

4 Projections and factors

For this section, H is a Hilbert space and $M \subseteq B(H)$ is a von Neumann algebra. We denote the set of projections of M by $P(M)$ and the group of unitaries in M by $U(M)$.

4.1 Compressions and ideals

Theorem 4.1.1. *Suppose $p \in P(M)$. Then $pMp \subseteq B(pH)$ is a von Neumann algebra with commutant $(pMp)' = M'p$.*

Proof. Clearly $pMp \subseteq B(pH)$ is an SOT-closed unital $*$ -subalgebra and thus a von Neumann algebra.

If $y \in M'$, then for all $x \in M$,

$$(yp)(pxp) = ypxp = pyxp = (pxp)(yp)$$

so $yp \in (pMp)'$. For the converse, we use a clever trick. First, it suffices to prove every unitary in $(pMp)'$ lies in $M'p$, as every element of $(pMp)'$ is a linear combination of 4 unitaries (why?). Suppose $u \in (pMp)'$ and set $K := \overline{MpH}$. Since K is both M and M' -invariant, $p_K \in M' \cap M = Z(M)$.

Claim. *We may extend u to K by $\tilde{u} \sum x_i p \xi := \sum x_i u p \xi$.*

Proof of claim. To see \tilde{u} is well-defined, we prove it is isometric:

$$\begin{aligned} \left\| \tilde{u} \sum x_i p \xi_i \right\|^2 &= \sum_{i,j} \langle x_i u p \xi_i, x_j u p \xi_j \rangle \\ &= \sum_{i,j} \langle p x_j^* x_i p u \xi_i, u \xi_j \rangle && ([u, p] = 0) \\ &= \sum_{i,j} \langle u p x_j^* x_i p \xi_i, u \xi_j \rangle && (u \in (pMp)') \\ &= \sum_{i,j} \langle p x_j^* x_i p \xi_i, \xi_j \rangle = \dots = \left\| \sum \sum x_i p \xi_i \right\|^2. \end{aligned} \quad \square$$

Now by construction, \tilde{u} commutes with the action of M on $K = \overline{MpH}$, and thus $\tilde{u} p_K \in M' \subseteq B(H)$; indeed, for all $x \in M$ and all $\xi \in H$,

$$\tilde{u} p_K x \xi = \tilde{u} x \underbrace{p_K \xi}_{\in K} = x \tilde{u} p_K \xi.$$

Finally, we claim that $u = \tilde{u}p_Kp \in M'p$; indeed, as $u = up \in M'p$, for all $\xi \in H$,

$$\tilde{u}p_Kp\xi = \tilde{u}(1_Mp\xi) = up\xi = u\xi. \quad \square$$

Definition 4.1.2. We call pMp , $M'p$ corners, compressions, or reductions of M , M' respectively.

Lemma 4.1.3. *If $J \subseteq M$ is a σ -WOT closed left ideal, then $J = Mp$ for a unique projection $p \in M$.*

Proof. If p is any projection such that $J = Mp$, then since $(xp)p = xp$ for all $x \in M$, $yp = y$ for all $y \in J$. It follows that if $Mq = J = Mp$, then $p \leq q$ and $q \leq p$, so $p = q$. This also tells us how to construct p : find the largest projection in J .

If $x \in J$, then so are $|x| = u^*x$ and $\chi_{[\varepsilon, \|x\|]}(|x|)$ for all $\varepsilon > 0$. Since $\chi_{[\varepsilon, \|x\|]}(|x|) \nearrow \text{supp}(|x|) = \text{supp}(x)$ as $\varepsilon \searrow 0$, $\text{supp}(x) \in J$.

Now observe that if there is a maximal projection p in J , then $p \geq \text{supp}(x)$ for all $x \in J$, so $x = x \cdot \text{supp}(x) \cdot p = xp$ for all $x \in J$. We thus have $J \subseteq Mp \subseteq J$, and thus equality holds.

Finally, to construct the maximal projection, since J is σ -WOT closed, it is a norm-closed left ideal, and thus contains a right approximate identity (e_i) such that $0 \leq e_i \leq 1$ for all i , $i \leq j$ implies $e_i \leq e_j$, and $\|x - xe_i\| \rightarrow 0$ for all $x \in J$. Since J is σ -WOT closed, $p := \bigvee e_i \in J$, which is automatically self-adjoint. Since $\|p - pe_i\| \rightarrow 0$, we see that $p = p^2$, so p is a projection, and since $\|x - xe_i\| \rightarrow 0$, $x = xp$ for all $x \in J$. We conclude that p is the largest projection in J . \square

Corollary 4.1.4. *A left ideal $J \subseteq M$ is SOT/WOT-closed if and only if it is σ -SOT/ σ -WOT closed.*

Proof. If J is σ -WOT closed, then $J = Mp$ for some projection $p \in J$, so J is WOT closed. The converse is trivial as WOT-closed sets are σ -WOT closed. \square

Corollary 4.1.5. *If $J \subseteq M$ is a σ -WOT closed 2-sided ideal, then $J = Mz$ for some projection $z \in Z(M)$.*

Proof. Since J is σ -WOT closed, it is also WOT and hence SOT-closed. By Lemma 4.1.3, $J = Mz$ for some projection $z \in M$. But as J is 2-sided, for every unitary $u \in M$, $J = uJu^*$. It follows that $J = uJu^* = uMzu^* = uMu^*(uzu^*) = Mu zu^*$, so $z = uzu^*$ by the uniqueness statement in Lemma 4.1.3. We conclude $z \in M' \cap M = Z(M)$. \square

4.2 Central support of a projection

Definition 4.2.1. A factor is a von Neumann algebra with trivial center, i.e., $Z(M) = M' \cap M = \mathbb{C}1$.

Remark 4.2.2. By Corollary 4.1.5, factors have no non-trivial σ -WOT closed 2-sided ideals.

Just as von Neumann algebras come in pairs M, M' , so do factors as $Z(M) = M' \cap M = Z(M')$.

Recall that for $p, q \in P(M) \subseteq B(H)$, $p \wedge q$ is the projection onto $pH \cap qH$ and $p \vee q$ is the projection onto $\overline{pH + qH}$. Observe we have the relation

$$p \vee q = 1 - (1 - p) \wedge (1 - q). \quad (4.2.3)$$

For homework, you will show that $p \wedge q, p \vee q \in M$. Thus $P(M)$ is a lattice under these operations.

Lemma 4.2.4. For $p, q \in P(M)$ and $u \in U(M)$, $u^*(p \vee q)u = u^*pu \vee u^*qu$ and $u^*(p \wedge q)u = u^*pu \wedge u^*qu$.

Proof. Observe that $\xi \in pH \cap qH$ if and only if $u^*\xi \in u^*puH \cap u^*quH$, and $\eta \perp pH \cap qH$ if and only if $u^*\eta \perp u^*puH \cap u^*quH$. Thus $u^*(p \wedge q)u = u^*pu \wedge u^*qu$. Now apply (4.2.3) to get $u^*(p \vee q)u = u^*pu \vee u^*qu$. \square

Definition 4.2.5. Given $p \in P(M)$, we define its central support

$$z(p) := \bigvee_{u \in U(M)} u^*pu := \text{lub } p_F$$

where $p_F := \bigvee_{u \in F} u^*pu$ for finite subsets $F \subset U(M)$, ordered by inclusion. By Lemma 4.2.4, for all $w \in U(M)$,

$$w^*p_Fw = \bigvee_{u \in F} w^*u^*puw = \bigvee_{v \in Fw} v^*pv = p_{Fw}.$$

As $z(p)$ is the SOT-limit of the p_F and multiplication is separately SOT-continuous,

$$w^*z(p)w = w^*(\lim^{SOT} p_F)w = \lim^{SOT} w^*p_Fw = \lim^{SOT} p_{Fw} = z(p).$$

This means $wz(p) = z(p)w$ for all $w \in U(M)$, so $z(p) \in M' \cap M = Z(M)$.

Lemma 4.2.6. Suppose $p \in P(M)$.

1. For $x \in M$, $xup = 0$ for all $u \in U(M)$ if and only if $xz(p) = 0$.
2. For $y \in M'$, $yp = 0$ if and only if $yz(p) = 0$. Hence the map $M'z(p) \rightarrow M'p$ given by multiplication by p is a $*$ -isomorphism.

Proof.

1. If $xup = 0$ for all $u \in U(M)$, then $xupu^* = 0$ for all such u . Then $xp_F = 0$ where $p_F = \bigvee_{u \in F} upu^*$ for any finite $F \subset U(M)$,^a and taking SOT limits, we have $xz(p) = x \lim^{SOT} p_F = 0$.

Conversely, if $xz(p) = 0$, then $x(upu^*)u = xz(p)(upu^*)u = 0$ for all $u \in U(M)$.

2. Since $yp = ypz(p) = yz(p)p$, $yz(p) = 0$ implies $yp = 0$. Conversely, if $yp = 0$, then $yupu^* = 0$ for all $u \in U(M)$ as $y \in M'$. The argument from (1) shows $yp_F = 0$ for all finite $F \subset U(M)$, so taking SOT limits, $yz(p) = 0$. \square

^aIf $u_1, \dots, u_n \in U(M)$ and $\xi_i \in u_i p u_i^* H$ for $i = 1, \dots, n$, then $x \sum \xi_i = \sum x \xi_i = \sum x u_i p u_i^* \xi_i = 0$, so $x p_F = 0$.

Proposition 4.2.7. *For a von Neumann algebra M and $p, q \in P(M) \setminus \{0\}$, the following are equivalent.*

1. $z(p)z(q) \neq 0$,
2. there is a $u \in U(M)$ such that $puq \neq 0$, and
3. there is a non-zero partial isometry $v \in M$ such that $vv^* \leq p$ and $v^*v \leq q$.

Proof.

$\neg(2) \Rightarrow \neg(1)$: If $puq = 0$ for all $u \in U(M)$, then $pz(q) = 0$ by Lemma 4.2.6(1). But then $0 = qz(p)u = quz(p)$ for all $u \in U(M)$, so by (the adjoint of) Lemma 4.2.6(1) again, $z(q)z(p) = 0$.

$(2) \Rightarrow (3)$: If $puq \neq 0$, consider the polar decomposition $puq = v|puq|$. By construction, $vv^*H = vH = \overline{puqH} \subset pH$, so $vv^* \leq p$. Since $\ker(v) = \ker(puq) \supset \ker(q)$, we have $v^*v = p_{\ker(v)^\perp} \leq p_{\ker(q)^\perp} = q$.

$(3) \Rightarrow (1)$: We prove that if $z(p)z(q) = 0$ and $v \in M$ is a partial isometry such that $vv^* \leq p$ and $v^*v \leq q$, then $v = 0$. Since $vv^* \leq p \leq z(p)$, $vv^* = vv^*z(p)$. Since $v^*v \leq q \leq z(q)$, $v^*v = v^*vz(q)$. Then

$$v = vv^*v = z(p)vv^*vz(q) = vz(q)z(p)v = 0. \quad \square$$

Corollary 4.2.8 (Ergodic property of factors). *Suppose M is a factor and $p, q \in P(M) \setminus \{0\}$. There is a unitary $u \in U(M)$ such that $puq \neq 0$.*

Proof. Since $p, q \neq 0$, $z(p) = z(q) = 1$. Now apply Proposition 4.2.7. \square

Corollary 4.2.9. *Suppose M is a factor and $p, q \in P(M) \setminus \{0\}$. There is a non-zero partial isometry $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. Moreover, we can find $u \in M$ such that $uu^* = p$ or $u^*u = q$.*

Proof. The first part is immediate as $z(p) = 1 = z(q)$. Consider the set of partial isometries $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. We can order this set by $u \leq v$ if $uu^* \leq vv^*$, $u^*u \leq v^*v$, and $v|_{uu^*H} = u$.

Claim. *Any increasing chain has an upper bound.*

Proof of Claim. If (v_i) is an increasing chain, then the operator $w : \bigcup v_i^*v_i H \rightarrow \bigcup v_j v_j^* H$ given by $\xi \mapsto v_k \xi$ whenever $\xi \in v_k^* v_k H$ is well-defined and unitary. It thus extends to an isometry $K := \overline{\bigcup v_i^* v_i H} \rightarrow H$, and thus to a partial isometry on H by defining $w|_{K^\perp} = 0$. Clearly $ww^* \leq p$, $w^*w \leq q$, and $v_i \leq w$ for all i . \square

We claim a maximal element satisfies $uu^* = p$ or $u^*u = q$. Indeed, if $p - uu^* \neq 0 \neq q - u^*u$, then there is a non-zero partial isometry $w \in M$ such that $ww^* \leq p - uu^*$ and $w^*w \leq q - u^*u$. Observe then that $u + w$ is a partial isometry (why?) with $(u + w)(u + w)^* \leq p$ and $(u + w)^*(u + w) \leq q$ contradicting maximality. \square

Exercise 4.2.10. Show that the central support $z(p)$ is the smallest projection in $Z(M)$ such that $p \leq z(p)$.

Corollary 4.2.11. $Z(pMp) = Z(M)p$.

Proof (Dixmier). Clearly $Z(M)p = p(M' \cap M)p \subset Z(pMp)$. Suppose $x \in Z(pMp) = pMp \cap M'p$. Then there is a $y \in M'$ such that $x = yp$. Since $p = z(p)p$, replacing y with $yz(p)$, we may assume $y = yz(p)$. We claim that $y \in Z(Mz(p))$ so that $y \in M' \cap Mz(p) \subset M' \cap M = Z(M)$. Indeed, the map $M'z(p) \rightarrow M'p$ given by multiplication by p is an isomorphism by Lemma 4.2.6(2), and thus maps the center onto the center. Since $yp = x \in Z(pMp)$, we conclude $y = yz(p) \in Z(Mz(p))$, as desired. \square

4.3 Classification of type I factors and their subfactors

Definition 4.3.1. A (nonzero) projection $p \in P(M)$ is called:

- minimal if $q \in P(M)$ with $q \leq p$ implies $q \in \{0, p\}$,
- abelian if pMp is abelian, and
- diffuse if there is no minimal projection $q \leq p$.

Examples 4.3.2. Here are examples of such projections.

1. The minimal projections in $B(H)$ are the rank 1 projections.
2. Every projection is diffuse in $L^\infty([0, 1], \lambda)$ where λ is Lebesgue measure.

Exercise 4.3.3. Suppose μ is a regular finite Borel measure on a compact Hausdorff space X . Show that the minimal projections of $L^\infty(X, \mu)$ correspond to atoms of X , i.e., $x \in X$ such that $\mu(\{x\}) > 0$.

Exercise 4.3.4. Suppose $p \in P(M)$ is minimal and $u \in M$ is a non-zero partial isometry such that $uu^* \leq p$. Show that $uu^* = p$ and that u^*u is a minimal projection.

Definition 4.3.5. A von Neumann algebra M is called type I if for all $z \in P(Z(M)) \setminus \{0\}$, there is an abelian $p \in P(M) \setminus \{0\}$ such that $p \leq z$, i.e., every non-zero central projection majorizes an abelian projection.

Examples 4.3.6. Examples of type I von Neumann algebras include abelian von Neumann algebras and $B(H)$.

Exercise 4.3.7. Here are some exercises on minimal projections.

1. $p \in P(M)$ is minimal if and only if $pMp = \mathbb{C}p$.
2. If M is a factor and p is abelian, then p is minimal.
3. If M is a factor, then M is type I if and only if M has a minimal projection.

Theorem 4.3.8 (Classification of type I factors). *If M is a type I factor acting on a Hilbert space H , there are Hilbert spaces K, L and a unitary $u \in B(K \otimes L \rightarrow H)$ such that $uMu^* = B(K) \otimes 1$.*

To prove this theorem, we will construct a system of matrix units for M , i.e., a family $\{e_{ij} | i, j \in I\}$ such that

- $e_{ij}^* = e_{ji}$,
- $e_{ij}e_{kl} = \delta_{j=k}e_{il}$, and
- $\sum e_{ii} = 1$ converging in SOT.

Lemma 4.3.9. *If $\{e_{ij}\}_{i,j \in I}$ is a system of matrix units in $B(H)$, then setting $K := e_{11}H$ which should be viewed as a ‘multiplicity space,’ there is a unitary $u: \ell^2 I \otimes K \rightarrow H$ such that $u^*e_{ij}u = |\delta_i\rangle\langle\delta_j| \otimes 1$ for all i, j . Thus $u^*(\{e_{ij}\}'')u = B(\ell^2 I) \otimes 1$, and $\dim(H) = |I| \dim(K)$.*

Proof. Let $\{\xi_j\}_{j \in J}$ be an ONB of $K = e_{11}H$. Since e_{1i} may be viewed as a unitary from $e_{ii}H$ onto $e_{11}H$, we see that $\{e_{i1}\xi_j | j \in J\}$ is an ONB for $e_{ii}H$. Since $H = \bigoplus e_{ii}H$, we see that $\{e_{i1}\xi_j | i \in I, j \in J\}$ is an ONB of H . Thus the map $u: \ell^2 I \otimes K \rightarrow H$ by $\delta_i \otimes \xi_j \mapsto e_{i1}\xi_j$ is a unitary isomorphism. Finally, we calculate

$$u^*e_{ij}u(\delta_k \otimes \xi_\ell) = u^*e_{ij}e_{k1}\xi_\ell = \delta_{j=k}u^*e_{i1}\xi_\ell = \delta_{j=k}(\delta_i \otimes \xi_\ell),$$

so $u^*e_{ij}u = |\delta_i\rangle\langle\delta_j| \otimes 1$ as claimed. \square

Remark 4.3.10. Observe that if $\{p_i\}$ is a family of mutually orthogonal projections such that $\sum p_i = 1$ SOT, and $\{e_{1j}\}_{j \neq 1}$ is a family of partial isometries such that $e_{1j}e_{1j}^* = p_1$, and $e_{1j}^*e_{1j} = p_j$, then setting $e_{11} := p_1$ and $e_{ij} := e_{1i}^*e_{1j}$ for all i, j with $i \neq 1$ completes $\{e_{1j}\}$ to a system of matrix units.

Proof of Theorem 4.3.8. Since M is a type I factor, it has a minimal projection p_1 . Let $\{p_i\}$ be a maximal family of mutually orthogonal minimal projections.

Claim. $\sum p_i = 1$ SOT.

Proof. Otherwise, by Corollary 4.2.9, there is a non-zero partial isometry $u \in M$ such that $uu^* \leq p_1$ and $u^*u \leq 1 - \sum p_i$, so $u^*u \perp p_i$ for all i . By minimality, $uu^* = p_1$, so u^*u is also minimal. Then $\{p_i\} \subsetneq \{p_i\} \cup \{u^*u\}$, contradicting maximality. \square

Now by Corollary 4.2.8, for each i , there is a non-zero partial isometry e_{1i} such that $e_{1i}e_{1i}^* \leq p_1$ and $e_{1i}^*e_{1i} \leq p_i$. By minimality, we must have $e_{1i}e_{1i}^* = p_1$ and $e_{1i}^*e_{1i} = p_i$. Setting $e_{ii} := p_i$ for all i , we can construct a system of matrix units $\{e_{ij}\}$ as in Remark 4.3.10.

Claim. $M = \{e_{ij}\}''$.

Proof. If $x \in M$, then $x = (\sum p_i)x(\sum p_j) = \sum_{ij} p_i x p_j$ SOT. But by minimality, each

$$p_i x p_j = e_{1i}^* e_{1i} x e_{1j}^* e_{1j} = e_{1i}^* \underbrace{p_1 e_{1i} x e_{1j} p_1}_{=: \lambda_{ij} p_1 \in \mathbb{C} p_1} e_{1j} = \lambda_{ij} e_{1i}^* p_1 e_{1j} = \lambda_{ij} e_{ij}.$$

Hence $x = \sum_{ij} \lambda_{ij} e_{ij}$, and $M = \{e_{ij}\}''$. \square

The final claim follows now from Lemma 4.3.9 \square

Definition 4.3.11. We say a type I factor M is type I_n if $M \cong B(H)$ with $\dim(H) = n$.

Fact 4.3.12. If u, v are two partial isometries with $uu^* \perp vv^*$ and $u^*u \perp v^*v$, then $u^*v = 0 = uv^*$ and $u + v$ is a partial isometry.

Corollary 4.3.13. *Suppose M, N are two type I subfactors of $B(H)$. Let $p \in M$ and $q \in N$ be minimal projections. The following are equivalent.*

1. *There is a unitary $u \in U(H)$ such that $u^*Mu = N$.*
2. *There are minimal $p \in P(M)$ and $q \in P(N)$ and a $u \in U(H)$ such that $u^*pu = q$.*
3. *There are minimal $p \in P(M)$ and $q \in P(N)$ and a partial isometry $v \in B(H)$ such that $vv^* = p$ and $v^*v = q$. (Note that this v is a unitary isomorphism between the multiplicity spaces pH and qH for M and N respectively.)*

Proof.

(1) \Rightarrow (2): If $p \in P(M)$ is minimal, then so is $u^*pu \in P(N)$.

(2) \Rightarrow (3): Take $v = pu$.

(3) \Rightarrow (1): Extend $\{p\}$ and $\{q\}$ to systems of matrix units $\{e_{ij}\}_{i,j \in I}$ for M with $e_{11} = p$ and $\{f_{k,\ell}\}_{k,\ell \in K}$ for N with $f_{11} = q$ respectively. Observe that for each $i \in I$ and $k \in K$,

$$(e_{i1}vf_{1k})(e_{i1}vf_{1k})^* = e_{i1} \underbrace{vqv^*}_{=p} e_{i1}^* = e_{ii} \quad \text{and} \quad (e_{i1}vf_{1k})^*(e_{i1}vf_{1k}) = f_{1k}^* \underbrace{v^*pv}_{=q} f_{1k} = f_{kk}.$$

Since $\sum e_{ii} = 1 = \sum f_{kk}$, we see that $|I| = |K|$, and we may identify the two index sets. By Fact 4.3.12, $u := \sum e_{i1}vf_{1i}$ is a unitary such that $uf_{ij}u^* = e_{ij}$ for all i, j . \square

4.4 Comparison of projections

Definition 4.4.1. For $p, q \in P(M)$, we say $p \preceq q$ if there is a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u \leq q$. We say $p \approx q$ if there is a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u = q$.

Example 4.4.2. For $x \in M$ and $x = u|x|$ the polar decomposition, $u \in M$ with $u^*u = \text{supp}(x)$ and $uu^* = \text{Range}(x)$. Hence $\text{supp}(x) \approx \text{Range}(x)$.

Example 4.4.3. Suppose u is a partial isometry such that $uu^* = p$. Then for all $q \leq p$, qu is a partial isometry such that $quu^*q = qpq = q$, so $u^*qu \approx q$.

Exercise 4.4.4. Show that \approx is an equivalence relation on $P(M)$ up to \approx .

Theorem 4.4.5. \preceq is a partial order on $P(M)$.

Proof.

reflexive: $p \preceq p$ via partial isometry p .

transitive: Suppose $uu^* = p$, $u^*u \leq q = vv^*$, and $v^*v \leq r$. Then

$$\begin{aligned} uvv^*u^* &= uqu^* = uu^*uqu^* = uu^*uu^* = uu^* = p \quad \text{and} \\ v^*u^*uv &\leq v^*qv = v^*vv^*v = v^*v \leq r. \end{aligned}$$

anti-symmetric: Suppose $p \preceq q$ and $q \preceq p$. Let $u, v \in M$ be partial isometries such that $uu^* = p$, $u^*u \leq q$, $vv^* = q$, and $v^*v \leq p$. Then for each $p' \leq p$,

$$u^*p'u \leq u^*pu = u^*uu^*u = u^*u \leq q,$$

and similarly, for each $q' \leq q$, $v^*q'v \leq p$. That is, we have order preserving maps

$$\{\text{projections} \leq p\} \xrightleftharpoons[\text{Ad}(v)]{\text{Ad}(u)} \{\text{projections} \leq q\}.$$

It immediately follows that inductively defining

$$\begin{aligned} p_{n+1} &:= v^*q_n v & p_0 &:= p \\ q_{n+1} &:= u^*p_n u & q_0 &:= q \end{aligned}$$

yields two decreasing sequences of projections in M . Define $p_\infty := \lim^{SOT} p_n = \bigwedge p_n$ and $q_\infty := \lim^{SOT} q_n = \bigwedge q_n$, the orthogonal projections onto $\bigcap p_n H$ and $\bigcap q_n H$ respectively. The clever trick here is to write $p = p_0$ and $q = q_0$ as telescoping sums of mutually orthogonal projections, which converge SOT:

$$\begin{array}{ccccccc} p &= & (p_0 - p_1) &+ & (p_1 - p_2) &+ & (p_2 - p_3) &+ & (p_3 - p_4) &+ & \cdots &+ & p_\infty \\ & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & & & \updownarrow \\ q &= & (q_0 - q_1) &+ & (q_1 - q_2) &+ & (q_2 - q_3) &+ & (q_3 - q_4) &+ & \cdots &+ & q_\infty \end{array}$$

We then pair up projections and sum up the partial isometries with orthogonal domains and ranges.

First, since multiplication is separately SOT-continuous,

$$v^*q_\infty v = v^*(\lim^{SOT} q_n)v = \lim^{SOT} v^*q_n v = \lim^{SOT} p_n = p_\infty.$$

Moreover, since $q_\infty \leq q$, $q_\infty = q_\infty q q_\infty = q_\infty v v^* q_\infty$. Hence $p_\infty \approx q_\infty$ via the partial isometry $q_\infty v$. Finally, observe that

$$\begin{aligned} \text{Ad}(u)(p_n - p_{n+1}) &= u^*(p_n - p_{n+1})u = u^*p_n u - u^*p_{n+1}u = q_{n+1} - q_{n+2} \\ \text{Ad}(v)(q_n - q_{n+1}) &= p_{n+1} - p_{n+2}. \end{aligned}$$

Thus $(p_n - p_{n+1})u$ is a partial isometry witnessing $p_n - p_{n+1} \approx q_{n+1} - q_{n+2}$, and $(q_n - q_{n+1})v$ is a partial isometry witnessing $q_n - q_{n+1} \approx p_{n+1} - p_{n+2}$. \square

Corollary 4.4.6. *If M is a factor, then \preceq is a total order up to \approx .*

Proof. This is a restatement of Corollary 4.2.9. □

Definition 4.4.7. A projection $p \in P(M)$ is called:

- finite if for all projections $q \leq p$, $q \approx p$ implies $q = p$.
- infinite if there is a $q \leq p$ with $q \neq p$ such that $q \approx p$ (not infinite). An infinite projection is called:
 - purely infinite if there is no non-zero finite $q \leq p$, and
 - properly infinite if for all $z \in P(Z(M))$ such that $zp \neq 0$, zp is infinite.

A von Neumann algebra M is called finite or (purely/properly) infinite if 1_M is respectively.

Exercise 4.4.8. Prove that abelian von Neumann algebras are finite. Deduce that p abelian implies p is finite.

Definition 4.4.9. A von Neumann algebra M is called:

- type III if M is purely infinite.
- type II if M has no abelian projections and any non-zero central projection majorizes a non-zero finite projection. In this case, we call M :
 - type II_1 if M is finite, and
 - type II_∞ if there is no non-zero finite central projection.

Remark 4.4.10. The above definition is rather hard to parse, so here is another way to say it. We will informally say that a von Neumann algebra M has *sufficiently many* projections with property (P) if every non-zero central projection of M majorizes a non-zero projection with property (P). Then M is:

- type I if M has sufficiently many abelian projections,
- type II if M has no abelian projections, but has sufficiently many finite projections. In this case, M is:
 1. type II_1 if M is finite and
 2. type II_∞ if has no non-zero finite central projections.
- type III if M has no abelian projections and no non-zero finite projections.

4.5 $L\Gamma$ is a II_1 factor when Γ is icc

Let Γ be a countable discrete group. Recall

$$L\Gamma := \{\lambda_g | g \in \Gamma\}'' \subset B(\ell^2\Gamma) \quad \text{where} \quad (\lambda_g \xi)(h) := \xi(g^{-1}h).$$

The functions $\delta_g(h) := \delta_{g=h}$ give a distinguished orthonormal basis of $\ell^2\Gamma$. Observe $\lambda_g \delta_h = \delta_{gh}$. We also have a right Γ action on $\ell^2\Gamma$ by $(\rho_g \xi)(h) := \xi(hg)$. Notice that $\rho_g \in U(\ell^2\Gamma) \cap L\Gamma'$.

Facts 4.5.1. We compute some basic properties about $L\Gamma$.

($L\Gamma 1$) For all $x \in L\Gamma$, there a sequence $(x_g) \in \ell^2\Gamma$ such that $x\delta_e = \sum x_g \delta_g$.

($L\Gamma 2$) For all $x \in L\Gamma$ and $h \in \Gamma$,

$$x\delta_h = x\rho_h \delta_e = \rho_h x\delta_e = \rho_h \sum x_g \delta_g = \sum_g x_g \delta_{gh} = \sum_g x_{gh^{-1}} \delta_g.$$

($L\Gamma 3$) $x^* \delta_e = \sum \overline{x_{g^{-1}}} \delta_g$ since for all $h \in \Gamma$,

$$\langle x^* \delta_e, \delta_h \rangle = \langle \delta_e, x\delta_h \rangle \stackrel{(L\Gamma 2)}{=} \sum \overline{x_{gh^{-1}}} \langle \delta_e, \delta_g \rangle = \overline{x_{h^{-1}}}.$$

($L\Gamma 4$) If $x\delta_e = \sum x_g \delta_g$ and $y\delta_e = \sum y_g \delta_g$, then $xy\delta_e = \sum_g \left(\sum_h x_h y_{h^{-1}g} \right) \delta_g$. Thus the convolution product $(x_g) * (y_h) \in \ell^2\Gamma$.

Proof. For all $g \in \Gamma$,

$$\langle xy\delta_e, \delta_g \rangle = \langle y\delta_e, x^* \delta_g \rangle \stackrel{(L\Gamma 3)}{=} \sum_{h,k} x_{h^{-1}} y_k \langle \delta_k, \rho_{g^{-1}} \delta_h \rangle = \sum_{h,k} x_{h^{-1}} y_k \delta_{k=hg},$$

which simplifies to $\sum_{h^{-1}} x_{h^{-1}} y_{hg}$. This is the claimed formula swapping h with h^{-1} as the index of summation. \square

($L\Gamma 5$) δ_e is a cyclic and separating vector for $L\Gamma$.

Proof. Clearly $\mathbb{C}[\Gamma]\delta_e \subset L\Gamma\delta_e$ is dense in $\ell^2\Gamma$, so δ_e is cyclic. If $x \in L\Gamma$ such that $x\delta_e = 0$, then $x\delta_g = \rho_{g^{-1}} x\delta_e = 0$ for all g , and $x = 0$. Thus δ_e is separating. \square

($L\Gamma 6$) $\text{tr} := \langle \cdot, \delta_e \rangle$ is a faithful σ -WOT continuous tracial state on $L\Gamma$ with $\text{tr}(x) = x_e$.

Proof. First, we have the tracial property as

$$\langle xy\delta_e, \delta_e \rangle \stackrel{(L\Gamma 4)}{=} \sum_h x_h y_{h^{-1}} = \sum_h y_h x_{h^{-1}} \stackrel{(L\Gamma 4)}{=} \langle yx\delta_e, \delta_e \rangle.$$

Next, $\text{tr}(x^*x) = \sum_g |x_g|^2 = 0$ if and only if $x_g = 0$ for all g if and only if $x = 0$, so tr is faithful. \square

(L Γ 7) All projections in $L\Gamma$ are finite.

Proof. Suppose $uu^* = p$ and $u^*u = q \leq p$. Then $\text{tr}(p-q) = \text{tr}(uu^*) - \text{tr}(u^*u) = 0$ which implies $p-q = 0$ as tr is faithful by (L Γ 6). \square

Example 4.5.2. If H is infinite dimensional, then $B(H)$ does not admit a trace.

Proposition 4.5.3. If Γ is icc (infinite and all nontrivial conjugacy classes infinite), then $L\Gamma$ is a II_1 factor.

Proof. If $z \in Z(L\Gamma)$, then

$$\sum z_g \delta_g = z\delta_e = \lambda_{h^{-1}} z \lambda_h \delta_e = \sum z_{h^{-1}gh} \delta_g,$$

so $(z_g) \in \ell^2\Gamma$ is constant on conjugacy classes. Since Γ is icc, $z_g = 0$ for $g \neq e$, so $z \in \mathbb{C}1$ by (L Γ 6), and $L\Gamma$ is a factor.

Since $L\Gamma$ is infinite dimensional and admits a trace, it cannot be type I by Exercise 4.5.2. Since $L\Gamma$ is finite by (L Γ 7) $L\Gamma$ is type II_1 . \square

4.6 II_1 factor basics

This subsection follows Jones' von Neumann algebra notes quite closely.

Above, we exploited the trace on $L\Gamma$ to prove Proposition 4.5.3. For this subsection, we assume a II_1 factor comes equipped with a σ -WOT continuous tracial state. We will construct such a trace in Corollary 4.8.5 below.

Facts 4.6.1. Here are some elementary facts about a factor M equipped with a tracial state tr , which is sometimes assumed to be faithful or σ -WOT continuous.

(tr1) A σ -WOT continuous tracial state on a factor M is faithful.

Proof. Let $J = \{x \in M \mid \text{tr}(x^*x) = 0\}$. Since $x^*y^*yx \leq \|y^*y\|x^*x$, J is a left ideal. But since tr is a trace, J is a 2-sided ideal. By Cauchy-Schwarz, $\text{tr}(x^*x) = 0$ if

and only if $\text{tr}(xy) = 0$ for all y , so

$$J = \bigcap_{y \in M} \ker(\underbrace{\text{tr}(\cdot y)}_{\sigma\text{-WOT cts}})$$

is σ -WOT closed. By Corollary 4.1.5, M has no non-trivial σ -WOT closed 2-sided ideals, so $\ker(\text{tr}) = 0$. \square

(tr2) If M is a factor with a faithful tracial state, then M is finite.

Proof. The proof of (LF7) applies verbatim. \square

(tr3) An infinite dimensional factor M with a σ -WOT continuous tracial state is type II_1 .

Proof. The second part of the proof of Proposition 4.5.3 applies verbatim. \square

(tr4) Suppose M is a factor and tr is faithful.

1. $p \preceq q$ if and only if $\text{tr}(p) \leq \text{tr}(q)$.
2. $p \approx q$ if and only if $\text{tr}(p) = \text{tr}(q)$.

Proof. For the forward direction, suppose $p = uu^*$ and $u^*u \leq q$. Then

$$\text{tr}(p) = \text{tr}(uu^*) = \text{tr}(u^*u) \leq \text{tr}(q)$$

with equality if and only if $q = u^*u$ as tr is faithful.

Conversely, suppose $\text{tr}(p) \leq \text{tr}(q)$. Since M is a factor, then $p \preceq q$ or $q \preceq p$. If $q \preceq p$, then by the forward step, $\text{tr}(q) \leq \text{tr}(p)$, in which case $\text{tr}(p) = \text{tr}(q)$ and $p = uu^*$ by faithfulness of tr . Thus $p \approx q$. \square

Lemma 4.6.2. *Suppose M is a II_1 factor with a faithful trace. For every non-zero $p \in P(M)$ and $0 < \varepsilon < \text{tr}(p)$, there is a $q \in P(M)$ with $0 \leq q \leq p$ and $0 < \text{tr}(q) < \varepsilon$.*

Proof. Let

$$\delta := \inf \{ \text{tr}(q) \mid q \in P(M) \setminus \{0\} \text{ such that } q \leq p \}.$$

If $0 < \delta \leq \text{tr}(p)$, there is a non-zero $q \in P(M)$ such that $q \leq p$ and $\text{tr}(q) < 2\delta$ by the definition of the inf. Since M is not type I, q is not minimal, so there is a non-zero projection $r \leq q$ with $0 \neq r \neq q$. Then $\delta \leq \text{tr}(r)$, but

$$\text{tr}(q - r) = \text{tr}(q) - \text{tr}(r) \leq \text{tr}(q) - \delta < 2\delta - \delta = \delta,$$

a contradiction. \square

Proposition 4.6.3. *Suppose M is a II_1 factor with a faithful trace. Then $\text{tr}(P(M)) = [0, 1]$.*

Proof. Fix $r \in (0, 1)$, and consider $\{p \in P(M) \mid 0 < \text{tr}(p) \leq r\}$ which is non-empty by Lemma 4.6.2. Ordering this set by \leq , every ascending chain has an upper bound, so by Zorn's Lemma, there is a maximal element p . Suppose for contradiction that $\text{tr}(p) < r$. Again by Lemma 4.6.2, there is a projection $q \leq 1 - p$ with $0 < \text{tr}(q) < r - \text{tr}(p)$. But then $p + q$ is a projection such that $\text{tr}(p) < \text{tr}(p) + \text{tr}(q) < r$, a contradiction. \square

Exercise 4.6.4. Give a better description of a projection of arbitrary trace in $[0, 1]$ in $L\mathbb{F}_2$ and LS_∞ .

Exercise 4.6.5. Let M be a II_1 factor with σ -WOT continuous tracial state tr .

1. Show that if $p \in M$ is a non-zero projection, then for every $0 < r < \text{tr}(p)$, there is a projection $q \in M$ with $q \leq p$ and $\text{tr}(q) = r$.
2. For every $n \in \mathbb{N}$, there is a unital subfactor $N \subseteq M$ with $N \cong M_n(\mathbb{C})$.
3. M is algebraically simple, i.e., M has no 2-sided ideals.

Proposition 4.6.6. *A finite von Neumann algebra M with a faithful σ -WOT continuous tracial state tr is a II_1 factor if and only if for any other σ -WOT continuous tracial state φ , $\varphi = \text{tr}$.*

Proof. Suppose M is a II_1 factor. It suffices to prove both traces agree on projections. By Exercise 4.6.5(2), the traces must agree on every subfactor $N \cong M_n(\mathbb{C})$ for all $n \in \mathbb{N}$. For an arbitrary projection $p \in M$, we can build a sequence (p_i) of mutually orthogonal projections such that $p = \sum p_i$ SOT (and thus also σ -WOT) and $\text{tr}(p_i) = \frac{1}{n_i}$ for some $n_i \in \mathbb{N}$ for every i using Exercise 4.6.5(1). Suppose now M is not a factor, and choose projection $z \in Z(M)$ such that $0 \neq z \neq 1$. Then $\varphi(x) := \frac{1}{\text{tr}(z)} \text{tr}(xz)$ is a σ -WOT continuous tracial state distinct from tr as $\varphi(1 - z) = 0 \neq \text{tr}(1 - z)$. \square

4.6.1 The hyperfinite II_1 factor

We now use Proposition 4.6.6 to construct a II_1 factor R which can be well approximated by finite dimensional subalgebras.

For $n \in \mathbb{N}$, let $A_n := \bigotimes^n M_2(\mathbb{C})$. Include $A_n \hookrightarrow A_{n+1}$ by $x \mapsto x \otimes 1$, and let $A_\infty := \varinjlim A_n = \bigotimes^\infty M_2(\mathbb{C})$. Since $A_n \cong M_{2^n}(\mathbb{C})$ has a unique normalized faithful tracial state tr_n , $\text{tr}_\infty := \varinjlim \text{tr}_n$ is the unique faithful trace on A_∞ , and it is positive definite in that $\text{tr}_\infty(x^*x) \geq 0$ for all $x \in A_\infty$ with equality if and only if $x = 0$. We can thus attempt to apply the GNS construction, where there are several things we must check along the way. We define H to be the completion of A_∞ in $\|\cdot\|_2$ under the sesquilinear form $\langle x, y \rangle := \text{tr}_\infty(y^*x)$. We write $\Omega \in H$ for the image of $1 \in A_\infty$ and $a\Omega \in H$ for the image of $a = a1 \in A_\infty$.

(R1) A_∞ acts faithfully on the left of H by bounded operators by $x(a\Omega) = xa\Omega$. We can thus define $R := (A_\infty)'' \subset B(H)$.

Proof. Since $x^*x \leq \|x^*x\|_{A_n}$ for all $x \in A_n$, and since the inclusions $A_n \hookrightarrow A_{n+k}$ are all injective and thus norm-preserving, we have

$$\|xa\Omega\|^2 = \text{tr}_\infty(a^*x^*xa) \leq \|x^*x\|_{A_n} \cdot \text{tr}_\infty(a^*a) = \|x\|_{A_n}^2 \cdot \|a\Omega\|^2.$$

Faithfulness of the action follows as Ω is separating for A_∞ by faithfulness of tr_∞ on A_∞ . \square

(R2) $\text{tr}_R(x) := \langle x\Omega, \Omega \rangle$ is a σ -WOT continuous tracial state on R such that $\text{tr}_R|_{A_\infty} = \text{tr}_\infty$.

Proof. For $x \in A_\infty$, $\text{tr}_R(x) = \langle x\Omega, \Omega \rangle = \text{tr}_\infty(x)$. Since tr_R is a vector state, it is both SOT-continuous and σ -WOT continuous. For $x, y \in R$, by the Kaplansky Density Theorem, we may pick bounded nets $(x_i), (y_i) \subset A_\infty$ with $x_i \rightarrow x$ and $y_i \rightarrow y$ SOT. Since multiplication is jointly SOT-continuous on bounded sets, $x_i y_i \rightarrow xy$ and $y_i x_i \rightarrow yx$ SOT. We thus have

$$\text{tr}_R(xy) = \lim^{\text{SOT}} \text{tr}_\infty(x_i y_i) = \lim^{\text{SOT}} \text{tr}_\infty(y_i x_i) = \text{tr}_R(yx). \quad \square$$

(R3) A_∞ acts on the *right* of H by bounded operators by $x(a\Omega) = ax\Omega$.

Proof. This is the step that uses that tr is a trace:

$$\begin{aligned} \|ax\Omega\|^2 &= \text{tr}_\infty(x^*a^*ax) = \text{tr}_\infty(axx^*a^*) \leq \|xx^*\|_{A_n} \cdot \text{tr}_\infty(aa^*) \\ &= \|x^*x\|_{A_n} \cdot \text{tr}_\infty(a^*a) = \|x\|_{A_n}^2 \cdot \|a\Omega\|^2. \end{aligned} \quad \square$$

(R4) tr_R is faithful on R so that R is a II_1 factor by Proposition 4.6.6.

Proof. Suppose $\text{tr}_R(x^*x) = 0$. Since the right A_∞ -action is bounded and commutes with the left A_∞ -action on H and thus also commutes with R , for all $a \in A_\infty$,

$$\|xa\Omega\|^2 = \|xR_a\Omega\|^2 = \|R_a x\Omega\|^2 \leq \|R_a\|^2 \cdot \|x\Omega\|^2 = \|R_a\|^2 \cdot \text{tr}_R(x^*x) = 0.$$

Since $A_\infty\Omega$ is dense in H , $x = 0$. \square

Exercise 4.6.7. Build a projection of arbitrary trace in $[0, 1]$ in R .

4.7 Useful results on comparison of projections

Our next task is to prove every finite von Neumann algebra admits a tracial state. We begin with some general results on projections in a von Neumann algebra. For this section, unless stated otherwise, M is a von Neumann algebra and $p, q \in P(M)$.

Facts 4.7.1. Here are some basic facts about comparison of projections.

(\preceq 1) (Kaplansky's formula) $p \vee q - p \approx q - p \wedge q$.

Proof. Consider $x = (1 - p)q$. Then $\ker(x) = \ker(q) \oplus (p \wedge q)H$, so

$$p_{\ker(x)} = (1 - q) + p \vee q \quad \text{and} \quad \text{Range}(x^*) = 1 - p_{\ker(x)} = q - p \wedge q.$$

Since $x = [(1 - (1 - q))(1 - p)]^*$, the above argument also tells us that

$$\text{Range}(x) = (1 - p) - (1 - p) \wedge (1 - q) = (1 - p - (1 - p \vee q)) = p \vee q - p.$$

Since $\text{Range}(x^*) = \text{supp}(x)$, these projections are equivalent by Example 4.4.2. \square

(\preceq 2) If $p_1 \preceq q_1$, $p_2 \preceq q_2$, and $q_1 q_2 = 0$, then $p_1 \vee p_2 \preceq q_1 + q_2$.

Proof. By (\preceq 1), $p_1 \vee p_2 - p_2 \approx p_1 - p_1 \wedge p_2 \preceq q_1$ so $p_1 \vee p_2 = (p_1 \vee p_2 - p_2) + p_2 \preceq q_1 + q_2$. \square

(\preceq 3) (Comparison Theorem) There is a $z \in P(Z(M))$ such that $pz \preceq qz$ and $q(1 - z) \preceq p(1 - z)$.

Proof. By Zorn's Lemma, there are maximal families of mutually orthogonal projections $\{p_i\}, \{q_i\}$ such that $\sum p_i \leq p$, $\sum q_i \leq q$, and $p_i \approx q_i$ for all i . Set $z_1 := z(p - \sum p_i)$ and $z_2 := z(q - \sum q_i)$. By maximality, $z_1 z_2 = 0$, so

$$\begin{aligned} \left(p - \sum p_i\right) \leq z_1 \leq 1 - z_2 &\implies z_2 \left(p - \sum p_i\right) = 0 \\ \left(q - \sum q_i\right) \leq z_2 &\implies (1 - z_2) \left(q - \sum q_i\right) = 0. \end{aligned}$$

Since $\sum p_i \approx \sum q_i$, we see

$$\begin{aligned} z_2 p &= z_2 \sum p_i \approx z_2 \sum q_i \leq z_2 q \\ (1 - z_2) q &= (1 - z_2) \sum q_i \approx (1 - z_2) \sum p_i \leq (1 - z_2) p. \end{aligned} \quad \square$$

(\preceq 4) If p, q are finite, so is $p \vee q$.

We omit the proof, which is quite technical. There is a much simpler proof when p, q are central in addition, which you will do on homework.

- ($\preceq 5$) If p, q are finite and $p \approx q$, then $1 - p \approx 1 - q$. Hence there is a $u \in U(M)$ such that $u^*pu = q$.

Remark 4.7.2. The proof below only uses ($\preceq 4$) to reduce to the case that M is finite. Since we will only use ($\preceq 5$) for finite von Neumann algebras, the rest of these notes is still self-contained without a proof of ($\preceq 4$) above.

Proof. By ($\preceq 4$), $p \vee q$ is finite, so replacing M by $(p \vee q)M(p \vee q)$, we may assume M is finite. By ($\preceq 3$), there is a central projection $z \in P(Z(M))$ such that $(1 - p)z \preceq (1 - q)z$ and $(1 - q)(1 - z) \preceq (1 - p)(1 - z)$. Since we can consider Mz and $M(1 - z)$ separately, we may assume $1 - p \approx r \leq 1 - q$. Since $1 = (1 - p) + p \approx r + q$, and M is finite, $r + q = 1$, so $1 - p \approx r = 1 - q$. Now if $vv^* = p$, $v^*v = q$ and $ww^* = 1 - p$, $w^*w = 1 - q$, then $u = v + w$ is a unitary satisfying $u^*pu = q$. \square

- ($\preceq 6$) Suppose $p, q \in P(M)$ finite with $p, q \leq r$.

($\preceq 6f$) If $p \approx q$, then $r - p \approx r - q$.

($\preceq 6f$) If $p \preceq q$, then $r - q \preceq r - p$.

Remark 4.7.3. Again, in the proof below, we will only use ($\preceq 4$) to pass to the case M is finite and $r = 1$.

Proof. Since $p, q \leq r$ implies $p \vee q \leq r$, passing to $(p \vee q)M(p \vee q)$, we may assume M is finite and $r = 1$ by ($\preceq 4$). Now ($\preceq 6$) follows immediately from ($\preceq 5$). For ($\preceq 6f$), let $s \in P(M)$ with $p \approx s \leq q$. By ($\preceq 5$) $1 - p \approx 1 - s \geq 1 - q$. \square

- ($\preceq 7$) If (q_n) is an increasing sequence of finite projections and $p \in P(M)$ such that $q_n \preceq p$ for all n , then $\bigvee q_n \preceq p$.

Proof. We inductively define a sequence of mutually orthogonal projections $p_n \leq p$ such that $p_0 = q_1$ and for all $n \in \mathbb{N}$, $p_n \approx q_{n+1} - q_n$. Then

$$\bigvee_{n=1}^{\infty} q_n = q_1 + \sum_{n=1}^{\infty} (q_{n+1} - q_n) \approx \sum_{n=0}^{\infty} p_n \leq p.$$

By assumption, $q_1 \preceq p$, so there is a $p_0 \leq p$ such that $q_1 \approx p_0$. Suppose we have p_0, p_1, \dots, p_n .

Claim. $q_{n+2} - q_{n+1} \preceq p - \sum_{i=0}^n p_i$.

Proof of Claim. Observe $q_{n+2} \preceq p$, so there is a partial isometry v such that $vv^* = q_{n+2}$ and $e_{n+2} := v^*v \leq p$. Since $q_{n+2} \geq q_{n+1}$,

$$e_{n+1} := v^*q_{n+1}v \leq v^*q_{n+2}v = v^*vv^*v = v^*v \leq p$$

and $e_{n+1} \approx q_{n+1}$. Then

$$v^*(q_{n+2} - q_{n+1})v = e_{n+2} - e_{n+1} \quad \text{and} \quad (q_{n+2} - q_{n+1})vv^*(q_{n+2} - q_{n+1}) = q_{n+2} - q_{n+1},$$

so $q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1}$. By the induction hypothesis,

$$e_{n+1} \approx q_{n+1} = (q_{n+1} - q_n) + (q_n - q_{n-1}) + \cdots + (q_2 - q_1) + q_1 \approx \sum_{i=0}^n p_i \leq p.$$

Since q_{n+2}, q_{n+1} are finite, so are $e_{n+2}, e_{n+1} \approx \sum_{i=0}^n p_i$. We calculate

$$q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1} = (p - e_{n+1}) - (p - e_{n+1}) \leq p - e_{n+1} \stackrel{(\preceq 6)}{\approx} p - \sum_{i=0}^n p_i,$$

proving the claim. \square

By the claim, we can find a projection $q_{n+2} - q_{n+1} \approx p_{n+1} \leq p - \sum_{i=0}^n p_i$, so we can inductively build the sequence as claimed. \square

- ($\preceq 8$) Suppose M is a finite von Neumann algebra and (p_n) is an infinite sequence of mutually orthogonal projections. Suppose (q_n) is another sequence of projections with $p_n \approx q_n$ for each n . Then $q_n \rightarrow 0$ SOT.

Proof. By induction using ($\preceq 2$), for all $m \leq n$,

$$\bigvee_{i=m}^n q_i \preceq \sum_{i=m}^n p_i \leq \sum_{i=m} p_i.$$

Since $\bigvee_{i=m}^n q_i$ is increasing in n , $\bigvee_{i \geq m} q_i \preceq \sum_{i \geq m} p_i$ by ($\preceq 7$). Let $p_0 = 1 - \sum_{i=0}^{\infty} p_i$. By ($\preceq 6$),

$$p_0 + \sum_{i=1}^{m-1} p_i = 1 - \sum_{i \geq m} p_i \preceq 1 - \bigvee_{i \geq m} q_i \leq 1 - \bigwedge_{m=1}^{\infty} \bigvee_{i \geq m} q_i.$$

Again by ($\preceq 7$), we can conclude that

$$1 = p_0 + \sum_{i=1}^{\infty} p_i \preceq 1 - \bigwedge_{m=1}^{\infty} \bigvee_{i \geq m} q_i.$$

Since M is finite, we must have

$$0 = \bigwedge_{m=1}^{\infty} \underbrace{\bigvee_{i \geq m} q_i}_{\text{decreasing}} = \text{SOT} - \lim \underbrace{\bigvee_{i \geq m} q_i}_{\geq q_m}.$$

Hence for all $\xi \in H$,

$$\|q_m\|^2 = \langle q_m \xi, \xi \rangle \leq \left\langle \bigvee_{i \geq m} q_i \xi, \xi \right\rangle = \left\| \bigvee_{i \geq m} q_i \xi \right\|^2 \xrightarrow{m \rightarrow \infty} 0,$$

and thus $q_m \rightarrow 0$ SOT. □

4.8 Existence of a trace on a finite von Neumann algebra

For this section, M is a finite von Neumann algebra. Recall that the σ -WOT on M is the weak* topology induced by M_* . Thus we may identify M_* with the σ -WOT continuous linear functionals on M .

Definition 4.8.1. Let $S(M) \subset M_*$ be the set of σ -WOT continuous states of M . Note that $U(M)$ acts on $S(M)$ by $u \cdot \varphi := \varphi(u^* \cdot u)$.

Lemma 4.8.2. Let M be a von Neumann algebra and $\varphi \in M^*$ a state. The following are equivalent.

1. φ is tracial, i.e., $\varphi(xy) = \varphi(yx)$ for all $x, y \in M$.
2. For all $x \in M$, $\varphi(xx^*) = \varphi(x^*x)$.
3. For all $u \in U(M)$, $\varphi(u^*xu) = \varphi(x)$.

Proof.

(1) \Rightarrow (2) : Obvious.

(2) \Rightarrow (3) : For $x \geq 0$, $\varphi(u^*xu) = \varphi(u^*x^{1/2}x^{1/2}u) = \varphi(x^{1/2}uu^*x^{1/2}) = \varphi(x)$. Now use that every $x \in M$ is a linear combination of 4 positive operators.

(3) \Rightarrow (1) : Replacing x with ux , we have $\varphi(xu) = \varphi(ux)$ for all $x \in M$ and $u \in U(M)$. Now use that every $y \in M$ is a linear combination of 4 unitaries. □

So to construct a trace in $S(M)$ for M finite, we will find a fixed point in $S(M)$ under the $U(M)$ -action. To do this, we will use the Ryll-Nardzewski Fixed Point Theorem. Our approach here follows the proof of Jacob Lurie.

Theorem 4.8.3 (Ryll-Nardzewski). *Let X be a Banach space and $K \subset X$ a weakly compact convex subset. Suppose $G \subset B(X)$ is a group of isometries with $GK \subseteq K$. Then there is an $x \in K$ such that $gx = x$ for all $g \in G$.*

For $u \in U(M)$, we define $\pi_u \in B(M_*)$ by $\pi_u \varphi := \varphi(u^* \cdot u)$. Hence for our purposes, $G = \pi(U(M)) \subset B(M_*)$.

The following theorem is the main result of this section.

Theorem 4.8.4. *Suppose M is a finite von Neumann algebra and fix $\varphi \in S(M)$. Define*

$$K_0 := \pi(U(M))\varphi = \{\varphi(u^* \cdot u) | u \in U(M)\} \subset S(M),$$

and let K be the weakly closed convex hull of K_0 in M_ . Then K is weakly compact.*

Before proving this theorem, observe that combining it with the Ryll-Nardzewski Fixed Point Theorem 4.8.3 yields the desired result.

Corollary 4.8.5. *There exists a σ -WOT continuous tracial state on a finite von Neumann algebra.*

Proof. Let $\varphi \in S(M)$. By Theorem 4.8.4, the weakly closed convex hull $K \subset S(M)$ of $\pi(U(M))\varphi$ is weakly compact. As K is clearly $\pi(U(M))$ -invariant, by the Ryll-Nardzewski Fixed Point Theorem 4.8.3, there is a $\pi(U(M))$ -fixed point $\text{tr} \in K \subset S(M)$, which is a tracial state by Lemma 4.8.2. \square

Lemma 4.8.6. *For a positive linear functional $\varphi \in M^*$, the following are equivalent.*

1. φ is σ -WOT continuous.
2. φ is normal: for all increasing nets of positive operators $x_i \nearrow x$ in M , $\varphi(x_i) \nearrow \varphi(x)$.
3. φ is completely additive: for every family (p_i) of mutually orthogonal projections in M , $\varphi(\sum p_i) = \sum \varphi(p_i)$.

Proof. Homework. \square

Remark 4.8.7. Suppose (p_i) is a family of mutually orthogonal projections in M . For all positive $\varphi \in M^*$, and for all finite subsets $F \subset I$, $\sum_{i \in F} \varphi(p_i) = \varphi(\sum_{i \in F} p_i) \leq \varphi(\sum p_i)$, so $\sum \varphi(p_i) \leq \varphi(\sum p_i)$. Hence φ is completely additive if and only if for every family of mutually orthogonal projections (p_i) in M , for all $\varepsilon > 0$, there is a finite $F \subset I$ such that $\varphi(\sum_{i \notin F} p_i) \leq \varepsilon$. Indeed,

$$\begin{aligned} \sum \varphi(p_i) &= \sup_{F \subset I} \sum_{i \in F} \varphi(p_i) = \sup_{F \subset I} \varphi\left(\sum_{i \in F} p_i\right) = \sup_{F \subset I} \varphi\left(\sum p_i\right) - \varphi\left(\sum_{i \notin F} p_i\right) \\ &= \varphi\left(\sum p_i\right) - \inf_{F \subset I} \varphi\left(\sum_{i \notin F} p_i\right). \end{aligned}$$

Proof of Theorem 4.8.4. Recall that the relative weak* topology on $X \subseteq X^{**}$ is the weak topology. To show $K \subset M_*$ is weakly compact, by the Banach-Alaoglu Theorem, it suffices to prove $K \subseteq M_*^{**} = M^*$ is weak* closed, as $K \subseteq (M^*)_1$ which is weak* compact.

Let $\psi \in \overline{K}$, the weak* closure of K in M^* . We show ψ is completely additive, and thus $\psi \in M_*$, so $\psi \in K$. Suppose for contradiction that ψ is not completely additive. Then there is a family $(p_i)_{i \in I}$ of mutually orthogonal projections and an $\varepsilon > 0$ such that for all finite $F \subset I$, $\psi\left(\sum_{i \notin F} p_i\right) > \varepsilon$.

Claim. *If $F \subset I$ is any finite set, there is a $\phi \in K_0$ and a finite set $G \subset I \setminus F$ such that $\phi(\sum_{i \in G} p_i) > \varepsilon$.*

Proof. The convex hull $\text{conv}(K_0)$ is weakly dense in K , which is weak* dense in \overline{K} , so $\text{conv}(K_0)$ is weak* dense in \overline{K} . Thus for all $\delta > 0$, the weak* open neighborhood

$$\left\{ \phi \in M^* \mid \left| (\psi - \phi) \left(\sum_{i \notin F} p_i \right) \right| < \delta \right\}$$

of ψ has non-empty intersection with $\text{conv}(K_0)$, so pick ϕ in this intersection. Since $\psi(\sum_{i \notin F} p_i) > \varepsilon$, choosing δ small, we have $\phi(\sum_{i \notin F} p_i) > \varepsilon$. Now if $\phi = \sum_{k=1}^n \lambda_k \phi_k$ is a convex combination of $\phi_k \in K_0$, there must be a particular k so that $\phi_k(\sum_{i \notin F} p_i) > \varepsilon$. Now since ϕ_k is completely additive, there is a finite $G \subset I \setminus F$ such that $\phi_k(\sum_{i \in G} p_i) > \varepsilon$. \square

Claim. *There is a sequence (F_n) of disjoint finite subsets of I and a sequence of states $(\phi_n) \subset K_0$ such that for all $n \in \mathbb{N}$,*

$$\phi_n \left(\sum_{i \in F_n} p_i \right) > \varepsilon.$$

Proof. We induct on n . Since $\psi(\sum p_i) > \varepsilon$, by the first claim, there is a $\phi_1 \in K_0$ and a finite set $F_1 \subset I$ such that $\phi_1(\sum_{i \in F_1} p_i) > \varepsilon$. Now suppose we have disjoint sets $F_1, \dots, F_n \subset I$ and states $\phi_1, \dots, \phi_n \in K_0$ such that $\phi_k(\sum_{i \in F_k} p_i) > \varepsilon$ for all $k = 1, \dots, n$. Since ψ is not completely additive,

$$\psi \left(\sum_{i \notin \coprod_{j=1}^n F_j} p_i \right) > \varepsilon,$$

so again by the first claim, there is a $\phi_{n+1} \in K_0$ and a set $F_{n+1} \subset I \setminus \coprod_{j=1}^n F_j$ such that $\phi_{n+1}(\sum_{i \in F_{n+1}} p_i) > \varepsilon$. \square

Now by the above claim, for each $\phi_n \in K_0$, there is a unitary $u_n \in U(M)$ such that $\phi_n = \varphi(u_n^* \cdot u_n)$. Moreover, setting $q_n := \sum_{i \in F_n} p_i$, we have a sequence (q_n) of mutually orthogonal projections such that $\varphi(u_n^* q_n u_n) > \varepsilon$ for all n . We now have our desired contradiction. Since the F_n are disjoint, the q_n are mutually orthogonal. Since $u_n^* q_n u_n \approx q_n$ for all n , $u_n^* q_n u_n \rightarrow 0$ SOT (and thus also σ -WOT) by [\(8\)](#). But $\varphi \in S(M)$ is σ -WOT continuous and $\varphi(u_n^* q_n u_n) > \varepsilon$ for all n , a contradiction. \square

4.9 The proof of Ryll-Nardzewski

In this section, we prove the Ryll-Nardzewski Fixed Point Theorem [4.8.3](#) following Lurie's proof.

<https://www.math.ias.edu/~lurie/261ynotes/lecture26.pdf>.

We begin by restating (a version of) the Ryll-Nardzewski Fix Point Theorem.

Theorem (Ryll-Nardzewski, Theorem [4.8.3](#)). *Let X be a Banach space and $K \subset X$ a weakly compact convex subset. Suppose $G \subset B(X)$ is a group of isometries with $GK \subseteq K$. Then there is an $x \in K$ such that $gx = x$ for all $g \in G$.*

Remark 4.9.1. Without loss of generality, we may assume G is finitely generated. Indeed, write $G = \bigcup G_i$ where each G_i is finitely generated. Then $K^G = \bigcap K^{G_i}$. By compactness of K and the finite intersection property, $\bigcap K^{G_i} \neq \emptyset$ for all i implies $K^G \neq \emptyset$.

Fix a Banach space X and a weakly compact convex subset $K \subset X$. We begin with the following warmup.

Lemma 4.9.2. *Suppose $T \in B(X)$ such that $TK \subseteq K$. There is an $x \in K$ such that $Tx = x$.*

Proof. For $n \in \mathbb{N}$, let $T_n := \frac{1}{n} \sum_{k=0}^{n-1} T^k$ and $K_n = T_n K \subseteq K$ as K is convex. We claim that $\{K_n\}$ has the finite intersection property. Indeed,

$$K_{n_1} \cap \cdots \cap K_{n_k} \supseteq T_{n_1} \cdots T_{n_k} K$$

as $T_m T_n = T_n T_m$ for all m, n .

Now let $x \in \bigcap K_n \neq \emptyset$. For each $n \in \mathbb{N}$, there is a $y \in K$ such that $x = T_n y$, so

$$Tx - x = (T - 1)T_n y = \frac{1}{n}(T - 1) \sum_{k=0}^{n-1} T^k y = \frac{1}{n}(T^n y - y) \in \frac{1}{n}(K - K).$$

Since K is weakly compact, so is $K - K$, and in particular, $K - K$ is bounded.^a Thus for every open neighborhood U of $K - K$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n}(K - K) \subset U$. But this means $Tx - x \in U$ for every open neighborhood U of 0, so $Tx = x$. \square

^aIf $S \subset X \subseteq X^{**}$ is weakly compact, then each $s \in S$ is pointwise bounded as a map on X^* by compactness. Now apply the Uniform Boundedness Principle.

The strategy of the proof will be to take our finitely generated group $G = \langle g_1, \dots, g_n \rangle \subseteq B(X)$ of isometries and find a candidate fixed point $x \in K$ for G using Lemma 4.9.2. We will prove by contradiction that this candidate $x \in K$ satisfies $g_i x = x$ for each generator. We thank Saúl Rodríguez for helping distill the conceptual statement in Lemma 4.9.3 below, which is the second main ingredient to achieve our contradiction.

Lemma 4.9.3. *Suppose X is a separable Banach space and $C \subset X$ is a weakly closed convex set with $\text{diam}(C) > 0$. For every $0 < \varepsilon < \text{diam}(C)$, there is a nonempty weakly closed convex subset $D = D(\varepsilon) \subsetneq C$ such that $\text{diam}(C \setminus D) \leq \varepsilon$.*

Assuming this technical lemma, we can now prove Theorem 4.8.3.

Proof of Theorem 4.8.3. Set $T = \frac{1}{n} \sum g_i \in B(X)$. By the warmup Lemma 4.9.2, there is an $x \in K$ such that $Tx = x$. If $g_i(x) = x$ for all i , we have our fixed point proving Theorem 4.8.3. Otherwise, relabelling the g_i , there is a $1 \leq k \leq n$ such that $g_i(x) \neq x$ for all $i = 1, \dots, k$ and $g_i(x) = x$ for all $i = k+1, \dots, n$. Then

$$x = Tx = \frac{1}{n} \sum_{i=1}^n g_i(x) = \frac{1}{n} \sum_{i=1}^k g_i(x) + \frac{n-k}{n} x,$$

which immediately implies that

$$x = T'x = \frac{1}{k} \sum_{i=1}^k g_i(x) \quad \text{for} \quad T' := \frac{1}{k} \sum_{i=1}^k g_i.$$

Now consider the norm closure X' of

$$\text{span} \left\{ g_{i_1} \cdots g_{i_m} x \mid m \in \mathbb{N} \text{ and } 1 \leq i_1, \dots, i_m \leq k \right\},$$

which is a *separable* Banach space, and let C be the weakly closed convex hull of $\langle g_1, \dots, g_k \rangle x \subseteq K$, which is again weakly compact. Let $\varepsilon > 0$ such that $\|g_i(x) - x\| > \varepsilon$ for all $i = 1, \dots, k$. By Lemma 4.9.3, there is a nonempty weakly compact convex subset $C(\varepsilon) \subsetneq C$ such that $\text{diam}(C \setminus C(\varepsilon)) \leq \varepsilon$. Since $\emptyset \neq C(\varepsilon) \neq C$, there is an $h \in G$ such that $hx \notin C(\varepsilon)$. Since $T'x = x$, we have

$$hx = hT'x = \frac{1}{k} \sum_{i=1}^k hg_i(x) \notin C(\varepsilon),$$

and since $C(\varepsilon)$ is convex, there must be some $1 \leq i \leq k$ such that $hg_i(x) \notin C(\varepsilon)$. Now both $hx, hg_i(x) \notin C(\varepsilon)$, but since h is an isometry, we have

$$\|x - g_i(x)\| = \|hx - hg_i(x)\| \leq \text{diam}(C \setminus C(\varepsilon)) \leq \varepsilon,$$

a contradiction. □

We now prove the lemma.

Proof of Lemma 4.9.3. Let $E = \partial_{\text{ext}} C \subseteq C$ be the set of extreme points. By the Krein-Milman Theorem, C is the weak closed convex hull of E . Let $\overline{E} \subseteq C$ be the weak closure of E , and let $B = \overline{B_{\varepsilon/3}}(0)$ be the closed ball of radius $\varepsilon/3$. Since B is convex and norm closed, B is also weakly closed as the norm and weak topology have the same closed convex sets. Since X is separable, there is a sequence $(y_j) \subset X$ such that $(y_j + B)$ covers X . Thus $((y_j + B) \cap \overline{E})$ is a cover of the weakly compact set \overline{E} . By the Baire Category Theorem, there is a j such that $(y_j + B) \cap \overline{E}$ has non-empty interior U in \overline{E} with respect to the relative weak topology on \overline{E} .

Now define

$$\begin{aligned} C_1 &:= \text{weakly closed convex hull of } \overline{E} \setminus U \\ C_2 &:= \text{weakly closed convex hull of } (y_j + B) \cap \overline{E}, \end{aligned}$$

which are both nonempty (why?) weakly closed convex subsets of C . Since C is the closed convex hull of

$$E \subseteq (\overline{E} \setminus U) \cup ((y_j + B) \cap \overline{E}),$$

E is the *convex join* of C_1 and C_2 , i.e., $C = \text{im}(\theta)$ for

$$\theta : C_1 \times C_2 \times [0, 1] \rightarrow X \quad \text{given by} \quad (a, b, t) \mapsto ta + (1 - t)b.$$

We now consider the sets $C(\delta) := \text{im}(\theta|_{C_1 \times C_2 \times [\delta, 1]})$.

Step 1: Each $C(\delta)$ is a weakly closed convex subset of C .

Closed: Since θ is continuous from the (weak, weak, standard) product topology to the weak topology as X with the weak topology is a topological vector space, $K(\delta)$ is weakly compact, and thus closed.

Convex: First, note that for all $0 < \delta \leq 1$, $\delta C_1 + (1 - \delta)C_2$ is convex. We claim that

$$\theta(C_1 \times C_2 \times [\delta, 1]) = \theta(C_1 \times (\delta C_1 + (1 - \delta)C_2) \times [0, 1]),$$

which is manifestly convex.

\subseteq : If $t \in [\delta, 1]$, $ta + (1 - t)b = sa + (1 - s)(\delta a + (1 - \delta)b)$ for $s \in [0, 1]$ such that $(1 - s)(1 - \delta) = (1 - t)$. This condition is equivalent to $t = \delta + s(1 - \delta)$.

\supseteq : If $s \in [0, 1]$, then $sa_1 + (1 - s)[\delta a_2 + (1 - \delta)b] = ta + (1 - t)b$ for $t = s + (1 - s)\delta = \delta + s(1 - \delta) \in [\delta, 1]$ as before and

$$a = \frac{sa_1 + (1 - s)\delta a_2}{s + (1 - s)\delta} \in C_1.$$

Step 2: For $\delta > 0$ sufficiently small, $\text{diam}(C \setminus C(\delta)) \leq \varepsilon$.

Since C is weakly compact, it is bounded, so $C \subset B_R(0)$ for some $R > 0$. If $y, y' \in C \setminus C(\delta)$, then there are $0 \leq t, t' < \delta$, $a, a' \in C_1$, and $b, b' \in C_2$ such that

$$y = ta + (1 - t)b \quad \text{and} \quad y' = t'a' + (1 - t')b'.$$

Then

$$\begin{aligned} \|y - y'\| &= \|t(a - b) + b - t'(a' - b') - b'\| \\ &\leq t(\|a\| + \|b\|) + t'(\|a'\| + \|b'\|) + \underbrace{\|b - b'\|}_{b, b' \in C_2} \\ &\leq 4\delta R + \frac{2}{3}\varepsilon \end{aligned}$$

as $b, b' \in C_2 \subset y_j + B$ which has diameter $2/3 \cdot \varepsilon$. Now choose $\delta < \frac{\varepsilon}{12R}$.

Step 3: For δ as in Step 2 above, $C(\delta) \neq C$.

Since $U \subseteq \overline{E}$ is a non-empty open subset, there is a $y \in E \cap U$. We claim that $y \notin C(\delta)$. Since $y \in E$ is an extreme point of C , it suffices to prove $y \notin C_1$. (Indeed, if $y \notin C_1$ and $y = ta + (1 - t)b$ for $a \in C_1$ and $b \in C_2$, since y is extreme, $y = a = b$. But since $a \in C_1$ and $y \notin C_1$, we must have $t = 0$. Thus y cannot be written as $ta + (1 - t)b$ for $a \in C_1$, $b \in C_2$, and $t \in [\delta, 1]$.) Since X with the weak topology is locally convex, there is a weakly open convex neighborhood V of 0 such that the weak closure \overline{V} satisfies $(y - \overline{V}) \cap \overline{E} \subseteq U$. (Indeed, we can use here that \overline{E} is weakly compact and thus weakly normal.)

Now since $\overline{E} \setminus U$ is weakly compact, it admits a weakly open cover $\{z_i + V\}_{i=1}^k$ where each $z_i \in \overline{E} \setminus U$. Thus C_1 is contained in the closed convex hull of

$$\bigcup_{i=1}^k (z_i + V) \cap \overline{E} \supseteq \overline{E} \setminus U.$$

In turn, $\bigcup_{i=1}^k (z_i + V) \cap \overline{E}$ is contained in the convex join of the $(z_i + \overline{V}) \cap C$. If $y \in C_1$, then $y \in (z_i + \overline{V}) \cap C$ for some i . But then $z_i \in (y - \overline{V}) \cap \overline{E} \subseteq U$, a contradiction to $z_i \in \overline{E} \setminus U$.

Thus if $\delta > 0$ is sufficiently small, we can take $C' = C(\delta) \subsetneq C$. □