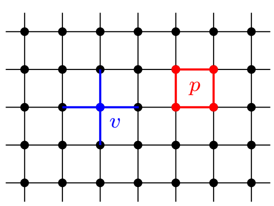
7. Phase transitions

In this section, we discuss various notions of quantum phase transitions between topological phases, including *gapped spatial* phase transitions, and *gapless critical value* phase transitions for a local Hamiltonian.

We begin this section by describing the two most widely known quantum phase transitions: gapped boundaries of toric code, and an application of the transverse-field Ising model to a quantum phase transition for the toric code. We will then do a systematic study of phase transitions, first starting with the notions of (de)equivariantiazation for fusion categories, and condensation and gauging for braided fusion categories.

7.1. **Gapped boundaries of the toric code.** Our discussion of the two gapped boundaries for toric code follows [KK12, §2], but with the Pauli X and Z operators swapped, which swaps the roles of e, m.

Recall that we defined the toric code lattice model on a genus g surface Σ_g by picking some cellulation, giving *vertex terms* $A_v = \prod_{\ell \sim v} Z_\ell$ and *plaquette terms* $B_p = \prod_{\ell \sim p} X_\ell$,

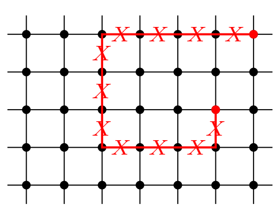


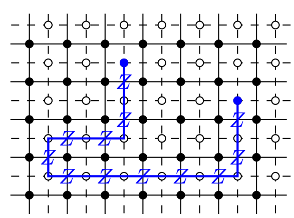
$$A_v = \bigotimes_{\ell \sim v} Z_\ell \qquad B_p = \bigotimes_{\ell \sim p} X_\ell.$$

and setting the Hamiltonian equal to

$$H_{\mathrm{TC}} := -\sum_v A_v - \sum_p B_p.$$

The lowest energy *excitations* of the toric code are when exactly two of the A_v or exactly two of the B_p are violated, which creates a pair of e or m quasiparticle excitations respectively.



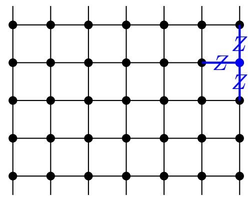


Above, we have *string operators* which create these quasiparticle excitations from the ground state on a torus with periodic boundary conditions.

We can define two *gapped boundaries* for the toric code to the vaccuum. For convenience, we will work with square lattices.

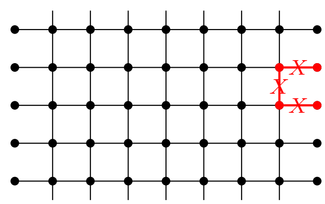
(1) The *smooth* boundary is a line (which really closes into a circle) of trivalent vertices in our lattice, where we do *not* have a plaquette term for the 'missing' face. In the

diagram below, we create two smooth boundaries on the left and right, and we have periodic vertical boundary conditions.



On the boundary, we have modified vertex terms A_v' with only three Pauli Z operators attached to each vertex. Observe that the Hamiltonian with these new truncated vertex terms at the boundary is still commuting projector.

(2) The *rough* boundary is a line (circle) of singly valent vertices in our lattice, where there is no vertex term for these vertices. In the diagram below, we create two rough boundaries on the left and right, and we have periodic vertical boundary conditions.



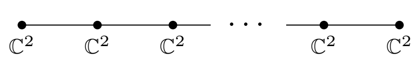
On the boundary, we have modified plaquette terms B_p' with only three Pauli X operators around each partial plaquette. Again, this new Hamiltonian is still commuting projector.

The effect of these boundaries is that certain quasi-particle excitations can be *created* or *destroyed* for free, or even *negative* energy cost.

(1) On the smooth boundary, an m excitation can completely vanish, but e particles remain, giving boundary excitations of $1, e$. The particles $1, m$ both map to 1, and the particles e, ϵ both map to e. Since these four excitations represent the simple objects in $D(\mathbb{Z}/2) = Z(\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2))$ where $\mathbb{Z}/2 = \{1, e\}$ and m represents the sign representation of $\mathbb{Z}/2$, we see that the behavior of anyons at the boundary corresponds to the forgetful functor $F : Z(\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2)) \to \mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2)$.

(2) On the rough boundary, an e particle can complete vanish, but m particles remain, giving boundary excitations $1, m$. The behavior is similar to the smooth boundary with e, m swapped. That is, as $D(\mathbb{Z}/2) = Z(\mathsf{Rep}(\mathbb{Z}/2))$, we have another forgetful functor $F : D(\mathbb{Z}/2) \to \mathsf{Rep}(\mathbb{Z}/2)$ which describes the behavior of anyons at the boundary.

7.2. **Application of the transverse-field Ising model to toric code.** Recall from the Lattice Models section that the 1D transverse-field Ising model has local Hamiltonian given by



$$H_{\mathrm{tIs}} = H_{\mathrm{tIs}}(B) := -\sum_j Z_j Z_{j+1} - B \sum_j X_j$$

on a 1D lattice with L sites, where B denotes the external magnetic field. This model has a well-understood *quantum phase transition* at $B = 1$, where the gap in the spectrum of H_{tIs}

closes in the thermodynamic limit (as $L \to \infty$). The 2D transverse-field Ising model has a similar Hamiltonian, but on a 2D lattice.

We can drive the toric code through a quantum phase transition by changing the local Hamiltonian in two different ways, which are analogous to the two different smooth and rough boundaries for toric code discussed in the previous section. We will do the case of condensing the m particle, and the case for the e particle is similar.

We can add terms of the form C_ℓ for each ℓ in our 2D lattice on Σ_g corresponding to a single Pauli Z operator, giving the Hamiltonian

$$H(B) := -\sum_v A_v - \sum_p B_p - B\sum_\ell C_\ell.$$

Observe that this Pauli Z operator for C_ℓ commutes with all A_v, but does not commute with the B_p such that $\ell \in p$.

When $B = 0$, we have H_{TC}, and when B is large the energetics of the C_ℓ term dominates, in which m particles can be freely created and destroyed everywhere on the lattice. Thus the m particle has been *condensed*. This is a good indicator that we have a quantum phase transition.

To make this rigorous, we can map this system onto the transverse-field Ising model as follows. First, observe that each C_ℓ term commutes with each of the A_v terms, so we may pass into the ground state space $\mathcal{G}_A$ of $-\sum A_v$ to analyze the phase transition. Recall that the space $\mathcal{G}_A$ is spanned by string nets where exactly an even number of edges meeting each vertex are 'on,' i.e., $|1\rangle \in \mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$.

Exercise 7.2.1. Show that the ground state space for A_v is 2^L dimensional, where L is the number of vertices.

Now on the 2-torus, the Euler characteristic $V - E + F = 0$, and each edge connects to 2 vertices, and each vertex is 4-valent. Hence there are exactly L vertices, $2L$ edges, and L plaquettes. Thus we may represent the space $\mathcal{G}_A$ as the tensor product of one qubit sitting on every *plaquette* as opposed to a subspace of the total Hilbert space with a qubit on every *edge*.

For each qubit sitting on a plaquette, we can represent the operator B_p on $\mathcal{G}_A$ as a Pauli X matrix X_p. The operator C_ℓ can then be represented by Z_pZ_q where p, q are the two plaquettes incident to ℓ. Thus under this change of basis, $H(B)$ restricted to $\mathcal{G}_A$ can be represented by

$$H'(B) = -\sum X_p - B\sum_{p\sim q} Z_pZ_q,$$

where $p \sim q$ means the plaquettes p and q are incident on the lattice. This is exactly the 2D transverse-field Ising model! We conclude that as we increase B, we force the toric code through a quantum phase transition.

Exercise 7.2.2. Adapt the above procedure to condense the m particle of the toric code.

7.3. **De-equivariantization and condensation.** Recall that given a unital algebra object (A, μ) in a fusion category $\mathcal{C}$, we can form the left $\mathcal{C}$-module category $\mathcal{C}_A := \mathsf{Mod}_{\mathcal{C}}(A)$ of right (unital) A-modules in $\mathcal{C}$. In the presence of more structure on A, we get more structure on the $\mathcal{C}$-module category $\mathcal{C}_A$.

Exercise 7.3.1 ([KO02, Fig. 4])**.** Recall that a right A-module is called *free* if it is of the form $c \otimes A$ for some $c \in \mathcal{C}$. The *free module functor* $\mathcal{C} \to \mathsf{Mod}_{\mathcal{C}}(A)$ is given by $c \mapsto c \otimes A$. Find a natural isomorphism

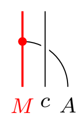
$$\mathcal{C}_A(a \otimes A \to b \otimes A) \cong \mathcal{C}(a \to b \otimes A).$$

Exercise 7.3.2. A unital algebra A in a fusion category $\mathcal{C}$ is called *connected* if $\dim(\mathcal{C}(1_{\mathcal{C}} \to A)) = 1$.

(1) Show that if A is connected, $\operatorname{End}_{-A}(A) = \mathbb{C}\,\mathrm{id}_A$.
(2) Show that if A is connected and separable, then the splitting s of μ as an $A - A$ bimodule is unique.

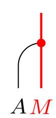
Definition 7.3.3. A *lift* of A to $Z(\mathcal{C})$ is a half-braiding $\sigma_{A,\bullet}$ for A such that the multiplication and unit of A are morphisms in $Z(\mathcal{C})$.

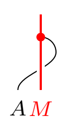
When A lifts to $Z(\mathcal{C})$, the category $\mathcal{C}_A$ gets a canonical $\mathcal{C}-\mathcal{C}$ *bimodule* structure by defining the right $\mathcal{C}$-action by $M \lhd c := M \otimes c$, where the right A-action is given by



$$= (\rho_M \otimes \mathrm{id}_c) \circ (\mathrm{id}_M \otimes \sigma_{A,c}).$$

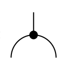
Moreover, observe that we can define a left A-action on M by

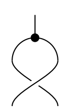
$$\lambda_M =$$


$$:=$$


$$= \rho_M \circ \sigma_{A,M}^{-1}.$$

Exercise 7.3.4. Prove that the left action λ_M commutes with ρ_M for every right A-module M if and only if the lift of A to $Z(\mathcal{C})$ is *commutative*, i.e.,

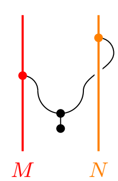
$$\mu =$$


$$=$$


$$= \mu \circ \sigma_{A,A}.$$

Hint: Consider A as a right A-module.

Now when $(A, \sigma_{A,\bullet})$ is a separable commutative algebra in $Z(\mathcal{C})$, we can canonically equip $\mathcal{C}_A$ with the structure of a tensor category where the tensor product is given by splitting the canonical separability idempotent:

$$p_{M,N} := \tag{7.3.5}$$


Exercise 7.3.6. Prove that $p_{M,N}$ from (7.3.5) above is an idempotent.

Moreover, since $\mathcal{C}_A$ is finitely semisimple, it is multifusion, provided we can prove it is rigid.

Theorem 7.3.7. *Suppose $\mathcal{C}$ is a fusion category and A is a unital algebra. The following are equivalent.*

(1) A is separable,

(2) $\mathcal{C}_A$ *is semisimple,*
(3) the category ${}_A\mathcal{C}_A$ *of* $A - A$ *bimodules in* $\mathcal{C}$ *is semisimple,*
(4) A *admits the structure of a special Frobenius algebra, and*
(5) the category ${}_A\mathcal{C}_A$ *of* $A - A$ *bimodules in* $\mathcal{C}$ *is rigid and semisimple.*

Proof. The equivalence of (1)-(3) is [DMNO13, Prop. 2.7]. For the rest of the proof, without loss of generality, we may assume A is connected.
(1) $\Rightarrow$ (4): We claim that (A, μ, i) can be uniquely endowed with the structure of a special Frobenius algebra Δ, ϵ, up to a scalar normalization. First, up to scalar, ϵ must be the unique left inverse of $i \in \mathcal{C}(1 \to A)$, which exists by semisimplicity of $\mathcal{C}$. The pairing $\epsilon \circ m$ is non-degenerate by [Ost03, Prop. 3.1.ii]. There is a unique comultiplication Δ making $(A, \mu, i, \Delta, \epsilon)$ a Frobenius algebra by [FRS02, Lem. 3.7], [FS08, Prop. 8]. Finally, A is automatically special (see [GS16, Thm. 2.6]). Indeed, since A is separable, there is a splitting $s \in {}_A\mathcal{C}_A(A \to A \otimes A) \cong \mathbb{C}$ by Exercise 7.3.2. Hence $\Delta = \lambda s$ for some $\lambda \in \mathbb{C}^\times$ as $\Delta \neq 0$. Thus $\mu \circ \Delta = \lambda(\mu \circ s) = \lambda\, \mathrm{id}_A$.
(4) $\Rightarrow$ (5): Exercise left to the reader.
(5) $\Rightarrow$ (3): Trivial. □

Exercise 7.3.8. Extend the proof of the equivalence of (3)-(5) of Theorem 7.3.7 to the non-connected case.
Hint: Use the following steps.

(1) Every algebra is a direct sum of simple algebras which satisfy $\mathrm{End}({}_A A_A) = \mathbb{C}\,\mathrm{id}_A$.
(2) For separable algebras, every simple algebra is Morita equivalent to a connected separable algebra.
(3) The property 'admits a special Frobenius algebra structure' is preserved under taking Morita equivalence and taking direct sums.

Exercise 7.3.9. Prove (4) $\Rightarrow$ (5) of Theorem 7.3.7.

Definition 7.3.10. Given a fusion category $\mathcal{C}$ and a separable unital algebra A that lifts to a commutative algebra in $Z(\mathcal{C})$, the multifusion category $\mathcal{C}_A$ is called the *de-equivariantization* of $\mathcal{C}$ by A.

Exercise 7.3.11. Suppose the separable algebra A lifts to a commutative algebra in $Z(\mathcal{C})$. Prove that $\mathcal{C}_A$ is fusion if and only if $\mathrm{End}_{A-A}(A) = \mathbb{C}\,\mathrm{id}_A$. Deduce from Exercise 7.3.2 that if A is connected, then $\mathcal{C}_A$ is fusion.

Exercise 7.3.12. Suppose the separable algebra A lifts to a commutative algebra in $Z(\mathcal{C})$. Show how to endow the free module functor $- \otimes A : \mathcal{C} \to \mathcal{C}_A$ with the structure of a tensor functor. Prove $- \otimes A$ is *dominant*, i.e., every simple object of $\mathcal{C}_A$ is isomorphic to a summand of a free module.
Hint: Use that all free modules are projective.

Definition 7.3.13. Suppose $\mathcal{C}$ is a braided fusion category. A unital algebra $A \in \mathcal{C}$ is called *étale* if it is both commutative and separable. By the above discussion, $\mathcal{C}_A$ for a connected étale algebra is a fusion category.

Exercise 7.3.14. Find all connected étale algebras in $D(\mathbb{Z}/2)$.

Exercise 7.3.15 ([BN11])**.** Suppose $\mathcal{C}, \mathcal{D}$ are fusion categories and $F : \mathcal{C} \to \mathcal{D}$ is a tensor functor. Let $I : \mathcal{D} \to \mathcal{C}$ be the right adjoint of F.

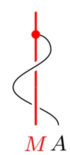
(1) Show that $I(1_{\mathcal{D}})$ has an organic unital algebra structure in $\mathcal{C}$.
(2) Show that $I(1_{\mathcal{D}})$ lifts to a commutative algebra in $Z(\mathcal{C})$.
(3) Assume in addition that F is *dominant*, i.e., every simple object in $\mathcal{D}$ is isomorphic to a summand of an object of the form $F(c)$. Construct an equivalence $\mathcal{D} \cong \mathcal{C}_A$ which takes F to the free-module functor.

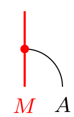
Exercise 7.3.16. Suppose $\mathcal{C}$ is a ribbon fusion category and $A \in \mathcal{C}$ is a (connected) étale algebra. Find a condition on A which allows us to endow $\mathcal{C}_A$ with a spherical structure.

Example 7.3.17 (Text adapted from [BJLP19, §2.2])**.** Suppose $\iota : \mathsf{Rep}(G) \to \mathcal{Z}(\mathcal{C})$ is a fully faithful braided tensor functor such that the composite $F \circ \iota : \mathsf{Rep}(G) \to \mathcal{C}$ is still fully faithful, where $F : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor. (Recall such an inclusion $\mathsf{Rep}(G) \subset \mathcal{Z}(\mathcal{C})$ is called a *Tannakian* subcategory.) Let $\mathcal{O}(G) \in \mathsf{Rep}(G)$ denote the algebra object of functions $G \to \mathbb{C}$ whose multiplication is given by $\chi_g \cdot \chi_h = \delta_{g=h}\chi_g$ where $\chi_g(h) = \delta_{h=g}$ for $g, h \in G$. Then $\iota(\mathcal{O}(G))$ is an étale algebra object in $Z(\mathcal{C})$ whose category of right modules $\mathcal{C}_G := \mathcal{C}_{\iota(\mathcal{O}(G))}$ is a fusion category, called the *de-equivariantization* of $\mathcal{C}$ by G.

Now when $\mathcal{C}$ is a braided fusion category and A is an étale algebra, $\mathcal{C}_A$ is multifusion but not braided in general.

Definition 7.3.18. A right A-module (M, ρ_M) is called *local* if

$\rho_M \circ \beta_{M,A} \circ \beta_{A,M} =$ 

$=$ 

$= \rho_M.$

The full subcategory category of local right A-modules, denoted $\mathcal{C}_A^{\mathrm{loc}}$, is called the *condensation* of $\mathcal{C}$ by A, which is again a multifusion category. In Exercise 7.3.19 below, we will see that the braiding β descends to $\mathcal{C}_A^{\mathrm{loc}}$, which is a braided multifusion category.

Exercise 7.3.19. Suppose $(\mathcal{C}, \beta)$ is a braided fusion category and $A \in \mathcal{C}$ is étale. Prove that β gives a well-defined braiding on $\mathcal{C}_A^{\mathrm{loc}}$.

Theorem 7.3.20 ([DMNO13, Cor. 3.30])**.** *Suppose $\mathcal{C}$ is a non-degenerate braided fusion category and $A \in \mathcal{C}$ is connected étale. There is a canonical equivalence*

$$Z(\mathcal{C}_A) \cong \mathcal{C} \boxtimes (\mathcal{C}_A^{\mathrm{loc}})^{\mathrm{rev}} \tag{7.3.21}$$

where rev *denotes the same (multi)fusion category with the* reverse *braiding.*

Definition 7.3.22 (Text adapted from [BJLP19, §2.2])**.** Let $\mathcal{C}$ be a braided fusion category. An invertible object $g \in \mathcal{C}$ is called a *boson* or *simple-current* if $\beta_{g,g} = \mathrm{id}_{g \otimes g}$.

Exercise 7.3.23. Show that if $B \subset \mathrm{Inv}(\mathcal{C})$ is a subgroup consisting of bosons, then the full subcategory of $\mathcal{C}$ generated by B is braided equivalent to $\mathsf{Rep}(\widehat{B})$, where $\widehat{B}$ is the dual group of B.

Definition 7.3.24. In the case of Exercise 7.3.23, we call $\mathcal{O}(\widehat{B})$ the étale algebra induced by the group of bosons B. The condensation $\mathcal{C}^{\mathrm{loc}}_{\mathcal{O}(\widehat{B})}$ is also referred to as the braided tensor category obtained by *condensing the bosons* B.

Definition 7.3.25. A connected étale algebra A in a non-degenerate braided fusion category $\mathcal{C}$ is called *Lagrangian* if $\mathcal{C}_A^{\mathrm{loc}} \cong \mathsf{Vec_{fd}}$. Observe that if $A \in \mathcal{C}$ is Lagrangian, then $\mathcal{C} \cong Z(\mathcal{C}_A)$ by (7.3.21).

7.4. **Equivariantization and gauging.** This section is adapted from [BJLP19, §2.2].

The inverse process to de-equivariantization is *equivariantization*. Suppose $\mathcal{C}$ is a fusion category. Recall that $\mathrm{Aut}_{\otimes}(\mathcal{C})$ is the 2-group (monoidal category where all objects and morphisms are invertible) whose objects are tensor automorphisms of $\mathcal{C}$ and whose morphisms are monoidal natural isomorphisms.

Notation 7.4.1. Suppose $\mathrm{B}G \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ is an action, where we still denote by g the tensor automorphism on the right hand side. We denote the tensorator of g by $\psi^g = \{\psi^g_{a,b} : g(a) \otimes g(b) \xrightarrow{\sim} g(a \otimes b)\}_{a,b\in\mathcal{C}}$. We denote the tensorator by $\mu = \{\mu_{g,h} : g \circ h \to (gh)\}_{g,h\in G}$, which is a family of monoidal natural isomorphisms satisfying associativity and unitality axioms.

By [Gal17, Thm. 1.1], one may assume that the action is *strict*, so that $g \circ h = gh$ for all $g, h \in G$, but for the sake of generality, we will only assume *strict unitality* of the action:

- Each monoidal functor (g, ψ^g) is *unital* [Gal17, Prop. 3.1], i.e., for all $g \in G$, $g(1_{\mathcal{C}}) = 1_{\mathcal{C}}$ and $g(\mathrm{id}_{1_{\mathcal{C}}}) = \mathrm{id}_{1_{\mathcal{C}}}$, and
- $e = \mathrm{id}_{\mathcal{C}}$, $e \circ g = g \circ e = g$ and $\mu_{g,e} = \mu_{e,g} = \mathrm{id}_g$ for all $g \in G$.

Definition 7.4.2. A *G-equivariant object* is a pair (c, λ) where $c \in \mathcal{C}$ and $\lambda = \{\lambda^g : g(c) \to c\}_{g\in G}$ is a family of isomorphisms such that the following diagram commutes for all $g, h \in G$:

$$\begin{tikzcd} g(h(c)) \arrow[r, "g(\lambda^h)"] \arrow[d, "\mu^c_{g,h}"'] & g(c) \arrow[d, "\lambda^g"] \\ gh(c) \arrow[r, "\lambda^{gh}"] & X. \end{tikzcd} \tag{7.4.3}$$

Given G-equivariant objects $(c, \lambda), (d, \kappa)$, we call a morphism $f \in \mathcal{C}(c \to d)$ a *G-equivariant morphism* if the following diagram commutes for all $g \in G$:

$$\begin{tikzcd} g(c) \arrow[r, "\lambda^g"] \arrow[d, "g(f)"'] & c \arrow[d, "f"] \\ g(d) \arrow[r, "\kappa^g"] & d. \end{tikzcd} \tag{7.4.4}$$

The *equivariantization* $\mathcal{C}^G$ is the category whose objects are G-equivariant objects and whose morphisms are G-equivariant morphisms. The tensor product in $\mathcal{C}^G$ is given by

$$(c, \lambda) \otimes (d, \kappa) := (c \otimes d, (\lambda^g \otimes \kappa^g) \circ (\psi^g_{c,d})^{-1}) \tag{7.4.5}$$

and the unit object is $(1_{\mathcal{C}}, \mathrm{id}_{1_{\mathcal{C}}})$.

Remark 7.4.6. Observe that when the G-action is strictly unital, the commutativity of (7.4.3) with $g = h = e$ shows that any G-equivariant object (c, λ) must have $\lambda^e = \mathrm{id}_c$.

Fact 7.4.7. For fusion categories, equivariantization $\mathcal{C} \mapsto \mathcal{C}^G$ and de-equivariantization $\mathcal{D} \mapsto \mathcal{D}_G$ are mutually inverse up to equivalence; we refer the reader to [EGNO15, Rem. 8.23.5] for more details.

We can also equivariantize a G-action on a braided fusion category by braided tensor automorphisms.

Exercise 7.4.8. Suppose $\mathcal{C}$ is a braided fusion category and $\mathrm{B}G \to \mathrm{Aut}_\otimes^{\mathrm{br}}(\mathcal{C})$ is a G-action by braided tensor automorphisms. Show that the braiding on $\mathcal{C}$ descends to a braiding on $\mathcal{C}^G$.

Even better, we can equivariantize the G-action on a G-crossed braided fusion category!

Exercise 7.4.9. Suppose $\mathcal{C}$ is a G-crossed braided fusion category, which is equipped with a G-action as part of its data. Show that the G-crossed braiding descends to an honest braiding on the equivariantization of $\mathcal{C}^G$.

Theorem 7.4.10 ([EGNO15, Thm. 8.24.3], following [Kir01] and [Müg04])**.** *Equivariantization and de-equivariantization are inverse procedures between equivalence classes of:*

- *braided fusion categories containing a symmetric* $\mathsf{Rep}(G)$ *fusion subcategory, and*
- *G-crossed braided fusion categories*

The inverse procedure to condensing a connected étale algebra of the form $\mathcal{O}(G) \in \mathsf{Rep}(G) \subset \mathcal{C}$ is given by *gauging* [CGPW16, BBCW19], which is the two step process:

(1) find a G-crossed braided extension $\mathcal{E}$ of $\mathcal{C}$, which comes equipped with a G-aciton $\mathrm{B}G \to \mathrm{Aut}_\otimes(\mathcal{E})$, and
(2) take the equivariantization $\mathcal{E}^G$.

Fact 7.4.11. When $\mathcal{C}$ is a braided fusion category with a strict G-action (every G-action is equivalent to a strict G-action by [Gal17, Thm. 1.1]), every G-crossed braided extension $\mathcal{E}$ of $\mathcal{C}$ is equivalent to a strict G-crossed braided extension of $\mathcal{C}$ by [Gal17, Thm. 5.6]. That is, if we only consider strict G-actions, we do not lose any G-crossed braided extensions.

Fact 7.4.12. For non-degenerately braided fusion categories, condensing $\mathcal{O}(G)$ and gauging a second level categorical G-symmetry (taking the equivariantization of a G-crossed braided extension) are mutually inverse; we refer the reader to [DGNO10, §4] and [CGPW16] for more details.

Remark 7.4.13. As mentioned above, we can condense any étale algebra in a nondegenerately braided fusion category, not just one of the form $\mathcal{O}(G)$. It is an important open question to find the inverse process to this more general condensation. The recent article [CZW18] provides an interesting step in this direction.

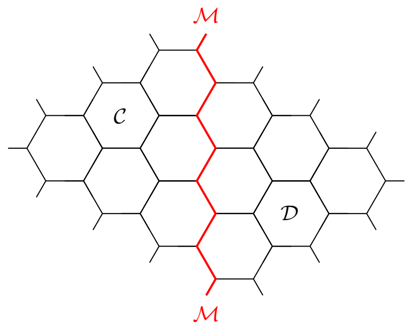
7.5. **Generalized model for spatial phase transition in Levin-Wen systems.** Suppose $\mathcal{C}, \mathcal{D}$ are unitary fusion categories and $\mathcal{M}$ is a finitely semisimple unitary $\mathcal{C}-\mathcal{D}$ bimodule category. We can write down a commuting projector generalized Levin-Wen model with an $\mathcal{M}$ defect wall between the $\mathcal{C}$ and $\mathcal{D}$ sides of the lattice. This gives a gapped spatial phase transition between topological phases whose topological orders are described by $Z(\mathcal{C})$ and $Z(\mathcal{D})$ respectively.

Exercise 7.5.1. Suppose $\mathcal{C}, \mathcal{D}$ are fusion categories and $\mathcal{M}$ is a Morita equivalence $\mathcal{C} - \mathcal{D}$ bimodule category, i.e., $\mathcal{D} = \mathrm{End}_{\mathcal{C}-}(\mathcal{M})$. Show how to endow

$$\begin{pmatrix} \mathcal{C} & \mathcal{M} \\ \mathcal{M}^{\mathrm{op}} & \mathcal{D} \end{pmatrix} \tag{7.5.2}$$

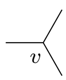
with the structure of a 2×2 multifusion category. When $\mathcal{C}, \mathcal{D}, \mathcal{M}$ are unitary, show how to equip (7.5.2) with a canonical unitary structure as well.

First, we build a hexagonal 2D lattice with a 1D defect line, where the region to the left of the defect is built from morphisms in $\mathcal{C}$, and the region to the right is built from morphisms in $\mathcal{D}$ as before. The nodes which meet the defect line are built from morphisms in $\mathcal{M}$.

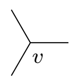


(7.5.3)

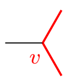
TODO: change convention from before to match this one.



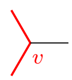
$= \bigoplus_{a,b,c\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to b \otimes c)$



$= \bigoplus_{a,b,c\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b \to c)$ or $\mathcal{D}$

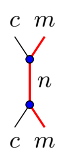


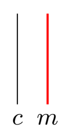
$= \bigoplus_{\substack{m,n\in \mathrm{Irr}(\mathcal{M})\\ c\in \mathrm{Irr}(\mathcal{C})}} \mathcal{M}(m \to c \rhd n)$



$= \bigoplus_{\substack{m,n\in \mathrm{Irr}(\mathcal{M})\\ d\in \mathrm{Irr}(\mathcal{D})}} \mathcal{M}(m \lhd d \to n)$

As before, we have edge and plaquette terms in the $\mathcal{C}$ and $\mathcal{D}$ bulk. We modify these terms when the edges lie in the defect or when the plaquette meets the defect. The edges in the defect just match up the simple objects. The plaquettes which meet the defect do the same gluing operation, but we use a fusion relation for $\mathcal{M}$:

$\sum_{\substack{m,n\in \mathrm{Irr}(\mathcal{M})\\ c\in \mathrm{Irr}(\mathcal{C})}} \sqrt{d_n}$


$= \sqrt{d_c d_m} \cdot$


(7.5.4)

Here, the dimensions d_m are defined as the unique unitary spherical dimensions in the 2×2 unitary multifusion category

$$\begin{pmatrix} \mathcal{C} & \mathcal{M} \\ \mathcal{M}^{\mathrm{op}} & \mathrm{End}_{-\mathcal{C}}(\mathcal{M}) \end{pmatrix}. \tag{7.5.5}$$

For plaquettes meeting the right hand side, we use similar formulas, replacing $\mathcal{C}$ with $\mathcal{D}$.

Exercise 7.5.6. In this exercise, we will prove that the dimensions imposed on $\mathcal{M}$ from (7.5.5) agree with those imposed on $\mathcal{M}$ from a similar unitary multifusion category built with $\mathcal{D}$ instead of $\mathcal{C}$.

(1) Suppose $\mathcal{M}$ is a unitary $\mathcal{C} - \mathcal{D}$ bimodule category. Show that we get commuting representations of $K_0(\mathcal{C})$ and $K_0(\mathcal{D}^{\mathrm{mp}})$ on $K_0(\mathcal{M}) = \mathbb{C}[\mathrm{Irr}(\mathcal{M})]$.

(2) We say X is a *self-dual generator* for $\mathcal{C}$ if $X \cong X^\vee$ and every object of $\mathcal{C}$ is isomorphic to a summand of a tensor power of X. Show that there exists a self-dual generator for $\mathcal{C}$.

(3) Choose self-dual generators $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Show that $[X] \in K_0(\mathcal{C})$ and $[Y] \in K_0(\mathcal{D}mp)$ are self-adjoint commuting matrices. Deduce they may be simultaneously diagonalized.

(4) Show that $(d_m)_{m\in \mathrm{Irr}(\mathcal{M})} \in \mathbb{C}[\mathrm{Irr}(\mathcal{M})]$ is a simultaneous eigenvector for both $[X]$ and $[Y]$. Deduce from the Frobenius-Perron Theorem that the dimensions imposed from $\mathcal{C}$ agree with the dimensions imposed from $\mathcal{D}$.

Since this model is commuting projector, it is gapped. On the left of the defect, the topological order is described by $Z(\mathcal{C})$, and on the right hand side, the topological order is described by $Z(\mathcal{D})$. On the boundary, the lowest energy excitations are described by the multifusion category $\mathrm{End}_{\mathcal{C}-\mathcal{D}}(\mathcal{M})$.

Exercise 7.5.7. Construct braided tensor functors $Z(\mathcal{C}) \to Z(\mathrm{End}_{\mathcal{C}-\mathcal{D}}(\mathcal{M}))$ and $Z(\mathcal{D})^{\mathrm{rev}} \to Z(\mathrm{End}_{\mathcal{C}-\mathcal{D}}(\mathcal{M}))$ whose images centralize each other.

Exercise 7.5.8. Prove that $Z(\mathrm{End}_{\mathcal{C}-\mathcal{D}}(\mathcal{M})) \cong Z(\mathcal{C}) \boxtimes Z(\mathcal{D})^{\mathrm{rev}}$.
Hint: $\mathcal{M}$ is a Morita equivalence bimodule between $\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{mp}}$ and $\mathrm{End}_{\mathcal{C}=\mathcal{D}}(\mathcal{M})$, and $Z(\mathcal{D}^{\mathrm{mp}}) \cong Z(\mathcal{D})^{\mathrm{rev}}$.

By the previous exercise, we see that the $\mathcal{C}-\mathcal{D}$ bimodule/$\mathcal{C}\boxtimes\mathcal{D}^{\mathrm{mp}}$-module $\mathcal{M}$ is equivalent to $[Z(\mathcal{C})\boxtimes Z(\mathcal{D})^{\mathrm{rev}}]_L$ for some Lagrangian algebra $L \in Z(\mathcal{C})\boxtimes Z(\mathcal{D})^{\mathrm{rev}}$. The following ansatz of Kong says that all 1D gappeddefects between 2D topological orders are described by Lagrangian algebras.

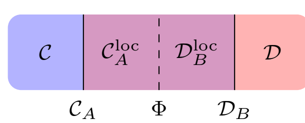
Ansatz 7.5.9. A 1D gapped defect between 2D topological orders described by the UMTCs $\mathcal{C}, \mathcal{D}$ corresponds to a Lagrangian algebra in $\mathcal{C} \boxtimes \mathcal{D}^{\mathrm{rev}}$.

We will see in §7.7 below that gapped 1D defects between 2D topological orders correspond to invertible 1-morphisms in the 4-category of braided fusion categories.

Now such Lagrangian algebras in $\mathcal{C} \boxtimes 1_{\mathcal{D}^{\mathrm{rev}}}$ have been completely classified in [DNO13, Thm. 3.6] by the following data:

- the condensable algebras $A := L \cap \mathcal{C} \boxtimes 1_{\mathcal{D}^{\mathrm{rev}}}$ and $B := L \cap 1_{\mathcal{C}} \boxtimes \mathcal{D}^{\mathrm{rev}}$, and
- a unitary braided equivalence $\Phi : \mathcal{C}_A^{\mathrm{loc}} \to \mathcal{D}_B^{\mathrm{loc}}$.

Hence any gapped spatial phase transition $\mathcal{C}$ to $\mathcal{D}$ is a composite of 3 phase transitions, comprised of 2 condensations and a braided equivalence:



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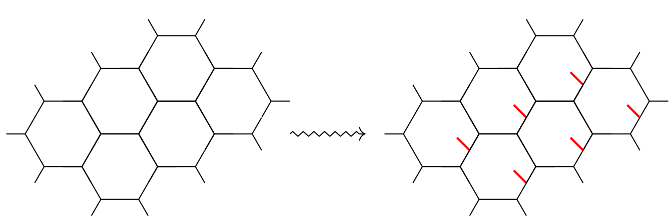
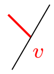
7.6. **Generalized model for condensation in Levin-Wen systems.** We now briefly describe the construction of [CGHP] based on ideas of Corey Jones which provides a lattice model to perform the critical value gapless condensation for any (connected) unitary étale algebra A in $Z(\mathcal{C})$ for a fusion category $\mathcal{C}$ (*unitary* étale means commutative unitarily separable).

First, we augment the usual 2D hexagonal lattice by adding 'vertical' edges corresponding to the algebra A, depicted in red below:



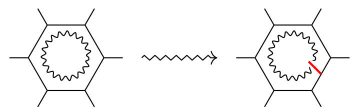
These new edges should be viewed as emanating vertically out of the plane. The usual vertices of the plaquette are assigned the usual hom spaces of the Levin-Wen model, and the new trivalent vertices correspond to the Hilbert space



$$\longleftrightarrow \qquad \bigoplus_{a,b\in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(F(A)\otimes a \to b)$$

where $F : Z(\mathcal{C}) \to \mathcal{C}$ is the forgetful functor.

The Hamiltonian has four terms: edge, plaquette, unit, and condensation. The edge term A_ℓ is as before, which checks simple labels agree. The plaquette term B_p is similar to before, but now, we use the half-braiding for A with $\mathrm{Irr}(\mathcal{C})$ in order to glue the regular element into the boundary.



The unit terms C_v at the vertices with vertical edges implement applying the adjoint $F(i^\dagger)$ of the unit map $F(i) : 1_\mathcal{C} \to F(A)$, which effectively chops off the vertical edges and puts us back into the usual Levin-Wen string net model. The condensation terms $D_{v,w}$ between two 'neighboring' vertices v, w with vertical edges implements the canonical separability projector $\mu^\dagger\mu$, which again uses the half-braiding for $F(A)$ with $\mathrm{Irr}(\mathcal{C})$.

The overall Hamiltonian is then given by

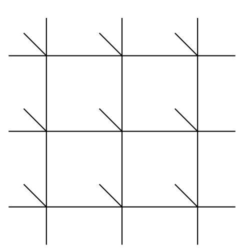
$$H = -\sum_\ell A_\ell - \sum_p B_p - K\left((1-t)\sum_v C_v + t\sum_{v,w} D_{v,w}\right),$$

where $K \gg 1$.

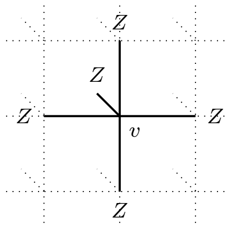
Remark 7.6.1. This model has been shown to produce the condensation quantum phase transitions for the $(\mathbb{Z}/n)$ toric code a doubled semion models as t goes from 0 to 1. This analysis is achieved similar to before by mapping our model onto the 2D transverse-field Ising model. Although the total Hilbert space is larger for our model, the analysis is easier, as we no longer need to pass to a dual lattice; simply passing to the ground state space of the edge A_ℓ terms and the plaquette B_p terms yields exactly the 2D transverse-field Ising model for the $\mathbb{Z}/2$ toric code.

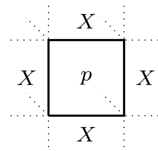
We now give two explicit examples of this model, one for condensing e in $\mathbb{Z}/2$ toric code, and one for condensing the boson m in the doubled semion model $Z(\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/2, \omega))$.

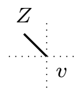
Example 7.6.2 ($\mathbb{Z}/2$ toric code [CGHP])**.** Our system has a qubit $\mathbb{C}^2$ on every edge of the following 2D lattice:

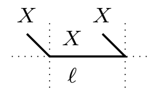


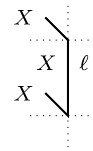
Our operators which comprise our local Hamiltonian are given by:

$A_v :=$ 

$B_p :=$ 

$C_v :=$ 

$D_\ell :=$ 

or 

.

Our Hamiltonian is given by:

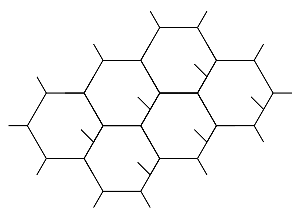
$$H_t := -V\left(\sum_v A_v + \sum_p B_p\right) - K\left((1-t)\sum_v C_v + t\sum_\ell D_\ell\right),$$

where $V > 0$ is a constant and $K \gg V$ is a large constant.

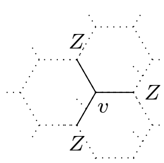
Exercise 7.6.3. Adapt the above example to condense m.

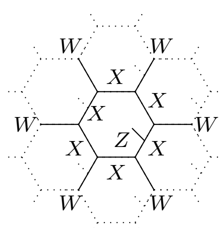
Exercise 7.6.4. Adapt the above example to condense the e or m particles in $Z(\mathsf{Hilb}_{\mathsf{fd}}(\mathbb{Z}/n))$.

Example 7.6.5 (Doubled semion [CGHP])**.** Our system has a qubit $\mathbb{C}^2$ on every edge of the following 2D lattice:

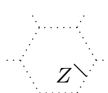


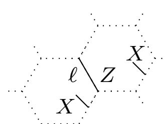
where by convention, there is a single qubit on the edge attached to the vertical edge. Our operators which comprise our local Hamiltonian are given by:

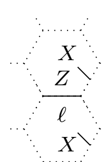
$A_v :=$ 

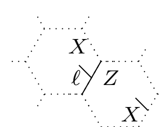
$B_p :=$ 

$\cdot\, \mathbf{P}$ where $W := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$C_p :=$ 

$D_\ell :=$ 

, 

, or 

where $\mathbf{P}$ is the orthogonal projection onto the $+1$ eigenstate of $\prod A_v$. Our Hamiltonian is given by a similar formula as before.

Exercise 7.6.6. Verify the claims made in Remark 7.6.1.

7.7. **The 4-category of braided fusion categories and the Witt group.** This section is basically copied from [JMPP19, §2.3].

By [Hau17, JFS17], there is a 4-category of braided tensor categories BrTens, and the sub-4-category BrFus of braided fusion categories is 4-dualizable by [BJS18, Thm. 1.19]. Following [BJS18, JMPP19], we describe the n-morphisms and the composition operations of the 4-category BrFus.

- 0-morphisms are braided fusion categories.
- 1-morphisms $\mathsf{BrFus}_1(\mathcal{A} \to \mathcal{B})$ are multifusion categories $\mathcal{C}$ together with a braided monoidal functor $F_{\mathcal{C}} : \mathcal{A} \boxtimes \mathcal{B}^{\mathrm{rev}} \to Z(\mathcal{C})$ called a *central structure*. Sometimes we denote $\mathcal{C} \in \mathsf{BrFus}_1(\mathcal{A} \to \mathcal{B})$ by ${}_{\mathcal{A}}\mathcal{C}_{\mathcal{B}}$.

  The composite of ${}_{\mathcal{A}_1}\mathcal{C}_{\mathcal{A}_2}$ and ${}_{\mathcal{A}_2}\mathcal{D}_{\mathcal{A}_3}$ is defined as follows. First, we look at the Deligne tensor product $\mathcal{C}\boxtimes\mathcal{D}$, which comes equipped with a braided monoidal functor $F : \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \to Z(\mathcal{C} \boxtimes \mathcal{D})$. We define $\mathcal{C} \boxtimes_{\mathcal{A}_2} \mathcal{D}$ to be $(\mathcal{C} \boxtimes \mathcal{D})_L$, the category of left L-modules in $\mathcal{C} \boxtimes \mathcal{D}$, where $L \in \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2$ is the commutative algebra obtained by taking $I(1_{\mathcal{A}_2})$, where I is the left adjoint to the canonical tensor product functor $\otimes : \mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \to \mathcal{A}_2$, given by $\otimes(a \boxtimes b) := a \otimes b$ and using the braiding for the tensorator. This algebra is commutative since $\otimes$ is a central functor [DMNO13, Lemma 3.5]. If $\mathcal{A}_2$ is nondegenerate, this algebra is identified with the canonical Lagrangian algebra under the standard equivalence $\mathcal{A}_2^{\mathrm{rev}} \boxtimes \mathcal{A}_2 \cong Z(\mathcal{A}_2)$.

  To see that $\mathcal{C} \boxtimes_{\mathcal{A}_2} \mathcal{D}$ has the structure of a 1-morphism in $\mathsf{BrFus}_1(\mathcal{A}_1 \to \mathcal{A}_3)$, we observe that $Z((\mathcal{C} \boxtimes \mathcal{D})_L) \cong Z(\mathcal{C} \boxtimes \mathcal{D})_L^{\mathrm{loc}}$, the L-local modules in $Z(\mathcal{C} \boxtimes \mathcal{D}) \cong Z(\mathcal{C}) \boxtimes Z(\mathcal{D})$ by [DMNO13, Thm. 3.20]. Since $\mathcal{A}_1$ centralizes $F_{\mathcal{A}_2^{\mathrm{rev}}}(\mathcal{A}_2^{\mathrm{rev}}) \boxtimes Z(\mathcal{D})$ and $\mathcal{A}_3^{\mathrm{rev}}$ centralizes $Z(\mathcal{C}) \boxtimes F_{\mathcal{A}_2}(\mathcal{A}_2)$ in $Z(\mathcal{C}) \boxtimes Z(\mathcal{D})$, we get a braided monoidal functor $\mathcal{A}_1 \boxtimes \mathcal{A}_3^{\mathrm{rev}} \to Z(\mathcal{C} \boxtimes \mathcal{D})_L^{\mathrm{loc}} \cong Z((\mathcal{C} \boxtimes \mathcal{D})_L)$.

  An explicit example calculation of the composite $\mathrm{Ad}\, E_8 \boxtimes_{\mathsf{Fib}} \mathrm{Ad}\, E_8'$ appears in [Row19].
- 2-morphisms in $\mathsf{BrFus}_2(\mathcal{C} \to \mathcal{D})$ are finitely semisimple $\mathcal{C} - \mathcal{D}$ bimdodule categories $\mathcal{M}$ together with natural isomorphisms $\eta_{a,m} : m \triangleleft F_{\mathcal{D}}(a) \to F_{\mathcal{C}}(a) \triangleright m$ for $a \in \mathcal{A}\boxtimes\mathcal{B}^{\mathrm{rev}}$ and $m \in \mathcal{M}$ called a $\mathcal{A} \boxtimes \mathcal{B}^{\mathrm{rev}}$-*centered structure* such that the following diagrams commute (here we suppress names of arrows):

$$
\begin{tikzcd}
 & F_{\mathcal{C}}(a) \triangleright (c \triangleright m) \arrow[r] & (c \triangleright m) \triangleleft F_{\mathcal{D}}(a) \arrow[dr] & \\
(F_{\mathcal{C}}(a) \otimes c) \triangleright m \arrow[ur] \arrow[dr] & & & c \triangleright (m \triangleleft F_{\mathcal{D}}(a)) \\
 & (c \otimes F_{\mathcal{C}}(a)) \triangleright m \arrow[r] & c \triangleright (F_{\mathcal{C}}(a) \triangleright m) \arrow[ur] &
\end{tikzcd}
\tag{7.7.1}
$$

$$\begin{tikzcd}[column sep=small]
 & F_{\mathcal{C}}(a) \triangleright (m \triangleleft d) \arrow[r] & (m \triangleleft d) \triangleleft F_{\mathcal{D}}(a) \arrow[dr] & \\
(F_{\mathcal{C}}(a) \triangleright m) \triangleleft d \arrow[ur] \arrow[dr] & & & m \triangleleft (d \otimes F_{\mathcal{D}}(a)) \\
 & (m \triangleleft F_{\mathcal{D}}(a)) \triangleleft d \arrow[r] & m \triangleleft (F_{\mathcal{D}}(a) \otimes d) \arrow[ur] &
\end{tikzcd} \tag{7.7.2}$$

$$\begin{tikzcd}
F_{\mathcal{C}}(a \otimes b) \triangleright m \arrow[r] \arrow[d] & m \triangleleft F_{\mathcal{D}}(a \otimes b) \arrow[r] & m \triangleleft (F_{\mathcal{D}}(a) \otimes F_{\mathcal{D}}(b)) \arrow[d] \\
(F_{\mathcal{C}}(a) \otimes F_{\mathcal{C}}(b)) \triangleright m \arrow[d] & & (m \triangleleft F_{\mathcal{D}}(a)) \triangleleft F_{\mathcal{D}}(b) \arrow[d] \\
F_{\mathcal{C}}(a) \triangleright (F_{\mathcal{C}}(b) \triangleright m) \arrow[r] & F_{\mathcal{C}}(a) \triangleright (m \triangleleft F_{\mathcal{D}}(b)) \arrow[r] & (F_{\mathcal{C}}(a) \triangleright m) \triangleleft F_{\mathcal{D}}(b)
\end{tikzcd} \tag{7.7.3}$$

The definitions of horizontal and vertical composition of 2-morphisms are given in [BJS18, p. 41-42]. Vertical composition is the relative Deligne tensor product ${}_{\mathcal{C}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}_{\mathcal{E}}$. As described in [BJS18, Def. Prop. 3.13], when $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are equipped with central structures $F_{\mathcal{C}}, F_{\mathcal{D}}, F_{\mathcal{E}}$ respectively and $\mathcal{M}, \mathcal{N}$ are equipped with $\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$-centered structures $\eta^{\mathcal{N}}, \eta^{\mathcal{M}}$ satisfying (7.7.1), (7.7.2), (7.7.3), the $\mathcal{C} - \mathcal{E}$ bimodule category $\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}$ is equipped with the $\mathcal{A} \boxtimes \mathcal{B}^{\text{rev}}$-centered structure

$$m \boxtimes_{\mathcal{D}} (n \triangleleft F_{\mathcal{E}}(a)) \cong m \boxtimes_{\mathcal{D}} (F_{\mathcal{D}}(a) \triangleright n) \cong (m \triangleleft F_{\mathcal{D}}(a)) \boxtimes_{\mathcal{D}} n \cong (F_{\mathcal{C}}(a) \triangleright m) \boxtimes_{\mathcal{D}} n. \tag{7.7.4}$$

- Let $\mathcal{M}$ and $\mathcal{N}$ be two 2-morphisms with source $\mathcal{C}$ and target $\mathcal{D}$. Then a 3-morphism is a bimodule functor $G : \mathcal{M} \to \mathcal{N}$ such that the following diagram commutes:

$$\begin{tikzcd}
G(m \triangleleft F_{\mathcal{D}}(a)) \arrow[r, "G(\eta_{a,m})"] \arrow[d] & G(F_{\mathcal{C}}(a) \triangleright m) \arrow[d] \\
G(m) \triangleleft F_{\mathcal{D}}(a) \arrow[r, "\eta_{a,G(m)}"] & F_{\mathcal{C}}(a) \triangleright G(m)
\end{tikzcd} \tag{7.7.5}$$

- 4 morphisms are bimodule natural transormations with no extra compatibility required!

Remark 7.7.6. Observe that we may consider a multifusion category $\mathcal{C} \in \mathsf{BrFus}_1(\mathsf{Vec} \to \mathsf{Vec})$ where we suppress the obvious braided central functor $\mathcal{F}^{\mathsf{Z}} : \mathsf{Vec} \to Z(\mathcal{C})$. In more detail, we expect that $\mathsf{BrFus}_1(\mathsf{Vec} \to \mathsf{Vec}) \cong \mathsf{MultFusCat}$, the 3-category of multifusion categories.

Definition 7.7.7. Non-degenerate braided fusion categories $\mathcal{A}, \mathcal{B}$ are said to be *Witt equivalent* [DMNO13, Def. 5.1 and Rem. 5.2] if there exist multifusion categories $\mathcal{C}, \mathcal{D}$ such that $\mathcal{A} \boxtimes Z(\mathcal{C}) \cong \mathcal{B} \boxtimes Z(\mathcal{D})$.

Theorem 7.7.8. *Suppose $\mathcal{A}, \mathcal{B}$ are non-degenerate braided fusion categories. The following are equivalent.*

(1) $\mathcal{A}$ and $\mathcal{B}$ are Witt equivalent.
(2) There is a fusion category $\mathcal{C}$ and a braided equivalence $F_{\mathcal{C}} : \mathcal{A} \boxtimes \mathcal{B}^{\text{rev}} \to Z(\mathcal{C})$.
(3) There is an invertible 1-morphism between $\mathcal{A}$ and $\mathcal{B}$ in BrFus.

Proof.
(1) $\Leftrightarrow$ (2): This follows from [DMNO13, Rem. 5.2 and Cor. 5.8].
(2) $\Leftrightarrow$ (3): This is [JMPP19, Thm. 2.18]. □

Combined with Ansatz 7.5.9, we should expect the existence of a 1D gapped defect between two 2D topological orders if and only if they are Witt equivalent.

7.8. Generalized condensation/gauging. TODO:

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