5. Tracial von Neumann algebras and the crossed product construction

These notes follow Chapters 9 and 11 of Jones' notes on von Neumann algebras quite closely.

5.1. Tracial von Neumann algebras.

Definition 5.1.1. A *tracial von Neumann algebra* is a von Neumann algebra M equipped with a faithful normal tracial state tr.

Facts 5.1.2. We rapidly recall some basic facts about a tracial von Neumann algebra (M, tr) that we have already proven, or which follow easily from facts we have already proven.

(tr1) Tracial von Neumann algebras are finite.
(tr2) Every isometry in a tracial von Neumann algebra is a unitary.
TODO: more?

Definition 5.1.3. Given a tracial von Neumann algebra (M, tr), the Gelfand-Naimark-Segal (GNS) Hilbert space $L^2(M, \mathrm{tr})$ is the completion of M under $\|\cdot\|_2$ coming from the inner product

$$\langle x, y\rangle := \mathrm{tr}(y^*x).$$

We typically write $\Omega \in L^2(M, \mathrm{tr})$ for the image of $1 \in M$. When M is a tracial factor, the trace is unique, and we simply write L^2M.

We have the following facts, building on how we constructed the hyperfinite II_1 factor R.

(J1) The left action $\lambda_a x\Omega := ax\Omega$ of M on $L^2(M, \mathrm{tr})$ is by bounded operators and $\lambda_a^* = \lambda_{a^*}$.
(J2) The right action $\rho_b x\Omega := xb\Omega$ of M on $L^2(M, \mathrm{tr})$ is also by bounded operators and $\rho_b^* = \rho_{b^*}$.
(J3) The map $J : M\Omega \to M\Omega$ given by $x\Omega \mapsto x^*\Omega$ is a *conjugate-linear* unitary such that $J^2 = 1$.
(J4) The map J satisfies $\langle Jx\Omega, Jy\Omega\rangle = \langle x^*\Omega, y^*\Omega\rangle = \mathrm{tr}(yx^*) = \mathrm{tr}(x^*y) = \langle y\Omega, x\Omega\rangle$ for all $x, y \in M$. Hence $\langle J\eta, J\xi\rangle = \langle \xi, \eta\rangle$ for all $\eta, \xi \in L^2(M, \mathrm{tr})$.
(J5) For all $x \in B(L^2(M, \mathrm{tr}))$, $(JxJ)^* = Jx^*J$.

Proof. For all $a, b \in M$,

$$\begin{aligned}\langle a\Omega, JxJb\Omega\rangle &\underset{(J3)}{=} \langle J^2a\Omega, JxJb\Omega\rangle \underset{(J4)}{=} \langle xJb\Omega, Ja\Omega\rangle = \langle Jb\Omega, x^*Ja\Omega\rangle \\ &\underset{(J3)}{=} \langle Jb\Omega, J^2x^*Ja\Omega\rangle \underset{(J4)}{=} \langle Jx^*Ja\Omega, b\Omega\rangle\end{aligned}$$

By density of $M\Omega$ in $L^2(M, \mathrm{tr})$, $(JxJ)^* = Jx^*J$. □

(J6) The map J satisfies $J\lambda_a J = \rho_{a^*}$ and $J\rho_b J = \lambda_{b^*}$ for all $a, b \in M$. Typically, we abbreviate $JaJ := J\lambda_a J$. In particular, $(JaJ)^* = Ja^*J$ and $JMJ \subseteq M'$.
(J7) For all $x \in M'$, $Jx\Omega = x^*\Omega$.

Proof. For all $a \in M$, we have

$$\begin{aligned}\langle Jx\Omega, a\Omega\rangle &\underset{(J3)}{=} \langle Jx\Omega, J^2a\Omega\rangle \underset{(J4)}{=} \langle Ja\Omega, x\Omega\rangle = \langle a^*\Omega, x\Omega\rangle \\ &= \langle \Omega, ax\Omega\rangle = \langle \Omega, xa\Omega\rangle = \langle x^*\Omega, a\Omega\rangle.\end{aligned}$$

By density of $M\Omega$ in $L^2(M, \mathrm{tr})$, $Jx\Omega = x^*\Omega$. □

(J8) For all $x, y \in M'$, $JxJy = yJxJ$. Hence $JM'J \subseteq M'' = M$.

Proof. For all $a, b \in M$,

$$\begin{aligned}\langle JxJya\Omega, b\Omega\rangle &\underset{(J5)}{=} \langle ya\Omega, Jx^*Jb\Omega\rangle = \langle ya\Omega, Jx^*b^*\Omega\rangle = \langle ay\Omega, Jb^*x^*\Omega\rangle \\ &\underset{(J7)}{=} \langle ay\Omega, Jb^*Jx\Omega\rangle \underset{(J6)}{=} \langle JbJay\Omega, x\Omega\rangle \underset{(J6)}{=} \langle aJbJy\Omega, x\Omega\rangle \\ &= \langle JbJy\Omega, a^*x\Omega\rangle = \langle JbJy\Omega, xa^*\Omega\rangle = \langle JbJy\Omega, xJa\Omega\rangle \\ &\underset{(J3)}{=} \langle JbJy\Omega, J^2xJa\Omega\rangle \underset{(J4)}{=} \langle JxJa\Omega, by^*\Omega\rangle = \langle JxJa\Omega, y^*b\Omega\rangle \\ &= \langle yJxJa\Omega, b\Omega\rangle.\end{aligned}$$

By density of $M\Omega$ in $L^2(M, \mathrm{tr})$, $JxJy = yJxJ$. □

We may summarize the above results as follows.

Theorem 5.1.4. *Given a tracial von Neumann algebra (M, tr), the commutant of the left action of M in the GNS representation is given by the right action: $M' = JMJ$.*

Corollary 5.1.5. *The commutant of $L\Gamma$ acting on $\ell^2\Gamma$ is $R\Gamma$, the right regular group von Neumann algebra.*

Exercise 5.1.6. Show that the map between elements $x \in L\Gamma$ and their corresponding ℓ^2-vectors (x_g) such that $x\delta_e = \sum x_g\delta_g$ has image

$$\left\{(y_g) \in \ell^2\Gamma \middle| y * z \in \ell^2\Gamma \text{ for all } z \in \ell^2\Gamma\right\}$$

where $(y * z)_g = \sum_h y_h z_{h^{-1}g}$. That is, $L\Gamma$ corresponds to all the ℓ^2-sequences whose convolutions with all other ℓ^2-sequences are again ℓ^2.

5.2. **Conditional expectation.** In probability theory, there is a notion of a conditional expectation of a random variable (measurable function $f : (X, \mathcal{M}) \to \mathbb{C}$) with respect to a σ-subalgebra $\mathcal{N} \subset \mathcal{M}$. In more detail, given a probability measure $\mu : \mathcal{M} \to [0, \infty]$, it restricts to a probability measure $\mu|_{\mathcal{N}} : \mathcal{N} \to [0, \infty]$, and we have a natural inclusion of von Neumann algebras $L^\infty(X, \mathcal{N}, \mu|_{\mathcal{N}}) \subset L^\infty(X, \mathcal{M}, \mu)$. The *conditional expectation* of $f \in L^\infty(X, \mathcal{M}, \mu)C$ with respect to $\mathcal{N}$, denoted $\mathbb{E}_{\mathcal{N}}(f)$ is the unique element of $L^\infty(X, \mathcal{N}, \mu|_{\mathcal{N}})$ such that for all $A \in \mathcal{N}$,

$$\int_A f\, d\mu = \int f\chi_A\, d\mu = \int \mathbb{E}_{\mathcal{N}}(f)\chi_A\, d\mu = \int_A \mathbb{E}_{\mathcal{N}}(f)\chi_A\, d\mu.$$

We will show the existence and uniqueness of $\mathbb{E}_{\mathcal{N}}(f)$ in more general setting, namely a tracial von Neumann algebra (M, tr_M) and a von Neumann subalgebra $N \subseteq M$.

Facts 5.2.1. Suppose (M, tr_M) is a tracial von Neumann algebra and $N \subseteq M$ is a von Neumann subalgebra.

(E1) The inclusion $N\Omega \hookrightarrow M\Omega \subset L^2M$ is isometric with respect to $\|\cdot\|_2$. We thus get a canonical isometry $i_N : L^2N \to L^2M$ such that $n\Omega_N \mapsto n\Omega_M$.

(E2) The isometry i_N is N-N bilinear, i.e., for all $x, n \in N$,

$$i_N(x \cdot n\Omega_N) = i_N(xn\Omega_N) = xn\Omega_M = x \cdot n\Omega_M \qquad \text{and}$$
$$i_N(n\Omega_N) \cdot x = J_M x^* J_M \iota_N n\Omega_N = nx\Omega_M = i_N(nx\Omega_N) = i_N J_N x^* J_N n\Omega_N = i_N(n\Omega_N \cdot x).$$

(E3) The adjoint $i_N^* : L^2M \to L^2N$ is also N-N bilinear.[1]

Proof. Since $ni_N = i_N n$ for all $n \in N$, taking adjoints, $i_N^* n^* = n^* i_N^*$ for all $n \in N$. Since $J_M n^* J_M i_N = i_N J_N n^* J_N$ for all $n \in N$, taking adjoints,

$$i_N^* J_M n J_M = i_N^*(J_M n^* J_M)^* = (J_N n^* J_N)^* i_N^* = J_N n J_N i_N^*$$

for all $n \in N$. The result follows. □

(E4) For $m \in M$, the operator $E_N(m) := i_N^* m i_N \in B(L^2N)$ commutes with the right N-action and thus lies in $(J_N N J_N)' = N$.

(E5) $E_N(m)$ is the unique element of N such that $\operatorname{tr}_N(E_N(m)n) = \operatorname{tr}_M(mn)$ for all $n \in N$.

Proof. If $x \in N$ such that $\operatorname{tr}_N(xn) = \operatorname{tr}_M(mn)$ for all $n \in N$, then

$$\begin{aligned}\langle x\Omega_N, n\Omega_N\rangle_{L^2N} &= \operatorname{tr}_N(xn^*) = \operatorname{tr}_M(mn^*) = \langle m\Omega_M, n\Omega_M\rangle_{L^2M} \\ &= \langle m\iota_N\Omega_N, \iota_N n\Omega_N\rangle_{L^2M} = \langle \iota_N^* m\iota_N\Omega_N, n\Omega_N\rangle_{L^2N}\end{aligned}$$

for all $n \in N$, and thus $x = \iota_N^* m \iota_N = E_N(m)$. □

(E6) $E_N(amb) = aE_N(m)b$ for all $a, b \in N$ and $m \in M$. In particular, $E_N|_N = \mathrm{id}_N$.

Proof. Immediate from i_N, i_N^* being $N - N$ bilinear. □

(E7) $E_N : M \to N$ is a normal unital completely positive (ucp) map. In particular, $E_N(m^*) = E_N(m)^*$ for all $m \in M$.

Proof. The formula $E_N(m) = i_N^* m i_N$ is manifestly ucp (recall the Stinespring Theorem). In particular, since E_N sends positive elements to positive elements, writing a self-adjoint $x \in M$ as $x_+ - x_-$, we see that $E_N(x)$ is also self adjoint. The final statement now follows by taking real and imaginary parts:

$$E_N(m) = E_N(\operatorname{Re}(m) + i\operatorname{Im}(m)) = E_N(\operatorname{Re}(m)) + iE_N(\operatorname{Im}(m))$$

which implies

$$E_N(m^*) = E_N(\operatorname{Re}(m) - i\operatorname{Im}(m)) = E_N(\operatorname{Re}(m)) - iE_N(\operatorname{Im}(m)) = E_N(m)^*. \ \square$$

[1] In more generality, if $\pi_H : A \to B(H)$ and $\pi_K : A \to B(K)$ are two $*$-representations of a $*$-algebra A and $x \in B(H \to K)$ such that $x\pi_H(a) = \pi_K(a)x$ for all $a \in A$, then $\pi_H(a)x^* = x^*\pi_K(a)$ for all $a \in A$.

(E8) For all $m \in M$, $\|E_N(m)\| \leq \|m\|$,

> *Proof.* Since i_N is an isometry, $\|E_N(m)\| = \|i_N^* m i_N\| \leq \|m\|$. $\square$

(E9) For all $m \in M$, $E_N(m)^* E_N(m) \leq E_N(m^*m)$ and $E_N(m^*m) = 0$ implies $m = 0$.

> *Proof.* Since i_N is an isometry, $i_N i_N^* \leq 1_{L^2M}$. In particular,
> $$E_N(m)^* E_N(m) = i_N^* m^* i_N i_N^* m i_N \leq i_N^* m^* m i_N = E_N(m^*m).$$
> Finally, if $E_N(m^*m) = 0$, then $\operatorname{tr}_M(m^*m) = \operatorname{tr}_N(E_N(m^*m)) = 0$, so $m = 0$. $\square$

5.3. **Outer, ergodic, and free actions.** In this section, G denotes a group and M denotes a von Neumann algebra.

Definition 5.3.1. An *action* of G on M is a group homomorphism $\alpha : G \to \operatorname{Aut}(M)$, where $\operatorname{Aut}(M)$ is the group of $*$-algebra isomorphisms of M.

Exercise 5.3.2. Prove that every $*$-algebra isomorphism of M is σ-WOT continuous.

Example 5.3.3. Suppose $u : G \to U(H)$ such that for all $g \in G$, $u_g M u_g^* = M$. Then $\alpha : G \to \operatorname{Aut}(M)$ by $\alpha_g = \operatorname{Ad}(u_g)$ is an action.

Definition 5.3.4. An automorphism Φ of $M \subseteq B(H)$ is said to be *implemented* by a unitary $u \in U(H)$ if $\Phi(x) = uxu^*$ for all $x \in M$.

We call $\Phi \in \operatorname{Aut}(M)$ *inner* if it is implemented by a unitary $u \in U(M)$. If Φ is not inner, it is called *outer*.

An action $\alpha : G \to \operatorname{Aut}(M)$ is called *outer* if α_g is only inner when $g = e$.

Exercise 5.3.5. Show that every trace-preserving $*$-automorphism of a tracial von Neumann algebra (M, tr_M) can be implemented on $L^2(M, \operatorname{tr})$. Deduce that every $*$-automorphism of a II_1 factor can be implemented on L^2M.

Exercise 5.3.6. Prove that every $*$-automorphism of $B(H)$ is inner.

Exercise 5.3.7. Consider $\mathbb{F}_2 = \langle a, b \rangle$. Show that the swap $a \leftrightarrow b$ extends to a $*$-automorphism of $L\mathbb{F}_2$. Prove it is outer.

Example 5.3.8. Let (X, μ) be a measure space and $T : X \to X$ a bijection preserving the measure class of μ, i.e., $\mu(A) = 0$ iff $\mu(T^{-1}A) = 0$ for all measurable A. Then T gives an automorphism α_T of $L^\infty(X, \mu)$ by $(\alpha_T f)(x) := f(T^{-1}x)$.

Moreover, if T preserves μ, i.e., $\mu(A) = \mu(T^{-1}A)$ for all measurable A, then α_T is implemented by the unitary $(u_T\xi)(x) := \xi(T^{-1}x)$ for $\xi \in L^2(X, \mu)$. Indeed, one computes $(u_T^*\xi)(x) = \xi(Tx)$ and we observe

$$(u_T M_f u_T^* \xi)(x) = (M_f u_T^* \xi)(T^{-1}x) = f(T^{-1}x)(u_T^*\xi)(T^{-1}x) = f(T^{-1}x)\xi(x) = (M_{\alpha_T(f)}\xi)(x).$$

Exercise 5.3.9. Suppose (X, μ) is a measure space and ν is a measure *equivalent* to μ, i.e., $\mu(A) = 0$ if and only if $\nu(A) = 0$ for all measurable A. Explain why we may identify $L^\infty(X, \mu) = L^\infty(X, \nu)$ as von Neumann algebras.

Definition 5.3.10. A measurable bijection T of X is called *ergodic* if A measurable with $TA = A$ implies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Proposition 5.3.11. *T is ergodic if and only if $L^\infty(X,\mu)^{\alpha_T} = \mathbb{C}1$.*

Proof. Note that $TA = A$ if and only if $\alpha_T(\chi_A) = \chi_A$. Hence T is ergodic iff $P(L^\infty(X,\mu)^{\alpha_T}) = \{0,1\}$ iff $L^\infty(X,\mu)^{\alpha_T} = \mathbb{C}1$. □

Definition 5.3.12. We say an action $\alpha : G \to \mathrm{Aut}(M)$ is *ergodic* if $M^G = \mathbb{C}1$.

Lemma 5.3.13. *Suppose α is an action of a countable group Γ on (X,μ) preserving the measure class of μ. Consider the following statements.*

- (Γ1) *α is* essentially transitive*, i.e., there is an $x \in X$ such that $\mu(X \setminus \Gamma x) = 0$.*
- (Γ2) *α is* essentially countable*, i.e., there is a countable set $Y \subseteq X$ such that $\mu(X\setminus Y) = 0$ and $\mu(\{y\}) > 0$ for all $y \in Y$.*
- (Γ3) *There is an atom $x \in X$, i.e., there is an $x \in X$ with $\mu(\{x\}) > 0$.*

Then (Γ1) implies (Γ2) implies (Γ3). If α is ergodic, then (Γ3) implies (Γ1).

Proof. The only interesting part is proving (Γ3) implies (Γ1) when α is ergodic. If $x \in X$ is an atom, then $\Gamma x \subseteq X$ is a Γ-invariant subset with $\mu(\Gamma x) > 0$. By ergodicity, $\mu(X \setminus \Gamma x) = 0$. □

Remark 5.3.14. Really, an atom of (X,μ) is a measurable set $A \subseteq X$ such that $\mu(A) > 0$ and for all measurable $B \subseteq A$, $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. Thus atoms of (X,μ) exactly correspond to minimal projections of $L^\infty(X,\mu)$. By Lemma 5.3.13 (applied to an equivalent measure space where all atoms have been collapsed to points), if an ergodic action of a countable Γ on (X,μ) preserving the measure class of μ is not essentially transitive, then $L^\infty(X,\mu)$ has no minimal projections.

Definition 5.3.15. An automorphism Φ of M is called *free* or *properly outer* if

$$m \in M \text{ and } m\alpha(x) = xm \quad \forall\, x \in M \qquad \Longrightarrow \qquad m = 0.$$

An action $\alpha : G \to \mathrm{Aut}(M)$ is called *free* if α_g not free implies $g = e$.

Exercise 5.3.16. Show that if $M = L^\infty(X,\nu)$ where X is countable and ν is a weighted counting measure (without loss of generality, we may assume there are no points with mass zero), and $\alpha = \alpha_T \in \mathrm{Aut}(L^\infty(X,\mu))$ for some bijection $T : X \to X$, then α is free if and only if T has no fixed points.

Exercise 5.3.17. Suppose X is compact Hausdorff and μ is a Radon (finite non-negative regular Borel) measure on X. Let $T : X \to X$ be a homeomorphism preserving the measure class of μ. Then α_T is free iff $\mu(\{x \in X | Tx = x\}) = 0$.

Proposition 5.3.18. *If M is a factor, then every outer automorphism is free.*

Proof. We prove the contrapositive. Suppose $\Phi \in \mathrm{Aut}(M)$ and there is an $m \in M\setminus\{0\}$ such that $m\Phi(x) = xm$ for all $x \in M$. If $m \in U(M)$, then $\Phi = \mathrm{Ad}(m)$ and we are finished. Otherwise, taking adjoints, we have $m^*x = \Phi(x)m^*$ for all $x \in M$, and thus

$$mm^*x = m\Phi(x)m^* = xmm^* \quad \forall\, x \in M \qquad \Longrightarrow \qquad mm^* \in Z(M).$$

Similarly, $m^*m \in Z(M)$. Since $Z(M) = \mathbb{C}1$ and $m \neq 0$, $mm^* = r$ and $m^*m = s$ for some non-zero $r, s \in \mathbb{R}_{>0}$. Since $rm = mm^*m = sm$ and $m \neq 0$, $r = s$. Thus $u := r^{-1/2}m \in U(M)$ and $\Phi = \mathrm{Ad}(u)$ is inner. □

5.4. **The crossed product.** The crossed product can be defined for a locally compact group, but we will present a simplified version for discrete groups acting on tracial von Neumann algebras. In this section, Γ is a discrete group.

The crossed product of a group action $\alpha : \Gamma \to \mathrm{Aut}(M)$ is a von Neumann algebra containing M in which the group action is implemented by unitaries.

Definition 5.4.1. Suppose $\alpha : \Gamma \to \mathrm{Aut}(M)$ is a group action where $M \subseteq B(H)$. Form the Hilbert space

$$\ell^2(\Gamma, H) := \left\{ \xi : \Gamma \to H \middle| \sum_g \|\xi(g)\|^2 < \infty \right\}.$$

We define actions of Γ and M on $\ell^2(\Gamma, H)$ by

$$(u_g\xi)(h) := \xi(g^{-1}h) \qquad \text{and} \qquad (\pi_m\xi)(h) := \alpha_{h^{-1}}(m)\xi(h).$$

The *crossed product* $M \rtimes_\alpha \Gamma$ is the von Neumann algebra generated by the π_m and the u_g acting on $\ell^2(\Gamma, H)$.

Example 5.4.2. When $M = L^\infty(X, \mu)$ and $\alpha : \Gamma \to \mathrm{Aut}(M)$ comes from an action of Γ on (X, μ) preserving the measure class of μ, we call $L^\infty(X, \mu) \rtimes \Gamma$ the *group measure space construction*

Exercise 5.4.3. Prove the following facts about the crossed product $M \rtimes_\alpha \Gamma$.

(1) $\pi : M \to B(\ell^2(\Gamma, H))$ is an injective normal σ-WOT continuous $*$-homomorphism. Thus $\pi(M) = \pi(M)'' \cong M$ as von Neumann algebras.
(2) $u_g\pi_m u_g^* = \pi_{\alpha_g(m)}$, i.e., the α_g-action on M is implemented by the u_g.
(3) Finite linear combinations $\sum x_g u_g$ where $x_g \in M$ form a σ-WOT dense unital $*$-subalgebra.

Exercise 5.4.4. Find a unitary isomorphism $v : \ell^2(\Gamma, H) \to \ell^2\Gamma \otimes H$ such that $vu_gv^* = \lambda_g \otimes 1$ and $(v\pi_m v^*)(\delta_h \otimes \xi) = \delta_h \otimes \alpha_{h^{-1}}(m)\xi$.

We now provide sufficient conditions for the crossed product to be a factor.

Lemma 5.4.5. *If $\alpha : \Gamma \to \mathrm{Aut}(M)$ is free, then $M' \cap (M \rtimes_\alpha \Gamma) \subseteq Z(M)$.*

Proof. **TODO: give general proof** Suppose $y \in M' \cap (M \rtimes_\alpha \Gamma)$ and let $(y_g) \subset M$ such that $y(\Omega \otimes \delta_e) = \sum y_g \Omega \otimes \delta_g$. For $x \in M$, we calculate $y(x\Omega \otimes \delta_e)$ in two ways:

$$\sum xy_g \otimes \delta_g = xy(\Omega \otimes \delta_e) = yx(\Omega \otimes \delta_e) = y(x\Omega \otimes \delta_e) \underset{(\rtimes 3)}{=} \sum y_g\alpha_g(x) \otimes \delta_g.$$

Hence $xy_g = y_g\alpha_g(x)$ for all $x \in M$. By freeness, $y_g = 0$ unless $g = e$, so $y = y_e$ as $\Omega \otimes \delta_e$ is separating by ($\rtimes$5). Hence $y \in M' \cap M = Z(M)$. □

Remark 5.4.6. Lemma 5.4.5 immediately implies that

$$Z(M) = M' \cap M \subseteq M' \cap (M \rtimes_\alpha \Gamma) \subseteq Z(M)$$

so all inclusions are equalities, and

$$Z(M \rtimes_\alpha \Gamma) \subseteq M' \cap (M \rtimes_\alpha \Gamma) \subseteq Z(M).$$

Thus if M is a factor and α is free, then $M \rtimes_\alpha \Gamma$ is a factor.

Corollary 5.4.7. *If $\alpha : \Gamma \to \mathrm{Aut}(M)$ is free and ergodic, then $M \rtimes_\alpha \Gamma$ is a factor.*

Proof. Suppose $x \in Z(M \rtimes_\alpha \Gamma)$. Since α is free, by Remark 5.4.6, $x \in Z(M)$. Since x commutes with every u_g, $\alpha_g(x) = u_g x u_g^* = x$ for all $g \in \Gamma$, and $x \in M^\Gamma$. Since α is ergodic, $M^\Gamma = \mathbb{C}1$, and thus $Z(M \rtimes_\alpha \Gamma) = \mathbb{C}1$. $\square$

Corollary 5.4.8. *Consider a group measure space construction $L^\infty(X,\mu) \rtimes \Gamma$. If the action α is free, then $L^\infty(X,\mu) \subset L^\infty(X,\mu) \rtimes \Gamma$ is maximal abelian.*

Proof. Since α is free,

$$L^\infty(X,\mu)' \cap (L^\infty(X,\mu) \rtimes \Gamma) \subseteq Z(L^\infty(X,\mu)) = L^\infty(X,\mu)$$

by Lemma 5.4.5. Hence if $L^\infty(X,\mu) \subseteq A \subseteq L^\infty(X,\mu) \rtimes \Gamma$ with A abelian, then $A \subseteq L^\infty(X,\mu)' \cap (L^\infty(X,\mu) \rtimes \Gamma) \subseteq L^\infty(X,\mu)$. $\square$

We now give examples of free and ergodic actions.

Example 5.4.9. $\Gamma = \mathbb{Z}$ acts by translation on $(\mathbb{Z}, \nu)$ where ν is counting measure. We have $L^\infty(\mathbb{Z}, \nu) \rtimes \mathbb{Z} \cong B(\ell^2\mathbb{Z})$.

Example 5.4.10. An irrational rotation of the torus is free and ergodic. That is, consider $(X,\mu) = (\mathbb{T}, d\theta)$ and $\Gamma = \mathbb{Z}$ generated by $tz = e^{i\alpha} z$ where $\frac{\alpha}{2\pi} \notin \mathbb{Q}$.

Example 5.4.11 (Bernoulli shift)**.** Let Γ be infinite and countable, and let (X,μ) be a standard probability space. Consider $(X,\mu)^\Gamma$ with product measure. Then Γ acts on $(X,\mu)^\Gamma$ by $(g \cdot A)(h) = A(g^{-1}h)$, where $A : \Gamma \to (X,\mu)$ is a measurable function.

One can also do the action of Γ on $\bigotimes^\Gamma (M, \mathrm{tr})$ by $h \cdot (x_{g_1} \otimes x_{g_2} \otimes \cdots) = x_{hg_1} \otimes x_{hg_2} \otimes \cdots$.

Example 5.4.12. $SL(2,\mathbb{Z})$ acts on $\mathbb{R}^2$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$.

Example 5.4.13. The "$ax+b$" group $\mathbb{Q} \rtimes \mathbb{Q}^\times$ acts on $\mathbb{R}$ by $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ 1 \end{pmatrix}$.

5.5. **The crossed product when the action preserves a trace.** We now give a second equivalent definition of $M \rtimes_\alpha \Gamma$ in the setting where (M, tr) is a tracial von Neumann algebra and $\mathrm{tr} \circ \alpha_g = \mathrm{tr}$ for all $g \in \Gamma$.

First, form the Hilbert space $L^2M \otimes \ell^2\Gamma$. We have an amplified left M-action $x(m\Omega \otimes \xi) = xm\Omega \otimes \xi$ and a left Γ-action given by $u_g(m\Omega \otimes \delta_h) := \alpha_g(m)\Omega \otimes \delta_{gh}$. In other words, if $v_g \in U(L^2M)$ is the unitary $v_g m\Omega := \alpha_g(m)\Omega$ implementing α_g, then $u_g = v_g \otimes \lambda_g$. We define $M \rtimes_\alpha \Gamma$ as the von Neumann algebra generated by the operators $x \otimes 1$ for $x \in M$ and the u_g for $g \in \Gamma$ acting on $L^2M \otimes \ell^2\Gamma$.

Exercise 5.5.1. Find a unitary isomorphism $w : L^2M\otimes\ell^2\Gamma \to \ell^2\Gamma\otimes L^2M$ which intertwines the two above definitions of $M \rtimes_\alpha \Gamma$. [[is there an op issue here?]]

There is also a commuting right action of M and Γ on $L^2M \otimes \ell^2\Gamma$ by defining

$$(m\Omega\otimes\delta_h)x := m\alpha_h(x)\Omega\otimes\delta_h \qquad\text{and}\qquad (m\Omega\otimes\delta_h)g := m\Omega\otimes\delta_{hg}.$$

Note that these right actions commute with the left action of $M \rtimes_\alpha \Gamma$. We will eventually show that $M\rtimes_\alpha\Gamma$ carries a canonical normal faithful tracial state under which $L^2(M\rtimes_\alpha\Gamma) \cong L^2M \otimes \ell^2\Gamma$, allowing us to identify the left and right actions as the canonical ones.

Facts 5.5.2. Here are some basic facts about $M \rtimes_\alpha \Gamma$.

($\rtimes$1) For finite linear combinations $\sum x_g u_g \in M \rtimes_\alpha \Gamma$, $(\sum x_g u_g)(\Omega\otimes\delta_e) = \sum_g x_g\Omega\otimes\delta_g$.

($\rtimes$2) For every $x \in M \rtimes_\alpha \Gamma$, there is a unique sequence (x_g) in

$$\ell^2(\Gamma, M) := \left\{ m : \Gamma \to M \middle| \sum \|m_g\Omega\|^2_{L^2M} < \infty \right\}$$

such that $x(\Omega\otimes\delta_e) = \sum x_g\Omega\otimes\delta_g$.

Proof. For $g\in\Gamma$, define $p_g : L^2M\otimes\ell^2\Gamma \to L^2M$ by $m\Omega\otimes\eta \mapsto \langle\eta,\delta_g\rangle\alpha_{g^{-1}}(m)\Omega$. Then $p_g^* m\Omega = \alpha_g(m)\Omega\otimes\delta_g$. Observe that p_g^* is right M-linear, so p_g is as well by Footnote 1. Hence for all $x \in M \rtimes_\alpha \Gamma$, $p_g x p_e^* \in (JMJ)' \cap B(L^2M) = M$. Define $x_g := \alpha_g(p_g x p_e^*) \in M$. We then compute that for all $m \in M$ and $g \in \Gamma$,

$$\begin{aligned}\langle x(\Omega\otimes\delta_e), m\Omega\otimes\delta_g\rangle &= \langle xp_e^*\Omega, p_g^*\alpha_{g^{-1}}(m)\Omega\rangle = \langle p_g x p_e^*\Omega, \alpha_{g^{-1}}(m)\Omega\rangle_{L^2M}\\ &= \operatorname{tr}_M(\alpha_{g^{-1}}(m)^* p_g x p_e^*) = (\operatorname{tr}_M\circ\alpha_g)(\alpha_{g^{-1}}(m)^* p_g x p_e^*)\\ &= \operatorname{tr}_M(m^*\alpha_g(p_g x p_e^*)) = \operatorname{tr}_M(m^* x_g)\\ &= \sum_h \langle x_h\Omega\otimes\delta_h, m\Omega\otimes\delta_g\rangle.\end{aligned}$$

We conclude $x(\Omega\otimes\delta_e) = \sum_h x_h\Omega\otimes\delta_h$ and $x : \Gamma\to M$ lies in $\ell^2(\Gamma, M)$. □

($\rtimes$3) For all $g\in\Gamma$ and $m \in M$ and $x \in M\rtimes_\alpha\Gamma$, $x(m\Omega\otimes\delta_g) = \sum_h x_h\alpha_h(m)\Omega\otimes\delta_{hg}$.

Proof. Note that $m\Omega\otimes\delta_g = (\Omega\otimes\delta_e)\cdot m\cdot g$. Since the left and right actions commute,

$$x(m\Omega\otimes\delta_g) = (x(\Omega\otimes\delta_e))\cdot m\cdot g = \left(\sum_h x_h\Omega\otimes\delta_h\right)\cdot m\cdot g = \sum_h x_h\alpha_h(m)\Omega\otimes\delta_{hg}$$

as claimed. □

($\rtimes$4) For $x \in M\rtimes_\alpha\Gamma$, $x^*(\Omega\otimes\delta_e) = \sum_h \alpha_h(x^*_{h^{-1}})\Omega\otimes\delta_h$.

Proof. We compute

$$\begin{aligned}\langle x^*(\Omega\otimes\delta_e), m\Omega\otimes\delta_g\rangle = \langle\Omega\otimes\delta_e, x(m\Omega\otimes\delta_g)\rangle &= \sum_h \langle\Omega\otimes\delta_e, x_h\alpha_h(m)\Omega\otimes\delta_{hg}\rangle \\ &= \delta_{h=g^{-1}}\langle\Omega, x_{g^{-1}}\alpha_{g^{-1}}(m)\Omega\rangle = \operatorname{tr}(x^*_{g^{-1}}\alpha_{g^{-1}}(m)^*) \\ &= (\operatorname{tr}\circ\alpha_g)(x^*_{g^{-1}}\alpha_{g^{-1}}(m)^*) = \operatorname{tr}(\alpha_g(x^*_{g^{-1}})m^*) \\ &= \langle\alpha_g(x^*_{g^{-1}})\Omega, m\Omega\rangle_{L^2M} = \sum_h\langle\alpha_h(x^*_{h^{-1}})\Omega\otimes\delta_e, m\otimes\delta_g\Omega\rangle.\end{aligned}$$

The result follows. ∎

($\rtimes$5) $\Omega\otimes\delta_e$ is cyclic and separating for $M\rtimes_\alpha\Gamma$.

Proof. First, $x=0$ iff $x_g=0$ for all $g\in\Gamma$, which implies $\Omega\otimes\delta_e$ is separating. Now for any finite linear combination $\sum_g m_g\Omega\otimes\delta_g\in L^2M\otimes\ell^2\Gamma$,

$$\sum_g m_g\Omega\otimes\delta_g = \left(\sum_g m_g u_g\right)(\Omega\otimes\delta_e),$$

so $\Omega\otimes\delta_e$ is cyclic. ∎

($\rtimes$6) The normal state $\omega_{\Omega\otimes\delta_e} = \langle\,\cdot\,\Omega\otimes\delta_e, \Omega\otimes\delta_e\rangle$ on $M\rtimes_\alpha\Gamma$ is faithful and tracial.

Proof. For $x,y\in M\rtimes_\alpha\Gamma$,

$$\begin{aligned}\langle xy(\Omega\otimes\delta_e),\Omega\otimes\delta_e\rangle &= \langle y(\Omega\otimes\delta_e), x^*(\Omega\otimes\delta_e)\rangle \\ &= \sum_{g,h}\langle y_g\Omega\otimes\delta_g, \alpha_h(x^*_{h^{-1}})\Omega\otimes\delta_h\rangle \\ &= \sum_g\langle y_g\Omega, \alpha_g(x^*_{g^{-1}})\Omega\rangle_{L^2M} \\ &= \sum_g\operatorname{tr}(\alpha_g(x_{g^{-1}})y_g) \\ &= \sum_g(\operatorname{tr}\circ\alpha_{g^{-1}})(\alpha_g(x_{g^{-1}})y_g) \\ &= \sum_g\operatorname{tr}(x_{g^{-1}}\alpha_{g^{-1}}(y_g)) \\ &= \sum_h\operatorname{tr}(\alpha_h(y_{h^{-1}})x_h) \\ &= \cdots = \langle yx(\Omega\otimes\delta_e),\Omega\otimes\delta_e\rangle.\end{aligned}$$

Faithfulness follows from the computation

$$\begin{aligned}\langle x^*x(\Omega\otimes\delta_e),\Omega\otimes\delta_e\rangle &= \langle x(\Omega\otimes\delta_e),x(\Omega\otimes\delta_e)\rangle = \sum_{g,h}\langle x_g\Omega\otimes\delta_g, x_h\Omega\otimes\delta_h\rangle\\ &= \sum_g \langle x_g\Omega, x_g\Omega\rangle_{L^2M} = \sum_g \operatorname{tr}(x_g^*x_g).\end{aligned}$$ $\square$

($\rtimes$7) The map $m\Omega\otimes\delta_g \mapsto mu_g\Omega$ is an $M\rtimes_\alpha\Gamma - M\rtimes_\alpha\Gamma$ bilinear unitary $L^2M\otimes\ell^2\Gamma \cong L^2(M\rtimes_\alpha\Gamma)$.

Proof. For finite linear combinations $\sum x_gu_g \in M\rtimes_\alpha\Gamma$,

$$\begin{aligned}\left\|\left(\sum x_gu_g\right)\Omega\right\|^2_{L^2(M\rtimes\Gamma)} &= \left\langle\left(\sum x_hu_h\right)^*\left(\sum x_gu_g\right)(\Omega\otimes\delta_e),\Omega\otimes\delta_e\right\rangle\\ &= \left\langle\left(\sum x_gu_g\right)(\Omega\otimes\delta_e),\left(\sum x_hu_h\right)(\Omega\otimes\delta_e)\right\rangle\\ &= \sum_{g,h}\langle x_g\Omega\otimes\delta_g, x_h\Omega\otimes\delta_h\rangle\\ &= \left\|\sum x_g\Omega\otimes\delta_g\Omega\right\|^2_{L^2M\otimes\ell^2\Gamma},\end{aligned}$$

and thus the map is isometric. We leave $M\rtimes_\alpha\Gamma$ bilinearity to the reader. $\square$

5.6. **The type of the crossed product.** Suppose $\alpha:\Gamma\to\mathrm{Aut}(M)$ is free and ergodic so that $M\rtimes_\alpha\Gamma$ is a factor. We further consider the special case of $M=L^\infty(X,\mu)$ coming from an action of Γ on (X,μ) preserving the measure class of μ. There are 4 types of free and ergodic actions of a countable discrete group Γ acting on (X,μ).

- (type I) Γ acts freely transitively so that X is a Γ-torsor.
- (type II_1) Γ preserves a finite measure on X.
- (type II_∞) Γ preserves an infinite measure on X.
- (type III) no measure on X equivalent to μ is preserved by Γ.

Theorem 5.6.1. *If α is a free ergodic, essentially transitive action, then $L^\infty(X,\mu)\rtimes\Gamma$ is type* I.

Proof. Since α is essentially transitive, by Lemma 5.3.13, $X=\Gamma x$ for some $x\in X$ up to null sets (where we have replaced atoms in (X,μ) by points). Thus we may identify μ with a weighted counting measure. Then $\chi_{\{x\}}\in L^\infty(X,\mu)$ is a minimal projection for every $x\in X$. We claim $\chi_{\{x\}}$ is also minimal in $L^\infty(X,\mu)\rtimes\Gamma$, showing it is type I. Since finite linear combinations $\sum y_gu_g$ form a σ-WWOT dense unital $*$-subalgebra, it suffices to prove that for every $h\neq e$ and $y\in L^\infty(X,\mu)$,

$$\chi_{\{x\}}yu_h\chi_{\{x\}}=0.$$

Indeed, for all $\xi\in L^2(\Gamma,L^2(X,\mu))$ and $g\in\Gamma$,

$$(\chi_{\{x\}}yu_h\chi_{\{x\}}\xi)(g) = \alpha_{g^{-1}}(\chi_{\{x\}})\alpha_{g^{-1}}(y)(u_h\chi_{\{x\}}\xi)(g)$$

$$
\begin{aligned}
&= \alpha_{g^{-1}}(\chi_{\{x\}})\alpha_{g^{-1}}(y)(\chi_{\{x\}}\xi)(h^{-1}g) \\
&= \alpha_{g^{-1}}(\chi_{\{x\}})\alpha_{g^{-1}}(y)\alpha_{g^{-1}h}(\chi_{\{x\}})\xi(h^{-1}g).
\end{aligned}
$$

Now as $L^\infty(X,\mu)$ is abelian, we see

$$\alpha_{g^{-1}}(\chi_{\{x\}})\alpha_{g^{-1}h}(\chi_{\{x\}}) = \chi_{g^{-1}x}\chi_{g^{-1}hx} = 0$$

as $h \neq e$ and α is free. □

Fact 5.6.2. Suppose (M, tr) is a tracial von Neumann algebra and $\alpha : \Gamma \to \mathrm{Aut}(M)$ is an action such that $\mathrm{tr} \circ \alpha_g = \mathrm{tr}$ for all $g \in \Gamma$. If α is free and ergodic, then $M \rtimes_\alpha \Gamma$ has a faithful normal tracial state by ($\rtimes$6), so it must be either type $\mathrm{I_n}$ for $n < \infty$ or type II_1.

Theorem 5.6.3. *If the action of Γ on (X,μ) is free, ergodic, non-transitive, and μ is a finite measure such that $\mu(gA) = \mu(A)$ for all measurable A, then $L^\infty(X,\mu) \rtimes \Gamma$ is type II_1.*

Proof. Since the action of Γ preserves the faithful normal tracial state $\int \cdot\, d\mu$, $L^\infty(X,\mu) \rtimes \Gamma$ is either finite dimensional or type II_1. So it suffices to prove that if $L^\infty(X,\mu) \rtimes \Gamma$ is finite dimensional and α is free and ergodic, then α is essentially transitive. If $L^\infty(X,\mu) \rtimes \Gamma$ is finite dimensional, then $L^\infty(X,\mu)$ is finite dimensional, and thus has minimal projections. Thus $(X,\mu) \cong (Y,\nu)$ for some finite measure space Y with ν a weighted counting measure. Indeed, by a maximality argument, we can write $1 = \sum_{i=1}^n \chi_{A_i}$ where each χ_{A_i} is minimal in $L^\infty(X,\mu)$ and the A_i are disjoint measurable subsets. We then define $\nu(\{i\}) := \mu(A_i)$. Finally, the action of Γ on the finite measure space (Y,ν) is free and ergodic, which implies it is transitive by Exercise 5.3.13. □

Exercise 5.6.4. A factor M is type II_∞ iff 1_M is infinite and there is a nonzero finite projection $p \in M$ such that pMp is type II_1.

Exercise 5.6.5. If $\{e_{ij}\} \subset M \subseteq B(H)$ is a system of matrix units, then there is a unitary $u : H \to e_{11}H \otimes \ell^2(I)$ such that $uMu^* = e_{11}Me_{11} \otimes B(\ell^2(I))$.

Lemma 5.6.6. *If M is a II_∞ factor, there is a II_1 factor N and a unital $*$-isomorphism $M \cong N \otimes B(\ell^2(I))$.*

Proof. By Exercise 5.6.4, there is a non-zero finite projection $p \in M$. Let $\{p_i\}_{i \in I}$ be a maximal family of mutually orthogonal projections such that $p_i \approx p$ for all $i \in I$.

Claim. $\sum p_i \approx 1$.

Proof of claim. Set $q = 1 - \sum p_i$. Since M is a factor, by maximality, $q \preccurlyeq p$. Since 1_M is infinite, there is an $i_0 \in I$ and a bijection $I \cong I \setminus \{i_0\}$. Then

$$1 = q + \sum p_i \approx q + \sum_{i \neq i_0} p_i \preccurlyeq p_{i_0} + \sum_{i \neq i_0} p_i = \sum p_i \preccurlyeq 1. \qquad \square$$

By the claim, we may assume that $\sum p_i = 1$; ortherwise, replace p_i with $u^* p_i u$ where $uu^* = \sum p_i$ and $u^*u = 1$. Now since $\sum p_i = 1$ and each $p_i \approx p$, for each j, we can choose a partial isometry e_{1j} such that $e_{1j}e_{1j}^* = p_1$ and $e_{1j}^*e_{1j} = p_j$. We then extend the e_{1j} to a system of matrix units in the usual way. Finally, the result follows from Exercise 5.6.5. $\square$

Theorem 5.6.7. *If the action of Γ on (X, μ) is free, ergodic, non-transitive, and μ is an infinite σ-finite measure such that $\mu(gA) = \mu(A)$ for all measurable A, then $L^\infty(X, \mu) \rtimes \Gamma$ is type* II_∞.

Proof. By Remark 5.3.14, there are no minimal projections in $L^\infty(X, \mu)$. As (X, μ) is σ-finite, there is a set $Y \subset X$ with $0 < \mu(Y) < \infty$. Consider the unit vector $\xi := \mu(Y)^{-1/2}\chi_Y \otimes \delta_e$ and the projection $p := \chi_Y$.

Claim. *The normal state ω_ξ on the factor $p(L^\infty(X, \mu) \rtimes \Gamma)p$ is tracial.*

Proof of claim. By a calculation similar to ($\rtimes$6), for all $x, y \in L^\infty(X, \mu) \rtimes \Gamma$,

$$\begin{aligned}\omega_\xi(pxppyp) &= \frac{1}{\mu(Y)} \sum_g \mathrm{tr}_M(\alpha_g(p)\alpha_g(x_{g^{-1}})py_g) \\ &= \frac{1}{\mu(Y)} \sum_g \mathrm{tr}_M(px_{g^{-1}}\alpha_{g^{-1}}(p)\alpha_{g^{-1}}(y_g)) = \omega_\xi(pyppxp). \qquad \square\end{aligned}$$

By the claim, $p(L^\infty(X, \mu) \rtimes \Gamma)p$ is a factor with no minimal projections and a tracial state, and thus is type II_1. But $L^\infty(X, \mu) \rtimes \Gamma$ is not type II_1 as it has an infinite family of non-zero mutually orthogonal projections (why?). Hence 1 is infinite and $L^\infty(X, \mu) \rtimes \Gamma$ is type $\mathrm{II})_\infty$ by Exercise 5.6.4. $\square$

We omit the proof that if Γ preserves no measure equivalent to μ, then $L^\infty(X, \mu) \rtimes \Gamma$ is type III.