One of the main techniques for unitary quantum algebra/topological phases is finite dimensional operator algebras. Here is a terse introduction for $M_n(\mathbb{C})$ via exercises. The presence of ($\star$) on a problem marks difficulty with respect to the exposition, and ($\star\star$) denotes well beyond the scope of these notes.

We denote by $*$ the conjugate transpose operation on $M_n(\mathbb{C})$. The matrix algebra $M_n(\mathbb{C})$ acts on the inner product (Hilbert) space $\mathbb{C}^n$ with inner product given by $\langle \eta, \xi \rangle := \sum_{j=1}^n \eta_j \overline{\xi_j}$ which is linear on the left.

Exercise 1. Show that for $a \in M_n(\mathbb{C})$ and $\eta, \xi \in \mathbb{C}^n$, $\langle a\eta, \xi \rangle = \langle \eta, a^*\xi \rangle$.

Exercise 2. Show that if $a \in M_n(\mathbb{C})$ commutes with all $b \in M_n(\mathbb{C})$, then $a = \lambda 1$ for some $\lambda \in \mathbb{C}$.

Definition 3. An element $a \in M_n(\mathbb{C})$ is called:

- *normal* if $aa^* = a^*a$
- *self-adjoint* if $a^* = a$
- *positive* if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in \mathbb{C}^n$
- a *projection* if $a^2 = a = a^*$
- a *partial isometry* if a^*a is a projection.
- a *unitary* if a is invertible with $a^{-1} = a^*$

Exercise 4. Show that positive implies self-adjoint implies normal.

Exercise 5. Suppose $p \in M_n(\mathbb{C})$ is a *minimal projection*, i.e., $pM_n(\mathbb{C})p = \mathbb{C}p$. Show that there are partial isometries $v_1, \ldots, v_n \in M_n(\mathbb{C})$ such that $\sum_{i=1}^n v_i p v_i^* = 1$.

Exercise 6. Prove that $M_n(\mathbb{C})$ has no non-trivial 2-sided ideals.

Exercise 7. Use Exercise 6 to show that any (not necessarily unital) $*$-algebra map out of $M_n(\mathbb{C})$ into another complex $*$-algebra is either injective or the zero map.

Exercise 8 (Spectral Theorem, $\star$)**.** Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

(1) a is normal.
(2) There is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors for a.
(3) There is a unitary $u \in M_n(\mathbb{C})$ ($uu^* = u^*u = 1$) such that u^*au is diagonal.

Definition 9 (Functional calculus)**.** Suppose $a \in M_n(\mathbb{C})$ is normal. Let $\operatorname{spec}(a)$ denote the *spectrum* of a, which is the set of eigenvalues. For $\lambda \in \operatorname{spec}(a)$, let $E_\lambda \subset \mathbb{C}^n$ denote the corresponding eigenspace, and let $p_\lambda \in M_n(\mathbb{C})$ be the orthogonal projection onto E_λ. Note that

$$a = \sum_{\lambda \in \operatorname{spec}(a)} \lambda p_\lambda,$$

as both operators agree on an orthonormal basis of $\mathbb{C}^n$, namely the orthonormal basis consisting of eigenvectors for a from Exercise 8. For $f : \operatorname{spec}(a) \to \mathbb{C}$, we define

$$f(a) := \sum_{\lambda \in \operatorname{spec}(a)} f(\lambda) p_\lambda \in M_n(\mathbb{C}).$$

Exercise 10. Suppose $a \in M_n(\mathbb{C})$ is normal, and let $C(\operatorname{spec}(a))$ denote the unital $*$-algebra of $\mathbb{C}$-valued functions on $\operatorname{spec}(a)$.

(1) Show that $C(\operatorname{spec}(a)) \ni f \mapsto f(a) \in M_n(\mathbb{C})$ is a unital $*$-algebra homomorphism.
(2) (Spectral mapping) Prove that $\operatorname{spec}(f(a)) = f(\operatorname{spec}(a))$.

Exercise 11. For $a \in M_n(\mathbb{C})$, define

$$\|a\| := \inf\left\{c > 0 \middle| \|a\xi\|_{\mathbb{C}^n} \leq c\|\xi\|_{\mathbb{C}^n} \text{ for all } \xi \in \mathbb{C}^n\right\}.$$

(1) Prove that $\|\cdot\|$ is a norm on $M_n(\mathbb{C})$.
(2) Prove that if $a \in M_n(\mathbb{C})$ is normal, then $\|a\| = \max_{\lambda \in \operatorname{spec}(a)} |\lambda|$.

Exercise 12. Suppose (a_n) is a sequence of normal matrices such that $a_n \to a$ in norm and U is an open neighborhood of $\operatorname{spec}(a)$.

(1) Suppose $K \subset \mathbb{C}$ is compact such that $\operatorname{spec}(a) \subset K^\circ$. Use the Spectral Mapping Theorem 10(2) and Exercise 11(2) to show that $\|f(a)\| \leq \|f\|_{C(K)}$ for all $f \in C(K)$, where $\|f\|_{C(K)} := \max_{k \in K} |f(k)|$.
(2) Show that eventually $\operatorname{spec}(a_n) \subset U$.
Hint: $GL_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$.
(3) Show that for every continuous $f : U \to \mathbb{C}$, $f(a_n) \to f(a)$.
Hint: Pick an open set V with $\overline{V}$ compact such that $\operatorname{spec}(a) \subset V \subset \overline{V} \subset U$, and approximate f uniformly by a polynomial p on $\overline{V}$. Then apply part (1) to $f - p$.

Exercise 13. Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

(1) $a \geq 0$.
(2) a is normal and all eigenvalues of a are non-negative.
(3) There is a $b \in M_n(\mathbb{C})$ such that $b^*b = a$.
(4) There is a $b \in M_{n\times k}(\mathbb{C})$ for some $k \in \mathbb{N}$ such that $b^*b = a$.

Exercise 14 ($\star\star$, [Pal01, Thm. 9.1.45])**.**

(1) Show that any involution $\dagger$ on $M_n(\mathbb{C})$ is of the form $a^\dagger = ha^*h^{-1}$ for some invertible $h \in M_n(\mathbb{C})$ such that $h = h^*$.
(2) Show that $(M_n(\mathbb{C}), \dagger) \cong (M_n(\mathbb{C}), *)$ as involutive algebras if and only if the corresponding h for $\dagger$ is positive or negative definite.

Definition 15. Let A be a unital complex $*$-algebra. We call a linear functional $\varphi : A \to \mathbb{C}$:

- a *trace* or *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a *state* if φ is positive and $\varphi(1) = 1$.
- *faithful* if φ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

Exercise 16. Prove that $M_n(\mathbb{C})$ has a unique trace such that $\operatorname{tr}(1) = 1$. In this case, prove that tr is positive (so tr is a state) and faithful.

Exercise 17. Let $A = \mathbb{C}^2$ with coordinate-wise multiplication and $(a, b)^\dagger := (\overline{b}, \overline{a})$. Prove that A has no states.

Exercise 18 ($\star$)**.** Prove that for any state φ on $M_n(\mathbb{C})$, there exists $d \in M_n(\mathbb{C})$ with $d \geq 0$ and $\operatorname{tr}(d) = 1$ such that $\varphi(a) = \operatorname{tr}(da)$ for all $a \in M_n(\mathbb{C})$. Prove that φ is a faithful if and only if d is also invertible.
The matrix d is called the density matrix *of φ with respect to* tr.

Let H denote a finite dimensional inner product (Hilbert) space. Denote by $B(H)$ the unital $*$-algebra of linear operators on H.

Exercise 19. Show that the identity

$$\langle a\eta, \xi\rangle = \langle \eta, a^*\xi\rangle \qquad \forall\, a \in B(H), \quad \forall\, \eta, \xi \in H$$

gives a well-defined linear map $a^* \in B(H)$.

Exercise 20. Show that a choice of orthonormal basis of H gives a unitary linear map $u : H \to \mathbb{C}^n$ such that $x \mapsto uxu^*$ is a unital $*$-algebra isomorphism $B(H) \to M_n(\mathbb{C})$.

Definition 21. For a subset $S \subset B(H)$, the *commutant* of S is

$$S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

Exercise 22. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$.

Exercise 23. Show that if $S \subset B(H)$, then $S' = S'''$.

Exercise 24 ($\star\star$)**.** Show that if $A \subset B(H)$ is a unital $*$-subalgebra, then $A = A''$.
Hint: See [Jon15, Thm. 3.2.1].

Exercise 25. Suppose φ is a faithful state on $M_n(\mathbb{C})$. Show that $\langle a, b\rangle := \varphi(b^*a)$ defines a positive definite inner product on $M_n(\mathbb{C})$ (thought of as a $\mathbb{C}$-vector space).

Definition 26. We define $L^2(M_n(\mathbb{C}), \varphi)$ to be $M_n(\mathbb{C})$ as an inner product (Hilbert) space with the inner product from Exercise 25. We denote the image of $1 \in M_n(\mathbb{C})$ in $L^2(M_n(\mathbb{C}), \varphi)$ by Ω, so $a\Omega$ is the image of $a \in M_n(\mathbb{C})$.

Exercise 27. Prove that if $a \in M_n(\mathbb{C})$, the map given by $b\Omega \mapsto ab\Omega$ defines a left multiplication operator $\lambda_a \in B(L^2(M_n(\mathbb{C}), \varphi))$. Prove that the adjoint of this operator is λ_{a^*} given by $b\Omega \mapsto a^*b\Omega$.

Exercise 28. Prove that if $a \in M_n(\mathbb{C})$, the map given by $b\Omega \mapsto ba\Omega$ defines a right multiplication operator $\rho_a \in B(L^2(M_n(\mathbb{C}), \varphi))$. Calculate the adjoint of ρ_a. When does $\rho_a^* = \rho_{a^*}$?

Exercise 29. Suppose φ is a faithful state on $M_n(\mathbb{C})$. Prove that the commutant of the *left* $M_n(\mathbb{C})$ action on $L^2(M_n(\mathbb{C}), \varphi)$ is the *right* $M_n(\mathbb{C})$ action.

Exercise 30. Suppose $\langle\,\cdot\,,\,\cdot\,\rangle$ is a positive definite inner product on the vector space $M_{m\times n}(\mathbb{C})$. Prove that the commutant of the *left* $M_m(\mathbb{C})$ action on $M_{m\times n}(\mathbb{C})$ is the *right* $M_n(\mathbb{C})$ action.

References

[Jon15] Vaughan F. R. Jones, *Von Neumann algebras*, 2015, https://math.vanderbilt.edu/jonesvf/VONNEUMANNALGEBRAS2015/VonNeumann2015.pdf.

[Pal01] Theodore W. Palmer, *Banach algebras and the general theory of $*$-algebras. Vol. 2*, Encyclopedia of Mathematics and its Applications, vol. 79, Cambridge University Press, Cambridge, 2001, $*$-algebras, MR1819503 DOI:10.1017/CBO9780511574757.003. MR 1819503