4. Quantum constructions from unitary fusion categories

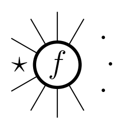
In this section, we take as input a unitary fusion category (UFC) $\mathcal{C}$, which we will view from the physical perspective as a fusion rule $(S, N_{\bullet\bullet}^{\bullet})$ together with a collection of unitary F-matrices. We will then use this data to produce quantum systems, including quantum spin chains and lattice models. We will also discuss the Turaev-Viro-Ocneanu topological quantum field theory (TQFT) constructed from our UFC.

4.1. **Quantum spin chains from UFCs.** Let $\mathcal{C}$ be a UFC. A *generator* for $\mathcal{C}$ is an object $c \in \mathcal{C}$ such that every object of $\mathcal{C}$ appears as a summand of $c^{\otimes n}$ for some $n \in \mathbb{N}$. For simplicity, we will assume $\mathcal{C}$ has a *real* generator, which is always possible by taking sums over $c \oplus c^\vee$ for $c \in \operatorname{Irr}(\mathcal{C})$.

Recall from the Lattice Models section that we constructed a quantum spin chain from the Temperley-Lieb algebras with periodic boundary conditions. In fact, that construction works for any real generator for a unitary fusion category.

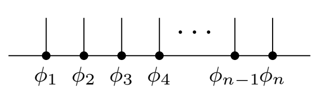
Example 4.1.1. Let ρ be a real generator of the unitary fusion category $\mathcal{C}$. There are many ways one might build a sequence of local Hilbert spaces from the pair $(\mathcal{C}, \rho)$.

- One might consider Hilbert spaces $\mathcal{H}_L := \mathcal{C}(1_\mathcal{C} \to \rho^{\otimes L})$ drawn in the 2D graphical calculus as follows:



$$\in \mathcal{C}(1_\mathcal{C} \to \rho^{\otimes L}) = \mathcal{H}_L.$$

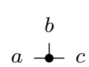
- One might consider the Hilbert spaces $\mathcal{K}_L := \mathcal{C}(1_\mathcal{C} \to \rho^{\otimes L+2})$, which should be viewed as a certain 'admissible' subspace. For computations, one usually picks a particular basis, like the left associated tree basis when working in $\mathcal{K}_L$. We usually draw these basis elements as follows:



$$\in \mathcal{C}(1_\mathcal{C} \to \rho^{\otimes L+2}) = \mathcal{K}_L \qquad (4.1.2)$$

  where we view $\rho^{L+2} = \rho \otimes (\rho \otimes (\rho \otimes \cdots (\rho \otimes \rho) \cdots))$ as right associated. However, it is not clear here how to deal with the boundary. Options include open boundary conditions (ignore the boundary) and periodic boundary conditions (glue the two ends).

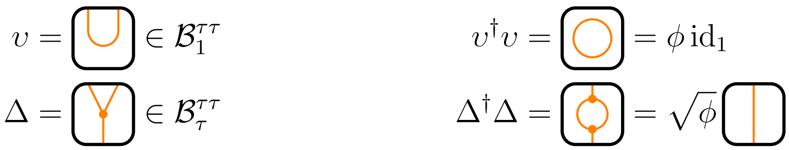
Exercise 4.1.3. Express $\mathcal{C}(1_\mathcal{C} \to \rho^{\otimes L})$ as a subspace of a tensor product of local on-site Hilbert spaces.
Hint: For each vertex, consider the local Hilbert space

$$\bigoplus_{a,b,c}$$


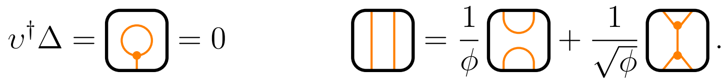
$$= \bigoplus_{a,b,c \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(a \to b \otimes c)\,.$$

Introduce a 2-local edge/link term H_ℓ *which projects to the subspace where the simple labels on adjacent vertices match.*

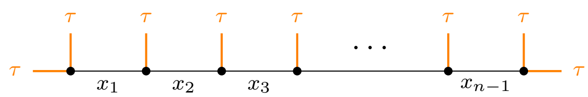
4.2. **Golden spin chain.** We now give the construction of [FTL+07] for a quantum spin chain based on Fib. Recall that Fib has two simples $1, \tau$ satisfying $\tau \otimes \tau = 1 \oplus \tau$. The standard normalized basis vectors are given by



with $\phi = \frac{1+\sqrt{5}}{2}$ subject to the additional relations



We consider the Hilbert spaces $\mathcal{K}_n := \mathsf{Fib}(1 \to \tau^{\otimes n+2})$ which we represent diagrammatically by



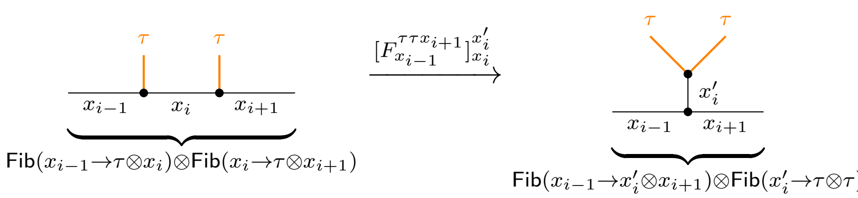
$\in \mathsf{Fib}(1 \to \tau^{\otimes n+2}) = \mathcal{K}_n,$

again, right associated. As Fib is multiplicity free, the labels $x_1, \ldots, x_{n-1} \in \{1, \tau\}$ on edges completely determine the morphism in $\mathcal{K}_n$.

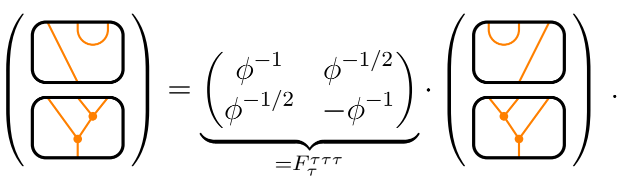
Remark 4.2.1. We may view $\mathcal{K}_n$ as the 'admissible' subspace of $(\mathbb{C}|1\rangle \oplus |\tau\rangle)^{\otimes n}$ spanned by elementary tensors with no consecutive $|1\rangle$s. (There are no maps from τ to 1 in Fib.)

Exercise 4.2.2. Compute $\dim(\mathcal{K}_n)$. That is, how many length n words on $\{1, \tau\}$ do not have consecutive 1s?

The 3-local Hamiltonian K_i acts as follows. First we apply the F-matrix to fuse two τ anyons.



Our convention for F-matrices gives the following formula for $x_{i-1} = x_{i+1} = \tau$:



We then assign the energies $E_\tau = 0$ for $x_i' = \tau$ and $E_1 = -1$ for $x_i' = 1$.

Exercise 4.2.3. Prove that in the basis $\{|1\tau1\rangle, |1\tau\tau\rangle, |\tau\tau1\rangle, |\tau1\tau\rangle, |\tau\tau\tau\rangle\}$ for $|x_{i-1}x_ix_{i+1}\rangle$, the operator K_i is given by

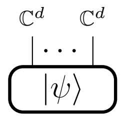
$$K_i = -\begin{pmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \phi^{-2} & \phi^{-3/2} \\ & & & \phi^{-3/2} & \phi^{-1} \end{pmatrix}$$

Exercise 4.2.4. Prove that viewing $\mathcal{K}_n$ as a subspace of $(\mathbb{C}|1\rangle \oplus |\tau\rangle)^{\otimes n}$ as in Remark 4.2.1, the local Hamiltonian K_i can be written as

$$(N_{i-1} + N_{i+1} - 1) - N_{i-1}N_{i+1}(\phi^{-3/2}X_i + \phi^{-3}N_i + 1 + \phi^{-2}) \tag{4.2.5}$$

where $N_j = \frac{1-Z_j}{2} = \operatorname{diag}(0,1)$ counts the τ particle on site j. That is, prove that (4.2.5) above fixes the subspace $\mathcal{K}_n$, and on this subspace, is equal to K_i.

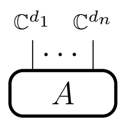
4.3. **Matrix product states.** Suppose we have a 1D quantum spin chain of N qudits, which gives the total space $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$. We will be agnostic as to the particular local Hamiltonian. A ground state is some vector in $\mathcal{H}$, which can be represented graphically by



$$= \sum_{i_1,\ldots,i_N} C_{i_1,\ldots,i_N} |i_1 \cdots i_N\rangle.$$

We thus need d^N numbers to describe this state; this quickly becomes far too large for even moderate system sizes.

Exercise 4.3.1. An *n-dimensional tensor* of *shape* $(d_1, \ldots, d_n)$ is a $d_1 \times \cdots d_n$ array of complex numbers. Observe these tensors form a complex vector space $T(d_1, \ldots, d_n)$. Find a linear isomorphism $T(d_1, \ldots, d_n) \cong \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$. Deduce we may represent an n-dimensional tensor A of shape $(d_1, \ldots, d_n)$ graphically as

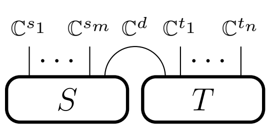


The area of *tensor networks* in theoretical condensed matter physics makes extensive use of the graphical calculus in the symmetric monoidal fusion category $\mathsf{Hilb}_{\mathsf{fd}}$. Here, we endow $\mathsf{Hilb}_{\mathsf{fd}}$ with its canonical spherical structure, where give a finite dimensional Hilbert space, its canonical evaluation and coevaluation are given by

$$\mathrm{ev}_H : \langle\eta| \otimes |\xi\rangle \mapsto \langle\eta|\xi\rangle \qquad\qquad \mathrm{coev}_H : |1_{\mathbb{C}}\rangle \mapsto \sum_\alpha |\alpha\rangle \otimes \langle\alpha|$$

where $\{|\alpha\rangle\}$ is some orthonormal basis of H. Observe that the quantum dimension is equal to the ordinary dimension of H.

As seen in Exercise 4.3.1, every n-dimensional tensor T can be represented as a coupon with n strands labelled by the shape of T. Given two tensors S and T whose shapes share a common number d, we may *contract* S and T along $\mathbb{C}^d$ by connecting their corresponding strands. In more detail, suppose S has shape $(s_1, \ldots, s_m, d)$ and T has shape $(d, t_1, \ldots, t_n)$. The contracted tensor $S \circ_d T$ has shape $(s_1, \ldots, s_m, t_1, \ldots, t_n)$ and can be represented graphically by



$$(S \circ_d T)_{i_1,\ldots,i_m,j_1,\ldots,j_n} = \sum_{i=1}^{d} S_{i_1,\cdots i_m,i} T_{i,j_1,\ldots,j_n}$$

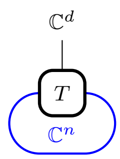
Since $\mathsf{Hilb}_{\mathsf{fd}}$ is spherical, we can rotate these strings around the boundary, and they may point in any direction we choose. (Although we will not need it here, $\mathsf{Hilb}_{\mathsf{fd}}$ is actually *symmetric*, which means we may freely cross strings, and these crossings satisfy very nice relations. We will discuss this further in the modular tensor category section.)

Certain states called *matrix product states* exploit networks of 3D tensors to describe states more efficiently with less data. Again, we will work in the total space $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$ corresponding to a 1D quantum spin chain.

Example 4.3.2. We can represent a 3D tensor T of shape (d, m, n) as an $m \times n$ matrix whose entries lie in $\mathbb{C}^d$.

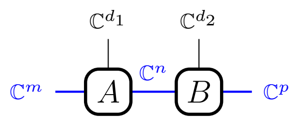
$$T = \left(|\psi_{ij}\rangle \in \mathbb{C}^d\right)_{i,j}$$

If T is of shape (d, n, n), then taking the trace by adding the diagonal entries in $\mathbb{C}^d$ corresponds to contracting the tensor along $\mathbb{C}^n$:



$$= \sum_{j=1}^{n} |\psi_{jj}\rangle.$$

Given A of shape (d_1, m, n) and B of shape (d_2, n, p), multiplying these matrices by tensoring the appropriate entries corresponds to contracting the tensor along $\mathbb{C}^n$:

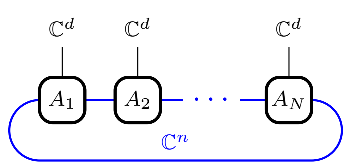


For example, if $d_1 = d_2 = 2$, $m = p = 1$, and $n = 2$, we have

$$\frac{1}{\sqrt{2}} \begin{pmatrix} |0\rangle & |1\rangle \end{pmatrix} \begin{pmatrix} |1\rangle \\ -|0\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{4.3.3}$$

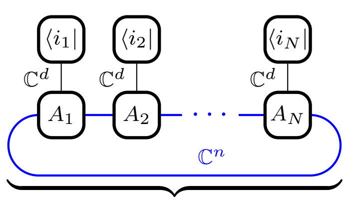
Definition 4.3.4. A *matrix product state* (MPS) is a state vector in $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$ of the form

$$|\psi\rangle =$$



where each $A_1, \ldots, A_N$ is a 3-dimensional tensor of shape (d, n, n), where n need not have anything to do with d. To express such an MPS in the computational basis, we simply take inner products. In the graphical calculus, this corresponds to the following formula:

$$|\psi\rangle = \sum_{i_1,\ldots,i_N}$$



$$|i_1 \cdots i_N\rangle$$

$=:\mathrm{Tr}(A_1^{(i_1)} A_2^{(i_2)} \cdots A_N^{(i_N)})$

where $\{i_0, \ldots, i_{d-1}\}$ is the computational basis of $\mathbb{C}^d$. This coefficient is typically denoted by $\mathrm{Tr}(A_1^{(i_1)} A_2^{(i_2)} \cdots A_N^{(i_N)})$ in the literature, but we will try to use the graphical notation whenever possible. We call n the *bond dimension* of the MPS. Typically, it will be uniform, but this need not be the case. For example, we could have A_1 of shape (d, n_1, n_2), A_2 of shape (d, n_2, n_3), $\cdots$, and A_N of shape (d, n_N, n_1).

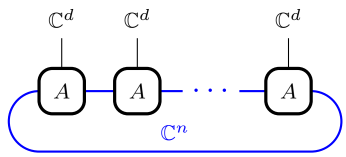
Every state can be expressed as an MPS, but note that an MPS representation of a state vector need not be unique due to the degrees of freedom afforded by the n which is not related to d.

Example 4.3.5 (Product state)**.** Every product state is an MPS where each A_i is a 3-dimensional tensor of shape $(d, 1, 1)$, i.e., a 1-dimensional tensor.

Example 4.3.6 (Singlet state)**.** Observe that (4.3.3) expresses a singlet state as an MPS.

Example 4.3.7 (GHZ state)**.** The state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\cdots 0\rangle + |1\cdots 1\rangle)$ can be written as

$$\frac{1}{\sqrt{2}} \operatorname{Tr} \begin{pmatrix} |0\rangle & 0 \\ 0 & |1\rangle \end{pmatrix}^N =$$



$$A = \frac{1}{2^{\frac{1}{2N}}} \begin{pmatrix} |0\rangle & 0 \\ 0 & |1\rangle \end{pmatrix}.$$

Exercise 4.3.8 (W state)**.** Write the W state $\frac{1}{\sqrt{5}}(|10000\rangle + |01000\rangle + |00100\rangle + |00010\rangle + |00001\rangle)$ as a matrix product state.

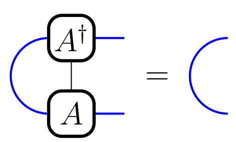
Exercise 4.3.9. Express an element of $\mathcal{C}(1_{\mathcal{C}} \to \rho^{\otimes L+2})$ as in (4.1.2) as an MPS.

Exercise 4.3.10 ($\star$, [BC17, §4])**.** Use the Schmidt decomposition to show that every state can be written as a matrix product state.

Remark 4.3.11. Unfortunately, Exercise 4.3.10 is not effective; the bond dimensions must grow extremely quickly to express some some states as MPSs. Moreover, a given MPS can be expressed in many different ways, with arbitrarily large bond dimensions. For example, for a given MPS with 3D tensor A of shape (d, n, n), for any left invertible $m \times n$ matrix M, we may replace A with MAM^{-1}.

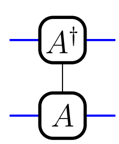
This means that an MPS which can be expressed with bounded or slowly growing bond dimensions is a very special state indeed.

Definition 4.3.12. An MPS built from from the 3D tensor A of shape (d, n, n) is in *left canonical form* if



(4.3.13)

We define the corresponding *transfer matrix* by

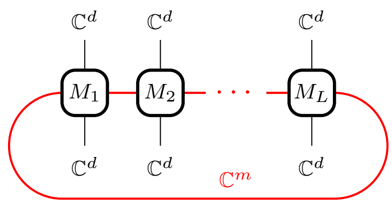


which is a 4D tensor of shape (n, n, n, n). Often, one combines the left and right indices and thinks of the transfer matrix as an $n^2 \times n^2$ matrix. We call an MPS in left canonical form *injective* if (4.3.13) spans the $+1$ eigenspace for the transfer matrix viewed as an $n^2 \times n^2$ matrix.

TODO: Connect MPS to transfer matrix method for solving 2D classical Ising model by mapping it to 1D transverse-field Ising.

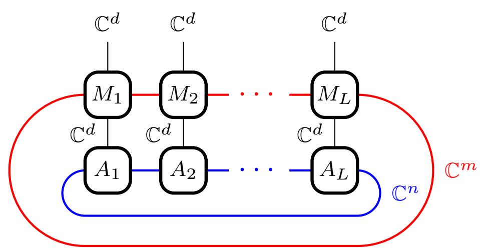
4.4. **Matrix product operators.** Fusion categories can act on quantum spin chains and MPSs via matrix product operators. In this section, we will construct an action of $\mathsf{Hilb}_{\mathsf{fd}}((\mathbb{Z}/d)^3, \omega)$ for a non-trivial 3-cocycle following [BW17, III.B and Appendix B].

Definition 4.4.1. A *matrix product operator* (MPO) on the total Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes L}$ is an operator of the form



We call m the *bond dimension* of the MPO. Similar to an MPS, an MPO need not have uniform bond dimension. If the bond dimension $m = 0$, the MPO is a product of operators, which is said to act *on-site*.

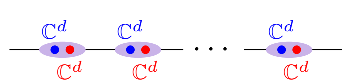
An MPO acts on an MPS via stacking as usual:



Observe that the action of an MPO on an MPS gives another MPS.

Exercise 4.4.2 (Transverse-field Ising MPO)**.** We saw in Example 4.3.5 that every product state can be written as an MPS. Express the transverse-field Ising 2-local Hamiltonian $H_{\text{tIs}} = -J \sum Z_j Z_{j+1} - h \sum X_j$ with open boundary conditions as an MPO.

Example 4.4.3 (CZX 1D boundary anomalous MPO action, [CLW11, BW17])**.** Consider a 1D spin chain $\mathcal{H} = (\mathbb{C}^d)^{\otimes 2L}$ where we group two quidits per site, with periodic boundary conditions. We color the left qudit blue and the right qudit red:



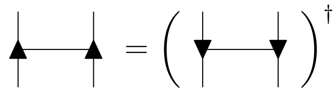
Let $\zeta := \exp(2\pi i/d)$, and define the generalized Pauli Z and X matrices for $\mathbb{C}^d$ by

$$X = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad Z = \operatorname{diag}(1, \zeta, \ldots, \zeta^{d-1}).$$

Observe that

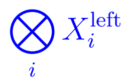
$$ZX = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \zeta & 0 & \cdots & 0 & 0 \\ 0 & \zeta^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta^{d-1} & 0 \end{pmatrix} = \zeta \begin{pmatrix} 0 & 0 & \cdots & 0 & \zeta^{d-1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & \zeta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \zeta^{d-2} & 0 \end{pmatrix} = \zeta XZ.$$

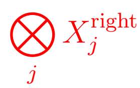
Define the *controlled Pauli Z operator*

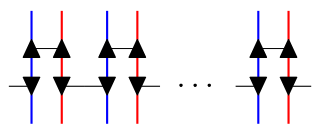


$$:= \frac{1}{d} \sum_{i,j=0}^{d-1} \zeta^{ij} Z_1^i Z_2^j.$$

We now have a $(\mathbb{Z}/d)^3$ action on $\mathcal{H}$ with generators given by

$(1,0,0) =$ 

$(0,1,0) =$ 

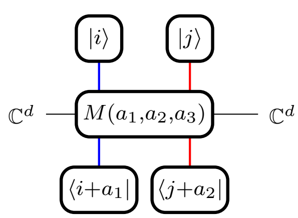
$(0,0,1) =$ 

(recall we enforce periodic boundary conditions). We will be agnostic to the Hamiltonian; we merely require the ground state $|\psi\rangle$ to be preserved by this $(\mathbb{Z}/d)^3$ symmetry.

Exercise 4.4.4. Verify the following facts about this $(\mathbb{Z}/d)^3$ action.

(1) $(1,0,0)^d$, $(0,1,0)^d$, and $(0,0,1)^d$ are all the identity operator on $\mathcal{H}$.
(2) $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ all pairwise commute.

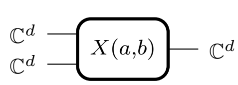
Exercise 4.4.5. Show that the $(\mathbb{Z}/d)^3$ action can be realized by a translationally invariant MPO with bond dimension N with on-site tensor determined by



$$= \sum_{k=0}^{d-1} \zeta^{ja_3(k-i)} |i\rangle\langle k|.$$

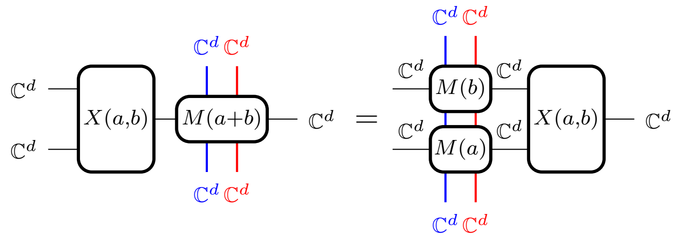
Note: Although we say this is the on-site tensor, the MPO action is not an on-site action, which would require the bond dimension to be zero.

Remark 4.4.6. Observe now that the vertical composite of $M(a) = M(a_1,a_2,a_3)$ and $M(b) = M(b_1,b_2,b_3)$ is simply not equal to $M(a+b) = M(a_1+b_1, a_2+b_2, a_3+b_3)$; indeed, the bond dimension has increased from d to d^2! To remedy this, we introduce the *reduction tensor* partial isometry in $\mathcal{B}(\mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^d)$ determined by

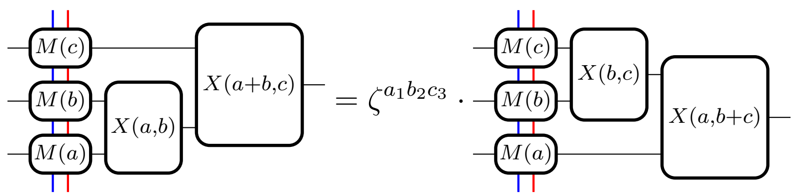


$$:= \sum_{k=0}^{d-1} \zeta^{-ka_2b_3} \left| \begin{matrix} k+a_1 \\ k \end{matrix} \right\rangle \langle k|.$$

TODO: Is this the only reduction tensor that would work for this MPO action?

Exercise 4.4.7. Prove that 

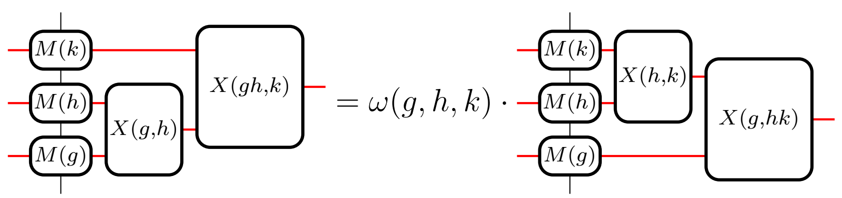
Exercise 4.4.8. Prove that $\omega(i,j,k) = \zeta^{ijk}$ is a 3-cocycle on $\mathbb{Z}^d$. Then show that



In light of this exercise, we see we have constructed an *anomalous* $(\mathbb{Z}/d)^3$-action on our quantum spin chain $\mathcal{H}$ where the *anomaly* corresponds to the non-trivial 3-cocycle $\omega \in Z^3((\mathbb{Z}/d)^3, U(1))$. Observe that if a group acts on-site, then there cannot be an anomaly; thus one should view this 3-cocycle anomaly as an obstruction to lifting the MPO action to an on-site symmetry action.

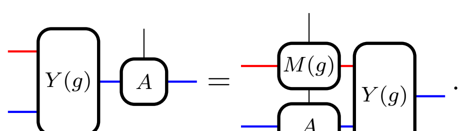
Exercise 4.4.9 ([BW17, Appendix B])**.** Suppose $|\psi\rangle$ is a translationally invariant MPS with 3D tensor A of shape (d, n, n), and suppose we have an MPO-action of the group G on $|\psi\rangle$ by translationally invariant on-site tensors $M(g)$ together with reduction tensors $X(g, h)$ for $g, h \in G$.

(1) Show that if $\omega \in Z^3(G, U(1))$, and the reduction tensor partial isometries satisfy

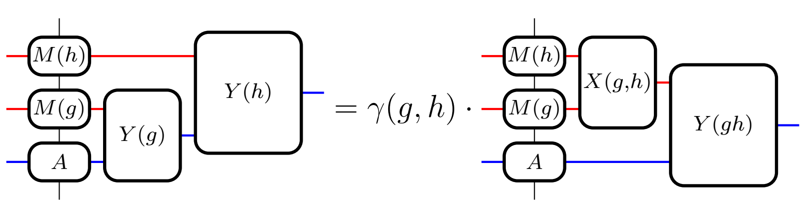


then $g \mapsto M(g)$ defines an associative G-action by MPOs.

(2) Suppose now each $M(g)$ acts on $|\psi\rangle$ via reduction tensor $Y(g)$:



Suppose further there is a scalar $\gamma(g, h) \in U(1)$ such that



Show that if $|\psi\rangle$ is injective, then $\omega = d\gamma$, i.e.,

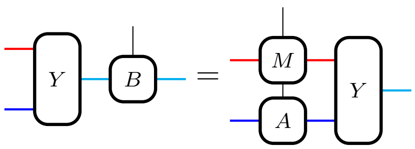
$$\omega(g,h,k) = \frac{\gamma(h,k)\gamma(g,hk)}{\gamma(gh,k)\gamma(g,h)}.$$

Deduce that no injective MPS admits an anomalous symmetry.

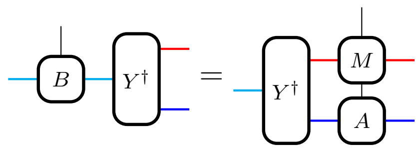
4.5. **MPS 2-category.** In joint work with Corey Jones, we now show that MPS forms a 2-category with 1-morphisms MPOs.

Definition 4.5.1. Suppose A and B are two MPS with bond dimensions d_A, d_B respectively. We define the hom category $\mathsf{MPO}(A \to B)$ by:

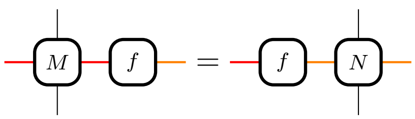
- Objects are MPOs M together with a reduction tensor $Y : \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_M} \to \mathbb{C}^{d_B}$ satisfying:



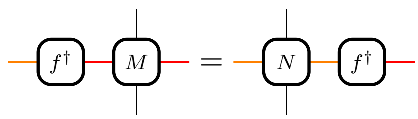
and



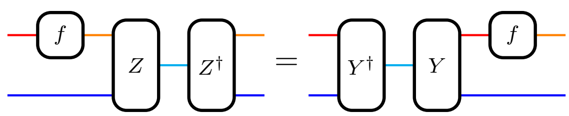
- A morphism $(M,Y) \to (N,Z)$ is a map $f : \mathbb{C}^{d_M} \to \mathbb{C}^{d_N}$ where d_M and d_N are the bond dimensions satisfying



and



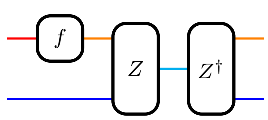
together with the following compatibility condition with the reduction tensors:



.

We compose morphisms by horizontal concatenation.

Observe $\mathsf{MPO}(A \to B)$ is a dagger category, but it need not be C* due to the possible presence of degenerate morphisms. We fix this by performing the quotient by the *negligible morphisms*, which satisfy



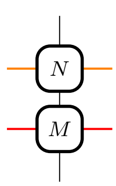
$= 0.$

TODO: do we really want to quotient? TODO: Karoubi complete

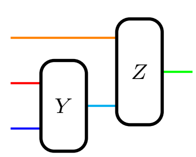
Exercise 4.5.2. Show that $\mathsf{MPO}(A \to B)$ is a unitary category.

Definition 4.5.3. We now discuss a 2-category structure on the categories $\mathsf{MPO}(A \to B)$. For MPOs $(M,Y) \in \mathsf{MPO}(A \to B)$ and $(N,Z) \in \mathsf{MPO}(B \to C)$, we define $(M,Y) \boxtimes (N,Z) \in \mathsf{MPO}(A \to C)$ by

$M \boxtimes N :=$



$Y \boxtimes Z :=$



For an MPS A, we define its identity MPO by

$\mathrm{id}_A :=$ 

Observe this 2-category is *strict*, i.e., all tensorators and unitors are identities.

Definition 4.5.4. An *action* of a UFC $\mathcal{C}$ on an MPS A is a unitary tensor functor $\mathcal{C} \to \mathsf{MPO}(A \to A)$.

Physics 4.5.5. The results from [BW17] cannot be expressed as an MPO action on an MPS for reasons that will be explained in the next section. In fact, UFC actions on MPS will most likely be completely uninteresting for the following reasons.

(1) Conjecturally, any non-group UFC cannot act on an injective MPS.
(2) In the next section, we will see that gapped ground states for 1D local Hamiltonians can be well-approximated by injective MPS whose bond dimensions do not grow too quickly.
(3) MPS are gapped, possibly with degeneracy, meaning the Hamiltonian may admit a finite dimensional space of ground states.

Combining the above, a ground state which admits an action of a UFC is probably either spontaneous symmetry breaking, in that the ground state is degenerate, or gapless. The interesting case would be a gapless state, which is not a MPS.

4.6. **No topological order for 1D gapped bosonic systems without symmetry.** For an injective MPS $|\psi\rangle$ whose bond dimensions do not grow too quickly, one can write down a gapped frustration free Hamiltonian such that $|\psi\rangle$ is unique ground state. There are three steps:

(1) Compute the RDM $\rho^{\mathrm{red}}_{[i-\ell,i+\ell]}$ of length 2ℓ interval centered at site i.
(2) Compute the projector P^i_ℓ onto the support of $\rho^{\mathrm{red}}_{[i-\ell,i+\ell]}$.
(3) For ℓ sufficiently large, set $H := \sum_i (1 - P^i_\ell)$.

One can also show that the correlations of local operators with respect to this gapped ground state decay exponentially [BC17].

Now given a gapped ground state $|\psi\rangle \in \mathcal{H} = (\mathbb{C}^d)^{\otimes L}$ for a 2-local Hamiltonian (such locality can always be achieved by coarse graining) $H = \sum_i H_i$ where each $\|H_i\| \leq M$ for some fixed $M > 0$ (H has *finite interaction strength*), Hastings proved in [Has07] that the von Neumann entropy is bounded, i.e., there is an $S_{\max > 0}$ such that for all $1 \leq j \leq L$, $S(\rho^{\mathrm{red}}_{[1,j]}) \leq S_{\max}$. He then uses this bound (or rather the proof of this bound) to prove that such a gapped ground state may be approximated to within trace norm ε by an MPS with with bond dimension depending polynomially on ε^{-1} and L. The argument proceeds by analyzing the Schmidt coefficients of $|\psi\rangle$ onto subsystems $[1, j]$ and $[j + 1, L]$ for some arbitrary fixed j. The upper bound $S_{\max}$ provides an important upper bound k_0 independent of choice of j such that the sum of the modulus squared of the largest k_0 Schmidt coefficients is at least $1/2$.

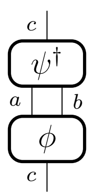
A recent article [DB19] improves upon this by showing that for any $k \in \mathbb{N}$ and $\varepsilon > 0$, there is an MPS $|\mu\rangle$ with bond dimension a polynomial in k and ε^{-1} such that the reduced densities of $|\psi\rangle$ and $|\mu\rangle$ on any interval of length k are ε-close in trace norm. This $|\mu\rangle$ is produced by a constant depth quantum circuit acting on an initial product (non-entangled) state. Their argument applies to a more general class of states $|\psi\rangle$ which have exponential decay of correlations.

We conclude that our gapped ground state $|\psi\rangle$ does not exhibit long range entanglement, since it can be approximated by an MPS obtained from a product state by a constant depth finite circuit. Since topological order is a property of long range entanglement of quantum many body systems, we conclude there is no topological order in 1D. Note that this argument only applies to gapped bosonic systems which are not protected by symmetry.

TODO: SPT

TODO: DMRG is a heuristic algorithm for computing ground state MPS for a 1D gapped Hamiltonian.

4.7. **Annular category, tube algebra, and skein modules.** Suppose $\mathcal{C}$ is a UFC and $\mathrm{Irr}(\mathcal{C})$ is a set of representatives for the isomorphism classes of $\mathcal{C}$. As in the Fusion Category notes, for each $a, b, c \in \mathrm{Irr}(\mathcal{C})$, choose an orthogonal basis $\mathcal{B}_c^{ab} \subset \mathcal{C}(c \to a \otimes b)$ normalized by

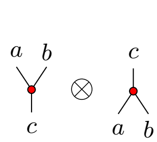


$$= \delta_{\phi=\psi} \sqrt{\frac{d_a d_b}{d_c}} \cdot \mathrm{id}_c \qquad (4.7.1)$$

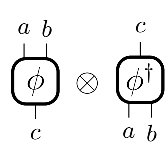
Notation 4.7.2 ([HP17, §2.5])**.** Consider $\mathcal{B}_c^{ab}$ and $(\mathcal{B}_c^{ab})^\dagger := \left\{\phi^\dagger \middle| \phi \in \mathcal{B}_c^{ab}\right\}$. We represent the canonical element (which is independent of the choice of $\mathcal{B}_c^{ab}$!)

$$\sum_{\phi \in \mathcal{B}_c^{ab}} \phi \otimes \phi^\dagger \in \mathcal{C}(c \to a \otimes b) \otimes \mathcal{C}(a \otimes b \to c)$$

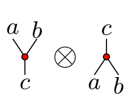
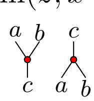
by a pair of trivalent nodes shaded by the same color:



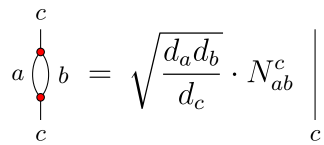
$$:= \sum_{\phi}$$



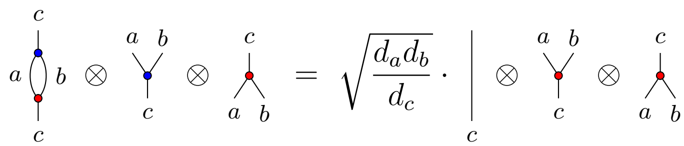
(4.7.3)

The element  lies in $\mathrm{Hom}(z, x \otimes y) \otimes \mathrm{Hom}(x \otimes y, z)\mathcal{C}(c \to a \otimes b) \otimes \mathcal{C}(a \otimes b \to c)$, and should not be confused with  $\in \mathcal{C}(c \otimes a \otimes b \to a \otimes b \otimes c)$.

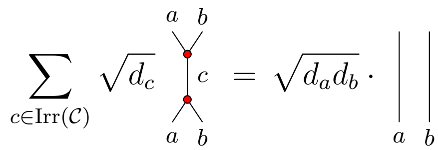
Exercise 4.7.4 ([HP17, Lem. 2.16])**.** Prove the following relations:



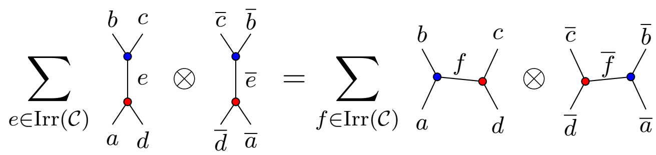
(Bigon 1)



(Bigon 2)



(Fusion)

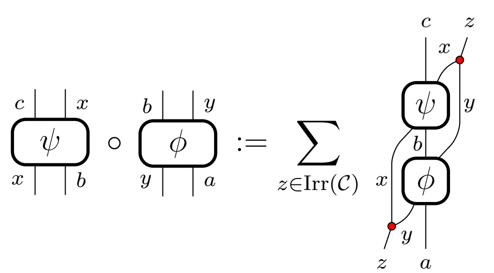


(I=H)

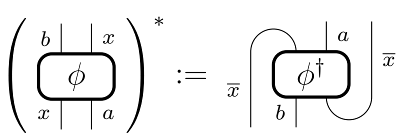
Definition 4.7.5. The (affine) *annular category* $\mathsf{Ann}(\mathcal{C})$ of $\mathcal{C}$ has the same objects as $\mathcal{C}$, with morphisms defined by **TODO: give definition more like skein module. Make exercise to show it is equivalent to this one.**

$$\mathsf{Ann}(\mathcal{C})(a \to b) := \bigoplus_{x \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(x \otimes a \to b \otimes x).$$

Composition in the annular category is given by the linear extension of



We define a †-structure on $\mathsf{Ann}(\mathcal{C})$ by

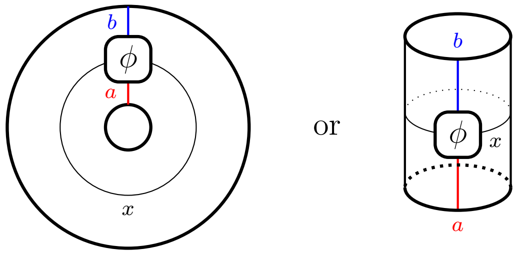


The *tube algebra* is the unital complex $*$-algebra defined by

$$\mathrm{Tube}(\mathcal{C}) := \bigoplus_{a,b \in \mathrm{Irr}(\mathcal{C})} \mathsf{Ann}(\mathcal{C})(a \to b)$$

with multiplication given by composition in $\mathsf{Ann}(\mathcal{C})$ in the usual 'matrix multiplication' style.

The reason the annular category and tube algebra have this name is that one may view the annular category as the category whose objects are standard circles S^1 labelled by a choice of object of $\mathcal{C}$, and whose morphisms are $\mathcal{C}$-diagrams drawn on an annulus, subject to the relation that we may apply any *local relations* in $\mathcal{C}$ to a *contractible region* in the annulus. (See Definition 4.7.10 below for a more precise definition.) We then represent morphisms in the annular category or elements of the tube algebra by

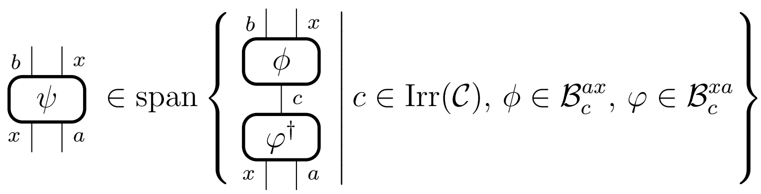


Composition/multiplication then corresponds to gluing annuli/tubes along their boundaries. Finally, the (Fusion) relation ensures that this graphical composition/multiplication agrees with the algebraic definition.

Warning 4.7.6. Even though the graphical calculus of $\mathcal{C}$ admits rotation by 2π as $\mathcal{C}$ admits a canonical unitary spherical structure, the boundary circles here are fixed and cannot rotate.

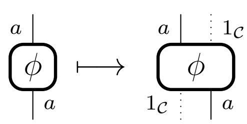
Exercise 4.7.7. Prove that $\mathsf{Ann}(\mathcal{C})(1_\mathcal{C} \to 1_\mathcal{C})$ is $*$-isomorphic to the fusion algebra $\mathcal{FA}(\mathcal{C})$.

Exercise 4.7.8. Show that for each $a, b \in \operatorname{Irr}(\mathcal{C})$, $\mathsf{Ann}(\mathcal{C})(a \to b)$ is finite dimensional. *Hint: Express each element of* $\mathsf{Ann}(\mathcal{C})(a \to b)$ *in terms of the* spine basis*:*



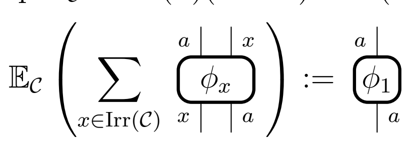
Exercise 4.7.9 (Adapted from [DGG14, §3] and [GJ16])**.** Let $a \in \mathcal{C}$ and consider the endomorphism $*$-algebra $\mathsf{Ann}(\mathcal{C})(a \to a)$.

(1) Prove that the map $\iota : \mathcal{C}(a \to a) \hookrightarrow \mathsf{Ann}(\mathcal{C})(a \to a)$ given by



is a unital $*$-algebra homomorphism.

(2) Show that the linear map $\mathbb{E}_\mathcal{C} : \mathsf{Ann}(\mathcal{C})(a \to a) \to \mathcal{C}(a \to a)$ given by



satisfies the following 'conditional expectation' properties:

(a) $\mathbb{E}_\mathcal{C}(\iota(x) \circ y \circ \iota(z)) = x \circ \mathbb{E}_\mathcal{C}(y) \circ z$ for all $y \in \mathsf{Ann}(\mathcal{C})(a \to a)$ and $x, z \in \mathcal{C}(a \to a)$,

(b) $\mathbb{E}_\mathcal{C}(x^*) = \mathbb{E}_\mathcal{C}(x)^\dagger$ for all $x \in \mathsf{Ann}(\mathcal{C})(a \to a)$, and

(c) $\mathbb{E}_\mathcal{C}(\iota(x)) = x$ for all $x \in \mathsf{Ann}(\mathcal{C})(a \to a)$.

(3) Show that the linear functional $\mathrm{tr}_a := d_a^{-1} \cdot \mathrm{tr}_a^{\mathcal{C}} \circ \mathbb{E}_{\mathcal{C}} : \mathsf{Ann}(\mathcal{C})(a \to a) \to \mathbb{C}$ is a faithful tracial state on $\mathsf{Ann}(\mathcal{C})(a \to a)$.
(4) Use Exercise 4.7.8 to deduce that $\mathsf{Ann}(\mathcal{C})$ is a unitary category and $\mathrm{Tube}(\mathcal{C})$ is a unitary algebra.
(5) Deduce that $\mathbb{E}_{\mathcal{C}}$ is the unique normalized trace-preserving conditional expectation.

Definition 4.7.10. Let Σ_g be a compact genus g orientable surface with boundary. [[discuss orientation/framing for boundary]] Let $\partial^{\mathrm{in}} := \Sigma_g^{\mathrm{in}}$ denote the ingoing boundary and $\partial^{\mathrm{out}} := \partial\Sigma_g^{\mathrm{out}}$ denote the outgoing boundary. Observe that $\partial^{\mathrm{in}}, \partial^{\mathrm{out}}$ are finite disjoint unions of circles.

A *boundary $\mathcal{C}$-decoration* of Σ_g consists of finitely many marked points on each connected component of ∂^{in} and ∂^{out}, and each point is labelled by an object of $\mathcal{C}$. A *$\mathcal{C}$-diagram* on Σ_g consists of a boundary $\mathcal{C}$-decoration and a string diagram drawn on Σ_g such that strings meet the boundary at exactly the marked points, whose labels match the labels of the strings.

Two $\mathcal{C}$-diagrams on Σ_g are said to be equivalent via a *local relation* if we can get from the first diagram to the second by either:

- Performing a smooth isotopy of the ambient manifold Σ_g, or
- substituting equal string diagrams in $\mathcal{C}$ into a contractible disk in $\mathcal{C}$ whose boundary only meets strings transversally in the $\mathcal{C}$-diagram.

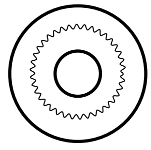
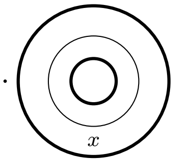
We say that two $\mathcal{C}$-diagrams are equivalent via local relations if there is a finite composite of local relations taking one to the other. Thus for each boundary $\mathcal{C}$-decoration δ on Σ_g, we get a vector space $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g, \delta)$ of $\mathcal{C}$-diagrams with boundary $\mathcal{C}$-decoration δ, modulo local relations.

The Σ_g *skein module over* $\mathcal{C}$ $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$ consists of the family of complex vector spaces $\{\mathrm{Skein}_{\mathcal{C}}(\Sigma_g, \delta)\}_\delta$ indexed over all boundary $\mathcal{C}$-decorations.

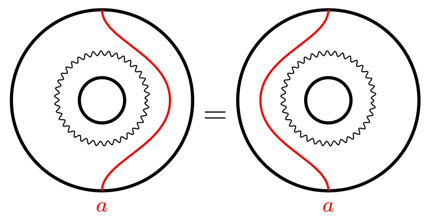
Exercise 4.7.11. Fix a boundary circle C of ∂^{in} of Σ_g. Show that gluing annuli into C turns $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$ into a module for $\mathsf{Ann}(\mathcal{C})$.

Exercise 4.7.12. Prove that for a fixed boundary $\mathcal{C}$-decoration δ on Σ_g, the space $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g, \delta)$ is finite dimensional.
Hint: Write Σ_g as a van Kampen diagram, and use the method of Exercise 4.7.8 to find a finite spine basis *for* $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g, \delta)$*. For spines on an n-holed disk, see* [Che14, Ch. 6].

Definition 4.7.13. Define the *global dimension* of $\mathcal{C}$ by $D_{\mathcal{C}} := \sum_{c \in \mathrm{Irr}(\mathcal{C})} d_c^2$. The *regular element* of $\mathsf{Ann}(\mathcal{C})(1_{\mathcal{C}} \to 1_{\mathcal{C}})$ is given by

$R :=$  $= \frac{1}{D_{\mathcal{C}}} \sum_{x \in \mathrm{Irr}(\mathcal{C})} d_x \cdot$  (4.7.14)

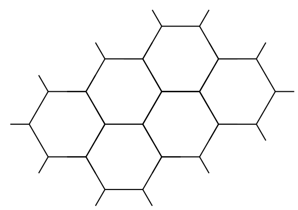
Exercise 4.7.15. Use the (Fusion) relation to prove that the regular element satisfies the relation



$$\forall\, a \in \mathrm{Irr}(\mathcal{C})\,.$$

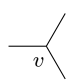
4.8. **Levin-Wen string net condensation.** We now present the Levin-Wen string net (2+1)D lattice model [LW05]. Our treatment will follow [KK12] as summarized in [Kon14]; when we make the reduction to the multiplicity free case, we will recover the original construction.

Let $\mathcal{C}$ be a unitary fusion category with fusion rule $(\mathrm{Irr}(\mathcal{C}), N_{\bullet\bullet}^{\bullet})$ and F-symbols $F_{\bullet}^{\bullet\bullet\bullet}$. Consider a hexagonal 2D lattice:

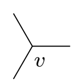


(4.8.1)

(The shape of the lattice is not really important, but the hexagonal/honeycomb lattice is easier to handle as it leads to smaller Hilbert spaces.) Every link in the lattice carries a *left to right* orientation. To every vertex, we assign the local Hilbert space $\mathcal{L}_+ := \bigoplus_{a,b,c\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to b\otimes c)$ or $\mathcal{L}_- := \bigoplus_{a,b,c\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\otimes b \to c)$ depending on the orientation, with the isometry/coisometry inner product.



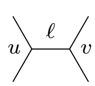
$$= \bigoplus_{a,b,c\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\to b\otimes c)$$



$$= \bigoplus_{a,b,c\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\otimes b\to c)$$

The total Hilbert spaces is thus $\mathcal{H} := \bigotimes_v \mathcal{L}$.

For each edge/link ℓ in the lattice connecting vertices u, v, there is a 2-local *edge term* A_ℓ which projects to the subspace of $\mathcal{L}_u \otimes \mathcal{L}_v$ whose simple labels match along ℓ:

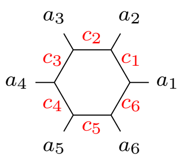


$$= \bigoplus_{a,b,c,d,e\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\otimes b\to e)\otimes \mathcal{C}(e\to c\otimes d)$$

$$\subset \left(\bigoplus_{a,b,e\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\otimes b\to e)\right)\otimes\left(\bigoplus_{c,d,f\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(f\to c\otimes d)\right)$$

Observe A_ℓ and $A_{\ell'}$ are commuting projections if $v\neq v'$. Thus to belong to the ground state space $\mathcal{G}_A$ of $-\sum A_\ell$, a vector must represent a valid $\mathcal{C}$-diagram drawn on the lattice. That is, if our lattice is drawn on the compact orientable genus g surface Σ_g with empty boundary, then there is a well-defined 'forgetful operation' from $\mathcal{G}_A$ to $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$ which forgets the lattice. One can view the operator A_ℓ as *condensing* the link ℓ from the vacuum between the two vertices u, v. That is, passing to the ground state of $-A_\ell$ connects the two vertices by a link.

We now define another part of the local Hamiltonian, but we only do so on the image $\mathcal{G}_A$ of the projector $P_A := -\prod A_\ell$. For each face/plaquette p in the lattice, there is a 6-local *plaquette term* B_p, which we will define in two ways; one uses skein theory, and the other uses the fusion relations from Exercise 4.7.4. First, given our plaquette p, we define the projector E_p to the hom space in $\mathcal{C}$ determined by the simples on the legs of p:



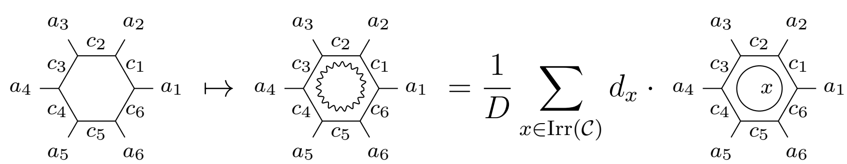
$$= \bigoplus_{c_1,\ldots,c_6} \begin{array}{l} \mathcal{C}(c_1 \otimes c_6 \to a_1) \otimes \mathcal{C}(c_2 \to a_2 \otimes c_1) \otimes \mathcal{C}(a_3 \otimes c_3 \to c_2) \\ \otimes \mathcal{C}(a_4 \to c_3 \otimes c_4) \otimes \mathcal{C}(c_4 \otimes a_5 \to c_5) \otimes \mathcal{C}(c_5 \to c_6 \otimes a_6) \end{array} \qquad (4.8.2)$$

$$\xmapsto{E_p} \mathcal{C}(a_3 \otimes a_4 \otimes a_5 \to a_2 \otimes a_1 \otimes a_6)$$

Exercise 4.8.3. Observe neither the source nor the target of the right hand side are necessarily simple. How do we endow the right hand side with an inner product? Can you do this in a way such that E_p is a partial isometry?
Hint: One can alternatively describe the (co)isometry inner product as some normalization of the trace inner product using the canonical unitary spherical structure of $\mathcal{C}$.

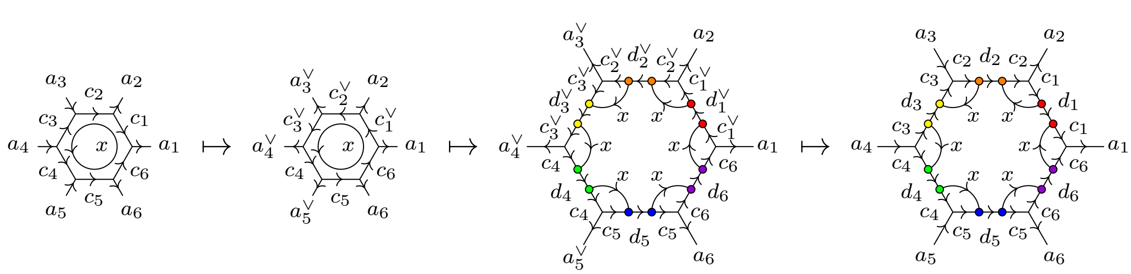
Using Exercise 4.8.3, we define the projector $B_p := E_p^\dagger E_p$.

We now give an alternative description of the projector B_p. Recall our UFC $\mathcal{C}$ consists of a fusion rule $(\mathrm{Irr}(\mathcal{C}), N_{\bullet\bullet}^{\bullet})$ together with the F-matrices. This means we implicitly have a trivalent basis for every hom space $\mathcal{C}(a \to b \otimes c)$ and $\mathcal{C}(a \otimes b \to c)$. In short, the operator B_p *glues in* the regular element R from (4.7.14) into the plaquette.



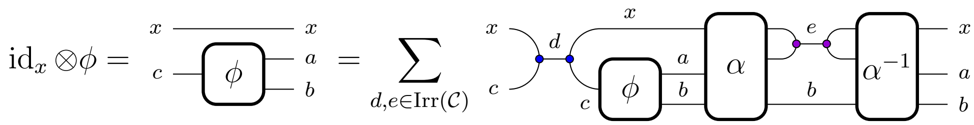
We then resolve the right hand side using the (Fusion) relation.

Exercise 4.8.4. Let $\mathcal{H}_{\hexagon}$ denote the Hilbert space (4.8.2). Consider the map $B_p^{\cdot)} : \mathrm{Irr}(\mathcal{C}) \to \mathcal{B}(\mathcal{H}_{\hexagon})$ given by x maps to the operator B_p^x given by



(4.8.5)

Here, we add orientations, since it is very confusing if we continue to omit them. Note that we switch orientations in the first and third arrows for ease of applying the (Fusion) relation. Observe that the right hand side in (4.8.5) is a linear combination of vectors in $\mathcal{H}_{\hexagon}$. We warn the reader that the second arrow requires the use of the F-matrices to re-associate in order to apply the (Fusion) relation. We give an explicit example below:



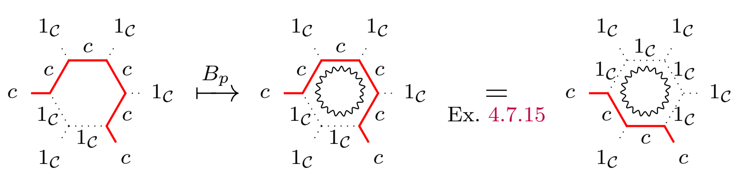
(1) Show that the map $x \mapsto B_p^x$ extends to an algebra homomorphism : $\mathcal{FA}(\mathcal{C}) \to \mathcal{B}(\mathcal{H}_{\hexagon})$.
Hint: See [Zha17, §5.4].
(2) Deduce that $B_p^2 = B_p$.
(3) Is this map a $*$-algebra map?

Exercise 4.8.6 ($\star\star$)**.** Prove that this definition of B_p agrees with the previous definition of B_p.

We now discuss the effect of these plaquette terms on $\mathcal{G}_A$ (the ground state space of $-\sum A_\ell$). Given a vector in $\mathcal{G}_A$, we saw there is a well-defined forgetful operation to $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$, giving a $\mathcal{C}$-diagram on Σ_g. The operator $\prod B_p$ implements an 'averaging procedure' over equivalent ways to write that $\mathcal{C}$-diagram on the lattice. That is, we can say two vectors in $\mathcal{G}_A$ represent the same $\mathcal{C}$-diagram if they forget to equal elements of $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$. We claim then that two such vectors are mapped to same vector under $\prod B_p$.

Exercise 4.8.7 ($\star\star$)**.** Prove that two vectors in the ground state space of $-\sum A_\ell$ which represent the same $\mathcal{C}$-diagram in the underlying skein module are mapped to the same vector under $\prod B_p$.

An extremely basic example of this phenomenon is when we consider one plaquette p and a vector in $\mathcal{G}_A$ which locally has only one c-string on the edges neighboring p, with all other edges labelled by $1_{\mathcal{C}}$:



Thus one can view the operator B_p as *condensing* the plaquette p from the vacuum from the empty hexagon condensed by the operators A_ℓ around the boundary. That is, passing to the ground state of $-B_p$ glues in a disk to the boundary hexagon.

Hence, the ground state of the 6-local Hamiltonian

$$H := -\sum A_\ell - P\left(\sum B_p\right)P$$

on $\mathcal{H}$ is isomorphic to $\mathrm{Skein}_{\mathcal{C}}(\Sigma_g)$ as a vector space. Note, however, that each ground state $\mathcal{C}$-diagram is 'spread out'/'averaged' over all possible ways to represent it on the lattice.

4.9. **Multiplicity free case.** In the multiplicity free case, we can give an equivalent formulation of the Levin-Wen string-net model where the *degrees of freedom* (translation: Hilbert spaces) live on edges instead of vertices. One can show these two formulations are equivalent by showing they are subspaces of a model where degrees of freedom live on both the edges and the vertices as in [Zha17].

Again, we fix a UFC $\mathcal{C}$ with fusion rule $(\mathrm{Irr}(\mathcal{C}), N_{\bullet\bullet}^{\bullet})$ and F-symbols $F_{\bullet}^{\bullet\bullet\bullet}$. Recall this necessitated fixing orthogonal bases $\mathcal{B}_a^{bc}$ for each Hilbert space $\mathcal{C}(a \to b \otimes c)$.

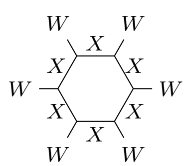
Consider the same hexagonal 2D lattice (4.8.1), where now the hexagonal/honeycomb structure is essential. Again, every link in the lattice carries a *left to right* orientation. To each edge in the lattice, we assign the Hilbert space $\mathbb{C}^{\mathrm{Irr}(\mathcal{C})}$. For each vertex v we define the 3-local projection A_v on the standard basis of $(\mathbb{C}^{\mathrm{Irr}(\mathcal{C})})^{\otimes 3}$ corresponding to the trivalent vertex as follows:

Basically, the point of A_v is to check whether the UFC $\mathcal{C}$ admits a non-zero morphism in the hom space determined by the three surrounding labelled edges. If there is no non-zero morphism, A_v returns zero; otherwise it preserves the vector.

Observe now that A_v and $A_{v'}$ commute for all vertices v, v' in the lattice. Hence $Q_A := \prod A_v$ is a projector, and again the ground state space of $-Q_A$ or $-\sum A_v$ can again be identified with the space of $\mathcal{C}$-diagrams drawn on the hexagonal honeycomb lattice. The definition of B_p remains unchanged.

Example 4.9.1 (Doubled semion model)**.** Consider the UFC $\mathsf{Hilb_{fd}}(\mathbb{Z}/2, \omega)$ for the non-trivial 3-cocycle ω satisfying $\omega(g, g, g) = -1$ and all other values are $+1$. The local Hilbert space is $\mathbb{C}^2 = \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$ for each edge, where we view $|0\rangle$ as 'off' and $|1\rangle$ as 'on' as in toric code. We define for each vertex v the vertex term

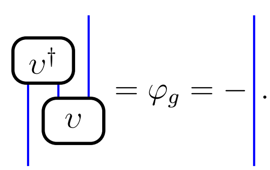
The plaquette term is the cutdown by P_A (the projector onto $\mathcal{G}_A$) of the operator



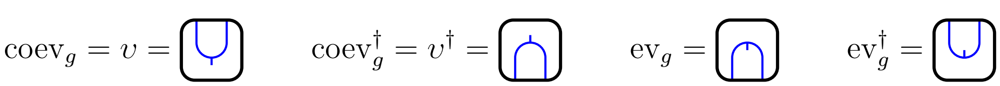
$$W = i^{\frac{1-Z}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Observe that on the ground state space $\mathcal{G}_A$ of $-\sum_v A_v$, every vertex has an even number of edges 'on', and so the operator B_p has only *real* eigenvalues.

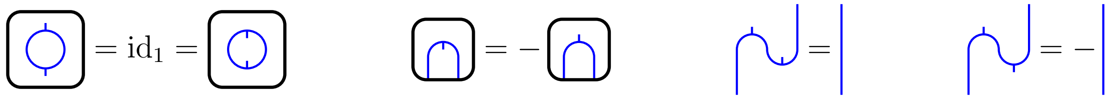
We claim that the ground state space of the 12-local Hamiltonian $H = -\sum_v A_v - \sum_p B_p$ is isomorphic to the skein module for $\mathsf{Hilb_{fd}}(\mathbb{Z}/2, \omega)$. Recall that the basis element $v \in \mathcal{B}_1^{gg}$, which is unique up to unique phase, satisfies $v^\dagger v = \mathrm{id}_1$. From a previous exercise, we saw that



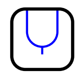
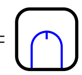
Thus $g \in \mathsf{Hilb_{fd}}(\mathbb{Z}/2, \omega)$ is pseudo-real. An elegant way to graphically represent the cup and cap of a psuedo-real object uses *disorientations* [CMW09], which are single co-oriented tags on strings:



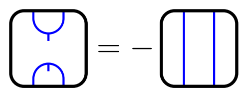
We then get the following graphical relations:



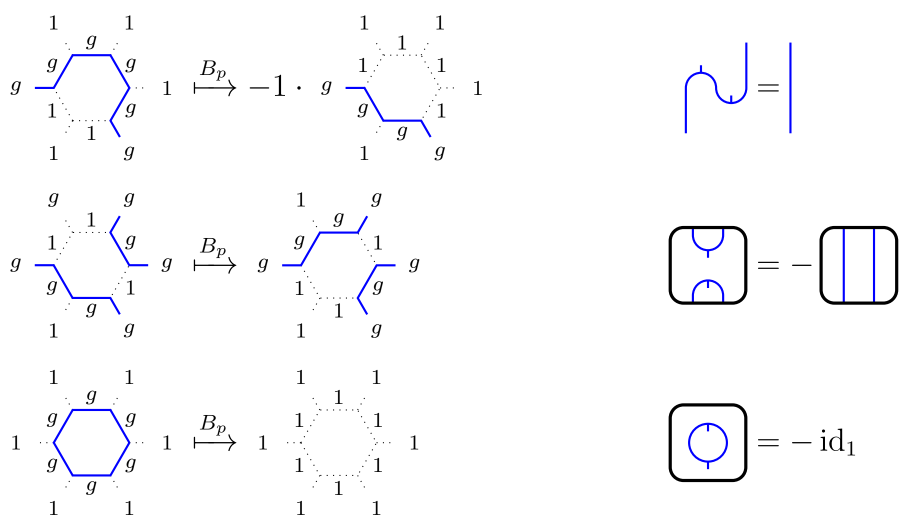
Now the skein module for $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2,\omega)$ for Σ_g is spanned by diagrams of closed loops drawn on Σ_g, where we now include small disorientations. We check that the operator $-B_p$ implements these skein relations. We use the convention that our $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2,\omega)$-diagrams drawn on our lattice in the ground state space $\mathcal{G}_A$ of $-\sum A_v$ have disorientations which point *downwards* at local minima, i.e., we always have

$\mathrm{coev}_g = \upsilon =$  $\qquad \mathrm{ev}_g =$ 

Exercise 4.9.2. Prove the following *recabling relations* in $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2,\omega)$:



We calculate that the action of B_p applies X to swap $|1\rangle$ and $|g\rangle$ on the hexagon, and applies W to multiply by i for every boundary edge turned on in the $|g\rangle$ state, e.g.:



Thus the operator $-B_p$ exactly implements the skein relations for $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2,\omega)$, and passing to the ground state of $-\sum B_p$ averages over these skein relations. The ground states for $H = -\sum A_v - \sum B_p$ are then computed similarly to the toric code.

4.10. **Quasiparticle excitations.** As we did with the toric code, we now discus the lowest energy excitations for the Levin-Wen string net models. As our local Hamiltonian is given as $H = -\sum A_\ell - \sum_p B_p$, we again see that $|\psi\rangle$ is in the ground state if and only if

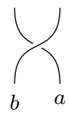
$$A_\ell|\psi\rangle = |\psi\rangle = B_p|\psi\rangle \qquad\qquad \forall\, \ell, p.$$

It turns out that quasiparticle excitations, which correspond to a minimal number of violations of these conditions [[how many?]] always occur in particle/antiparticle pairs. [[location of excitations corresponds to locations of these energy violations.]] As before, a pair of quasiparticle excitations is produced via a *string operator* on the lattice, but only the endpoints of the string are observable; we may topologically deform the string at zero energy cost.

We will now give an ansatz for computing the lowest energy excitations.

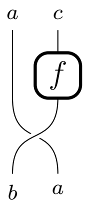
Ansatz 4.10.1. Excitations of the Levin-Wen string-net model correspond to pairs (a, σ_a) consisting of an object $a \in \mathcal{C}$ equipped with a *unitary half-braiding* σ_a.

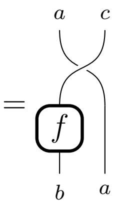
Definition 4.10.2. A *unitary half-braiding* for $a \in \mathcal{C}$ is a family unitary isomorphisms

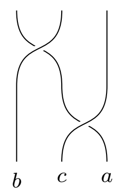
$$\sigma_a = \left\{ \quad \right.$$


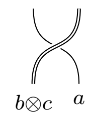
$$\left. = \sigma_{a,b} : b \otimes a \to a \otimes b \right\}_{b \in \mathcal{C}}$$

satisfying the following axioms:

- (naturality) for all $f \in \mathcal{C}(b \to c)$, 

$= (\mathrm{id}_a \otimes f) \circ \sigma_{a,b} = \sigma_{a,c} \circ (f \otimes \mathrm{id}_a) =$ 

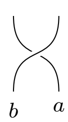
.
- (monoidality) For all $b, c \in \mathcal{C}$, 

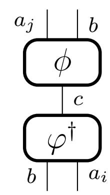
$=$ 

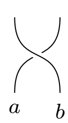
, where we have suppressed the associators. More formally, the following diagram should commute:

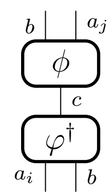
$$\begin{array}{ccccc} b \otimes (c \otimes a) & \xrightarrow{\mathrm{id}_b \otimes \sigma_{a,c}} & b \otimes (a \otimes c) & \xrightarrow{\alpha} & (b \otimes a) \otimes c \\ \downarrow \alpha & & & & \downarrow \sigma_{a,b} \otimes \mathrm{id}_c \\ (b \otimes c) \otimes a & \xrightarrow{\sigma_{a, b\otimes c}} & a \otimes (b \otimes c) & \xrightarrow{\alpha} & (a \otimes b) \otimes c \end{array} \tag{4.10.3}$$

We can also describe the half-braiding σ_a in terms of the Ω-*matrices*, which must satisfy a coherence condition with the F-matrices. Suppose (a, σ_a) is an excitation. Since $a \in \mathcal{C}$, we can write $a = \bigoplus a_i$ as a direct sum of simples. Observe that for every simple object $\sigma_{a,b} : a \otimes b \to b \otimes a$ we can decompose σ_a and $\sigma_a^{-1} = \sigma_a^\dagger$ into component composite maps



$$= \sum_{i,j} \sum_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ \phi \in \mathcal{B}_c^{b a_j} \\ \varphi \in \mathcal{B}_c^{a_i b}}} \sqrt{\frac{d_c}{d_b \sqrt{d_{a_i} d_{a_j}}}} [\Omega_{a,b}^{a_i c a_j}]_{\varphi, \phi} \cdot$$




$$= \sum_{i,j} \sum_{\substack{c \in \mathrm{Irr}(\mathcal{C}) \\ \phi \in \mathcal{B}_c^{b a_j} \\ \varphi \in \mathcal{B}_c^{a_i b}}} \sqrt{\frac{d_c}{d_b \sqrt{d_{a_i} d_{a_j}}}} [\overline{\Omega}_{a,b}^{a_i c a_j}]_{\varphi, \phi} \cdot$$


(These conventions are adapted from [LLB20, (42,43)].) The $\Omega^{\bullet\bullet\bullet}_{\bullet\bullet}$ must satisfy the following relations:

(Ω1) **TODO:**

(Ω2) $\overline{\Omega}_{a,b}^{a_i c a_j} = (\Omega_{a,b}^{a_j c a_i})^\dagger$ for all $a_i, a_j, c \in \mathrm{Irr}(\mathcal{C})$
(Ω3) $\overline{\Omega}_{a,b}^{a_i c a_j} \Omega_{a,b}^{a_k d a_\ell} = \delta_{c=d}\delta_{a_j=a_k}$ for all $a_i, a_j, a_k, a_\ell, c, d \in \mathrm{Irr}(\mathcal{C})$.

Exercise 4.10.4. Prove that:

(1) (4.10.3) and naturality of $\sigma_{a,b}$ are equivalent to (Ω1) for $\Omega_{a,b}$.
(2) unitarity of $\sigma_{a,b}$ is equivalent to (Ω2) and (Ω3).

We will see in the next module how these excitations form a *unitary modular tensor category*.

Example 4.10.5. We now calculate all quasi-particle excitations for the toric code and the doubled semion models simultaneously as both are built from $\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2, \omega)$ with ω a 3-cocycle determined by $\omega(g, g, g) = \pm 1$ and all other values $+1$. For notational simplicity, we simply write $\omega := \omega(g, g, g)$.

It turns out that for these models, it suffices to find the excitations whose underlying objects are simple. Since all simples here are invertible, this means that the Ω-tensors are simply complex numbers which only depend on 2 indices. Indeed, $a \otimes b$ and $b \otimes a$ are both simple when $a, b \in \mathrm{Irr}(\mathcal{C})$ and a is invertible. Thus there is at most a 1-dimensional space of maps $a \otimes b \to b \otimes a$.

- Suppose $a = 1 \in \mathbb{Z}/2$. For [[]], (Ω1) reduces to $\Omega_{1,1}^2 = \Omega_{1,1}$, so $\Omega_{1,1} = 1$. For [[]], (Ω1) reduces to $\Omega_{1,g}^2 = 1$, so $\Omega_{1,g} = \pm 1$. We see this gives 2 solutions for the Ω tensors for $a = 1$.
- Suppose $a = g \in \mathbb{Z}/2$. For [[]], (Ω1) reduces to $\Omega_{g,1}^2 = \Omega_{g,1}$, so $\Omega_{g,1} = 1$. For [[]], (Ω1) reduces to $\Omega_{1,g}^2 = \omega$.
  - When $\omega = 1$, we have $\Omega_{1,g} = \pm 1$. Our four excitations are exactly the $1, m, e, \epsilon$ lowest energy excitations of the toric code.
  - When $\omega = -1$, we have $\Omega_{1,g} = \pm i$. Our four excitations are then $1, b, s, \overline{s}$ of the doubled semion model.

4.11. **Topological quantum field theory.** The notion of topological quantum field theory (TQFT) we present here is due to Atiyah [Ati88] based on Segal's definition of a conformal field theory [Seg, Seg04]. The notion of *extended* TQFT appears to be from [Law93, Fre94, FQ93, BD95] and unpublished work of Kevin Walker [Wal91].

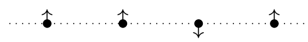
Definition 4.11.1. An (*adjectives*) *n-dimensional topological quantum field theory* is a symmetric monoidal functor from the symmetric monoidal category of n-dimensional (*adjectives*) bordisms $n\mathsf{Bord}^{(adj)}$ to the symmetric monoidal category Vec of vector spaces. Here, *adjectives* may include *framed, combed, oriented, Riemannian*, etc.

The symmetric monoidal category of n-dimensional (*adjectives*) bordisms $n\mathsf{Bord}^{(adj)}$ has as objects compact (*adjectives*) $(n-1)$-manifolds, and morphisms compact n-dimensional *bordisms*, i.e., n-manifolds with incoming and outgoing boundary $(n-1)$-manifolds ∂^{in} and ∂^{out} respectively. Composition is gluing of bordisms, and the symmetric monoidal structure is disjoint union.

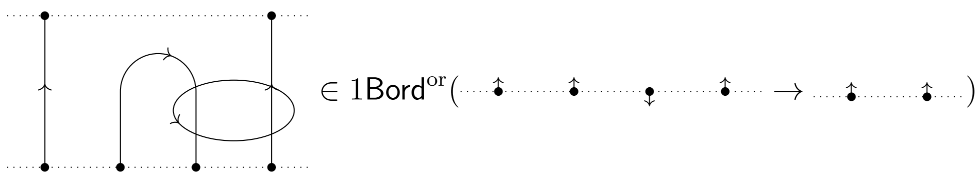
Remark 4.11.2. Really, the morphisms are not manifolds with boundary; they are closed manifolds with collars drawn on them which have the same structures as the *adjectives* provide. One then cuts along these collars in order to glue two such manifolds.

Remark 4.11.3. Sometimes in the literature, you may see topological field theories without the word 'quantum.' The word 'quantum' should really only be added when the receptacle for the field theory is enriched in vector/Hilbert spaces. There is nothing 'quantum' about a TFT in which one cannot take superpositions at the top categorical level.

Example 4.11.4. The 1D oriented bordism category $1\mathsf{Bord}^{\mathrm{or}}$ has objects finite disjoint unions of points which come with an orientation, up or down. We should think of these points as living along an imaginary horizontal line, e.g.:



A morphism is a finite disjoint union of oriented arcs and circles whose end points lie on these horizontal lines, where we read *bottom to top*, e.g.:



Exercise 4.11.5. Suppose $F : 1\mathsf{Bord}^{\mathrm{or}} \to \mathsf{Vec}$ is a 1D oriented TQFT.

(1) Prove that $F(\downarrow)$ and $F(\uparrow)$ are finite dimensional.
Hint: They are dualizable!

(2) Show that $F(\downarrow)$ completely determines the TQFT up to unique natural isomorphism.

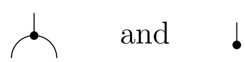
Exercise 4.11.6. Show that there is only one ε-thick circle up to diffeomorphism.

Exercise 4.11.7. Now consider a 2D oriented TQFT $F : 2\mathsf{Bord}^{\mathrm{or}} \to \mathsf{Vec}$ Observe that the objects of $2\mathsf{Bord}^{\mathrm{or}}$ are 1D compact manifolds, which must consist of a finite disjoint union of oriented circles, which are actually ε-thick with an orientation.

Consider such an ε-thick oriented circle C, and consider the vector space $F(C)$.

(1) Prove that $F(C)$ is finite dimensional.
Hint: Prove $F(C)$ is dualizable.

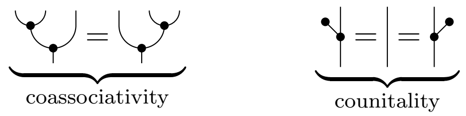
(2) Show that the TQFT F endows $F(C)$ with the structure of an *algebra*, i.e., there is a multiplication $m : F(C) \otimes F(C) \to F(C)$ and a unit map $i : 1_{\mathsf{Vec}} \to F(C)$ which we represent by a trivalent and univalent vertex respectively



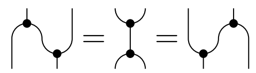
which satisfy the following relations:

(3) Show that the TQFT F endows $F(C)$ with the structure of a *coalgebra*, i.e., there is a comultiplication $\Delta : F(C) \to F(C) \otimes F(C)$ and a counit map $\epsilon : F(C) \to 1_{\mathsf{Vec}}$ which we also represent by a trivalent and univalent vertex respectively

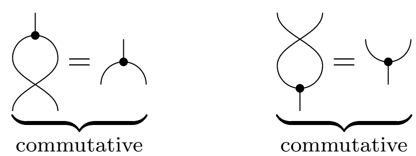
which satisfy the following relations:



(4) Show that the TQFT F endows $F(C)$ with the structure of a *Frobenius* algebra, i.e., the algebra structure (m, i) and the coalgebra structure (Δ, ϵ) satisfy the following additional relations:



(5) Prove that $F(C)$ is commutative and cocommutative, i.e.,



Definition 4.11.8. For $1 \leq k \leq n$, a $(k-1)$-*extended* n-dimensional TQFT is a symmetric monoidal k-functor from the symmetric monoidal k-category of n-dimensional (*adjectives*) bordisms with co-dimension $k-1$ corners to some symmetric monoidal k-category $k\mathsf{Vec}$ of *k-vector spaces*. [[check for off by one errors.]]

An $(n-1)$-extended n-dimensional TQFT is called *fully extended*.

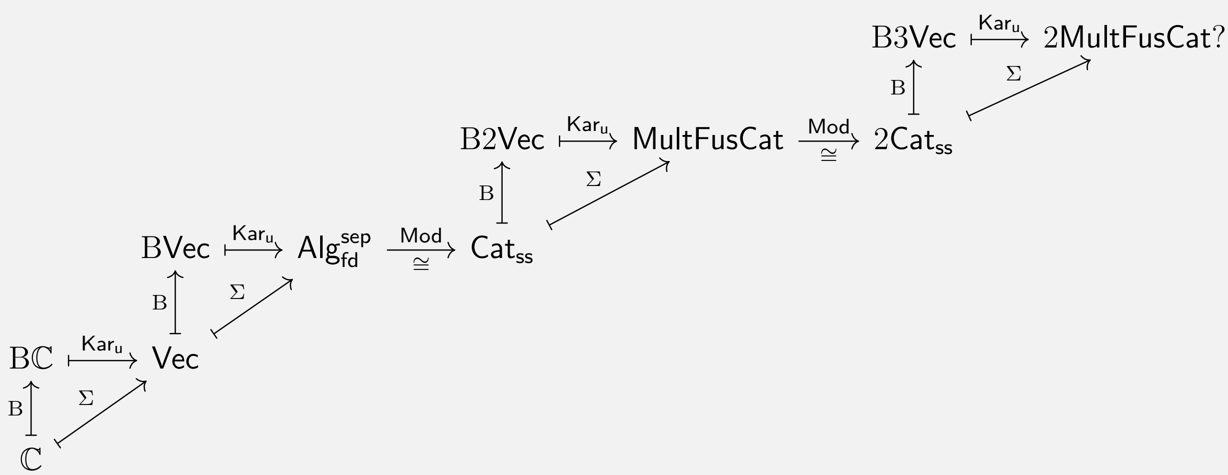
The term *k-vector spaces* is a bit nebulous, as it should be at this time. At first, researchers had a good idea about what $k\mathsf{Vec}$ ought to be, and made the following educated guesses.

Symmetric monoidal k-category $k\mathsf{Vec}$ of k-vector spaces.

$k\mathsf{Vec}$	name	description
$1\mathsf{Vec}$	$\mathsf{Vec_{fd}}$	finite dimensional complex vector spaces and linear maps
$2\mathsf{Vec}$	$\mathsf{Alg_{fd}^{sep}}$	finite dimensional separable unital algebras, unital bimodules, and intertwiners
$3\mathsf{Vec}$	$\mathsf{MultFusCat}$	multifusion categories, bimodule categories, bimodule functors, and bimodule natural transformations

Evidence for these definitions comes from the classification of 1D and 2D TQFTs and the existence of the fully extended Turaev-Viro-Ocneanu TQFT which we will describe in the next section. Recent progress in higher idempotent completions for n-categories [DR18, GJF19, JF20] give us a more-or-less precise notion of what $k\mathsf{Vec}$ is meant to be. For $k = 1, 2, 3$, we will see that the answer agrees with our guess above. Moreover, higher idempotent completion fits these examples into a sequence where $(k+1)\mathsf{Vec}$ arises formally from $k\mathsf{Vec}$. We will revisit these ideas in a future module.

Formal construction of kVec from $(k-1)$Vec.



Notation:

- B means take the *delooping* [BS10, §5.6], i.e., consider the monoidal k-category as a $(k+1)$-category with one object.
- $\mathsf{Kar_u}$ means take a unital higher Karoubi completion [GJF19].
- Σ is the composite $\mathsf{Kar_u} \circ \mathrm{B}$, called the *suspension*.
- Mod is the equivalence given by taking the 1- or 2-category of modules for the algebra/multifusion category respectively.

Fully extended n-dimesional TQFTs valued in a particular symmetric monoidal n-category are characterized by the Baez-Dolan-Lurie *Cobordism Hypothesis*.

Hypothesis 4.11.9 ([BD95, Lur09]). *Fully extended n-dimensional framed TQFTs are completely determined by where they send a point. In other words, $n\mathsf{Bord}^{\mathrm{fr}}$ is the free symmetric monoidal n-category on a fully dualizable dualizble object.*

This hypothesis motivates the following research question.

Question 4.11.10. *Compute the fully dualizable objects in your favorite symmetric monoidal n-category.*

Some classifications of fully dualizable objects in n-categories.

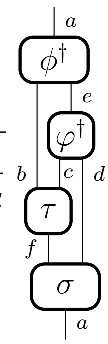
n	n-category	desciption	fully dualizable objects
1	Vec	complex vector spaces	finite dimensional complex vector spaces
2	KarLinCat	Karoubian linear categories	finitely semisimple categories
	$\mathsf{Alg_u}$	unital complex algebras	finite dimensional separable algebras [SP11]
3	2KarLinCat	Karoubian linear 2-categories	semisimple 2-categories [DR18, Dec20]
	(Mult)TensCat	(multi)tensor categories	(multi)fusion categories [DSPS13][a]
4	BrTensCat	braided tensor categories	braided fusion categories [BJS18] (is this all?)

[a]The article [DSPS13] works in the setting finite tensor categories, which are abelian and locally finite. In these notes, we work in the linear Karoubian setting. So there is possibly some discrepancy here, but everything can probably be pieced together from the existing liteature.

4.12. **Turaev-Viro TQFT.** We now define the 3D Turaev-Viro TQFT [TV92], which also owes some credit to Ocneanu [Ocn91, EK98] and Barrett-Westbury [BW99, BW96]. We will give the most accessible definition using 6j-symbols for a UFC $\mathcal{C}$, and later, we will discuss the fully extended TQFT afforded by the Cobordism Hypothesis [BD95, Lur09] and the fact that UFCs are fully dualzable objects in the 3-category of linear tensor categories [DSPS13].

As in [[]] above, we fix a set of representatives of simple object $\operatorname{Irr}(\mathcal{C})$ and orthogonal bases $\mathcal{B}_c^{ab}$ for $\mathcal{C}(c \to a \otimes b)$ for all simples $a, b, c \in \operatorname{Irr}(\mathcal{C})$. Recall that the 6j-symbols were defined as follows:

$$[F_a^{bcd}]_{(e,\phi,\varphi)}^{(f,\sigma,\tau)} \cdot \mathrm{id}_a = \sqrt{\frac{d_a}{d_b d_c d_d}} \qquad (4.12.1)$$



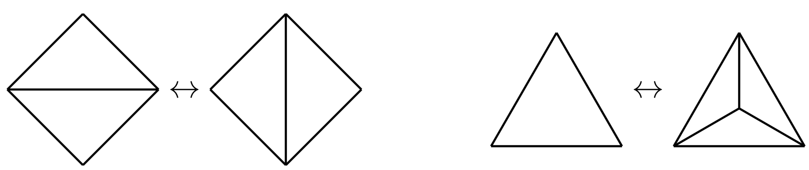
The Turaev-Viro TQFT is a *state sum model* which assigns a vector [[Hilbert?]] space $V(\Sigma)$ to every compact orientable surface Σ and a linear map $Z(W) : V(\Sigma^{\text{in}}) \to V(\Sigma^{\text{out}})$ to every 3-manifold W with boundary $\Sigma^{\text{in}} \amalg \Sigma^{\text{out}}$.

We build $V(\Sigma)$ as follows. First, pick some *triangulation* of Σ, i.e., some presentation of Σ as a CW-complex where every 2-cell is a non-degenerate triangle. Recall that all edges in a CW-complex are necessarily oriented.

TODO:

Later on, we will have to show that $V(\Sigma)$ is independent of this choice of triangulation. We will use the following theorem of [[]].

Theorem 4.12.2 ([[??]]). *Any two triangulations of Σ can be related to one another by a finite sequence of [[]] moves:*

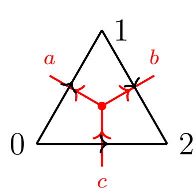


There are different moves corresponding to the possible orientations on the triangles above.

Given a triangulation, the *dual lattice* has one vertex for every triangle, and an edge between two vertices whenever the corresponding triangles share an edge.

TODO:

Observe that edges in the dual lattice inherit an orientation using the right-hand rule.



$\rightsquigarrow \qquad \mathcal{C}(c \to a \otimes b).$

A $\mathcal{C}$-*state* σ on Σ is an assignment of simple objects of $\mathcal{C}$ to every edge in the dual lattice. Given a state σ on Σ, the *state space* for each triangle is the hom space in $\mathcal{C}$ assigned to the vertex in the dual lattice corresponding to that triangle. The space $V(\Sigma)$ is given by the *state sum* formula

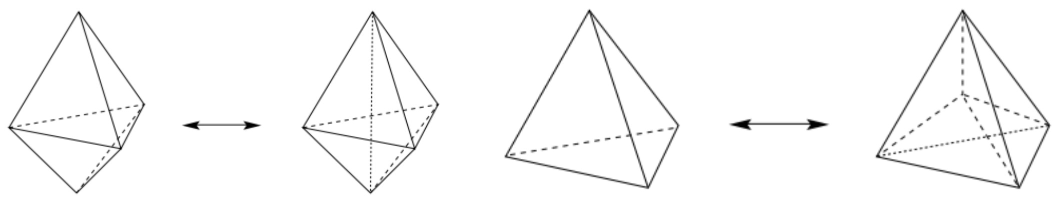
$$V(\Sigma) := \bigoplus_{\sigma} \bigotimes_{T} V_T$$

where we sum over $\mathcal{C}$-states σ and tensor over triangles T, where V_T is the hom space associated to each triangle. [[some kind of relations]]

TODO:

We will use the following theorem of Pachner.

Theorem 4.12.3 ([?]). *Any two triangluations of W can be related to one another by a finite sequence of Pachner moves:*



[BW96, Fig. 7 and 8]

4.13. **(1+1)D relative TQFT. TODO:**

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