3. von Neumann algebra basics

For this section, H is a Hilbert space.

3.1. Operator topologies.

Definition 3.1.1. The weak operator topology (WOT) is the locally convex TVS structure on $B(H)$ induced by the separating family of seminorms

$$\{x \mapsto |\langle x\eta, \xi\rangle| | \eta, \xi \in H\}.$$

Thus $x_i \to x$ WOT if and only if $\langle x_i\eta, \xi\rangle \to \langle x\eta, \xi\rangle$ for all $\eta, \xi \in H$.

The strong operator topology (SOT) is the locally convex TVS structure on $B(H)$ induced by the separating family of seminorms

$$\{x \mapsto \|x\xi\| | \xi \in H\}.$$

Thus $x_i \to x$ SOT if and only if $x_i\xi \to x\xi$ for all $\xi \in H$.

More operator topologies will be introduced later.

Facts 3.1.2. Here are some basic facts about these operator topologies.

(OT1) WOT $\subseteq$ SOT $\subseteq$ norm, with equality if and only if H is finite dimensional.

(OT2) $*$ is WOT-continuous, but not SOT-continuous (unless H is finite dimensional).

Proof. If $x_i \to x$ WOT, then $|\langle (x^* - x_i^*)\eta, \xi\rangle| = |\langle \eta, (x - x_i)\xi\rangle| \to 0$ for all η, ξ, so $x_i^* \to x^*$ WOT.
Now suppose (e_n) is an orthonormal sequence, and consider the unilateral shift $se_n = e_{n+1}$ for all n. Then $s^*e_n = e_{n-1}$ for $n \geq 2$ and $s^*e_1 = 0$. Then $(s^*)^n \to 0$ SOT, but $\|s^n\xi\| = \|\xi\|$ for all n. $\square$

(OT3) $*$ is SOT-continuous on the subset of normal elements.

Proof. Observe that x normal is equivalent to $\|x\xi\| = \|x^*\xi\|$ for all $\xi \in H$. If $x_i \to x$ SOT, then for all $\xi \in H$,

$$\begin{aligned}
\|(x^* - x_i^*)\xi\|^2 &= \langle (x - x_i)(x - x_i)^*\xi, \xi\rangle \\
&= \|x^*\xi\|^2 - \langle xx_i^*\xi, \xi\rangle - \langle x_ix^*\xi, \xi\rangle + \|x_i^*\xi\|^2. \\
&= \|x^*\xi\|^2 - \underbrace{\langle x_i^*\xi, x^*\xi\rangle}_{\to\langle x^*\xi, x^*\xi\rangle} - \underbrace{\langle x_ix^*\xi, \xi\rangle}_{\to\langle xx^*\xi,\xi\rangle} + \underbrace{\|x_i\xi\|^2}_{\to\|x\xi\|^2} \\
&= \|x^*\xi\|^2 - \|x\xi\|^2 = 0.
\end{aligned}$$

In the third equality above, we used normality of x_i. To get to the next line, we used that SOT-convergence implies WOT-convergence and that $*$ is WOT-continuous. The final equality follows from normality of x. $\square$

(OT4) Multiplication is separately WOT/SOT-continuous in each variable, but not jointly.

Example 3.1.3. $N := \{x \in B(H) | x^2 = 0\}$ is SOT dense in $B(H)$. Indeed, the sets

$$\{x \in B(H) | \|(x - x_0)\xi_i\| < \varepsilon, \ \forall\, i = 1, \ldots, n\}$$

indexed over fixed $x_0 \in B(H)$ and $\xi_1, \ldots, \xi_n \in H$ linearly independent form a base for the SOT. Each such set contains an element of N. To see this, choose $\eta_1, \ldots, \eta_n$ such that $S = \{\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n\}$ is linearly independent and $\|x_0\xi_i - \eta_i\| < \varepsilon$ for all i. Defining $x\xi_i = \eta_i$ and $x\eta_i = 0$ and $x = 0$ on $S^\perp$ gives such an element of N.

(OT5) Multiplication is jointly SOT-continuous on $B_r(0) \times B(H)$ for all $r > 0$. In particular, multiplication is jointly SOT-continuous on bounded sets.

Proof. If $x_i \to x$ and $y_i \to y$ SOT with $\|x_i\| < r$ for all i, then

$$\begin{aligned}\|(xy - x_iy_i)\xi\| &\le \|(xy - x_iy)\xi\| + \|(x_iy - x_iy_i)\xi\| \\ &\le \underbrace{\|(x - x_i)y\xi\|}_{\to 0} + \underbrace{\|x_i\|}_{\le r} \cdot \underbrace{\|(y - y_i)\xi\|}_{\to 0}.\end{aligned}$$

□

For Proposition 3.1.4 below, we will use the following trick.

Trick (Amplification)**.** Given a Hilbert space H, H^n is also a Hilbert space with

$$\langle (\eta_i), (\xi_i) \rangle_{H^n} := \sum_{i=1}^{n} \langle \eta_i, \xi_i \rangle_H.$$

Given $x \in B(H)$, x acts on H^n by $\alpha_x(\eta_i) := (x\eta_i)$, and $\|\alpha_x\|_{B(H^n)} = \|x\|_{B(H)}$.

Proposition 3.1.4. *For a functional $\varphi\colon B(H) \to \mathbb{C}$, the following are equivalent.*

(1) There are $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in H$ such that $\varphi(x) = \sum \langle x\eta_i, \xi_i \rangle$,
(2) φ is WOT-continuous, and
(3) φ is SOT-continuous.

Proof. That (1) ⇒ (2) ⇒ (3) is straightforward.
For (3) ⇒ (1), the strategy of the proof is as follows:

(a) use SOT-continuity to find $\eta_1, \ldots, \eta_n$,
(b) amplify the action and look at $\eta := (\eta_i)_{i=1}^n \in H^n$,
(c) φ gives a bounded functional on the cyclic subspace generated by $\eta \in H^n$, and
(d) use Hahn-Banach and Riesz Representation to find $\xi_1, \ldots, \xi_n$.

Suppose φ is SOT-continuous. Since $\varphi^{-1}(B_1^{\mathbb{C}}(0))$ is SOT-open, there are $\eta_1, \ldots, \eta_n \in H$ such that

$$\|x\eta_i\| < 1 \qquad \Rightarrow \qquad |\varphi(x)| < 1 \qquad\qquad \forall\, i = 1, \ldots, n, \ \forall\, x \in B(H).$$

This implication gives the following inequalities:[a]

$$|\varphi(x)| \leq \max_{i=1,\dots,n} \|x\eta_i\| \leq \left(\sum_{i=1}^n \|x\eta_i\|^2\right)^{1/2} \qquad \forall\, x \in B(H). \tag{3.1.5}$$

Consider the cyclic subspace generated by $\eta := (\eta_i)_{i=1}^n \in H^n$:

$$K := \{\alpha_x\eta = (x\eta_i)_{i=1}^n | x \in B(H)\} \subset H^n.$$

We claim $\psi(\alpha_x\eta) := \varphi(x)$ is a well-defined bounded linear functional on K. Indeed,

$$|\psi(\alpha_x\eta)| := |\varphi(x)| \underset{(3.1.5)}{\leq} \left(\sum_{i=1}^n \|x\eta_i\|^2\right)^{1/2} = \|\alpha_x\eta\|_K,$$

so $\alpha_x\eta = 0$ implies $\psi(\alpha_x\eta) = 0$, and $\psi \in K^*$. By Hahn-Banach, we can extend ψ to H^n, and by Riesz Representation, there is a $\xi = (\xi_i)_{i=1}^n \in H^n$ such that

$$\varphi(x) = \psi(\alpha_x\eta) = \langle \alpha_x\eta, \xi\rangle_K = \varphi(x) = \sum \langle x\eta_i, \xi_i\rangle \qquad \forall\, x \in B(H)$$

as desired. □

[a] WLOG, if $\|x\eta_1\| < |\varphi(x)|$ for some x, then for some $\lambda > 0$, $\|(\lambda x)\eta_1\| < 1 < |\varphi(\lambda x)|$. The other inequality is a standard fact about $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

Corollary 3.1.6. *Both the WOT and the SOT have the same closed convex sets.*

Proof. Apply the Separating Hyperplane Theorem to see that each closed convex set is an intersection of one side of the separating hyperplanes associated to the continuous linear functionals. Since the sets of continuous linear functionals agree, so does this intersection. □

Exercise 3.1.7. Suppose H is a Hilbert space, and (x_i) is a norm bounded, increasing net of self-adjoint operators in $B(H)$, i.e., $x_i = x_i^*$ and $\|x_i\| < K$ for all i, and $i \leq j$ implies $x_i \leq x_j$. Prove that the following are equivalent.

(1) $x_i \to x$ SOT.
(2) $x_i \to x$ WOT.
(3) For every $\xi \in H$, $\omega_\xi(x_i) = \langle x_i\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.
(4) There exists a dense subspace $D \subset H$ such that for every $\xi \in D$, $\omega_\xi(x_i) = \langle x_i\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.

We say an increasing net of positive operators (x_i) *increases to* $x \in B(H)_+$, denoted $x_i \nearrow x$, if any of the above equivalent conditions hold.

Hint: It suffices to prove (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (3). *For* (3) $\Rightarrow$ (1), *note that* $\sqrt{x - x_i} \geq 0$, *and use (OT5) and (SOT4) to show $x_i \to x$ SOT if and only if $\sqrt{x - x_i} \to 0$ SOT.*

3.2. **Bicommutant Theorem and first examples.**

Definition 3.2.1. For $S \subseteq B(H)$, define the commutant

$$S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

Exercise 3.2.2. Prove the following.

(1) $S \subseteq T$ implies $T' \subseteq S'$.
(2) $S \subseteq S''$
(3) $S' = S'''$.

Lemma 3.2.3. *Suppose $S \subseteq B(H)$ is $*$-closed and $K \subseteq H$ is a closed subspace. Then K is S-invariant ($sK \subset K$ for all $s \in S$) if and only if $p_K \in S'$.*

Proof. Immediate from the earlier exercise that K is s and s^*-invariant if and only if $[s, p_K] = 0$. □

Exercise 3.2.4. In this exercise, we work through the compatibility between commutant and amplification. Let H be a Hilbert space.

(1) Find a unital $*$-isomorphism $B(H^n) \cong M_n(B(H))$.
Hint: use orthogonal projections.
(2) Suppose $S \subseteq B(H)$, and let $\alpha\colon B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

Prove that:
(a) $\alpha(S)' = M_n(S')$, and
(b) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.
(c) Deduce that when $0, 1 \in S$, $\alpha(S)'' = \alpha(S'')$.

Lemma 3.2.5. *If $M \subseteq M_n(\mathbb{C})$ is a unital $*$-closed subalgebra, then $M = M''$.*

Proof. It suffices to prove $y \in M''$ implies $y \in M$. Fix $y \in M''$, and consider the amplified action $\alpha : M'' \to M_n(M_n(\mathbb{C})) \cong B(\bigoplus_{i=1}^n \mathbb{C}^n)$ and the vector $\xi = (e_i)_{i=1}^n \in \bigoplus_{i=1}^n \mathbb{C}^n$. Set $K = \alpha(M)\xi \subseteq \bigoplus_{i=1}^n \mathbb{C}^n$, and observe that $\alpha(M)K \subseteq K$. Since $M = M^*$, $p_K \in \alpha(M)' = M_n(M')$ by Exercise 3.2.4. So if $y \in M''$, then $\alpha(y) \in M_n(M')'$ commutes with p_K, and thus $\alpha(y)K \subseteq K$. Since $1 \in M$, $\xi \in K$, and thus $\alpha(y)\xi \in K = \alpha(M)\xi$. So there is an $x \in M$ such that $\alpha(y)\xi = \alpha(x)\xi$. Then for all $i = 1, \ldots, n$, $ye_i = xe_i$, so $y = x \in M$. □

Theorem 3.2.6 (von Neumann bicommutant)**.** *If $M \subset B(H)$ is a unital $*$-closed subalgebra, the following are equivalent:*
(1) $M = M''$,
(2) M is WOT-closed, and
(3) M is SOT-closed.

Such a unital $*$-closed subalgebra of $B(H)$ is called a von Neumann algebra.

Proof.
(1) $\Rightarrow$ (2): Commutants are WOT-closed, since if $x_i \to x$ WOT in M, then for all $y \in M'$ and $\eta, \xi \in H$,

$$\langle xy\eta, \xi\rangle \longleftarrow \langle x_i y\eta, \xi\rangle = \langle yx_i\eta, \xi\rangle \longrightarrow \langle yx\eta, \xi\rangle,$$

so $xy = yx$.
(2) $\Leftrightarrow$ (3): Since M is convex, M is WOT-closed if and only if it is SOT-closed by Corollary 3.1.6.
(3) $\Rightarrow$ (1): Suppose $y \in M''$, and consider a basic SOT-open neighborhood

$$\{x \in B(H) | \|(x-y)\xi_i\| < \varepsilon,\ \forall\, i = 1, \dots, n\}$$

of y where $\xi_1, \dots, \xi_n$ are linearly independent. To see that M intersects this neighborhood non-trivially, set $\xi = (\xi_i)_{i=1}^n \in \bigoplus_{i=1}^n H$, and consider the amplified representation of $B(H)$ on $\bigoplus_{i=1}^n H$. Define $K := \overline{\alpha(M)\xi} \subseteq \bigoplus_{i=1}^n H$, and observe K is $\alpha(M)$-invariant. Hence $p_K \in \alpha(M)' = M_n(M')$ which visibly commutes with $\alpha(y)$. Since $1 \in M$, $\alpha(y)\xi \in K$, and thus for every $\varepsilon > 0$, there is an $x \in M$ with $\|\alpha(x)\xi - \alpha(y)\xi\| < \varepsilon$. But then $\|x\xi_i - y\xi_i\| < \varepsilon$ for all i. $\square$

Examples 3.2.7. Here are some examples of von Neumann algebras.

(1) $M_n(\mathbb{C}) \cong B(H)$ for $\dim(H) = n$.
(2) Any finite dimensional unital $*$-closed subalgebra of $M_n(\mathbb{C})$.
(3) $B(H)$ itself.
(4) $L^\infty(X, \mu)$ for a σ-finite meansure space (X, μ).
(5) If $S = S^* \subset B(H)$, then S' is a von Neumann algebra.
(6) If $S \subset B(H)$, then $\langle S \rangle := (S \cup S^*)''$ is the von Neumann algebra generated by S.

Example 3.2.8 (Group von Neumann algebra)**.** Let Γ be a discrete group. Define

$$\ell^2\Gamma := \left\{ \xi : \Gamma \to \mathbb{C} \middle| \sum_g |\xi(g)|^2 < \infty \right\}$$

with inner product $\langle \eta, \xi \rangle := \sum_g \eta(g)\overline{\xi(g)}$. An ONB for $\ell^2\Gamma$ is given by $\{\delta_g : h \mapsto \delta_{g=h}\}_{g \in \Gamma}$. For all $g \in \Gamma$, we define a unitary operator $\lambda_g \in B(\ell^2\Gamma)$ by $(\lambda_g\xi)(h) := \xi(h^{-1}g)$. Then $\lambda_g\lambda_h = \lambda_{gh}$ and $\lambda_g^* = \lambda_{g^{-1}}$, so we get a group homomorphism $\lambda : \Gamma \to U(\ell^2\Gamma)$ called the *left regular representation*. The *group algebra* is $\mathbb{C}\Gamma := \operatorname{span} \lambda\Gamma$. Its norm closure is the *reduced group* C*-*algebra* $C_r^*\Gamma := \overline{\lambda\Gamma}^{\|\cdot\|}$. The *group von Neumann algebra* is $L\Gamma := (\lambda\Gamma)''$.

Open problem: Is $L\mathbb{F}_2 \cong L\mathbb{F}_3$?

Proposition 3.2.9. *Suppose $M \subseteq B(H)$ is a von Neumann algebra and $x = u|x|$ is the polar decomposition of $x \in M$. Then $u \in M$.*

Proof. Since $|x| \in M$, for all $v \in U(M')$, $x = v^*xv = v^*u|x|v = v^*uv|x|$. Moreover, $\ker(v^*uv) = v^*\ker(u) = v^*\ker(x)$. But since v^* commutes with x, $v^*\ker(x) = \ker(x)$. So by the uniqueness statement of the polar decomposition, $v^*uv = u$ for all $v \in U(M')$. Since the unitaries of M' linearly span M', $u \in M'' = M$. $\square$

3.3. **Strongly continuous functions and Kaplansky density.**

Facts 3.3.1. Here are some basic facts about SOT-continuous functions.

(SOT1) If $p \in \mathbb{C}[z, \overline{z}]$, then $x \mapsto p(x)$ is SOT-continuous on bounded sets of normal operators.

Proof. Multiplication is jointly SOT-continuous on bounded subsets, and $*$ is SOT-continuous on the subset of normal operators. $\square$

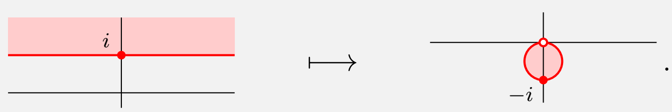
Remark 3.3.2. (SOT1) above holds on bounded sets of $B(H)$ for non-commutative polynomials $p \in \mathbb{C}\langle z, \overline{z}\rangle$.

(SOT2) If $f \in C(\mathbb{C})$, then $x \mapsto f(x)$ is SOT-continuous on bounded sets of normal operators.

Proof. Suppose (x_i) is a bounded net of normal operators and x is normal with $x_i \to x$ SOT. There is an $R > 0$ such that $\mathrm{sp}(x), \mathrm{sp}(x_i) \subseteq \overline{B_R^{\mathbb{C}}(0)}$. Then $f|_{B_R(0)}$ can be uniformly approximated by polynomials in $z, \overline{z}$. The result now follows from (SOT1) by a standard $\varepsilon/3$ argument. $\square$

(SOT3) The Cayley transform $x \mapsto (x-i)(x+i)^{-1}$ is SOT-countinuous $B(H)_{\mathrm{sa}} \to U(H)$.

Proof. First, observe that the map $z \mapsto z^{-1}$ on $\mathbb{C}$ maps



Hence by the Spectral Mapping Theorem, for x self-adjoint, $\mathrm{sp}((x+i)^{-1}) \subset \overline{B_1^{\mathbb{C}}(0)}$. Since $(x+i)^{-1}$ is normal, we know that $\|(x+i)^{-1}\| = r((x+i)^{-1}) \leq 1$. Now suppose $x_j \to x$ is an SOT-convergent net of self-adjoint operators (so x is self-adjoint). Then for all $\xi \in H$,

$$\begin{aligned}
&\|(x-i)(x+i)^{-1}\xi - (x_j - i)(x_j+i)^{-1}\xi\| \\
&\qquad = \|(x_j+i)^{-1} \underbrace{\big((x_j+i)(x-i) - (x_j-i)(x+i)\big)}_{2i(x-x_j)}(x+i)^{-1}\xi\| \\
&\qquad \leq 2\|(x-x_j)\underbrace{(x+i)^{-1}\xi}_{\in H}\| \longrightarrow 0.
\end{aligned}$$

$\square$

Remark 3.3.3. The Cayley transform is a Möbius transformation which sends $\mathbb{R} \to \mathbb{T} = S^1$, since

$$\frac{t-i}{t+i} \cdot \frac{t-i}{t-i} = \frac{(t-i)^2}{t^2+1} = \frac{t^2-1}{t^2+1} - i\frac{2t}{t^2+1},$$

and $(t^2-1)^2 + (2t)^2 = (t^2+1)^2$.

Alternatively, a Möbius transformation must map $\mathbb{R}$ onto a line or circle in $\mathbb{C}$, and we calculate

$$\begin{aligned} 0 &\mapsto \frac{-i}{i} = -1 \\ 1 &\mapsto \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = \frac{-2i}{2} = -i \\ -1 &\mapsto \frac{-1-i}{-1+i} = \frac{(-1-i)^2}{2} = \frac{2i}{2} = i. \end{aligned}$$

For $x \in B(H)_{\mathrm{sa}}$, by the Spectral Mapping Theorem, $\mathrm{sp}((x-i)(x+i)^{-1}) \subset \mathbb{T} = S^1$ and is normal, and is thus a unitary.

Since the inverse of the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ $(ad - bc \neq 0)$ is given by $z \mapsto \frac{dz-b}{-cz+a}$, the inverse of the Cayley transform is given by $u \mapsto i(1+u)(1-u)^{-1}$.

(SOT4) If $f \in C_0(\mathbb{R})$, then $x \mapsto f(x)$ is SOT-continuous on $B(H)_{\mathrm{sa}}$.

Proof. Let $f \in C_0(\mathbb{R})$. Define $g\colon \mathbb{T} \to \mathbb{C}$ by

$$g(t) := \begin{cases} f\left(i \cdot \frac{1+t}{1-t}\right) & \text{if } t \neq 1 \\ 0 & \text{if } t = 1 \end{cases} \qquad \text{so} \qquad g = f \circ c^{-1}$$

where c^{-1} is the inverse of the Cayley Transform. By (SOT2), g is SOT-continuous on $U(H)$. Now $f = g \circ c$ where c is the Cayley Transform. So by (SOT3), we have f is SOT-continuous as a composite of SOT-continuous maps. □

For $S \subset B(H)$, we write $(S)_1 := S \cap \overline{B_1(0)}$.

Theorem 3.3.4 (Kaplansky Density)**.** *Suppose $M \subseteq B(H)$ is a $*$-subalgebra.*

(1) $(M_{\mathrm{sa}})_1$ is SOT-dense in $(\overline{M}^{SOT}_{\mathrm{sa}})_1$.
(2) $(M_+)_1$ is SOT-dense in $(\overline{M}^{SOT}_+)_1$.
(3) $(M)_1$ is SOT-dense in $(\overline{M}^{SOT})_1$.

Proof. We proceed in several steps.
Step 0: We may assume M is a C*-algebra.

This reduction follows by noting:

(3′) $(M)_1$ is norm dense in $(\overline{M}^{\|\cdot\|})_1$. Indeed, for $x \in (\overline{M}^{\|\cdot\|})_1$, pick $(x_n) \subset M$ with $x_n \to x$ in $\|\cdot\|$. Then $\|x_n\| \to \|x\| \leq 1$, so passing to a subsequence if necessary, we may assume $\|x_n\| \leq 1 + \frac{1}{n}$. Then $\frac{n}{n+1}x_n \to x$ and $\|\frac{n}{n+1}x_n\| \leq 1$ for all n.

(1′) $(M_{\mathrm{sa}})_1$ is norm dense in $(\overline{M}^{\|\cdot\|}_{\mathrm{sa}})_1$. Indeed, for $x \in (\overline{M}^{\|\cdot\|}_{\mathrm{sa}})_1$, pick $(x_n) \subset (M)_1$ and $x_n \to x$ in $\|\cdot\|$. Then $\frac{x_n + x_n^*}{2} \to x$ as desired.

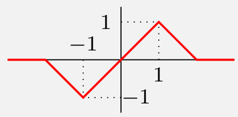
(2′) $(M_+)_1$ is norm dense in $(\overline{M}^{\|\cdot\|}_+)_1$. Indeed, for $x \in (\overline{M}^{\|\cdot\|}_+)_1$. we can write $x = y^*y$ where $y \in (\overline{M}^{\|\cdot\|})_1$. We can pick $(y_n) \subset (M)_1$ with $y_n \to y$, so $y_n^* y_n \to y^*y = x$ as desired.

Finally, we note that since SOT-closed sets are norm-closed, $\overline{M}^{\|\cdot\|} \subset \overline{M}^{SOT}$, and if $x_n \to x$ in $\|\cdot\|$, then $x_n \to x$ SOT. Hence if R is norm-dense in S and S is SOT-dense in T, then R is SOT-dense in T.

We now proceed with the rest of the proof assuming M is a C*-algebra.

(1) $(M_{\mathrm{sa}})_1$ is SOT-dense in $(\overline{M}_{\mathrm{sa}}^{SOT})_1$.

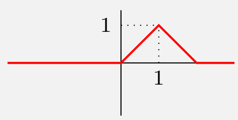
Suppose $x \in \overline{M}_{\mathrm{sa}}^{SOT}$. Let $x_i \to x$ SOT where $(x_i) \subset M$. Then $x_i \to x$ WOT, and since $*$ is continuous WOT, $x_i^* \to x^* = x$ WOT. Thus $\frac{x_i + x_i^*}{2} \to x$ WOT. Hence M_{sa} is WOT-dense in $\overline{M}_{\mathrm{sa}}^{SOT}$. But since M_{sa} is convex, we have $\overline{M_{\mathrm{sa}}}^{SOT} = \overline{M_{\mathrm{sa}}}^{WOT} = \overline{M}_{\mathrm{sa}}^{SOT}$. Now in addition, assume $\|x\| \leq 1$. There is some net $(x_i) \subset M_{\mathrm{sa}}$ such that $x_i \to x$ SOT. Consider $f \in C_0(\mathbb{R})$ such that $f(t) = t$ for all $|t| \leq 1$, e.g.,



By (SOT4), $f(x_i) \to f(x) = x$ SOT. By the Spectral Mapping Theorem, $\mathrm{sp}(f(x_i)) \subset [-1, 1]$, and thus $\|f(x_i)\| = r(f(x_i)) \leq 1$ for all i.

(2) $(M_+)_1$ is SOT-dense in $(\overline{M}_+^{SOT})_1$.

Suppose $x \in (\overline{M}_+^{SOT})_1$. By (1), there is a net $(x_i) \subset (M_{\mathrm{sa}})_1$ with $x_i \to x$ SOT. Let $f \in C_0(\mathbb{R})$ be any function which is zero on the negative reals and $f(t) = t$ for $0 \leq t \leq 1$, e.g.,



By (SOT4), $f(x_i) \to f(x) = x$ SOT. Again by Spectral Mapping, $\mathrm{sp}(f(x_i)) \subset [0, 1]$ and $f(x_i)$ is self-adjoint, and thus $f(x_i)$ is positive for all i.

(3) $(M)_1$ is SOT-dense in $(\overline{M}^{SOT})_1$.

First, we prove $M_2(M)$ is SOT-dense in $M_2(\overline{M}^{SOT})$ on H^2. Suppose $(x_{ij}) \in M_2(\overline{M}^{SOT})$, and let $(x_{ij}^k) \subset M$ such that $x_{ij}^k \to x_{ij}$ SOT. One then checks that $(x_{ij}^k) \to (x_{ij})$ SOT in $B(H^2)$.

Now suppose $x \in (\overline{M}^{SOT})_1$. Then

$$X := \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in (M_2(\overline{M}^{SOT})_{\mathrm{sa}})_1,$$

so by (1), there is an SOT-convergent net

$$\left(X_i := \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \right) \subset (M_2(M))_1$$

with $X_i \to X$ in SOT in $B(H^2)$. Then $\|b_i\| \le 1$ for all i, and $b_i \to x$ SOT in $B(H)$. $\square$

Remark 3.3.5. It is also true that the unitary group $U(M)$ is SOT-dense in $U(\overline{M}^{SOT})$ when M is a unital C*-algebra. As this uses the Borel functional calculus, we will postpone this until later.

3.4. **Predual.** In the homework, you proved that $B(H) \cong \mathcal{L}^1(H)^*$, implemented by Tr.

Definition 3.4.1. The σ-weak operator topology (σ-WOT) is the weak* topology induced by the predual $\mathcal{L}^1(H)$.

Corollary 3.4.2. *The unit ball of $B(H)$ is σ-WOT compact.*

Proof. Immediate from the Banach-Alaoglu Theorem. $\square$

Proposition 3.4.3. *For a functional $\varphi\colon B(H) \to \mathbb{C}$, the following are equivalent.*

1. *There are $(\eta_n), (\xi_n) \subset H$ such that $\sum \|\eta_n\|^2, \sum \|\xi_n\|^2 < \infty$ and $\varphi(x) = \sum \langle x\eta_n, \xi_n\rangle$ for all $x \in B(H)$,*
2. *There are $(\eta_n), (\xi_n) \subset H$, pairwise orthogonal, such that $\sum \|\eta_n\|^2, \sum \|\xi_n\|^2 < \infty$ and $\varphi(x) = \sum \langle x\eta_n, \xi_n\rangle$ for all $x \in B(H)$,*
3. *There is a $t \in \mathcal{L}^1(H)$ such that $\varphi(x) = \mathrm{Tr}(tx)$ for all $x \in B(H)$, and*
4. *φ is σ-WOT continuous.*

Proof.
(1) $\Rightarrow$ (3): Let $H_0 \subseteq H$ be the closed subspace generated by $(\eta_n), (\xi_n)$, and let (e_n) be an ONB for H_0. If $\dim(H_0) < \infty$, we may express each of η_n, ξ_n as a linear combination of the e_n to obtain scalars λ_{ij} such that

$$\varphi(x) = \sum \lambda_{ij}\langle xe_i, e_j\rangle = \mathrm{Tr}(xt) \qquad \text{where} \qquad t := \sum_{i,j} \lambda_{ij}|e_i\rangle\langle e_j|$$

is finite rank. If $\dim(H_0) = \infty$, define $t_1, t_2 \in B(H)$ by $t_i|_{H_0^\perp} = 0$ and

$$t_1 = \sum |\eta_n\rangle\langle e_n| \qquad \text{and} \qquad t_1 = \sum |\xi_n\rangle\langle e_n|$$

which are both bounded (by Cauchy-Schwarz). We calculate

$$\mathrm{Tr}(t_1^*t_1) = \sum \langle t_2^*t_1e_n, e_n\rangle = \sum \|\eta_n\|^2 < \infty$$

and similarly $\mathrm{Tr}(t_2^*t_2) = \sum \|\xi_n\|^2 < \infty$, so $t_1, t_2 \in \mathcal{L}^2(H)$. Thus $t = t_1t_2^* \in \mathcal{L}^1(H)$, and

$$\mathrm{Tr}(xt) = \mathrm{Tr}(xt_1t_2^*) = \mathrm{Tr}(t_2^*xt_1) = \sum \langle xt_1e_n, t_2e_n\rangle = \sum \langle x\eta_n, \xi_n\rangle = \varphi(x).$$

(3) $\Rightarrow$ (2): Let $t = u|t|$ be the polar decomposition so that $|t| = u^*t \in \mathcal{L}^1(H)_+$. Let $|t| = \sum \lambda_n|e_n\rangle\langle e_n|$ be a Schmidt decomposition, and note $\sum \lambda_n = \|t\|_1 < \infty$. Define $\eta_n := \lambda_n^{1/2}ue_n$ and $\xi_n := \lambda_n^{1/2}e_n$. Then the (η_n) are pairwise orthogonal as u is a partial isometry with $u^*ue_n = e_n$ for all n. Clearly the (ξ_n) are pairwise orthogonal, and we

calculate

$$\varphi(x) = \mathrm{Tr}(xt) = \sum \langle xte_n, e_n\rangle = \sum \lambda_n \langle xue_n, e_n\rangle = \sum \langle x\eta_n, \xi_n\rangle.$$

(2) $\Rightarrow$ (1): Obvious.

(3) $\Leftrightarrow$ (4): By a homework exercise, Tr implements the duality $\mathcal{L}^1(H)^* \cong B(H)$, so a linear functional is σ-WOT continuous if and only if it is of the form $x \mapsto \mathrm{Tr}(tx)$ for some $t \in \mathcal{L}^1(H)$. ☐

Corollary 3.4.4. *If φ is a σ-WOT continuous linear functional on $B(H)$ and $\varphi \geq 0$, then $\varphi(x) = \sum \langle x\xi_n, \xi_n\rangle$ for some orthogonal sequence $(\xi_n) \subset H$ with $\sum \|\xi_n\|^2 < \infty$.*

Proof. By the proposition, $\varphi = \mathrm{Tr}(\,\cdot\, t)$ for some $t \in \mathcal{L}^1(H)$. Now for all $\xi \in H$,

$$\langle t\xi, \xi\rangle = \mathrm{Tr}(|\xi\rangle\langle\xi|t) = \varphi(|\xi\rangle\langle\xi|) \geq 0,$$

so $t \geq 0$. Letting $t = \sum \lambda_n |e_n\rangle\langle e_n|$ be a Schmidt decomposition, $\xi_n := \lambda_n^{1/2} e_n$ works. ☐

Proposition 3.4.5. *On bounded subsets of $B(H)$, the σ-WOT and the WOT agree. In particular, the unit ball of $B(H)$ is WOT-compact.*

Proof. The identity map $(B(H), \sigma - \mathrm{WOT}) \to (B(H), \mathrm{WOT})$ is continuous and bijective. Restricting to the unit ball of $B(H)$, we get a continuous bijection from a compact space to a Hausdorff space, which is necessarily a homeomorphism. ☐

Lemma 3.4.6. *Suppose M is a von Neumann algebra. For any norm bounded increasing net $(x_i) \subset M$ of self-adjoint operators, there is a unique self-adjoint operator $x = \mathrm{lub}\, x_i \in M$ such that $x_i \leq x$ for all i, x is minimal with respect to this property, and $x_i \nearrow x$.*

Proof. Since the norm-closed ball of radius R is WOT-compact, there is a WOT-limit point x of (x_i). For every $\xi \in H$, we see $\langle x_i\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle$ as (x_i) is increasing, so $x_i \to x$ WOT. Since each x_i is self-adjoint and $*$ is WOT-continuous, $x = x^*$. Finally, if $y \in B(H)$ such that $x_i \leq y$ for all i, then $\langle x_i\xi, \xi\rangle \leq \langle y\xi, \xi\rangle$, and thus $\langle x\xi, \xi\rangle \leq \langle y\xi, \xi\rangle$ for all $\xi \in H$, so $x \leq y$. ☐

Corollary 3.4.7. *If $(p_i)_{i\in I}$ is a family of mutually orthogonal projections, then $\sum p_i$ converges as the increasing limit of finite sums to the orthogonal projection onto $\overline{\bigoplus p_i H}$.*

Proof. Consider the index set of finite subsets $F \subseteq I$ ordered by inclusion. Then $p_F := \sum_{i\in F} p_i$ defines an increasing net which is bounded above. Apply Lemma 3.4.6 to get $p_F \nearrow p$ for $p := \mathrm{lub}\, p_F \in B(H)$. Use (OT5) to see $p^2 = p$ and Exercise 3.1.7(2) to see $p^* = p$. Since $p_i \leq p$, we have $p_i H \subseteq pH$ for all i, and thus $\overline{\bigoplus p_i H} \subseteq pH$. Since $\overline{\bigoplus p_i H}$ is the smallest closed subspace containing each $p_i H$, the claim follows by minimality from Lemma 3.4.6. ☐

Remark 3.4.8. The σ-WOT is the WOT on $\alpha(B(H))$ where $\alpha\colon B(H) \to B(H \otimes \ell^2)$ is the countably infinite amplification.

Definition 3.4.9. The σ-SOT is the SOT on $\alpha(B(H))$. That is, $x_i \to x$ σ-SOT if and only if for all $(\xi_n) \subset H$ with $\sum \|\xi_n\|^2 < \infty$, $\sum_n \|(x - x_i)\xi_n\|^2 \to 0$.

The SOT* is generated by the seminorms $x \mapsto \|x\xi\| + \|x^*\xi\|$ for $\xi \in H$. The σ-SOT* is generated by the seminorms $x \mapsto \sum \|x\xi_n\|^2 + \|x^*\xi_n\|^2$ for $(\xi_n) \subset H$ with $\sum \|\xi_n\|^2 < \infty$. These locally convex topologies are like the SOT/σ-SOT, but they ensure $*$ is continuous.

Remark 3.4.10. We have:
$$\begin{array}{ccccccc} \sigma\text{-WOT} & \subset & \sigma\text{-SOT} & \subset & \sigma\text{-SOT}^* & \subset & \text{norm} \\ \cup & & \cup & & \cup & & \\ \text{WOT} & \subset & \text{SOT} & \subset & \text{SOT}^* & & \end{array}$$

Exercise 3.4.11. Show that a functional $\varphi\colon B(H) \to \mathbb{C}$ is σ-WOT continuous if and only if it is σ-SOT continuous.

Exercise 3.4.12. Show that for a unital $*$-subalgebra $M \subseteq B(H)$, the following are equivalent.

(1) $M = M''$
(2) M is σ-WOT closed
(3) M is σ-SOT closed
(4) M is SOT*-closed
(5) M is σ-SOT* closed

Exercise 3.4.13. Prove that on bounded subsets of $B(H)$, the σ-SOT and SOT agree.

Theorem 3.4.14. *Let $M \subseteq B(H)$ be a von Neumann algebra. There is a Banach space M_* such that M is isometrically isomorphic to $(M_*)^*$. Moreover, the σ-WOT on M is the weak* topology induced by M_*. We call M_* a* predual *of M. Any other predual of M inducing the σ-WOT topology on M is canonically isometrically isomorphic to M_*.*

Proof. We identify $B(H) = \mathcal{L}^1(H)^*$. Consider the pre-annihilator
$$M_\perp = \left\{ t \in \mathcal{L}^1(H) \middle| \operatorname{Tr}(mt) = 0 \text{ for all } m \in M \right\}.$$
Then $M_\perp \subseteq \mathcal{L}^1(H)$ is a $\|\cdot\|_1$-closed subspace, so $M_* := \mathcal{L}^1(H)/M_\perp$ is a Banach space with the quotient norm. Since $M \subseteq B(H)$ is σ-WOT (weak*) closed,
$$M = (M_\perp)^\perp = \{x \in B(H) | \operatorname{Tr}(xt) = 0 \text{ for all } t \in M_\perp\}.$$
We recall that for a closed subspace Y of a normed space X, there is a canonical isometric isomorphism $(X/Y)^* \cong Y^\perp$. Taking $X = \mathcal{L}^1(H)$ and $Y = M_\perp$ so that $X/Y = M_*$ yields $(M_*)^* \cong (M_\perp)^\perp = M$. It follows that the σ-WOT on M, which is the relative weak* topology on $M \subseteq B(H) = \mathcal{L}^1(H)^*$ is the the weak* topology induced by M_*.
Suppose now we have another predual X of M which also induces the σ-WOT on M. The images of the canonical isometric embeddings $X \hookrightarrow M^*$ and $M_* \hookrightarrow M^*$ agree, which gives an isometric isomorphism $X \cong M_*$. Indeed, the image of X (respectively M_*) is precisely the bounded linear functionals $M \to \mathbb{C}$ which are continuous with respect to the X-weak* (respectively M_*-weak*) topology, which is the σ-WOT. $\square$

Definition 3.4.15. A unital C*-algebra M is called a W*-*algebra* if it has a predual, i.e., there exists a Banach space M_* and an isometric isomorphism $M \cong (M_*)^*$.

By Theorem 3.4.14, every von Neumann algebra is a W*-algebra. The converse is also true by a result of Sakai, but it goes beyond this class.

3.5. **Borel functional calculus.**

Definition 3.5.1. Let $(X, \mathcal{M})$ be a measurable set ($\mathcal{M}$ is a σ-algebra on X), let H be a Hilbert space, and let $P(H)$ denote the set of orthogonal projections. A spectral measure is a function $E\colon \mathcal{M} \to P(H)$ satisfying

(0) $E(\emptyset) = 0$ and
(1) For all disjoint sequences $(S_n) \subset \mathcal{M}$, $\sum E(S_n) = E(\bigcup S_n)$, where the sum converges SOT.

Observe that for all $\eta, \xi \in H$, $\mu_{\eta,\xi}(S) := \langle E(S)\eta, \xi\rangle$ is a finite $\mathbb{C}$-valued measure. If X is LCH, $\mathcal{M}$ is the Borel σ-algebra, and every $\mu_{\eta,\xi}$ is regular, we call E a regular Borel spectral measure.

Example 3.5.2. Suppose X is a compact Hausdorff space and μ is a finite regular Borel measure (a.k.a. a Radon measure) on X. Then $S \mapsto \chi_S \in L^\infty(X, \mu) \subset B(L^2(X, \mu))$ defines a regular Borel spectral measure.

Facts 3.5.3. Here are some facts about spectral measures. All sets below are assumed measurable.

(E1) If $S \cap T = \emptyset$, then $E(S) \perp E(T)$.

Proof. Since $E(S \cup T) = E(S) + E(T)$ is a projection, the result follows from the following exercise.

Exercise 3.5.4. Suppose $p, q \in P(H)$ are projections. Then $p \perp q$ if and only if $p + q$ is a projection. □

(E2) $E(S \cap T) = E(S)E(T)$.

Proof. By (E1),

$$E(S)E(T) = \big(E(S \setminus T) + E(S \cap T)\big)\big(E(T \setminus S) + E(S \cap T)\big) = E(S \cap T). \quad \square$$

(E3) If $S \subset T$, then $E(S) \leq E(T)$ (which is equivalent to $E(S)E(T) = E(S)$).

Proof. Immediate from (E2). □

Definition 3.5.5. Let $E\colon (X, \mathcal{M}) \to P(H)$ be a spectral measure. We say a mesurable function f on X is *essentially bounded with respect to* E if there is a $c > 0$ such that $E(\{|f| > c\}) = 0$. For such f, we define

$$\|f\|_E := \inf \{c > 0 | E(\{|f| > c\}) = 0\}.$$

We denote by $L^\infty(E)$ the collection of (equivalence classes of) functions essentially bounded with respect to E.

Exercise 3.5.6. Show that $L^\infty(E)$ is a unital commutative C*-algebra.

Remark 3.5.7. Suppose $E\colon (X,\mathcal{M}) \to P(H)$ is a spectral measure. Consider $B^\infty(X)$, the bounded measurable functions on X. Observe there is a unital $*$-homomorphism from $B^\infty(X) \to L^\infty(E)$ such that $f_i \nearrow f$ in $B^\infty(X)$ implies $[f_i] \nearrow [f]$ in $L^\infty(E)$. (Here, increasing means pointwise, as neither algebra is a priori a von Neumann algebra acting on a Hilbert space.)

While the kernel is generally difficult to describe and is highly dependent on E, we claim this map is surjective. Indeed, suppose $[f] \in L^\infty(E)$, so that $E(\{|f| > c\}) = 0$ for some $c > 0$. Then consider the function $f\chi_{\{|f|\leq c\}} \in B^\infty(X)$. Observe that $[f\chi_{\{|f|\leq c\}}] = [f]$ since $\|f\chi_{\{|f|>c\}}\|_E = 0$. Indeed, for all $\varepsilon > 0$ (with $c > \varepsilon$), we have

$$E(\{|f\chi_{\{|f|>c\}}| > \varepsilon\}) = E(\{|f| > c\}) = 0.$$

Construction 3.5.8. Given a spectral measure $E\colon (X,\mathcal{M}) \to P(H)$, we construct an isometric unital $*$-homomorphism $\int \cdot \, dE\colon L^\infty(E) \to B(H)$.

Step 1: We first define it for simple functions $\int \sum_{i=1}^n c_i\chi_{S_i}\, dE := \sum_{i=1}^n c_iE(S_i)$.

Well-defined. Suppose $\sum_{i=1}^n c_i\chi_{S_i} = 0$. For $F \subseteq \{1,\dots,n\}$, let

$$S_F := \left(\bigcap_{i\in F} S_i\right) \setminus \left(\bigcup_{j\notin F} S_j\right).$$

Then the sets $\{S_F | F \subseteq \{1,\dots,n\}\}$ are mutually disjoint and $S_i = \bigcup_{i\in F} S_F$. We calculate

$$0 = \sum_{i=1}^n c_i\chi_{S_i} = \sum_{i=1}^n c_i\chi_{\bigcup_{i\in F} S_F} = \sum_{i=1}^n c_i \sum_{i\in F}\chi_{S_F} = \sum_F \left(\sum_{i\in F} c_i\right)\chi_{S_F},$$

so $\sum_{i\in F} c_i = 0$ for all F. Thus

$$\sum_{i=1}^n c_iE(S_i) = \sum_{i=1}^n c_iE\left(\bigcup_{i\in F} S_F\right) = \sum_{i=1}^n\sum_{i\in F} c_iE(S_F) = \sum_F \left(\underbrace{\sum_{i\in F} c_i}_{=0}\right) E(S_F) = 0.$$

□

Step 2: For all simple functions f, $\|\int f\, dE\|_{B(H)} = \|f\|_E$.

Proof. As in the proof of Step 1, $\sum_{i=1}^n c_i\chi_{S_i} = \sum_F \left(\sum_{i\in F} c_i\right)\chi_{S_F}$ and $\int f\, dE = \sum_F\left(\sum_{i\in F} c_i\right)E(S_F)$ where the S_F are disjoint. Both norms are equal to the largest $\left|\sum_{i\in F} c_i\right|$ such that $E(S_F) \neq 0$. □

Step 3: Since $\int \cdot\, dE$ is a linear isometry from simple functions in $L^\infty(E)$ to $B(H)$, and the simple functions are dense in $L^\infty(E)$, it extends uniquely to an isometry $L^\infty(E) \to B(H)$.

Facts 3.5.9. The unital $*$-homomorphism $\int \cdot\, dE$ satisfies the following properties. All functions below are assumed to be in $L^\infty(E)$.

($\int$1) $\int \overline{f}\, dE = \left(\int f\, dE\right)^*$.

Proof. The condition is clearly $L^\infty(E)$-norm closed and holds for simple functions, which are norm-dense in $L^\infty(E)$. $\square$

($\int$2) $\left(\int f\, dE\right)\left(\int g\, dE\right) = \left(\int fg\, dE\right)$

Proof. Again, this holds when f, g are simple functions, and we can approximate separately. $\square$

($\int$3) $\left\langle \left(\int f\, dE\right)\eta, \xi\right\rangle = \int f\, d\mu_{\eta,\xi}$

Proof. Again, use simple functions. $\square$

($\int$4) If $(f_i) \subset L^\infty(E)$ with $f_i \nearrow f \in L^\infty(E)$ pointwise, then $\int f_i\, dE \nearrow \int f\, dE$ SOT.

Proof. For $\xi \in H$, $\mu_{\xi,\xi}(S) = \langle E(S)\xi, \xi\rangle$, which is a non-negative finite measure on $(X, \mathcal{M})$. Since $f_i \nearrow f$ in $L^\infty(E)$ and $\mu_{\xi,\xi}$ is finite, $f \in L^1(\mu_{\xi,\xi})$. By the Monotone Convergence Theorem,

$$\left\langle \left(\int f_i\, dE\right)\xi, \xi\right\rangle = \int f_i\, d\mu_{\xi,\xi} \nearrow \int f\, d\mu_{\xi,\xi} = \left\langle \left(\int f\, dE\right)\xi, \xi\right\rangle.$$

Since ξ was arbitrary, $\int f_i\, dE \nearrow \int f\, dE$. $\square$

($\int$5) (Spectral Mapping) $\operatorname{sp}_{B(H)}(\int f\, dE) = \operatorname{ess.range}(f)$ in $L^\infty(E)$.

Proof. Suppose $\lambda \in \mathbb{C}$ and $\varepsilon > 0$ such that $E(S := \{|f-\lambda| < \varepsilon\}) = 0$. Define $g \in L^\infty(E)$ by

$$g(z) := \begin{cases} (f(z)-\lambda)^{-1} & \text{if } |f(z)-\lambda| \geq \varepsilon \quad \Leftrightarrow \quad z \notin S \\ 0 & \text{if } |f(z)-\lambda| < \varepsilon \quad \Leftrightarrow \quad z \in S, \end{cases}$$

and note that $\|g\|_E \leq \varepsilon^{-1}$. Then

$$\begin{aligned} \left(\int g\, dE\right)\left(\int f\, dE - \lambda\right) &= \int g(f-\lambda)\, dE \\ &= \underbrace{\int_S g(f-\lambda)\, dE}_{:=\int \underbrace{\chi_S g(f-\lambda)}_{=0}\, dE} + \underbrace{\int_{X\setminus S} g(f-\lambda)\, dE}_{:=\int \underbrace{\chi_S g(f-\lambda)}_{=1}\, dE} \\ &= E(X \setminus S) = 1, \end{aligned}$$

so $\lambda \notin \operatorname{sp}\left(\int f\, dE\right)$.

Conversely, suppose $E(S_\varepsilon := \{|f-\lambda| < \varepsilon\}) \neq 0$. Since $|f-\lambda|\chi_{S_\varepsilon} < \varepsilon\chi_{S_\varepsilon}$, for all unit vectors $\xi_\varepsilon \in E(S_\varepsilon)H$,

$$\begin{aligned}\left\|\left(\int f\,dE - \lambda\right)\xi_\varepsilon\right\| &= \left\|\left(\int (f-\lambda)\,dE\right)E(S_\varepsilon)\xi_\varepsilon\right\| \\ &= \left\|\left(\int \chi_{S_\varepsilon}(f-\lambda)\,dE\right)\xi_\varepsilon\right\| \\ &\leq \left\|\left(\int \chi_{S_\varepsilon}(f-\lambda)\,dE\right)\right\| \\ &\leq \varepsilon\,\|\chi_{S_\varepsilon}\| = \varepsilon.\end{aligned}$$

Thus λ is an approximate eigenvalue for $\int f\,dE$ and lies in its spectrum. $\square$

Theorem 3.5.10 (Spectral)**.** *Let $A \subseteq B(H)$ be a unital commutative* C*-*algebra. There is a unique regular Borel spectral measure E_x on $\widehat{A}$ such that $\int f\,dE_x = f(x)$ for all $f \in C(\widehat{A})$. Moreover, $\int \cdot\,dE_x$ is an isometric unital $*$-homomorphism $L^\infty(E_x) \to A'' \subset B(H)$.*

The proof proceeds in a series of steps.

Step 1: Construction of the candidate operator $E_x(S)$ for $S \subset \widehat{A}$ Borel.

Proof. For $\eta, \xi \in H$, $f \mapsto \langle f(x)\eta, \xi\rangle$ is a continuous linear functional on $C(\widehat{A})$. By the Riesz Representation Theorem, there is a unique finite regular Borel measure $\mu_{\eta,\xi}$ on $\widehat{A}$ such that $\langle f(x)\eta,\xi\rangle = \int f\,d\mu_{\eta,\xi}$ for all $f \in C(\widehat{A})$. Now $\mu_{\xi,\xi}$ is non-negative, and since $\widehat{A}$ is compact Hausdorff and thus normal, for every open $U \subset \widehat{A}$,

$$\mu_{\xi,\xi}(U) = \sup\left\{\int f\,d\mu_{\xi,\xi}\,\middle|\, f \in C(\widehat{A}),\ 0 \leq f \leq 1,\ \operatorname{supp}(f) \subset U\right\}. \tag{3.5.11}$$

We now observe that $(\eta,\xi) \mapsto \mu_{\eta,\xi}$ is linear in η, conjugate linear in ξ, and so by polarization,

$$\mu_{\eta,\xi} = \frac{1}{4}\sum_{k=0}^{3} i^k \mu_{\eta+i^k\xi,\eta+i^k\xi}.$$

By (3.5.11) and regularity, it follows that $\overline{\mu_{\eta,\xi}(S)} = \mu_{\xi,\eta}(S)$ for every Borel set S and $\eta,\xi \in H$. Fixing S Borel, the map $(\eta,\xi) \mapsto \mu_{\eta,\xi}(S)$ is sesquilinear and bounded. Hence there is a unique operator $E(S) \in B(H)$ such that $\mu_{\eta,\xi}(S) = \langle E(S)\eta,\xi\rangle$ for all $\eta,\xi \in H$. $\square$

Step 2: $E_x : (\widehat{A}, \text{Borel}) \to B(H)$ is a regular spectral measure which takes values in $P(A'')$.

Proof. This step of the proof proceeds in a series of sub-steps. In the proofs below, $\eta, \xi \in H$ are arbitrary.

(a) $E_x(\emptyset) = 0$ and $E_x(\widehat{A}) = 1$:

$$\langle E_x(\emptyset)\eta, \xi\rangle = \mu_{\eta,\xi}(\emptyset) = 0 \quad \text{and} \quad \langle E_x(\widehat{A})\eta, \xi\rangle = \mu_{\eta,\xi}(\widehat{A}) = \langle \eta, \xi\rangle.$$

(b) $E_x(S)^* = E(S)$ for all Borel $S \subset \widehat{A}$:

$$\langle E_x(S)^*\eta, \xi\rangle = \langle \eta, E_x(S)\xi\rangle = \overline{\mu_{\xi,\eta}(S)} = \mu_{\eta,\xi}(S) = \langle E_x(S)\eta, \xi\rangle.$$

(c) If $S_1 \cap S_2 = \emptyset$, $E(S_1 \cup S_2) = E(S_1) + E(S_2)$:

$$\begin{aligned}\langle E_x(S_1 \cup S_2)\eta, \xi\rangle &= \mu_{\eta,\xi}(S_1 \cup S_2)\\ &= \mu_{\eta,\xi}(S_1) + \mu_{\eta,\xi}(S_2)\\ &= \langle E_x(S_1)\eta, \xi\rangle + \langle E_x(S_2)\eta, \xi\rangle\\ &= \langle (E_x(S_1) + E_x(S_2))\eta, \xi\rangle\end{aligned}$$

(d) For all $g \in C(\widehat{A})$, $d\mu_{g(x)\eta,\xi} = g d\mu_{\eta,\xi}$:

$$\int f\, d\mu_{g(x)\eta,\xi} = \langle f(x)g(x)\eta, \xi\rangle = \int fg\, d\mu_{\eta,\xi} \qquad \forall f \in C(\widehat{A}).$$

(e) For all $g \in C(\widehat{A})$ and $S \subseteq \widehat{A}$ Borel, $d\mu_{E_x(S)\eta,\xi} = \chi_S d\mu_{\eta,\xi}$:

$$\begin{aligned}\int f\, d\mu_{E_x(S)\eta,\xi} &= \langle f(x)E_x(S)\eta, \xi\rangle = \langle E_x(S)\eta, f(x)^*\xi\rangle\\ &= \mu_{\eta, f^*(x)\xi}(S) = \overline{\mu_{\overline{f}(x)\xi,\eta}(S)}\\ &= \overline{\int \chi_S \overline{f}\, d\mu_{\xi,\eta}} = \int f\chi_S\, d\mu_{\eta,\xi} \qquad\qquad \forall f \in C(\widehat{A}).\end{aligned}$$

(f) $E_x(S_1 \cap S_2) = E_x(S_1)E_x(S_2)$. In particular $E_x(S)^2 = E_x(S)$:

$$\langle E_x(S_1)E_x(S_2)\eta, \xi\rangle = \mu_{E_x(S_2)\eta,\xi}(S_1) = \int \underbrace{\chi_{S_1}\chi_{S_2}}_{=\chi_{S_1 \cap S_2}} d\mu_{\eta,\xi} = \langle E_x(S_1 \cap S_2)\eta, \xi\rangle.$$

(g) If (S_n) is a sequence of disjoint Borel sets, then $E_x\left(S := \bigcup S_n\right) = \sum E_x(S_n)$ where the sum converges SOT:

Indeed, for all $N \in \mathbb{N}$,

$$E_x(S) - \sum_{n=1}^{N} E_x(S_n) = E_x(S) - E_x\left(\bigcup_{n=1}^{N} S_n\right) = E_x\left(S \setminus \bigcup_{n=1}^{N} S_n\right).$$

Then for all $\xi \in H$,

$$\left\|\left(E_x(S) - \sum_{n=1}^{N} E_x(S_n)\right)\xi\right\|^2 = \left\|E_x\left(S \setminus \bigcup_{n=1}^{N} S_n\right)\xi\right\|^2$$
$$= \left\langle E_x\left(S \setminus \bigcup_{n=1}^{N} S_n\right)\xi, \xi\right\rangle$$
$$= \mu_{\xi,\xi}\left(S \setminus \bigcup_{n=1}^{N} S_n\right)$$
$$\xrightarrow{n\to\infty} 0.$$

(h) For all Borel $S \subseteq \widehat{A}$, $E_x(S) \in A''$:

Indeed, for all $a \in A'$, $\mu_{a\eta,\xi} = \mu_{\eta,a^*\xi}$ since for all $f \in C(\widehat{A})$,

$$\int f\, d\mu_{a\eta,\xi} = \langle f(x)a\eta, \xi\rangle = \langle f(x)\eta, a^*\xi\rangle = \int f\, d\mu_{\eta,a^*\xi}.$$

Thus for all Borel $S \subseteq \widehat{A}$,

$$\langle E(S)a\eta, \xi\rangle = \mu_{a\eta,\xi}(S) = \mu_{\eta,a^*\xi}(S) = \langle E(S)\eta, a^*\xi\rangle = \langle aE(S)\eta, \xi\rangle. \qquad \square$$

Step 3: For all $f \in C(\widehat{A})$, $f(x) = \int f\, dE_x$. Thus $C(X)$ sits injectively inside $L^\infty(E_x)$, and thus for all non-empty open $U \subset \widehat{A}$, $E_x(U) \neq 0$ by Urysohn's Lemma.

Proof. Let $f \in C(\widehat{A})$. Then f also defines an element of $L^\infty(E_x)$. We simply check for all $\eta, \xi \in H$,

$$\langle f(x)\eta, \xi\rangle = \int f\, d\mu_{\eta,\xi} = \left\langle \left(\int f\, dE_x\right)\eta, \xi\right\rangle. \qquad \square$$

Step 4: E_x is the unique regular Borel spectral measure such that $f(x) = \int f\, dE_x$ for all $f \in C(\widehat{A})$.

Proof. Suppose F is another such regular Borel spectral measure so that for $\eta, \xi \in H$, $\nu_{\eta,\xi}(S) := \langle F(S)\eta, \xi\rangle$ is a regular Borel measure on $\widehat{A}$. Then for all $f \in C(\widehat{A})$,

$$\int f\, d\nu_{\eta,\xi} = \left\langle\left(\int f\, dF\right)\eta, \xi\right\rangle = \langle f(x)\eta, \xi\rangle = \left\langle\left(\int f\, dE_x\right)\eta, \xi\right\rangle = \int f\, d\mu_{\eta,\xi}$$

so $\nu_{\eta,\xi} = \mu_{\eta,\xi}$. We conclude that $E(S) = F(S)$ for all Borel $S \subseteq \widehat{A}$. $\square$

Construction 3.5.12 (Borel/L^∞ functional calculus (BFC))**.** Let $x \in B(H)$ be normal and consider the von Neumann algebra $\mathrm{W}^*(x) := \{x, x^*\}''$ generated by x. There is a unique regular Borel spectral measure E_x on $\mathrm{sp}(x)$ such that $\int \mathrm{id}\, dE_x = x$ where $\mathrm{id}(z) = z$ for all

$z \in \mathrm{sp}(x)$. Moreover, $\int f\, dE_x = f(x)$ for all $f \in C(\mathrm{sp}(x))$. We may thus unambiguously denote $\int f\, dE_x = f(x)$ for $f \in L^\infty(E_x)$.

Proposition 3.5.13. *Suppose $x \in B(H)$ is normal and $f \in L^\infty(E_x)$. Then for all $g \in L^\infty(E_{f(x)})$, $g \circ f \in L^\infty(E_x)$, and $(g \circ f)(x) = g(f(x))$.*

Proof. First, since $\mathrm{sp}(f(x)) = \mathrm{ess.\,range}(f)$ in $L^\infty(E_x)$ and g is Borel measurable on $\mathrm{sp}(f(x))$, $g \circ f$ is Borel measurable on $\mathrm{sp}(x)$, and defines an element of $L^\infty(E_x)$.
It suffices to prove that

$$G(S) := (\chi_S \circ f)(x) \in P(\mathrm{W}^*(x)) \qquad\qquad S \subset \mathrm{sp}(f(x)) \text{ Borel}$$

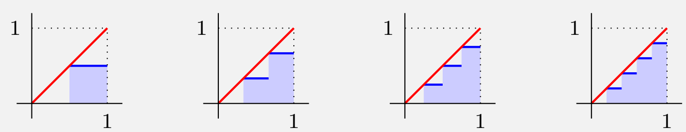
is a regular Borel spectral measure on $\mathrm{sp}(f(x))$ such that $\int \mathrm{id}\, dG = f(x)$. Note that $\chi_S \circ f = \chi_{f^{-1}(S)}$, so $G(S) = E(f^{-1}(S))$, and for all $\eta, \xi \in H$, $\mu^G_{\eta,\xi}$ on $\mathrm{ess.\,range}(f)$ is the pushforward of $\mu^E_{\eta,\xi}$ via $f : \mathrm{sp}(x) \to \mathrm{sp}(f(x))$. Hence $\mu^G_{\eta,\xi}$ is regular Borel. (Recall that any finite Borel measure on a second countable locally compact space is regular, and $\mathrm{ess.\,range}(f) \subset \mathbb{C}$ is compact.)
Now if we approximate $\mathrm{id} \in B^\infty(\mathrm{sp}(f(x)))$ by simple functions $g_n \to \mathrm{id}$ in $\|\cdot\|_\infty$, then $g_n \circ f \to f$ in $\|\cdot\|_\infty$ in $B^\infty(\mathrm{sp}(x))$, so $g_n \circ f \to f$ in $L^\infty(E)$. Finally,

$$\int \mathrm{id}\, dG = \lim \int g_n\, dG = \lim (g_n \circ f)(x) = \lim \int g_n \circ f\, dE = \int f\, dE = f(x). \qquad \square$$

Facts 3.5.14. Here are some elementary applications of the BFC. Let $M \subset B(H)$ be a von Neumann algebra.

(1) M is the norm-closure of the span of its projections.

Proof. It suffices to approximate any positive operator in the unit ball of M by a linear combination of projections. Just uniformly approximate the identity function on $[0,1]$ by a simple function.



etc. $\square$

(2) If $L \subseteq M$ is a non-zero left-ideal, then L contains a projection.

Proof. If $x \in L \setminus \{0\}$, then $x^*x \in L \setminus \{0\}$. Without loss of generality, $\|x^*x\| = 1$. Let $0 < \varepsilon < 1$ and consider $f(t) = t^{-1}\chi_{[\varepsilon,1]}(t)$. Then $f(x^*x)x^*x = \chi_{[\varepsilon,1]}(x) \in L$ is a non-zero projection. $\square$

(3) If $x \in M$ is normal, then $\chi_{\{0\}}(x) = p_{\ker(x)}$.

Proof. Clearly $x\chi_{\{0\}}(x) = 0$, so $\chi_{\{0\}}(x) \le p_{\ker(x)}$.
For $n \ge 0$, let $E_n := \mathrm{sp}(x) \setminus B_{1/(n+1)}(0)$; then set $F_0 = E_0$ and for $n \in \mathbb{N}$ inductively define $F_n := E_n \setminus E_{n-1}$. Then the projections $p_n := \chi_{F_n}(x)$ are mutually orthogonal, and $\sum_{n=0}^\infty p_n = \chi_{\mathrm{sp}(x)\setminus\{0\}}(x)$ converges SOT. For $n \ge 0$,

define $f_n : \mathrm{sp}(x) \to \mathbb{C}$ by

$$f_n(z) = \begin{cases} z^{-1} & \text{if } z \in F_n \\ 0 & \text{else,} \end{cases}$$

and observe that $p_n = f_n(x)x$. If $\xi \in \ker(x)$, then for all $n \geq 0$, $p_n\xi = f_n(x)x\xi = 0$. Thus $\chi_{\mathrm{sp}(x)\setminus\{0\}}(x)\xi = \sum_{n=0}^{\infty} p_n\xi = 0$, and thus $\xi = \chi_{\{0\}}(x)\xi$. Hence $p_{\ker(x)} \leq \chi_{\{0\}}(x)$, and so they are equal. □

(4) For all $x \in M$, $\mathrm{supp}(x)$ and $\mathrm{range}(x)$ lie in M.

Proof. Since $p_{\ker(x^*x)} = \chi_{\{0\}}(x^*x) \in M$ and $\ker(x) = \ker(x^*x)$, $\mathrm{supp}(x) = 1 - p_{\ker(x)} \in M$. Formally, $\mathrm{range}(x) = \mathrm{supp}(x^*) \in M$. □

(5) The unitary group $U(M)$ is path connected in the norm topology.

Proof. Let $u \in U(M)$ and let log be any branch of the logarithm. Then $u = \exp(\log(u)) = \exp(i(-i\log(u)))$ where $-i\log(u)$ is self-adjoint by the Spectral Mapping Theorem ($\int$5). Then $t \mapsto \exp(it(-i\log(u)))$ is a norm-continuous path of unitaries from u to 1 in $U(M)$. □

Corollary 3.5.15. *(Kaplansky) If $A \subset B(H)$ is a unital* C*-*algebra, then $U(A)$ is SOT-dense in $U(\overline{A}^{SOT})$.*

Proof. Suppose $u \in U(\overline{A}^{SOT})$. Let $x \in \overline{A}^{SOT}_{\mathrm{sa}}$ such that $u = \exp(ix)$. By the Kaplansky Density Theorem 3.3.4, there is a net $(x_i) \subset A_{\mathrm{sa}}$ with $\|x_i\| \leq \|x\|$ and $x_i \to x$ SOT. Let $f \in C_0(\mathbb{R})$ such that $f = \exp(it)$ on $[-\|x\|, \|x\|]$. Then $f(x_i) \in U(A)$ for all i, and since f is SOT-continuous by (SOT4), $f(x_i) \to f(x)$ SOT. □

Definition 3.5.16. Suppose $M \subset B(H)$ is a von Neumann algebra. A unital $*$-homomorphism $\Phi : M \to B(K)$ is called *normal* if $0 \leq x_i \nearrow x$ in M implies $\Phi(x_i) \nearrow \Phi(x)$. Observe that σ-WOT continuous unital $*$-homomorphisms are normal by

Example 3.5.17. Every σ-WOT continuous unital $*$-homomorphism is normal by Exercise 3.1.7 and Proposition 3.4.5.

Proposition 3.5.18. *Suppose $\Phi : M \to B(K)$ is a normal $*$-homomorphism and $x \in M$ is normal. For all $f \in B^\infty(\mathrm{sp}(x))$, $\Phi(f(x)) = f(\Phi(x))$.*

Proof. Since Φ is contractive, $\mathrm{sp}_{B(K)}(\Phi(x)) \subseteq \mathrm{sp}_M(x)$, and thus $f(\Phi(x))$ is well-defined. Since Φ is normal, $F(S) := \Phi(E_x(S))$ is a well-defined regular Borel spectral measure on $\mathrm{sp}(x)$. Moreover, as simple functions are dense in $L^\infty(F)$, for all $f \in C(\mathrm{sp}(x))$,

$$\int f\, dF = \Phi(f(x)).$$

Note that $\Phi(f(x)) = f(\Phi(x))$ whenever f is a polynomial in z and $\overline{z}$, so by Stone-Weierstrass, $\Phi(f(x)) = f(\Phi(x))$ for any $f \in C(\mathrm{sp}(x))$. Thus for all $f \in C(\mathrm{sp}(x))$ and $\eta, \xi \in K$,

$$\begin{aligned} \int f\, d\mu^F_{\eta,\xi} &= \left\langle \left(\int f\, dF \right) \eta, \xi \right\rangle = \langle \Phi(f(x))\eta, \xi\rangle \\ &= \langle f(\Phi(x))\eta, \xi\rangle = \left\langle \left(\int f\, dE_{\Phi(x)} \right) \eta, \xi \right\rangle = \int f\, d\mu^{E_{\Phi(x)}}_{\eta,\xi}. \end{aligned}$$

Hence $\mu^F_{\eta,\xi} = \mu^{E_{\Phi(x)}}_{\eta,\xi}$ for all $\eta, \xi \in K$, and thus $F(S) = E_{\Phi(x)}(S)$ for all Borel sets $S \subseteq \mathrm{sp}(x)$. The result follows. □

Corollary 3.5.19. *The partial isometry u in the polar decomposition $x = u|x|$ is independent of the choice of faithful σ-WOT continuous representation of M.*

Proof. Suppose K is another Hilbert space and $\pi\colon M \to B(K)$ is a faithful σ-WOT continuous unital $*$-homomorphism, which is automatically normal. Let $\pi(x) = v\pi(|x|)$ be the polar decompostion on K, where we have used that $\pi(|x|) = |\pi(x)|$ as $\pi(x^*x) = \pi(x)^*\pi(x)$ has a unique positive square root. By the uniqueness statement of the polar decomposition, it suffices to prove $\ker(\pi(u)) = \ker(v)$, which follows by the calculation

$$\begin{aligned} v^*v &= \mathrm{supp}(\pi(x)) = \chi_{\mathrm{sp}(\pi(x))\setminus\{0\}}(\pi(x)) = \chi_{\mathrm{sp}(x)\setminus\{0\}}(\pi(x)) \\ &= \pi(\chi_{\mathrm{sp}(x)\setminus\{0\}}(x)) = \pi(\mathrm{supp}(x)) = \pi(u^*u) = \pi(u)^*\pi(u). \end{aligned}$$ □

3.6. **Abelian von Neumann algebras and multiplicity theory.**

Exercise 3.6.1. Suppose $M \subset B(H)$ is a unital $*$-subalgebra. A vector $\xi \in H$ is called:

- *cyclic* for M if $M\xi$ is dense in H.
- *separating* for M if for every $x, y \in M$, $x\xi = y\xi$ implies $x = y$.

(1) Prove that ξ is cyclic for M if and only if ξ is separating for M'.
(2) Prove that H can be orthogonally decomposed into M-invariant subspaces $H = \bigoplus_{i\in I} K_i$, such that each K_i is cyclic for M (has a cyclic vector). Prove that if H is separable, this decomposition is countable.
(3) Prove that if M is abelian and H is separable, then there is a separating vector in H for M.

Exercise 3.6.2. Let H be a separable Hilbert space and $A \subseteq B(H)$ an abelian von Neumann algebra. Prove that the following are equivalent.

(1) A is maximal abelian, i.e., $A = A'$.
(2) A has a cyclic vector $\xi \in H$.
(3) For every norm separable SOT-dense C*-subalgebra $A_0 \subset A$, A_0 has a cyclic vector.
(4) There is a norm separable SOT-dense C*-subalgebra $A_0 \subset A$ such that A_0 has a cyclic vector.

(5) There is a finite regular Borel measure μ on a compact Hausdorff second countable space X and a unitary $u \in B(L^2(X,\mu), H)$ such that $f \mapsto uM_f u^*$ is an isometric $*$-isomorphism $L^\infty(X,\mu) \to A$.

Hints:
For (1) $\Rightarrow$ (2), *use Exercise 3.6.1.*
For (3) $\Rightarrow$ (4) *it suffices to construct a norm separable SOT-dense C*-algebra. First show that* $A_* = \mathcal{L}^1(H)/A_\perp$ *is a separable Banach space. Then show that* A *is* σ*-WOT separable, which implies SOT-separable. Take* A_0 *to be the unital C*-algebra generated by an SOT-dense sequence.*
For (4) $\Rightarrow$ (5) *show that* A_0 *separable implies* $X = \widehat{A}_0$ *is second countable. Define* $\mu = \mu_{\xi,\xi}$ *on* X*, and show that the map* $C(X) \to H$ *by* $f \mapsto \Gamma^{-1}(f)\xi$ *is a* $\|\cdot\|_2 - \|\cdot\|_H$ *isometry with dense range.*

Exercise 3.6.3. Suppose $E\colon (X,\mathcal{M}) \to P(H)$ is a spectral measure with H separable, and let $A \subset B(H)$ be the unital C*-algebra which is the image of $L^\infty(E)$ under $\int \cdot\, dE$. Suppose there is a cyclic unit vector $\xi \in H$ for A.

(1) Show that $\omega_\xi(f) = \langle (\int f dE)\xi, \xi\rangle$ is a faithful state on $L^\infty(E)$ ($\omega_\xi(|f|^2) = 0 \Longrightarrow f = 0$).
(2) Consider the finite non-negative measure $\mu = \mu_{\xi,\xi}$ on $(X,\mathcal{M})$. Show that a measurable function f on $(X,\mathcal{M})$ is essentially bounded with respect to E if and only if f is essentially bounded with respect to μ.
(3) Deduce that for essentially bounded measurable f on $(X,\mathcal{M})$, $\|f\|_E = \|f\|_{L^\infty(X,\mathcal{M},\mu)}$.
(4) Construct a unitary $u \in B(L^2(X,\mathcal{M},\mu), H)$ such that for all $f \in L^\infty(E) = L^\infty(X,\mathcal{M},\mu)$, $(\int f dE)u = uM_f$.
(5) Deduce that $A \subset B(H)$ is a maximal abelian von Neumann algebra.

Definition 3.6.4. A normal operator $x \in B(H)$ is called multiplicity free if one of the following equivalent conditions holds:

- $\mathrm{C}^*(x)$ has a cyclic vector or
- $\mathrm{W}^*(x) = \mathrm{W}^*(x)' (= \mathrm{C}^*(x)')$

Corollary 3.6.5. *Suppose* H*is separable and* $x \in B(H)$ *is normal and multiplicity free. There exist a regular Borel measure* μ *on* $\mathrm{sp}(x)$ *and a unitary* $u \in B(L^2(\mathrm{sp}(x),\mu) \to H)$ *such that*

(1) $L^\infty(X,\mu) = L^\infty(E_x)$*,*
(2) for all $f \in L^\infty(X,\mu)$*,* $uM_f u^* = \int f\, dE_x = f(x)$*, and*
(3) the map $L^\infty(X,\mu) \ni f \mapsto f(x) \in \mathrm{W}^*(x)$ *is an isometric* $*$*-isomorphism.*

Theorem 3.6.6. *Suppose* H *is separable and* $x \in B(H)$ *is normal.*

(1) There exists a sequence $(p_n) \subset \mathrm{C}^*(x)'$ *of mutually orthogonal projections such that* $\sum_n p_n = 1$ *SOT and* $p_n H$ *is a cyclic subspace for* $\mathrm{C}^*(x)$ *for all* n*.*
(2) For all n*, there is a finite regular Borel measure* μ_n *on* $\mathrm{sp}(x)$ *and a unitary* $u_n \in B(L^2(\mathrm{sp}(x),\mu_n) \to H)$ *such that*
  - *for all* $f \in B^\infty(\mathrm{sp}(x))$*,* $u_n M_f u_n^* = f(x)p_n = f(xp_n)$*, and*
  - *the map* $L^\infty(\mathrm{sp}(x),\mu_n) \ni f \mapsto f(x)p_n \in \mathrm{W}^*(x)p_n$ *is an isometric* $*$*-isomorphism.*

(3) Setting $\mu := \sum 2^{-n}\mu_n$*,*
  - *for all* $f \in L^\infty(\mathrm{sp}(x),\mu)$*,* $\sum u_n M_f u_n^* = \sum f(x)p_n = f(x)$ *SOT, and*

- *the map* $L^\infty(\mathrm{sp}(x), \mu) \ni f \mapsto f(x) \in \mathrm{W}^*(x)$ *is an isometric* $*$*-isomorphism.*