, and $f_n \to f$ locally uniformly on U, then $f_n(a) \to f(a)$ in A, and
- If $f(z) = \sum \alpha_k z^k$ is a power series with radius of convergence greater than $r(a)$, then $f(a) = \sum \alpha_k a^k$.

These two properties characterize this ring homomorphism. The HFC also satisfies:

(1) If $f(z) = \prod (z - z_j)^{m_j}$ is rational, then $f(a) = \prod (a - z_j)^{m_j}$.
(2) (spectral mapping) $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$, and
(3) if $g \in \mathcal{O}(\mathrm{sp}(a))$, then $g(f(a)) = (g \circ f)(a)$.

If A is unital and commutative, the Gelfand transform gives a norm-contractive unital algebra homomorphism $A \to C(\widehat{A})$ given by

$$a \mapsto [\mathrm{ev}_a : \varphi \mapsto \varphi(a)],$$

where $\widehat{A}$ is the set of algebra homomorphisms from $A \to \mathbb{C}$, also called characters or multiplicative linear functionals. The image of the Gelfand transform is a subalgebra of $C(\widehat{A})$ which separates points of $\widehat{A}$.

Fact 1. If A is unital and commutative and $a \in A$, then for all $\varphi \in \widehat{A}$, $\varphi(a) \in \mathrm{sp}(a)$.

2. C*- algebras

Let A be a unital C*-algebra, i.e., a unital Banach algebra with an involution satisfying $\|a^*a\| = \|a\|^2$ for all $a \in A$.

We call $a \in A$:

- self-adjoint if $a = a^*$,
- positive if $a = b^*b$ for some $b \in A$,
- normal if $aa^* = a^*a$,
- a projection if $a = a^* = a^2$,
- an isometry if $a^*a = 1$,
- a unitary if $a^*a = 1 = aa^*$ (equivalently, an invertible isometry),
- a partial isometry if a^*a is a projection.

Here are some elementary properties:

(1) If a is normal, then $\|a\| = r(a)$.
(2) If $\lambda \in \mathrm{sp}(a)$, then $\overline{\lambda} \in \mathrm{sp}(a^*)$.
(3) If u is unitary, then $\mathrm{sp}(u) \subset \partial\mathbb{D} = \mathbb{T} = S^1$.

(4) If $a = a^*$, then e^{ia} is unitary (defined by the HFC).
(5) If $a = a^*$, then $\mathrm{sp}(a) \subset \mathbb{R}$.
(6) Each a can be written as $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$ where $\mathrm{Re}(a) = \frac{a+a^*}{2}$ and $\mathrm{Im}(a) = \frac{a-a^*}{2i}$ are self-adjoint.

Fact 2. If A is commutative, then every $\varphi \in \widehat{A}$ is a $*$-homomorphism.