(Untwisted) Quantum Doubles

We want to understand $D(G) := Z(\mathrm{vec}[G])$

trivial cocycle

finite group

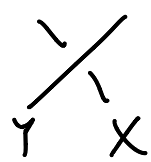
The Drinfel'd Center

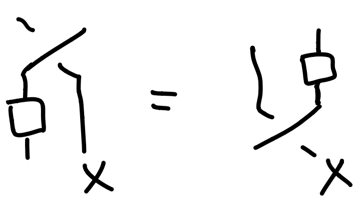
monoidally semisimple

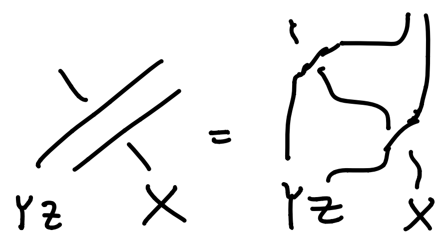
Objects in $Z_1(\mathcal{C})$: pairs (X, b)

X: object in $\mathcal{C}$

b: "$\frac{1}{2}$-braiding" natural isomorphism $\bullet X \to X \bullet$

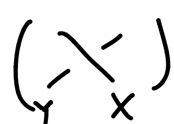
Written  $= b_Y$,

natural: 

monoidal 

Since $\mathcal{C}$ is semisimple, a natural isomorphism of functors

$\mathcal{C} \to -$ is determined by components for simple objects.

So a $\frac{1}{2}$-braiding is determined by  $_{Y \in \mathrm{Irr}(\mathcal{C})}$.

... Of Vec[G].

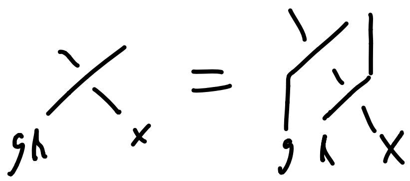
In Vec[G], simple objects form a group G under $\otimes$.

So $(X, b) \in D(G)$ $:= Z_1(\mathrm{Vec}[G])$ is :

X: object in $\mathcal{C}$ G-graded vectorspace

b: "$\frac{1}{2}$-braiding" natural isomorphism $\bullet X \longrightarrow X \bullet$

$\forall g \in G$ G-graded map $gX \longrightarrow Xg$

Monoidality: 

$$b_{gh} = b_g \circ b_h .$$

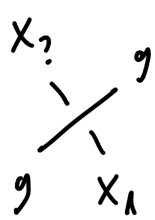
So b is representation of G.

$\frac{1}{2}$-braiding on X $\longrightarrow$ Representation of G on X.

How does b interact with grading?

$$X = \bigoplus_h X_h \longleftarrow h\text{-graded Component}$$

$gh = (ghg^{-1})g$

So  where $? = ghg^{-1}$

So X is a G-graded G-representation, where the action of G conjugates the grading.

So there are some low-hanging fruit!

If G is Abelian: conjugation never does anything.

so get G-graded G-representations, action of G preserves grading.

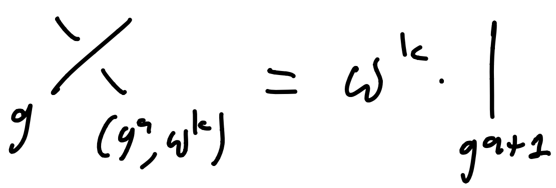
$$Z(\mathrm{Vec}[G]) \cong \mathrm{Vec}[G] \boxtimes \mathrm{Rep}(G)$$

For example, if $G = \mathbb{Z}/n = \langle g \mid g^n = 1 \rangle$.

Representations: Irreps are 1 dimensional,

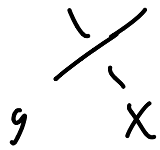
$g \longmapsto [\omega^k]$, ω a primitive n^{th} root of 1.

Simple Objects in $D(\mathbb{Z}/n)$ look like (g^a, ω^k)



$$X = \boxed{\bigoplus_a X_{g^a}}$$ as G-representations

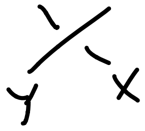
Look for g-eigenvector in X_{gq}: $v \in X_{gq}$ such that


$: e_g \otimes v \longmapsto \lambda (v \otimes e_g)$

$\uparrow$ guaranteed to be ω^a for some a.

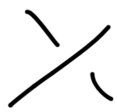
Braiding on $D(G)$:

If (X,ρ), $(Y,\lambda) \in D(G)$



$: \quad \underset{\uparrow \text{ grading}}{y_g} \otimes x_h \longmapsto \underbrace{\rho(g)x_h}_{ghg^{-1}\text{-graded}} \otimes y_g$

So on $\mathrm{Vec}[\mathbb{Z}/n]$:



$(g^a, \omega^k) \quad (g^b, \omega^j)$

$= \omega^{ja} \cdot$ 

In particular, if $n=2$, toric code:

$\omega = -1$

$1 \longleftrightarrow (1,1)$

$e \longleftrightarrow (g,1)$

$m \longleftrightarrow (1,-1)$

$\varepsilon \longleftrightarrow (g,-1)$

The General Case:

Notation: G - (finite) group

$[g]$ - conjugacy class $(\{hgh^{-1} : h \in G\})$

$Stab(g)$ - stabilizer $(\{h \in G : hgh^{-1} = g\})$

$R(g)$ - system of representatives of $G/Stab(g)$

i.e. $\forall\, k \in [g]\ \exists!\, h \in R(g) : hgh^{-1} = k$.

Arbitrary choice, except $1 \in R(g)$.

Say $(X, b(\rho)) \in D(G)$.

Then
$$\begin{tikzcd} \rho(g) : X_h \arrow[r, "\sim"] & X_{ghg^{-1}} \arrow[l, bend left, dashed, "\rho(g^{-1})"] \end{tikzcd}$$
so

- conjugate components are isomorphic
- $X \cong \bigoplus_{[g]} \Big(\underbrace{\bigoplus_{h \in [g]} X_h}_{\text{subobjects in } D(G)} \Big)$

Structure of a graded representation $X = X_{[g]}$:

$$\rho(h)(X_g) \subseteq X_g \iff h \in Stab(g),$$

So X_g is a $Stab(g)$ representation.

What about $k \notin Stab(g)$?

$$X_g \xrightarrow{h} X_{hgh^{-1}}$$

$$X = \bigoplus_{h \in R(g)} X_h.$$

For any k, $\exists!\, h \in R(g) : hgh^{-1} = kgk^{-1}$, So $k^{-1}h \in \mathrm{Stab}(g)$, $\rho(h) = \rho(k)\boxed{\rho(k^{-1}h)}$

Conclusion: $\rho|_{\mathrm{Stab}(g)}$ determines all of ρ.

Math Digression:

Induction and Restriction: If $H \leq G$,

Can restrict representations of G to H:

$$\mathrm{Res}^G_H : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$$

Adjoint functor: "induced representation"

$$\mathrm{Ind}^G_H : \mathrm{Rep}(H) \to \mathrm{Rep}(G)$$

$$\Big(\underset{\substack{\circlearrowleft \\ H}}{X}, \mathrm{Res}^G_H Y\Big) \cong \Big(\mathrm{Ind}^G_H X, \underset{\substack{\circlearrowleft \\ G}}{Y}\Big)$$

V an H representation $\iff$ $\mathbb{C}[H] \circlearrowright V$

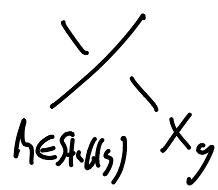
Induction: $V \longmapsto \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$

Basis for $\mathbb{C}[G]$ over $\mathbb{C}[H]$: representatives of G/H. $\quad (X = X_{[g]})$

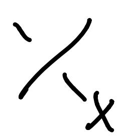
So as a G-representation, $X \cong \mathrm{Ind}^G_{\mathrm{Stab}(g)} X_g$.

Summary:

- Objects of $D(G)$ are G-graded G-representations
- Simple objects are: (g, ρ) where
  - $g \in G$
  - $\rho \in \mathrm{Rep}(\mathrm{Stab}(g))$



determines



— overall graded representation if

$$\left(\bigoplus_{h\in R(g)} \mathbb{C}_h\right)\otimes \rho$$

3-steps to Ind

A First Nonabelian Example

The Dihedral Group

$$D_n = \langle r, f \mid r^n = f^2 = 1,\ \underline{rf = fr^{-1}}\rangle$$

odd

Conjugacy

$$frf^{-1} = f\underline{r}f = f^2\underline{r^{-1}} = r^{-1}$$

$$[r^k] = \{r^k, r^{-k}\}$$

$[1] = \{1\}.$

$$r\underline{fr^{-1}} = r^2 f \qquad n \text{ is odd}$$

$$[f] = \{r^k f\}_{k=1}^{n}$$

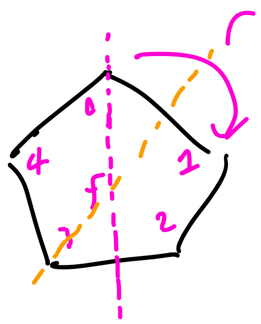
$$\mathrm{Stab}(r^k) = \langle r\rangle \cong \mathbb{Z}/n$$

normal

$k \neq 0$

$$\mathrm{Stab}(r^k f) = \langle r^k f\rangle \cong \mathbb{Z}/2$$

abnormal



$r : e_i \mapsto e_{i+1}$

$f : e_i \mapsto e_{-i}$

"Standard rep'n"

Representation Theory

Standard Rep'n Eigenbases:

primitive n^{th} root of 1

for r: "momentum" basis $\phi_j := \sum_k \omega^{-jk} e_k$, $\qquad \omega^j$

for f: $e_i + e_{-i}$, 1

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\qquad e_i - e_{-i}$, -1

$f\phi_j \sim \phi_{-j}$, so

$$\phi_i + \phi_{-i} = \sum_k \omega^{jk}(e_i + e_{-i}), \quad 1$$

$$\phi_i - \phi_{-i} = \sum_k \omega^{jk}(e_i - e_{-i}) \quad -1$$

Subrepresentations: on $\mathbb{C}\{\phi_j, \phi_{-j}\}$:

Pauli-esque

$$\sigma_j : r \longmapsto \begin{array}{c} \begin{array}{cc} \phi_j & \phi_{-j} \end{array} \\ \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix} \end{array} \quad f \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"Trivial" representation 1: $r \mapsto 1$, $f \mapsto 1$ (at ϕ_0)

"sign" representation ε: $r \mapsto 1$, $f \mapsto -1$ (not in standard)

Representations of $\mathbb{Z}/n$: the eigenvalue (± 1, or ω^j).

Simple objects of $D(D_n)$:

$(1, \rho)$: $\rho \in Rep(D_n)$ $\qquad Rep(D_n)$ $\qquad \dim(\rho)$

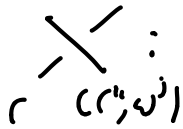
$r^k \oplus r^{-k}$

$(r^k, \omega^j) : k \leq \lfloor \frac{n}{2} \rfloor, \; j = 1 \ldots n$ 2    $: e_{r^k} \mapsto \omega^j e_{r^k}$

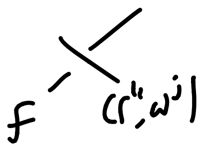
$(f, \pm 1)$ n

Unpacking (r^k, ω^j):

basis $e_{r^k}, e_{r^{-k}}$, representation



$: \; e_r \otimes e_{r^k} \longmapsto \omega^j e_{r^k} \otimes e_r$

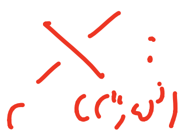


$e_f \otimes e_{r^k} \longmapsto e_{r^{-k}} \otimes e_f$

- Show that $\sigma_i \otimes \sigma_j \cong \sigma_{i+j} \oplus \sigma_{i-j}$

  Hint: $\sigma_0 := 1 \oplus \varepsilon$

- Show that $(r^k, \omega^j) \cong (r^{-k}, \omega^{-j})$, i.e.



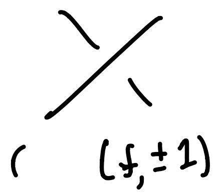
$: \; e_r \otimes e_{r^{-k}} \longmapsto \omega^{-j} e_{r^{-k}} \otimes e_r$

- Show that $(r^a, \omega^j) \otimes (r^b, \omega^k) \cong (r^{a+b}, \omega^{j+k}) \oplus (r^{a-b}, \omega^{j-k})$.

  Hint: $(r^0, \omega^j) := (1, \sigma_j)$

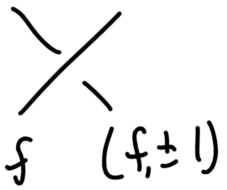
Unpacking $(f, 1)$:

basis $\{e_{r^k f} : k = 1 \dots n\}$.



: $e_r \otimes e_{r^k f} \mapsto e_{r^{k+2} f} \otimes e_r$

(Note: $\exists\, r^{1/2} \in D_n$, namely $r^{\frac{n+1}{2}}$)

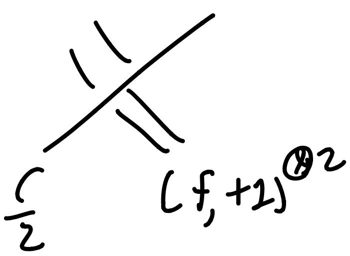


: $e_f \otimes e_{r^k f} \mapsto e_{r^{-k} f} \otimes e_f$

Understanding $(f, 1) \otimes (f, 1)$:

$$r^a f r^b f = r^{a-b} f^2 = r^{a-b}, \quad \text{so}$$

r^c-graded vectors include $e_{r^{c+k} f} \otimes e_{r^k f}$: $k = 1 \dots n$



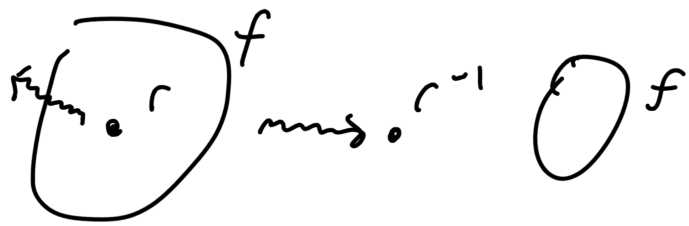
: $e_{\frac{r}{2}} \otimes e_{r^{c+k} f} \otimes e_{r^k f} \mapsto e_{r^{c+(k+1)} f} \otimes e_{r^{(k+1)} f}$

So $$(f, 1) \otimes (f, 1) \cong \bigoplus_{c=1}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{j=1}^{n} (r^c, \omega^j).$$

"Superposition of <u>all</u> $D(\mathbb{Z}/n)$ particles"

- What about $(r^q, \omega^k) \otimes (f, 1)$?
- How should we think about $(f, -1)$?
- What about D_{even}?

  Hint: you <u>can</u> slog through it, but there is a slick answer available.

Same interpretation: 

So $(r^k, \omega^j) \simeq e^k m^j \oplus e^{-k} m^{-j}$.

$$(f,1) \otimes (f,1) \cong 1 \oplus \left(\bigoplus_{\boxed{c=1}}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{j=1}^{n} (r^c, \omega^j) \right)$$

"Superposition of <u>most</u> $D(\mathbb{Z}/n)$ particles"