1 Banach algebras

1.1 Basics

Definition 1.1.1. A *Banach algebra* is a complete normed complex algebra, i.e., $(A, \|\cdot\|)$ is a Banach space with a multiplication $\cdot : A^2 \to A$ such that

$$\|ab\| \leq \|a\| \cdot \|b\| \qquad \forall\, a, b \in A.$$

We say A is *unital* if there is an element $1 \in A$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in A$.

We will typically only consider unital Banach algebras.

Remark 1.1.2. If A is a Banach algebra and $J \subseteq A$ is a closed 2-sided ideal, then J is also a Banach algebra, as is A/J with norm

$$\|a + J\| := \inf_{j \in J} \|a + j\|.$$

Examples 1.1.3. Here are some examples of Banach algebras.

1. Let X be any Banach space and define $x \cdot y = 0$ for all $x, y \in X$.
2. $M_n(\mathbb{C})$ is a Banach algebra for all $n \in \mathbb{N}$ with the operator norm.
3. If X is compact Hausdorff, then $C(X)$ is a Banach algebra algebra with norm
$$\|f\|_\infty := \max_{x \in X} |f(x)|.$$
4. If X is locally compact Hausdorff (LCH), then $C_0(X)$, the space of continuous functions which vanish at ∞, is a Banach algebra with the norm
$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$
5. If X is LCH, the space $C_b(X)$ of continuous bounded functions is a Banach algebra with norm
$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$
6. Let $U \subset \mathbb{C}$ be an simply connected open domain with simply connected compact closure K. (For example, $U = \mathbb{D}$, the unit disk works.) We will see in [[]] below that
$$A(K) := \{f \in C(K) | f|_U \text{ is holomorphic}\}$$
is a Banach subalgebra of $C(K)$.

7. $C^n[0,1]$, the n-times continuously differentiable functions is a Banach algebra with norm
$$\|f\| := \sum_{k=0}^{n} \frac{1}{k!} \cdot \|f^{(k)}\|_\infty.$$
8. $\ell^1(\mathbb{Z})$ with convolution multiplication
$$(x * y)(n) := \sum_{k=-\infty}^{\infty} x(n-k)y(k)$$
is a unital Banach algebra with unit $\delta_0(n) := \delta_{n=0}$.
9. $\ell^1(\mathbb{Z}_{\geq 0})$ with convolution
$$(x * y)(n) := \sum_{k=0}^{\infty} x(n-k)y(k)$$
is also a Banach algebra.
10. $L^1(\mathbb{R}^n)$ with convolution
$$(f * g)(x) := \int f(x-y)g(y)\,dy$$
is a non-unital Banach algebra.
11. If (X, μ) is a measure space, $L^\infty(X, \mu)$ is a Banach algebra.
12. $B(X)$, the space of all bounded linear maps on a Banach space X, is a unital Banach algebra.
13. $K(X)$, the compact operators is a Banach subalgebra, which is unital if and only if X is finite dimensional.
14. The *Calkin algebra* is the Banach algebra $B(X)/K(X)$.

Facts 1.1.4. Here are some basic facts about Banach algebras.

(B1) We may always *adjoin a unit* to any Banach algebra by setting $A_1 := A \oplus \mathbb{C}1$ with multiplication
$$(a, w) \cdot (b, z) := (ab + wb + za, wz)$$
and norm $\|(a, z)\|_1 := \|a\|_A + |z|$.

However, there are other choices of norms on A_1 which may appear more natural, e.g.,
$$\|(a, z)\| := \sup_{\|b\| \neq 0} \frac{\|ab + zb\|}{\|b\|}.$$
Thus without loss of generality, we may assume A is unital.

(B2) Given a Banach algebra A, the *left regular representation* is given by $\lambda : A \to B(A)$ by $\lambda_a b := ab$. Then λ is a norm-decreasing (continuous) homomorphism (exercise!) from A to $B(A)$ with the operator norm, and if A is unital,

$$\|a\| = \|\lambda_a 1\| \leq \|\lambda_a\| \cdot \|1\| \leq \|a\| \cdot \|1\|.$$

This implies that $\|\cdot\|_{B(A)}$ on λA is strongly equivalent to $\|\cdot\|_A$, and thus gives the same topology. Moreover, clearly $\|1\|_{B(A)}\| = 1$.

Thus without loss of generality, we may assume $\|1\|_A = 1$.

Exercise 1.1.5. Determine the correct norm on the unitization of $C_0(X)$ so that it is isometrically isomorphic to $C(X \cup \{\infty\})$, continuous functions on the one-point compactification.

Definition 1.1.6. An *approximate unit* for a Banach algebra A is a net $(e_\lambda) \subset A$ with $\|e_\lambda\| \leq 1$ for all λ such that

$$\lim e_\lambda a = a = \lim a e_\lambda \qquad\qquad \forall\, a \in A.$$

Exercise 1.1.7. Find approximate units in $C_0(X)$, $L^1(\mathbb{R}^n)$, and $K(H)$, the compact operators on a Hilbert space H.

Exercise 1.1.8. Show that if A has an approximate unit, then the left regular representation $\lambda : A \to B(A)$ is isometric whenever $\|1\|_A = 1$.

We will construct approximate units for certain Banach algebras using *functional calculus* later on.

1.2 Spectrum

For this section, A is a unital Banach algebra. We identify $\mathbb{C} \subset A$ by $\lambda \mapsto \lambda 1_A$. We denote by $A^\times$ the set of (multiplicatively) invertible elements in A, i.e.,

$$A^\times := \{a \in A | \text{there is a } b \in A \text{ such that } ab = 1 = ba\}.$$

Exercise 1.2.1. Show that if $a, b, c \in A$ such that $ab = 1 = bc$, then $a = c$. Deduce that the inverse of b is unique if it exists, and can unambiguously be denoted b^{-1}.

Facts 1.2.2. Here are some facts about the set of invertible elements $A^\times$.

($\times$1) If $\|a\| < 1$, then $1 - a \in A^\times$ since

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

where the partial sums converge in norm.

(×2) Observe that if $\|a\| < 1$, then we get the following norm bound:

$$\|(1-a)^{-1}\| = \left\|\sum_{n=0}^{\infty} a^n\right\| \leq \sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1-\|a\|}.$$

(×3) If $a \in A^\times$, then whenever $\|a-b\| < \|a^{-1}\|^{-1}$, since

$$b = a - (a-b) = \left(1 - (a-b)a^{-1}\right) a,$$

and $\|(a-b)a^{-1}\| < 1$, we see that

$$b^{-1} = a^{-1} \sum_{n=0}^{\infty} \left((a-b)a^{-1}\right)^n$$

where the partial sums converge in norm. Thus $A^\times$ is open in A.

(×4) Inversion is continuous on $A^\times$. Indeed,

$$\begin{aligned}
\|b^{-1} - a^{-1}\| &= \left\| a^{-1} \sum_{n=1}^{\infty} \left((a-b)a^{-1}\right)^n \right\| \\
&\underset{(\times 3)}{\leq} \|a^{-1}\| \cdot \sum_{n=1}^{\infty} \Big(\underbrace{\|a-b\| \cdot \|a^{-1}\|}_{<1} \Big)^n \\
&= \|a^{-1}\| \cdot \frac{\|a-b\| \cdot \|a^{-1}\|}{1 - \|a-b\| \cdot \|a^{-1}\|} \xrightarrow{a \to b} 0.
\end{aligned}$$

Definition 1.2.3. The *spectrum* of $a \in A$ is

$$\operatorname{sp}_A(a) := \left\{\lambda \in \mathbb{C} | \lambda - a \notin A^\times \right\}.$$

If A is non-unital, we define $\operatorname{sp}_A(a) := \operatorname{sp}_{A_1}((a, 0_{\mathbb{C}}))$.

Remark 1.2.4. The spectrum depends on the algebra (see §1.7). Observe that if we have a unital inclusion of Banach algebras $A \subset B$, then invertibility in A implies invertibility in B, or in other words,

$$\operatorname{sp}_B(a) \subseteq \operatorname{sp}_A(a). \tag{1.2.5}$$

More generally, if $\phi : A \to B$ is a unital algebra map between Banach algebras and $a \in A^\times$, then $\phi(a) \in B^\times$, so $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$.

Examples 1.2.6.

1. For X compact Hausdorff and $f \in C(X)$ $\operatorname{sp}(f) = f(X)$. This is also true in $C^n[0,1]$ and $A(K)$.

2. For X LCH and $f \in C_0(X)$, $\operatorname{sp}(f) = \overline{f(X)}$. This is also true in $C_b(X)$.

3. For $f \in L^\infty(X)$, $\operatorname{sp}(f)$ is the *essential range*, i.e.,

$$\begin{aligned}\text{ess range}(f) &:= \left\{\lambda \in \mathbb{C} \middle| z \mapsto \frac{1}{f(z)-\lambda} \text{ is in } L^\infty(X)\right\}\\ &= \left\{\lambda \in \mathbb{C} \middle| \forall\, \varepsilon > 0, \mu(f^{-1}(B_\varepsilon(\lambda))) > 0\right\}.\end{aligned}$$

4. For $\ell^1(\mathbb{Z})$, $\ell^1(\mathbb{Z}_{\geq 0})$ and $L^2(\mathbb{R}^n)$, one uses *Fourier analysis* methods.

We now return to the setting where A is unital.

Facts 1.2.7. Here are some basic facts about the spectrum.

(sp1) If $\|a\| < |z|$, then $z - a \in A^\times$ with $\|(z-a)^{-1}\| \leq \frac{1}{|z|-\|a\|}$ by applying (×1) and (×2) to a/z. Hence $\operatorname{sp}_A(a) \subset B_{\|a\|}(0)$.

(sp2) Suppose $z_0 \notin \operatorname{sp}_A(a)$. If $|z - z_0| < \|(z-a)^{-1}\|^{-1}$, then $z \notin \operatorname{sp}_A(a)$ by (×3). Hence $\operatorname{sp}_A(a)$ is compact for every $a \in A$.

(sp3) (First resolvent formula) For $a \in A$ and $w, z \notin \operatorname{sp}_A(a)$,

$$(w-a)^{-1} - (z-a)^{-1} = (z-w)(w-a)^{-1}(z-a)^{-1} = (z-w)(z-a)^{-1}(w-a)^{-1}.$$

Indeed, we can multiply both sides by $(z-a)(w-a)$, which is invertible.

(sp4) For every $\varphi \in A^*$, the function $z \mapsto \varphi((z-a)^{-1})$ is holomorphic on $\operatorname{sp}_A(a)^c$. Indeed,

$$\begin{aligned}\lim_{w\to z} \frac{\varphi((w-a)^{-1}) - \varphi((z-a)^{-1})}{w-z} &= \varphi\left(\lim_{w\to z} \frac{(w-a)^{-1} - (z-a)^{-1}}{w-z}\right)\\ &\underset{\text{(sp3)}}{=} -\varphi\left(\lim_{w\to z}(w-a)^{-1}(z-a)^{-1}\right)\\ &\underset{(\times 4)}{=} -\varphi((z-a)^{-2}).\end{aligned}$$

(sp5) $\operatorname{sp}_A(a)$ is always nonempty.

Proof. For $\varphi \in A^*$ and $|z| > \|a\|$,

$$|\varphi((z-a)^{-1})| \leq \|\varphi\| \cdot \left\|(z-a)^{-1}\right\| \underset{\text{(sp1)}}{\leq} \|\varphi\| \cdot \frac{1}{|z| - \|a\|} \xrightarrow{|z|\to\infty} 0.$$

Suppose for contradiction that $\operatorname{sp}_A(a) = \emptyset$. For every $\varphi \in A^*$, $z \mapsto \varphi((z-a)^{-1})$ would be bounded and entire, hence constant (see Liouville's Theorem (CA5) below). Since A^* separates points of A, we would be forced to conclude that $z \mapsto (z-a)^{-1}$ and thus $z \mapsto z - a$ were both constant, which is absurd. □

Theorem 1.2.8 (Gelfand-Mazur)**.** *The only normed division algebra over $\mathbb{C}$ is $\mathbb{C}$ itself.*

Proof. Suppose A is such an algebra. Its completion $\overline{A}$ is a unital Banach algebra. If $a \in A$, then

$$\emptyset \underset{\text{(sp5)}}{\neq} \operatorname{sp}_{\overline{A}}(a) \underset{(1.2.5)}{\subseteq} \operatorname{sp}_A(a).$$

If $\lambda \in \operatorname{sp}_A(a)$, then $a - \lambda$ is not invertible, so $a - \lambda = 0$ and $a \in \mathbb{C}$. □

1.3 Spectral radius

Let A be a unital Banach algebra.

Lemma 1.3.1. *For $a \in A$, the sequence $\|a^n\|^{1/n}$ converges to $\inf_n \|a^n\|^{1/n}$.*

Proof. Fix $m \in \mathbb{N}$. For $n \in \mathbb{N}$, write $n = qm + r$ with $q, r \in \mathbb{Z}_{\geq 0}$ and $r < m$ via the Euclian Algorithm. Then

$$\|a^n\| \leq \|a^m\|^q \cdot \|a\|^r \qquad \Longrightarrow \qquad \|a^n\|^{1/n} \leq \|a^m\|^{q/n} \cdot \|a\|^{r/n} \xrightarrow{n\to\infty} \|a^m\|^{1/m}.$$

Thus

$$\limsup \|a^n\|^{1/n} \leq \inf_m \|a^m\|^{1/m} \leq \liminf \|a^n\|^{1/n}.$$

□

Definition 1.3.2. For $a \in A$, we define its *spectral radius* as

$$r(a) := \lim_{n\to\infty} \|a^n\|^{1/n},$$

which exists by the previous lemma.

Proposition 1.3.3. *For $a \in A$, $\operatorname{sp}_A(a) \subset B_{r(a)}(0_{\mathbb{C}})$, and there is a $z \in \operatorname{sp}_A(a)$ with $|z| = r(a)$.*

Proof. First, we prove that if $|z| > r(a)$, then $z \notin \operatorname{sp}_A(a)$. If $|z| > r(a)$, then $\lim \|a^n\|^{1/n} < |z|$. Thus there is an $N > 0$ such that for all $n > N$, $\|a^n\|^{1/n} < |z|$. This means

$$\frac{\|a^n\|^{1/n}}{|z|} < 1,$$

so there is an $0 < r < 1$ such that that

$$\frac{\|a^n\|^{1/n}}{|z|} \leq r < 1 \qquad \Longleftrightarrow \qquad \frac{\|a^n\|}{|z|^n} \leq r^n \qquad \forall\, n > N.$$

Hence the formula

$$\frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{a^n}{z^n}$$

converges in norm, and one verifies directly that its limit is $(z-a)^{-1}$.
Second, it is enough to show that if $\operatorname{sp}_A(a) \subset \{|z| < r\}$, then $r \geq r(a)$. Fix such an $r > 0$. By (sp4), for $\varphi \in A^*$, the function $f_{a,\varphi}(z) := \varphi((z-a)^{-1})$ is holomorphic on $\operatorname{sp}_A(a)^c$. For $|z| > \|a\|$, we have

$$f_{a,\varphi} = \sum_{n=0}^{\infty} z^{-n-1} \varphi(a^n)$$

converges uniformly, so it must be the Laurent series of the holomorphic function $f_{a,\varphi}$. But since $f_{a,\varphi}$ is holomorphic on $\operatorname{sp}_A(a)^c \subset \{|z| \geq r\}$, this Laurent series must converge whenever $|z| \geq r$. This implies that

$$\sup_n |r^{-n-1}\varphi(a^n)| = \frac{1}{r} \sup_n \left|\varphi\left(\left(\frac{a}{r}\right)^n\right)\right| < \infty \qquad \forall \varphi \in A^*.$$

Now considering $\left\{\left(\frac{a}{r}\right)^n \middle| n \in \mathbb{N}\right\}$ as a collection of operators on A^*, by the Uniform Boundedness Principle, we have that this set is norm bounded, i.e., there is an $M > 0$ such that

$$\sup_n \left\|\frac{a^n}{r^n}\right\| < M < \infty.$$

It immediately follows that $\|a^n\|^{1/n} \leq r M^{1/n}$ for all n, and thus $r(a) \leq r$. □

1.4 Some complex analysis

In this section, we review some basic complex analysis that we will use for the Holomorphic Functional Calculus. For $U \subset \mathbb{C}$ open, let $H(U)$ denote the algebra of holomorphic functions on U. For a compact set $K \subset \mathbb{C}$, define

$$\mathcal{O}(K) := \{(U, f) | K \subset U \text{ is an open neighborhood and } f \in H(U)\} / \sim$$

where $(U_1, f_1) \sim (U_2, f_2)$ if $f_1 = f_2$ on $U_1 \cap U_2$. (Exercise: verify $\sim$ is an equivalence relation.)

Remark 1.4.1. If K is connected and has an accumulation point, then

$$\mathcal{O}(K) = \{f \in C(K) | \exists U \supset K \text{ open and } g \in H(U) \text{ such that } g|_K = f\},$$

but in general, these sets are not equal.

Definition 1.4.2. A *simple closed contour* γ in $\mathbb{C}$ is a non-intersecting finite family of injective piecewise C^1 maps $S^1 \to U$. We identify γ with its image, which inherits an orientation from the counter-clockwise orientation of S^1. Given $z \notin \gamma$, the *index/winding number* of γ about z is given by

$$\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{w-z}\, dw.$$

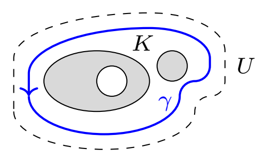
It takes values zero and one for a simple closed contour; the *inside* of γ is

$$\mathrm{ins}(\gamma) := \left\{ z \in \mathbb{C} \middle| \mathrm{ind}_\gamma(z) = 1 \right\}.$$

Facts 1.4.3. Here are some basic facts from Complex Analysis.

(CA1) (Jordan Curve Theorem) For every open set $U \subset \mathbb{C}$ and every nonempty compact $K \subset \mathbb{C}$, there is a simple closed contour $\gamma \subset U$ such that

$$\mathrm{ind}_\gamma(z) = \begin{cases} 1 & \text{if } z \in K \\ 0 & \text{if } z \notin U. \end{cases}$$



(CA2) (Cauchy-Goursat) If γ is a simple closed contour in U with $\mathrm{ins}(\gamma) \subset U$ and $f \in H(U)$, then

$$\int_\gamma f(z)\, dz = 0.$$

(CA3) For any two homotopic paths γ_1, γ_2 in U and holomorphic $f \in H(U)$,

$$\int_{\gamma_1} f(z)\, dz = \int_{\gamma_2} f(z)\, dz.$$

(CA4) (Cauchy integral formula) For every contour as in (CA1), for all $f \in H(U)$ and $z \in K$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dz.$$

(CA5) A bounded entire function is constant.

Proof. If $|f(z)| \le M$ for all $z \in \mathbb{C}$, use (CA4) to see that for R sufficiently large,

$$\begin{aligned}|f(z) - f(0)| &= \left| \frac{1}{2\pi i} \int_{|w|=R} \frac{f(w)}{w-z} - \frac{f(w)}{w}\, dw \right| \\ &= \left| \frac{1}{2\pi i} \int_{|w|=R} \frac{z f(w)}{(w-z)w}\, dw \right| \\ &\le \frac{1}{2\pi} \int_{|w|=R} \frac{M}{R} \cdot \underbrace{\frac{|z|}{|w-z|}}_{<\varepsilon}\, dw \\ &< M\varepsilon.\end{aligned}$$

Since ε can be made arbitrarily small, $f(z) = f(0)$. □

(CA6) (Morera) If $f : U \to \mathbb{C}$ is continuous and for every simple closed contour $\gamma \subset U$

$$\int_\gamma f(z)\, dz = 0,$$

then $f \in H(U)$, i.e., f is holomorphic.

(CA7) (Identity Theorem) Suppose U is open and connected and $f \in H(U)$. If (z_n) is a convergent sequence in U whose limit lies in U and $f(z_n) = 0$ for all n, then $f = 0$.

(CA8) (Maximum Modulus Principle) Suppose $U \subset \mathbb{C}$ is open and $z_0 \in U$. If $f \in H(U)$ and there is a $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset U$ and

$$|f(z_0)| \ge |f(z)| \qquad \forall\, z \in B_\varepsilon(z_0),$$

then f is constant.

Example 1.4.4. Generalize the proof of (CA5) to show that if f is entire and there is an $0 < r < 1$ and $a, b \ge 0$ such that $|f(z)| \le a|z|^r + b$ for all $z \in \mathbb{C}$, then f is constant.

Lemma 1.4.5. *If $(f_n) \subset H(U)$ is uniformly Cauchy on each compact $K \subset U$, then there is an $f \in H(U)$ such that $f_n \to f$ locally uniformly, i.e., uniformly on every compact $K \subset U$.*

Proof. Since points are compact, we can define f to be the pointwise limit of the f_n. Since U is open and locally compact, clearly f is continuous on U and $f_n \to f$ locally uniformly. Then for every simple closed contour $\gamma \subset U$,

$$\int_\gamma f(z)\, dz = \int_\gamma \lim f_n(z)\, dz = \lim \int_\gamma f_n(z)\, dz \underset{\text{(CA2)}}{=} 0.$$

By Morera's Theorem (CA6), $f \in H(U)$. □

Proposition 1.4.6. *The topology of local uniform convergence is a first countable Frechet topological vector space structure on $C(U)$, the continuous functions on an open subset $U \subset \mathbb{C}$, and $H(U) \subset C(U)$ is a closed subspace.*

Proof. Pick nested compact sets (K_n) such that $K_n \subset K_{n+1}^\circ$ and $\bigcup K_n = U$. Observe that we also have $U = \bigcup K_n^\circ$, so every compact subset of U is contained in some K_n. On $C(U)$, consider the separating family of seminorms

$$m_n(f) := \|f\|_{C(K_n)} = \sup_{x \in K_n} |f(x)|,$$

and let $\mathcal{T}$ be the locally convex vector space topology on $C(U)$ generated by the m_n. Observe that $\mathcal{T}$ is metrizable via the translation invariant metric

$$d(f, g) := \sum 2^{-n} \frac{m_n(f-g)}{1+m_n(f-g)},$$

and is thus a first countable Frechet TVS structure. Since $f_k \to f$ if and only if $m_n(f - f_k) \to 0$ for all n, we see that convergence in $\mathcal{T}$ is exactly local uniform convergence (and independent of the choice of (K_n)). Finally, $H(U)$ is a closed subspace of $C(U)$ by Lemma 1.4.5. □

Definition 1.4.7. Recall that a subset $S \subset \mathbb{C}$ is *simply connected* if both S and S^c are connected.

Lemma 1.4.8 (Runge). *Suppose $w \in \mathbb{C}$, $\zeta \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, and U is an open connected subset of $\widehat{\mathbb{C}}$ which contains a path from w to ζ. For any compact subset $K \subset U^c$, the function $f_w(z) := \frac{1}{w-z}$ can be uniformly approximated by rational functions whose only poles lie at ζ.*

Remarks 1.4.9.

1. If $\zeta = \infty$, then such rational functions are exactly polynomials.

2. Without loss of generality, we may assume that both U and K are simply connected. Indeed, U need only contain a path from w to ζ, so we may take U to be an ε-neighborhood of such a path. In this case, we can always find a simply connected compact set $L \subset U^c$ with $K \subset L$, so it suffices to uniformly approximate f_w on L.

Proof. Before we begin the general proof, observe that if $|w - \zeta|$ is very small, more precisely,

$$|\zeta - w| < \inf_{z\in K} |\zeta - z|,$$

then the convergence

$$\frac{1}{w-z} = \frac{1}{(\zeta - z) - (\zeta - w)} = \frac{1}{\zeta - z} \cdot \sum_{n=0}^{\infty} \left(\frac{\zeta - w}{\zeta - z}\right)^n \tag{1.4.10}$$

is uniform, as $\frac{\zeta - w}{\zeta - z} < 1$ for such w on K. Hence we can uniformly approximate f_w on K close to ζ by rational functions whose only pole is at ζ.
(The above argument is also valid if $\zeta = \infty$; indeed, for $|w|$ sufficiently large, the convergence

$$\frac{1}{w-z} = \frac{1}{w}\sum_{n=0}^{\infty} \frac{z^n}{w^n} \qquad \forall\, |w| > \sup_{z\in K} |z| \tag{1.4.11}$$

is uniform, as $|z/w| < 1$ for such w on K. Hence we can uniformly approximate f_w on K close to $\zeta = \infty$ by polynomials.)
We now use some functional analysis to finish the proof. Let $A \subset C(K)$ be the Banach subalgebra of uniform limits of rational functions whose only poles lie at ζ. By the Hahn-Banach Theorem, it suffices to show that $\varphi(f_w) = 0$ for every $\varphi \in C(K)^*$ such that $\varphi|_A = 0$. To show this, we need only prove that each function $g_\varphi(w) := \varphi(f_w)$ is holomorphic on K^c. Indeed, $g_\varphi(w) = 0$ is zero for w sufficiently close to ζ by (1.4.10,1.4.11), and since K^c is connected, $g_\varphi = 0$ on K^c by the Identity Theorem (CA7).
Now we show g_φ is holomorphic on K^c in two steps. First, as $w \in K^c$, as a function of $z \in K$, $h^{-1}(f_{w+h} - f_w)$ converges uniformly to $z \mapsto \frac{-1}{(w-z)^2}$ as $h \to 0$. Second, we consider the difference quotient:

$$\lim_{h\to 0} \frac{g_\varphi(w+h) - g_\varphi(w)}{h} = \varphi\left(\lim_{h\to 0} \frac{f_{w+h} - f_w}{h}\right) = \varphi\left(z \mapsto \frac{-1}{(w-z)^2}\right).$$

Hence $g'_\varphi(w)$ exists for all $w \in K^c$, and thus g_φ is holomorphic. □

Theorem 1.4.12 (Runge)**.** *Suppose $K \subset \mathbb{C}$ is compact and $S \subset \widehat{\mathbb{C}}$ contains an element from each connected component of K^c. Each $f \in \mathcal{O}(K)$ can be uniformly approximated on K by rational functions whose only poles lie in S.*

Proof. Let $U \subset \mathbb{C}$ be an open subset containing K on which f is holomorphic. Pick a simple closed contour $\gamma \subset U \setminus K$ as in (CA1). By (CA4),

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\, dw \qquad \forall z \in K.$$

This is a Riemann integral, which can be approximated uniformly on K (see Remark 1.4.13 below) by a finite sum which is a linear combination of functions of the form $f_w(z) := \frac{1}{w-z}$ for $w \in \gamma \subset U \setminus K$. By Lemma 1.4.8, each of these f_w can be uniformly approximated by rational functions whose only poles lie in S. The result follows. $\square$

Remark 1.4.13. The above uniform approximation of the Riemann integral on K follows from the following analysis. Suppose K is compact and $g : K \times [0,1] \to \mathbb{C}$ is continuous. Since $K \times I$ is compact, g is uniformly continuous, so there is a $\delta > 0$ such that

$$\|(w,s)-(z,t)\|_\infty < \delta \qquad \Longrightarrow \qquad |g(w,s)-g(z,t)| < \varepsilon.$$

Pick a partition $P\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ where $\Delta_i := t_i - t_{i-1} < \delta$ for all i. Then for each fixed $z \in K$ and $i = 1, \dots n$, setting

$$M_{z,i} := \max\{g(z,t) | t_{i-1} \le t \le t_i\} \qquad \text{and} \qquad m_{z,i} := \min\{g(z,t) | t_{i-1} \le t \le t_i\},$$

we have that

$$U(g(z,t),P) - L(g(z,t),P) = \sum_{i=1}^{n} (M_{z,i} - m_{z,i}) \cdot \Delta_i < \varepsilon.$$

This immediately implies that for every $z \in K$, the Riemann integral is uniformly approximated by the right endpoint Riemann sum,

$$\left| \int_0^1 g(z,t)\,dt - \sum_{i=1}^{n} g(z,t_i)\Delta_i \right| < \varepsilon \qquad \forall\, z \in K,$$

as both lie between the upper and lower sum.

Remark 1.4.14. If K is compact and K^c is connected, then choosing $S = \{\infty\}$, each $f \in \mathcal{O}(K)$ can be uniformly approximated on K by polynomials.

Example 1.4.15. The *Hardy space* $H^\infty(U)$ is the space of holomorphic functions $f : U \to \mathbb{C}$ which are uniformly bounded, which is a Banach algebra under the sup norm.

Sub-Example 1.4.16. The *disk algebra* $A(\mathbb{D})$ is $H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, i.e., the continuous functions on the closed unit disk which are holomorphic on the interior. It is exactly the uniform limit of the polynomials in $C(\overline{\mathbb{D}})$. Indeed, each such uniform limit of polynomials is clearly holomorphic on the interior by Morera's Theorem (CA6). Conversely, each $f \in A(\mathbb{D})$ has a Taylor series which converges locally uniformly on $\mathbb{D}$. Since f is uniformly continuous on $\overline{\mathbb{D}}$, for every $\varepsilon > 0$, there is a $\delta > 0$ so that $|w - z| < \delta$ implies $|f(w) - f(z)| < \varepsilon$. Fix $1 - \delta < r < 1$ so that $1 - r < \delta$ and $|z - rz| = (1-r) \cdot |z| < \delta$ for all $z \in \overline{\mathbb{D}}$. Then $|f(z) - f(rz)| < \varepsilon$ for all $z \in \overline{\mathbb{D}}$. Moreover, $z \mapsto f(rz)$ can be uniformly approximated by polynomials on $\overline{\mathbb{D}}$, and thus f can be as well.

Lemma 1.4.17. *The map $A(\mathbb{D}) \ni f \mapsto f|_{S^1} \subset C(S^1)$ is an isometric isomorphism onto the uniform closure of the polynomials in $C(S^1)$.*

Proof.

Isometric: It suffices to prove that every $f \in A(\mathbb{D})$ achieves its norm on S^1. To do so, we need only prove that if $|f|$ achieves its maximum on $\mathbb{D}$, then f is constant; this is immediate by the Maximum Modulus Principle (CA8).

Injective: Isometric maps between normed spaces are always injective.

Surjective: Suppose (p_n) is a sequence of polynomials on S^1 with $p_n \to f$ uniformly. By the Maximum Modulus Principle (CA8), (p_n) is uniformly Cauchy on $\overline{\mathbb{D}}$, and thus $f \in A(\mathbb{D})$ by Morera's Theorem (CA6). □

1.5 Banach-valued differentiation and integration

We now discuss the notion of a Banach-valued holomorphic function, and the Riemann integral for curves in a Banach space. For this section, X is a Banach space.

Definition 1.5.1. For an open set $U \subset \mathbb{C}$, we call $f : U \to X$:

- *weakly holomorphic* if for every $\varphi \in X^*$, $\varphi \circ f : U \to \mathbb{C}$ is holomorphic, and
- *strongly holomorphic* if for every $z \in U$,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

  exists, where the limit is taken in norm in X.

Example 1.5.2. The First Resolvent Formula (sp3) implies that the *resolvent function* $R_a(z) := (z-a)^{-1}$ is strongly holomorphic on $\operatorname{sp}_A(a)^c$. Indeed,

$$\lim_{w \to z} \frac{(w-a)^{-1} - (z-a)^{-1}}{w - z} = \lim_{w \to z} \frac{(z-w)(w-a)^{-1}(z-a)^{-1}}{w - z} = -(z-a)^{-2}.$$

Clearly strong holomorphicity implies weak holomorphicity by linearity and continuity of $\varphi \in X^*$. We immediately obtain the following generalization of Liouville's Theorem (CA5).

Corollary 1.5.3. *If $f : \mathbb{C} \to X$ is a Banach-valued strongly entire function which is norm bounded, then f is constant.*

Proof. Since f is norm bounded, $\varphi \circ f$ is norm-bounded and entire for every $\varphi \in X^*$, and thus constant by (CA5). So $\varphi(f(z)) = \varphi(f(0))$ for all $\varphi \in X^*$, and since X^* separates points, $f(z) = f(0)$. □

We are interested in proving the other direction, i.e., weakly holomorphic implies strongly holomorphic. The trick will be to define a version of the Cauchy Integral Formula (CA4) for functions $f : U \to X$. To do this, we would like to be able to integrate along curves valued in X, i.e., we want to define

$$\int_a^b \gamma(t)\, dt$$

for $[a, b] \subset \mathbb{R}$ and continuous $\gamma : [a, b] \to X$. Since X^* separates points of X, observe that there is at most one $x \in X$ such that

$$\varphi(x) = \int_a^b (\varphi \circ \gamma)(t)\, dt \qquad \forall\, \varphi \in X^*. \tag{1.5.4}$$

Exercise 1.5.5 (Homework)**.** Define

$$\int_a^b \gamma(t)\, dt = \lim_{\|P\| \to 0} x_{P,u}$$

where

$$x_{P,u} := \sum_{j=1}^n \underbrace{(t_j - t_{j-1})}_{=\Delta_j} \gamma(u_j)$$

where $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ is a partition, $\|P\| = \max_j \Delta_j$, and $t_{j-1} \le u_j \le t_j$ for each $1 \le j \le n$. Then show the limit satisfies:

($\int$1) $\varphi\left(\int_a^b \gamma(t)\, dt\right) = \int_a^b (\varphi \circ \gamma)(t)\, dt$, and

($\int$2) $\int_a^b : C([a, b], X) \to X$ is a bounded linear map.

Theorem 1.5.6. *Suppose X is a Banach space, $U \subset \mathbb{C}$ is open, and $f : U \to X$ is weakly holomorphic.*

1. *f is norm-continuous, i.e., if $z_n \to z$ in U, then $f(z_n) \to f(z)$ in norm.*
2. *The Cauchy-Goursat and Cauchy Integral Formula hold. That is, if $\gamma \subset U$ is a simple closed contour with $\operatorname{ind}_\gamma(z) = 0$ for all $z \notin U$, then*

$$\int_\gamma f(z)\, dz = 0 \qquad \text{and} \qquad f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z}\, dw \qquad \forall\, z \in \operatorname{ins}(\gamma).$$

3. *f is strongly holomorphic.*

Proof. Without loss of generality, we may assume $0 \in U$, and we need only prove (1) and (3) at 0.

Proof of 1: Let $r > 0$ so that $\overline{B_{2r}(0)} \subset U$. For $\varphi \in X^*$, since $\varphi \circ f$ is holomorphic, for all $0 < |z| < 2r$,

$$\begin{aligned}\frac{\varphi(f(z)) - \varphi(f(0))}{z} &\underset{\text{(CA4)}}{=} \frac{1}{2\pi i z}\int_{|w|=2r} \frac{\varphi(f(w))}{w-z} - \frac{\varphi(f(w))}{w}\, dw \\ &= \frac{1}{2\pi i}\int_{|w|=2r} \frac{\varphi(f(w))}{(w-z)w}\, dw.\end{aligned}$$

Set $M_\varphi := \max|\varphi \circ f|$ on $\overline{B_{2r}(0)}$. By the above formula, for $0 < |z| \le r$,

$$\begin{aligned}\left|\frac{\varphi(f(z)) - \varphi(f(0))}{z}\right| &\le \frac{1}{2\pi}\int_{|w|=2r} \left|\frac{\varphi(f(w))}{(w-z)w}\right| dw \\ &\le \frac{1}{2\pi}\int_{|w|=2r} \frac{M_\varphi}{2r^2}\, dw = \frac{M_\varphi}{r}\end{aligned}$$

Hence the set

$$\left\{\frac{f(z) - f(0)}{z}\middle| 0 < |z| \le r\right\}$$

is weakly bounded, and is thus bounded in norm by the Uniform Boundedness Principle[a] Thus there is an $R > 0$ such that whenever $0 \le |z| \le r$, $\|f(z) - f(0)\| \le |z| \cdot R \to 0$ as $z \to 0$.

Proof of 2: For $\gamma \subset U$ a simple closed contour and $\varphi \in X^*$,

$$\varphi\left(\int_\gamma f(z)\, dz\right) \underset{(\int 1)}{=} \int_\gamma \varphi(f(z))\, dz = 0$$

since f is weakly holomorphic. Thus $\int_\gamma f(z)\, dz = 0$ as X^* separates points by the Hahn-Banach Theorem. Similarly, for all $z \in \text{ins}(\gamma)$,

$$\begin{aligned}\varphi(f(z)) &\underset{\text{(CA4)}}{=} \frac{1}{2\pi i}\int_\gamma \frac{\varphi(f(w))}{w-z}\, dw = \frac{1}{2\pi i}\int_\gamma \varphi\left(\frac{f(w))}{w-z}\right) dw \\ &\underset{(\int 1)}{=} \varphi\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w))}{w-z}\, dw\right)\end{aligned}$$

Again as X^* separates points, we conclude that $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w))}{w-z}\, dw$.

Proof of 3: Choose r as in part (1) so that $\overline{B_{2r}(0)} \subseteq U$. By parts (1) and (2), we have

$$\frac{f(z)-f(0)}{z} = \frac{1}{2\pi i}\int_{|w|=2r} \frac{f(w)}{(w-z)w}\,dw \qquad \forall 0<|z|<2r.$$

Now for $0 < |z| \le r$, note that the functions $w \mapsto \frac{f(w)}{(w-z)w}$ converge uniformly to $w \mapsto \frac{f(w)}{w^2}$ in $C([0,1],X)$ as $z \to 0$. Hence

$$\begin{aligned}\lim_{z\to 0}\frac{f(z)-f(0)}{z} &= \lim_{z\to 0}\frac{1}{2\pi i}\int_{|w|=2r}\frac{f(w)}{(w-z)w}\,dw \\ &\underset{(\int 2)}{=} \frac{1}{2\pi i}\int_{|w|=2r}\lim_{z\to 0}\frac{f(w)}{(w-z)w}\,dw \\ &= \frac{1}{2\pi i}\int_{|w|=2r}\frac{f(w)}{w^2}\,dw\end{aligned}$$

exists in X as $w \mapsto \frac{f(w)}{w^2}$ is norm-continuous. □

[a] Consider the elements $\frac{f(z)-f(0)}{z}$ as bounded linear operators $X^* \to \mathbb{C}$.

Just as (CA3) is a corollary of (CA2), we have the following corollary.

Corollary 1.5.7. *Suppose $U \subset \mathbb{C}$ is open and $f : U \to X$ is Banach-valued holomorphic. For any two homotopic paths γ_1, γ_2 in U,*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

1.6 Holomorphic functional calculus

Let A be a unital Banach algebra.

Exercise 1.6.1. Show that if $a, b \in A$ with b invertible, then $[a,b]=0$ if and only if $[a,b^{-1}]=0$.

Construction 1.6.2 (Rational functional calculus)**.** If $a \in A$ and

$$q(z) = \prod_{j=1}^{k}(z-z_j)^{m_j}$$

is a rational function whose poles $z_j \notin \mathrm{sp}(a)$. (In other words, $q \in \mathcal{O}(\mathrm{sp}_A(a)) \cap \mathbb{C}(z)$.) We can unambiguously define

$$q(a) := \prod_{j=1}^{k} (a - z_j)^{m_j}$$

as all the terms pairwise commute by Exercise 1.6.1.

Exercise 1.6.3. Show that the map $\mathcal{O}(\mathrm{sp}_A(a)) \cap \mathbb{C}(z) \longrightarrow A$ given by $q \mapsto q(a)$ is a unital algebra homomorphism.

We now want to extend the map $\mathcal{O}(\mathrm{sp}_A(a)) \cap \mathbb{C}(z) \longrightarrow A$ to all of $\mathcal{O}(\mathrm{sp}_A(a))$.

Construction 1.6.4 (Holomorphic functional calculus)**.** For $a \in A$ and $f \in \mathcal{O}(\mathrm{sp}_A(a))$, let U be an open neighborhood of $\mathrm{sp}_A(a)$ on which f is holomorphic. Pick a simple closed contour $\gamma \subset U$ as in (CA1) for $K = \mathrm{sp}_A(a)$, i.e.,

$$\mathrm{ind}_\gamma(z) = \begin{cases} 1 & \text{if } z \in \mathrm{sp}_A(a) \\ 0 & \text{if } z \notin U. \end{cases}$$

Define

$$f(a) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\, dz,$$

i.e., the unique element of A satisfying

$$\varphi(f(a)) \underset{(\int 1)}{=} \frac{1}{2\pi i} \int_\gamma \varphi((z-a)^{-1}) f(z)\, dz \qquad\qquad \forall\, \varphi \in A^*.$$

Note that $f(a)$ is independent of the choice of open set U and contour γ by (CA3) or Corollary 1.5.7.

Example 1.6.5. Suppose $f(z) := \sum_k \alpha_k z^k$ is a power series with radius of convergence $R > \|a\|$ for $a \in A$. Then for any $z \in \mathbb{C}$ with $|z| > \|a\|$,

$$(z-a)^{-1} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n .$$

Thus fixing $\|a\| < r < R$, we have

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z-a}\, dz \\ &= \frac{1}{2\pi i} \int_{|z|=r} \sum_{k=0}^{\infty} \alpha_k z^k \frac{dz}{z-a} \\ &= \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{|z|=r} z^{k-1} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n dz \end{aligned}$$

$$= \sum_{k,n=0}^{\infty} \alpha^k a^n \underbrace{\frac{1}{2\pi i}\int_{|z|=r} \frac{1}{z^{n-k+1}}\,dz}_{=\delta_{n=k}}$$

$$= \sum_{k=0}^{\infty} \alpha^k a^k.$$

In particular, the constant function $1(z) = 1$ applied to a is always $1 \in A$, and the identity function $z \mapsto z$ applied to a is always $a \in A$.

Exercise 1.6.6. Suppose $a \in A$ and $K \subset \mathbb{C}$ is compact such that $\operatorname{sp}_A(a) \subset K^\circ$. Show there is an $M_K > 0$ such that for any $f \in H(K^\circ)$ which has a continuous extension to K, $\|f(a)\| \leq M_K \|f\|_{C(K)}$.

Theorem 1.6.7. *The HFC map* $\mathcal{O}(\operatorname{sp}_A(a)) \ni f \longmapsto f(a) \in A$ *satisfies:*

(HFC1) *The map* $f \mapsto f(a)$ *is a unital algebra homomorphism such that* $(z \mapsto z) \mapsto a$.

(HFC2) *If* $\operatorname{sp}_A(a) \subset U$ *and* $(f_n) \subset H(U)$ *with* $f_n \to f$ *locally uniformly, then* $f_n(a) \to f(a)$ *in norm in* A.

Proof. After Example 1.6.5, to finish the proof of (HFC1), it remains to prove $f \mapsto f(a)$ is an algebra homomorphism. Additivity is immediate from $(\int 2)$. To show multiplicativaty, if $f, g \in \mathcal{O}(\operatorname{sp}_A(a))$, choose an open set $U \supset \operatorname{sp}_A(a)$ on which f, g are both holomorphic. Then choose simple closed contours γ, σ in $U \setminus K$ such that $\operatorname{sp}_A(a) \subset \operatorname{ins}(\gamma)$ and $\gamma \cup \operatorname{ins}(\gamma) \subset \operatorname{ins}(\sigma)$. We then calculate

$$\begin{aligned}
f(a)g(a) &= \frac{-1}{4\pi^2}\int_\gamma \frac{f(z)}{z-a}\,dz \int_\sigma \frac{g(w)}{w-a}\,dw \\
&= \frac{-1}{4\pi^2}\int_\gamma\int_\sigma f(z)g(w) \underbrace{(z-a)^{-1}(w-a)^{-1}}_{=\frac{(z-a)^{-1}-(w-a)^{-1}}{w-z} \text{ by (sp3)}}\,dw\,dz \\
&= \frac{-1}{4\pi^2}\int_\gamma \frac{f(z)}{z-a} \underbrace{\left(\int_\sigma \frac{g(w)}{w-z}\,dw\right)}_{=2\pi i g(z) \text{ by (CA4)}} dz + \frac{1}{4\pi^2}\int_\sigma \frac{g(w)}{w-a} \underbrace{\left(\int_\gamma \frac{f(z)}{w-z}\,dz\right)}_{=0 \text{ by (CA2)}} dw \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f(z)g(z)}{z-a}\,dz \\
&= (fg)(a).
\end{aligned}$$

To prove (HFC2), suppose $f_n \to f$ in $H(U)$ (locally uniformly) where $\operatorname{sp}_A(a) \subset U$. By normality, there is a compact $K \subset U$ such that $\operatorname{sp}_A(a) \subset K^\circ$. By Exercise 1.6.6, since f and each f_n are holomorphic on K° and continuous on K, there is a constant

$M_K > 0$ such that

$$\|f(a) - f_n(a)\| = \|(f - f_n)(a)\| \leq M_K \|f - f_n\|_{C(K)} \xrightarrow{n\to\infty} 0. \qquad \square$$

Corollary 1.6.8. *The HFC extends the 'rational functional calculus' from Construction 1.6.2.*

Proof. By Example 1.6.5, for any polynomial $q(z) = \sum_{k=0}^{n} \alpha_k z^k$, $q(z) = \sum_{k=0}^{n} \alpha_k a^k$. Moreover, if $q(z) \neq 0$ for all $z \in \mathrm{sp}_A(a)$, then $\frac{1}{q} \in \mathcal{O}(\mathrm{sp}_A(a))$, and by (HFC1),

$$1 = \left(q \cdot \frac{1}{q}\right)(a) = q(a) \cdot \left(\frac{1}{q}\right)(a).$$

This means we can write the inverse of $q(a)$ in two ways: (1) using Construction 1.6.2 negating all the multiplicities, and (2) as $\left(\frac{1}{q}\right)(a)$. Since inverses are unique, these two definitions must be equal. Again using (HFC1), for any rational function p/q where q is a polynomial which does not vanish on $\mathrm{sp}_A(a)$, we see that

$$\left(\frac{p}{q}\right)(a) = \frac{p(a)}{q(a)}$$

agrees with the definition from Construction 1.6.2. $\square$

Theorem 1.6.9. *Properties (HFC1) and (HFC2) uniquely characterize the HFC. That is, if $\Phi : \mathcal{O}(\mathrm{sp}_A(a)) \to A$ is another unital algebra homomorphism such that*

(Φ1) $\Phi(z \mapsto z) = a$, *and*

(Φ2) *If $\mathrm{sp}_A(a) \subset U$ and $(f_n) \subset H(U)$ with $f_n \to f$ locally uniformly, then $\Phi(f_n) \to \Phi(f)$ in norm in A,*

then $\Phi(f) = f(a)$ for all $f \in \mathcal{O}(\mathrm{sp}_A(a))$.

Proof. An argument similar to Corollary 1.6.8 shows that (Φ1) implies that $\Phi\left(\frac{p}{q}\right) = \left(\frac{p}{q}\right)(a)$ for every rational function with poles off of $\mathrm{sp}_A(a)$. For arbitrary $f \in \mathcal{O}(\mathrm{sp}_A(a))$, pick an open set U such that $f \in H(U)$. By normality, there is an open set V with compact closure such that $\mathrm{sp}_A(a) \subset V \subset \overline{V} \subset U$. By Runge's Theorem 1.7.4, there is a sequence of rational functions $(f_n) \subset H(U)$ with $f_n \to f$ locally uniformly on V. We conclude that

$$\Phi(f) \underset{(\Phi 2)}{=} \lim \Phi(f_n) = \lim f_n(a) \underset{(\text{HFC2})}{=} f(a). \qquad \square$$

Theorem 1.6.10 (Spectral mapping). *If $a \in A$ and $f \in \mathcal{O}(\mathrm{sp}_A(a))$, then $\mathrm{sp}_A(f(a)) = f(\mathrm{sp}_A(a))$.*

$\subseteq$: *Proof.* If $\lambda \notin f(\mathrm{sp}_A(a))$, then $g(z) := (f(z) - \lambda)^{-1} \in \mathcal{O}(\mathrm{sp}_A(a))$. Then

$$g(a) \cdot (f - \lambda)(a) = g(a)(f(a) - \lambda) = 1,$$

so $\lambda \notin \mathrm{sp}_A(f(a))$.

$\supseteq$: If $\lambda \in \mathrm{sp}_A(a)$, then there is a $g \in \mathcal{O}(\mathrm{sp}_A(a))$ such that $f(z) - f(\lambda) = (z - \lambda)g(z)$. If $f(\lambda) \notin \mathrm{sp}_A(f(a))$, then

$$1 = (z - \lambda)g(z) \cdot \frac{1}{f(z) - f(\lambda)} \qquad \Longrightarrow \qquad 1 = (a - \lambda)g(a) \cdot \frac{1}{f(a) - f(\lambda)},$$

a contradiction. $\square$

Corollary 1.6.11. *If $a \in A$, $f \in \mathcal{O}(\mathrm{sp}_A(a))$, and $g \in \mathcal{O}(f(\mathrm{sp}_A(a)))$, then $(g \circ f)(a) = g(f(a))$.*

Proof. The map $f^* : \mathcal{O}(\mathrm{sp}_A(f(a))) \to \mathcal{O}(\mathrm{sp}(a))$ given by $g \mapsto g \circ f$ is a unital algebra homomophism such that

- $(\mathrm{id} : z \mapsto z) \mapsto (f : z \mapsto f(z))$, and
- if $g_n \to g$ locally uniformly on $U \supset \mathrm{sp}_A(f(a))$, then $g_n \circ f \to g \circ f$ locally uniformly on $f^{-1}(U) \supset \mathrm{sp}_A(a)$.

Thus the composite $\Phi : \mathcal{O}(\mathrm{sp}_A(f(a))) \to \mathcal{O}(\mathrm{sp}(a)) \to A$ given by $g \mapsto g \circ f \mapsto (g \circ f)(a)$. satisfies both ($\Phi$1) and ($\Phi$2) for $f(a) \in A$. By Theorem 1.6.9, we conclude $(g \circ f)(a) = g(f(a))$. $\square$

We end this section with some applications of the HFC.

Proposition 1.6.12. *If $0_{\mathbb{C}}$ is in the unbounded component of $\mathbb{C} \setminus \mathrm{sp}_A(a)$, then a has a logarithm in A.*

Proof. Take a simple curve $\gamma \in \mathbb{C} \setminus \operatorname{sp}_A(a)$ connecting $0_\mathbb{C}$ and ∞. Then $\mathbb{C} \setminus \gamma$ is simply connected and open, and does not contain $0_\mathbb{C}$. Hence there is an $f \in H(\mathbb{C} \setminus \gamma)$ such that $\exp(f(z)) = z$. Then $f(a) \in A$ and $\exp(f(a)) = a$. □

Construction 1.6.13. Suppose $\operatorname{sp}_A(a)$ is a disjoint union $K_1 \amalg K_2$ of non-empty compact sets K_1, K_2. Since $\mathbb{C}$ is normal, there are disjoint open sets U_1, U_2 with $K_i \subset U_i$. Then $\operatorname{sp}_A(a) \subset U := U_1 \amalg U_2$, and

$$\chi_{U_1}(z) = \begin{cases} 1 & \text{if } z \in U_1 \\ 0 & \text{if } z \in U_2 \end{cases}$$

is holomorphic on U. Then $e := \chi_{U_1}(a)$ is a nontrivial *idempotent*, i.e., $0 \neq e \neq 1$ and $e^2 = e$. Moreover, $[a, e] = 0$, as $z\chi_{U_1}(z) = \chi_{U_1}(z)z$ for all $z \in U$. By the Spectral Mapping Theorem 1.6.10, $\operatorname{sp}_A(ae) = K_1 \cup \{0\}$.

Remark 1.6.14. If X is a Banach space and $T \in B(X)$ with $\operatorname{sp}_{B(X)}(T) = K_1 \amalg K_2$ as in Construction 1.6.13, setting $Y := eX$ and $Z := (1-e)X$, we have that (Y, Z) are complementary subspaces of X, i.e., $X = Y \oplus Z$. Moreover, since $[e, T] = 0$ and $[1-e, T] = 0$, both Y, Z are T-invariant.

Question 1.6.15. *Does every operator in $B(H)$ for a Hilbert space H have a non-trivial invariant subspace? (This fails for Banach spaces due to an example of Enflo.)*

1.7 Dependence of the spectrum on the algebra

Example 1.7.1. Consider the inclusion of the disk algebra $A(\mathbb{D})$ into $C(S^1)$. The identity function $\mathrm{id} : z \mapsto z$ is invertible in $C(S^1)$, but not in $A(\mathbb{D})$. In fact,

$$\operatorname{sp}_{C(S^1)}(\mathrm{id}) = S^1 = \partial\mathbb{D} \subset \overline{\mathbb{D}} = \operatorname{sp}_{A(\mathbb{D})}(\mathrm{id}).$$

Indeed, if $\lambda \in \overline{\mathbb{D}}$, then $\mathrm{id} - \lambda$ is not invertible as it vanishes on $\overline{\mathbb{D}}$, so $\overline{\mathbb{D}} \subseteq \operatorname{sp}_{A(\mathbb{D})}(\mathrm{id})$. Conversely, since $\|\mathrm{id}\|_{A(\mathbb{D})} = 1$, we know $\operatorname{sp}_{A(\mathbb{D})}(\mathrm{id}) \subset \overline{\mathbb{D}}$.

Definition 1.7.2. For $K \subset \mathbb{C}$ compact, the *polynomially convex hull* of K is defined as

$$\operatorname{conv}_{\mathrm{poly}}(K) := \{z \in \mathbb{C} \,|\, |p(z)| \leq \|p\|_K \text{ for all polynomials } p\}.$$

We say K is *polynomially convex* if $K = \operatorname{conv}_{\mathrm{poly}}(K)$.

Clearly $K \subseteq \operatorname{conv}_{\mathrm{poly}}(K)$. But $\operatorname{conv}_{\mathrm{poly}}(K)$ can be quite different from K.

Example 1.7.3. Observe that $\operatorname{conv}_{\mathrm{poly}}(S^1) = \overline{\mathbb{D}}$ by the Maximum Modulus Principle (CA8).

For the next proposition, for a bounded set $S \subset \mathbb{C}$, we will say that a *hole* of B is a bounded connected component of S^c.

Proposition 1.7.4. *For $K \subset \mathbb{C}$ compact, $\operatorname{conv}_{\mathrm{poly}}(K)^c$ is the unbounded component of K^c. Thus K is polynomially convex if and only if K^c is connected, i.e., K has no holes.*

Proof. Enumerate $\pi_0(K^c) = \{U_n\}_{n\geq 0}$ where U_0 is the unbounded component, and set

$$L := K \cup \bigcup_{n\geq 1} U_n.$$

Note that for $n \geq 1$, each U_n is a bounded open set and $\partial U_n \subseteq K$. By the Maximum Modulus Principle (CA8), $U_n \subseteq \operatorname{conv}_{\text{poly}}(K)$, so $L \subset \operatorname{conv}_{\text{poly}}(K)$.
Conversely, if $\lambda \in U_0$, then $f_\lambda(z) := (\lambda - z)^{-1}$ lies in $\mathcal{O}(L)$. Since U_0 is connected, by Runge's Theorem and Remark 1.4.14, we can uniformly approximate f_λ on L by a sequence of polynomials (p_n). Then the sequence of polynomials $q_n := p_n \cdot (z - \lambda)$ converges uniformly on L to 1. If $N > 0$ such that $\|q_n - 1\| < 1/2$ for all $n \geq N$, then since $K \subseteq L$ and

$$|(q_N - 1)(\lambda)| = 1 > \|q_N - 1\|_K,$$

we have $\lambda \notin \operatorname{conv}_{\text{poly}}(K)$. □

Proposition 1.7.5. *Suppose* $1 \in A \subset B$ *is a unital inclusion of Banach algebras and* $a \in A$.

(sp1) $\operatorname{sp}_B(a) \subseteq \partial \operatorname{sp}_A(a)$,

(sp2) $\partial \operatorname{sp}_A(a) \subseteq \partial \operatorname{sp}_B(a)$, *and*

(sp3) $\operatorname{conv}_{\text{poly}}(\operatorname{sp}_A(a)) = \operatorname{conv}_{\text{poly}}(\operatorname{sp}_B(a))$.

Proof. The first statement (sp1) is (1.2.5).
To prove (sp2), suppose for contradiction that $\lambda \in \partial \operatorname{sp}_A(a) \cap \operatorname{sp}_B(a)^c$. Pick a sequence $(\lambda_n) \subset \operatorname{sp}_A(a)^c$ such that $\lambda_n \to \lambda$, so $a - \lambda_n \to a - \lambda$. Then $a - \lambda_n \in A^\times$, so $a - \lambda_n \in B^\times$, and thus $\lambda_n \notin \operatorname{sp}_B(a)$ for all n. Since we assumed $\lambda \notin \operatorname{sp}_B(a)$ and inversion is continuous on $B^\times$, we have $(a - \lambda_n)^{-1} \to (a - \lambda)^{-1} \in B$. But A is complete, so $(a - \lambda)^{-1} \in A$, a contradiction.
For (sp3), for any polynomial p, (sp1) implies that $\|p\|_{\operatorname{sp}_B(a)} \leq \|p\|_{\operatorname{sp}_A(a)}$. But (sp2) together with the Maximum Modulus Principle (CA8) imply that

$$\|p\|_{\operatorname{sp}_A(a)} \leq \|p\|_{\operatorname{sp}_B(a)}.$$

Hence $z \in \operatorname{conv}_{\text{poly}}(\operatorname{sp}_A(a))$ if and only if $z \in \operatorname{conv}_{\text{poly}}(\operatorname{sp}_A(a))$. □

Corollary 1.7.6. *Suppose* $1 \in A \subset B$ *is a unital inclusion of Banach algebras and* $a \in A$. *For each hole* $H \subset \operatorname{sp}_B(a)^c$, *either* $H \subset \operatorname{sp}_A(a)$ *or* $H \cap \operatorname{sp}_A(a) = \emptyset$.

Proof. Set $H_1 := H \cap \operatorname{sp}_A(a)$ and $H_2 := H \setminus \operatorname{sp}_A(a)$ so that $H = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$. Clearly H_2 is open. Since $\partial \operatorname{sp}_A(a) \subseteq \operatorname{sp}_B(a)$ by (sp2) and $H \cap \operatorname{sp}_B(a) =$

$\emptyset$, we must have $H_1 = H \cap \mathrm{sp}_A(a)^\circ$, which is also open. But H is connected, so one of H_1 or H_2 must be empty. $\square$

The previous corollary tells us that $\mathrm{sp}_A(a)$ is obtained from $\mathrm{sp}_B(a)$ by filling in some of the holes. This leads to the following obvious question: how do we fill in some holes and perhaps not others? Can we fill in all the holes? We will answer the second question and leave the first an an exercise.

Example 1.7.7. Suppose B is a unital Banach algebra and $a \in B$. Letting $A \subseteq B$ be the norm-closure of the space of polynomials in a,

$$\mathrm{sp}_A(a) = \mathrm{conv}_{\mathrm{poly}}(\mathrm{sp}_B(a)).$$

To see this, we already know that

$$\mathrm{sp}_B(a) \underset{(1.2.5)}{\subseteq} \mathrm{sp}_A(a) \subseteq \mathrm{conv}_{\mathrm{poly}}(\mathrm{sp}_A(a)) \underset{(\mathrm{sp3})}{=} \mathrm{conv}_{\mathrm{poly}}(\mathrm{sp}_B(a)).$$

Suppose for contradiction that $\lambda \in \mathrm{conv}_{\mathrm{poly}}(\mathrm{sp}_B(a))$, but $\lambda \notin \mathrm{sp}_A(a)$. Then $f_\lambda(z) := (\lambda - z)^{-1}$ lies in $\mathcal{O}(\mathrm{sp}_A(a))$, so $f_\lambda(a) \in A \subseteq B$. By definition, there is a sequence of polynomials (p_n) such that $p_n(a) \to f_\lambda(a)$ in norm in B. As in the proof of Proposition 1.7.4, defining $q_n(z) := (z - \lambda)p_n(z)$, we have $q_n - 1 \to 0$. But then

$$\begin{aligned}
\|q_n(a) - 1\| &\geq r(q_n(a) - 1) \\
&= \sup\left\{|z - 1| \middle| z \in \mathrm{sp}_B(q_n(a))\right\} \\
&= \sup\left\{|q_n(z) - 1| \middle| z \in \mathrm{sp}_B(q_n(a))\right\} && \text{(Spectral Mapping Thm. 1.6.10)} \\
&= \|q_n - 1\|_{C(\mathrm{sp}_B(a))} \\
&\geq |q_n(\lambda) - 1| && (\lambda \in p\,\mathrm{conv}(\mathrm{sp}_B(a))) \\
&= 1,
\end{aligned}$$

a contradiction.

1.8 Gelfand theory

Given a unital *commutative* Banach algebra A, we will construct a canonical compact Hausdorff space $\widehat{A}$ together with a continuous unital algebra homomorphism $\widehat{\cdot} : A \to C(\widehat{A})$. Without loss of generality, we assume $\|1_A\| = 1$.

Definition 1.8.1. A *multiplicative linear functional* or *(algebra) character* on A is a non-zero linear map $\varphi : A \to \mathbb{C}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. The set of characters is denoted $\widehat{A}$.

Example 1.8.2. If X is compact Hausdorff and $x \in X$, then $\mathrm{ev}_x : C(X) \to \mathbb{C}$ given by $\mathrm{ev}_x(f) := f(x)$ is a character. Thus $X \subseteq \widehat{C(X)}$.

Remark 1.8.3. Suppose $\varphi \in \widehat{A}$. Observe that for all $a \in A$, $\varphi(a) = \varphi(a \cdot 1) = \varphi(a) \cdot \varphi(1)$. Since $\varphi \neq 0$, we must have that $\varphi(1) = 1$.

We can also discuss characters on A in terms of maximal 2-sided ideals of A. If A is a unital Banach algebra and $J \subset A$ is a proper ideal, then for all $a \in J$, $\|a - 1\| \geq 1$. Indeed, If $\|1 - a\| < 1$, then $a \in A^\times$ by (×3). It immediately follows that:

- $\overline{J} \subset A$ is also a proper ideal, and
- all maximal ideals are closed.

Moreover, for every maximal ideal $M \subset A$, M/A is again a Banach algebra which is also a division ring, and thus $A/M \cong \mathbb{C}$ by the Gelfand-Mazur Theorem 1.2.8. Summarizing, we have the following immediate proposition.

Proposition 1.8.4. *The map $\varphi \mapsto \ker(\varphi)$ gives a bijection*

$$\widehat{A} \longrightarrow \{\text{maximal ideals of } A\}$$

with inverse $M \longmapsto (a \mapsto a + M \in A/M \cong \mathbb{C})$.

Lemma 1.8.5. *If $\varphi \in \widehat{A}$, then $\|\varphi\| = 1$.*

Proof. By Proposition 1.8.4, $\ker(\varphi)$ is closed, so $\varphi \in A^*$. More constructively, if $\varphi(a) \neq 0$, then $1 - \varphi(a)^{-1}a \in \ker(\varphi)$, and thus by (×3),

$$1 \leq \|1 - (1 - \varphi(a)^{-1}a)\| = \frac{\|a\|}{|\varphi(a)|} \qquad \Longrightarrow \qquad |\varphi(a)| \leq \|a\|.$$

This implies $\|\varphi\| \leq 1$, and since $\varphi(1_A) = 1 = \|1_A\|$, we conclude $\|\varphi\| = 1$. □

Lemma 1.8.6. *$\widehat{A} \subset A^*$ is compact in the relative weak* topology.*

Proof. By the Banach-Alaoglu Theorem, it suffices to prove $\widehat{A}$ is closed. If $(\varphi_i) \subset \widehat{A}$ with $\varphi_i \to \varphi$ weak*, then for all $a, b \in A$,

$$\varphi(ab) = \lim_i \varphi_i(ab) = \lim_i \varphi_i(a)\varphi_i(b) = \varphi(a)\varphi(b).$$ □

Exercise 1.8.7. Suppose A is commutative Banach algebra which might be non-untial and $A_1 = A \oplus \mathbb{C}1$.

1. Prove that for all $\varphi \in \widehat{A}$, the non-zero multiplicative linear functionals, there is a unique $\varphi_1 \in \widehat{A_1}$ such that $\varphi_1|_A = \varphi$.
2. Observe that if $\varphi \in \widehat{A_1}$, then either $\varphi|_A = 0$ or $\varphi|_A \in \widehat{A}$.
3. Deduce that the map $\iota : \widehat{A} \to \widehat{A_1}$ by $\varphi \mapsto \varphi_1$ hits all but one element of $\widehat{A_1}$.
4. Prove that $\widehat{A_1}$ is the one point compactification of $\widehat{A}$, i.e., the relative topology on $\iota(\widehat{A})$ in $\widehat{A_1}$ is the relative weak* topology.

Lemma 1.8.8. *For a unital commutative Banach algebra A and $a \in A$, the following are equivalent.*

1. $a \notin A^\times$,

2. there is a maximal ideal $M \subset A$ such that $a \in M$, and

3. there is a $\varphi \in \widehat{A}$ such that $\varphi(a) = 0$.

(1)⇒(2): *Proof.* If $a \notin A^\times$, then $Aa \subset A$ is a non-trivial ideal which is contained in a maximal ideal by Zorn's Lemma.

(2)⇒(3): The map $A \to A/M \cong \mathbb{C}$ given by $x \mapsto x + M$ works.

¬(1)⇒ ¬(3): If $a \in A^\times$, then for all $\varphi \in \widehat{A}$, $1 = \varphi(a) \cdot \varphi(a^{-1})$, so $\varphi(a) \neq 0$. □

Corollary 1.8.9. *For all $a \in A$,* $\operatorname{sp}_A(a) = \left\{\varphi(a) \middle| \varphi \in \widehat{A}\right\}$.

Proof. By the previous proposition, $\lambda \in \operatorname{sp}_A(a)$ if and only if $\lambda - a \notin A^\times$ if and only if there is a $\varphi \in \widehat{A}$ such that $\lambda = \varphi(a)$. □

Construction 1.8.10 (Gelfand transform)**.** Suppose A is a unital commutative Banach algebra. The map $\hat{a} := \operatorname{ev}_a : \widehat{A} \to \mathbb{C}$ given by $\hat{a}(\varphi) := \varphi(a)$ is continuous as $\widehat{A}$ has the relative weak* topology. We thus get a unital algebra homomorphism by

$$\hat{\cdot} : A \longrightarrow C(\widehat{A}) \qquad \text{by} \qquad a \longmapsto \hat{a}$$

called the *Gelfand transform* which is norm-continuous as

$$\|\hat{a}\| = \sup\left\{|\varphi(a)| \middle| \varphi \in \widehat{A}\right\} \underset{\text{(Cor. 1.8.9)}}{=} \sup\left\{|\lambda| \middle| \lambda \in \operatorname{sp}_A(a)\right\} \underset{\text{(Prop. 1.3.3)}}{=} r(a) \leq \|a\|.$$

We thus see that the kernel of the Gelfand transform is exactly the ideal of *quasi-nilpotent* elements, i.e., the $a \in A$ such that $\operatorname{sp}_A(a) = \{0\}$.

Remark 1.8.11. The relative weak* topology on $\widehat{A}$ is the weakest topology such that each $\hat{a} : \widehat{A} \to \mathbb{C}$ is continuous. Indeed, the weak* topology on $\widehat{A}$ is determined by $\varphi_i \to \varphi$ if and only if $\hat{a}(\varphi_i) = \varphi_i(a) \to \varphi(a) = \hat{a}(\varphi)$ for each $a \in A$.

Theorem 1.8.12 (Gelfand)**.** *Suppose X is compact Hausdorff.*

1. *For each $\varphi \in \widehat{C(X)}$, $\varphi(\overline{f}) = \overline{\varphi(f)}$.*

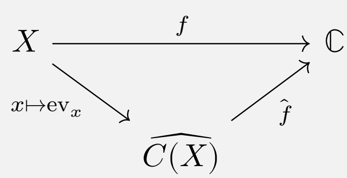
2. *The map $X \to \widehat{C(X)}$ given by $x \mapsto (\mathrm{ev}_X : f \mapsto f(x))$ is a homeomorphism.*

3. *The Gelfand transform $f \mapsto \hat{f}$ is an isometric isomorphism.*

Proof. To prove (1), we observe that if $f \in C(X)$ is real-valued, then $\mathrm{sp}(f) \subset \mathbb{R}$, and thus $\varphi(f) \in \mathbb{R}$. Writing $f = \Re(f) + i\Im(f)$, we see that

$$\overline{\varphi(f)} = \overline{\underbrace{\varphi(\Re(f))}_{\in\mathbb{R}} + i\underbrace{\varphi(\Im(f))}_{\in\mathbb{R}}} = \varphi(\Re(f)) - i\varphi(\Im(f)) = \varphi(\Re(f) - i\Im(f)) = \varphi(\overline{f}).$$

To prove (2), since X is compact and $\widehat{C(X)}$ is Hausdorff, it suffices to prove the map $x \mapsto \mathrm{ev}_x$ is a continuous bijection. For continuity, observe that if $x_i \to x$ in X, then for all $f \in C(X)$, $f(x_i) \to f(x)$, and thus $\mathrm{ev}_{x_i} \to \mathrm{ev}_x$ weak*. Injectivity follows by Urysohn's Lemma. For surjectivity, if $\varphi \in \widehat{C(X)}$, then $\ker(\varphi) \subset C(X)$ is a complex subalgebra closed under complex conjugation by (1). Moreover, it separates points by Urysohn's Lemma, so it must be all of $C(X)$ by the Stone-Weierstrass Theorem.
To prove (3), the map



clearly commutes, establishing the isomorphism. To see it is isometric, observe that

$$\|\hat{f}\| = \sup_{\varphi \in \widehat{C(X)}} |\varphi(f)| = \sup_{x \in X} |f(x)| = \|f\|. \qquad \square$$

Exercise 1.8.13. Use Exercise 1.8.7 to prove that Theorem 1.8.12 holds for LCH spaces X replacing $C(X)$ with $C_0(X)$.