Main points from intro [ZCZW19]:

- Classical phases of matter described by Landau's theory of symmetry breaking
- Fractional quantum hall (FQH) states discovered in 1989 led to discovery of new topological phases/order with emergent particle physics with fractional charge
- Entanglement has no classical counterpart, and topological order comes from many-body entanglement
- Quantum theory unifies information and matter

Main points from intro `http://web.math.ucsb.edu/~zhenghwa/data/course/cbms.pdf`:

- Beyond Shor's factoring algorithm, quantum computers could be used to simulate quantum many-body systems and quantum field theories
- Emergent quasiparticles from quantum systems like FQH are called *anyons* as they need not have integer (*boson*), half-integer (*fermion*), or quarter-integer (*semion*) statistics.
- Anyons can be used to perform topological quantum computation, which is manifestly fault-tolerant. Moreover, some anyonic systems are universal for quantum computation.
- Mathematically, anyons are simple objects in a *unitary modular tensor category*

Basic outline of the course:

- Basics of quantum information theory [ZCZW19, Ch. 1-3]
- Local Hamiltonian lattice models (Ising, Potts, transverse field Ising), quantum error correction codes (Toric code), Gapped quantum systems, entanglement area law, [ZCZW19, Ch. 4-5]
- Basics of tensor categories [EGNO15]
- UMTC of localized excitations [SKK20, **?**], topological order and ground state degeneracy, string-net condensation [ZCZW19, Ch. 6]
- Higher linear algebra [HV19], condensations in higher categories [GJF19, JF20]
- Phase transitions: anyon condensation, gauging, condensing domain walls

## 1. Basics of quantum information

To understand entanglement of quantum many-body systems, we must have a basic understanding of quantum information.

1.1. **Basics of probability.** We rapidly recall some basic notions from probability theory.

- A *probability space* is a measure space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}(\Omega) = 1$.
- A *random variable* is a (Borel measurable) function $X : \Omega \to \mathbb{C}$.
- The *push-forward measure* of $\mathbb{P}$ to $\mathbb{C}$ via $X$ is given by $p_X(E) := \mathbb{P}(X^{-1}(E))$.
- The *expected value* of $X$ is $\mathbb{E}(X) := \int X(\omega)\, \mathrm{d}\mathbb{P}(\omega)$.
- The *correlation* of $X, Y$ is $C(X, Y) := \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

We will concern ourselves with finite spaces $\Omega$, which drastically simplifies the above.

**Definition 1.1.1.** A *bit* is a random variable which only takes the values $0, 1$.

**Notation 1.1.2.** Sometimes we will identify a random variable $X : \Omega \to \mathbb{C}$ with its image $X \subset \mathbb{C}$. We then write $p_X(x) := \mathbb{P}(X^{-1}(\{x\}))$ for the push-forward measure, and by convention, we can remove from $X$ any points for which $p_X(x) = 0$. If there is only one random variable $X$ around, we just write $p$. In this case, $\mathbb{E}(X) = \sum_{x \in X} xp(x)$.

**Example 1.1.3.** Suppose we roll a fair die, so $\Omega = \{1, \ldots, 6\}$, and $\mathbb{P}(1) = \cdots = \mathbb{P}(6) = 1/6$. Let $X$ denote the number of 6's in 3 rolls. Then $p_X(j) = \binom{3}{j}(5/6)^{3-j}(1/6)^j$ for $j = 0, 1, 2, 3$, which is the binomial distribution.

Often in quantum information, there are two 'players' Alice and Bob who are performing joint experiments. Classically, Alice ($A$) has probability space $(\Omega_A, \mathbb{P}_A)$ , and Bob ($B$) has probability space $(\Omega_B, \mathbb{P}_B)$. The *joint probability space* is $\Omega_A \times \Omega_B$, on which we may consider any *joint probability distribution* $\mathbb{P}_{AB}$. The *conditional probability* of $\omega_A$ given $\omega_B$ is

$$\mathbb{P}_{A|B}(\omega_A|\omega_B) := \frac{\mathbb{P}_{AB}(\omega_A, \omega_B)}{\mathbb{P}_B(\omega_B)},$$

and similarly for $\mathbb{P}_{B|A}$.

**Exercise 1.1.4.** The following are equivalent for $(\Omega_A, \mathbb{P}_A)$, $(\Omega_B, \mathbb{P}_B)$, and $(\Omega_A \times \Omega_B, \mathbb{P}_{AB})$:

(1) $\mathbb{P}_{AB}(\omega_A, \omega_B) = \mathbb{P}_A(\omega_A)\mathbb{P}_B(\omega_B)$ for all $\omega_A \in \Omega_A$ and $\omega_B \in \Omega_B$.
(2) $\mathbb{P}_{A|B}(\omega_A|\omega_B) = \mathbb{P}_{A|B}(\omega_A|\omega'_B)$ for all $\omega_A \in \Omega_A$ and $\omega_B, , \omega'_B \in \Omega_B$
(3) Similar to (2) with roles of $A$ and $B$ swapped.
(4) For all random variables $X_A$ on $\Omega_A$ and $X_B$ on $\Omega_B$, $\mathbb{E}_{AB}(X_A X_B) = \mathbb{E}_A(X_A)\mathbb{E}_B(X_B)$, i.e., $C(X_A, X_B) = 0$.

In this case, we say that $\mathbb{P}_{AB}$ does not have correlation.

1.2. **Shannon entropy and mutual information.** Suppose you roll 2 dice and you measure the sum of the values. Observe that you learn more information from an outcome of 2 than you do from an outcome of 7. If we wanted to define an 'information function' $h : (0, 1] \to \mathbb{R}_{\geq 0}$ that can applied to a probability distribution $\{p_i\}$, it is natural to ask for the following properties:

(I1) $h$ is *strictly decreasing* with $\lim_{p \to 0+} h(p) = \infty$ and $h(1) = 0$, as observing an unlikely event gives more information than observing a likely event. Observing an event that is certain to happen yields no information, and one cannot measure an event that has no change of happening.
(I2) $h(pq) = h(p) + h(q)$, since the information obtained from observing independent events should be the sum of the information of observing each event individually.

**Exercise 1.2.1.**

(1) Show that any continuous homomorphism $g : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is of the form $g(x) = tx$ for some $t \in \mathbb{R}$. Deduce this result still holds for the semigroup $(\mathbb{R}_{>0}, +)$.
(2) Show that the only function $h : (0, 1] \to \mathbb{R}_{\geq 0}$ satisfying (I1) and (I2) which is continuous on $(0, 1]$ is $h(p) = -c \cdot \ln(p)$ where $c \in (0, \infty)$.
*Hint: Consider* $h|_{(0,1)} \to \mathbb{R}_{>0}$.

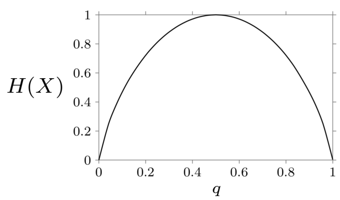
Since the constant $c > 0$ above is equivalent to a choice of base for the logarithm, we define $h(p) := -\log(p)$. The entropy of the distribution $\{p_i\}$ will be defined as $-\sum_i p_i \log(p_i)$, which is the average/expected information gained from observing a measurement. More entropy then means you gain more information from taking a measurement, which means the system is more disordered, i.e., in a more uncertain state.

**Definition 1.2.2.** Given a random variable $X$ on a finite probability space $(\Omega, \mathbb{P})$, the *(Shannon) entropy* of $X$ is

$$H(X) := -\sum_{x \in X} p(x) \log p(x),$$

where log means base 2. It is a measure of *uncertainty* of the random variable $X$ which only depends on $p$, and not the values of $X$.

**Example 1.2.3.** Suppose $X$ is a bit with probability distribution $p(0) = q$ and $p(1) = 1 - q$, like counting the number of heads in one flip of a weighted coin, or successes in one Bernoulli trial. Then $H(X) = -q \log q - (1-q)\log(1-q)$, called the *binary entropy function*.




[arXiv:1508.02595, p. 11]

**Exercise 1.2.4.** Suppose $\{p_i\}_{i=1}^N$ is some probability distribution on a finite set with $N$ elements. Prove that $H(\{p_i\}_{i=1}^N)$ is maximized by the uniform distribution.

**Definition 1.2.5.** Given random variables $X_A$ on $(\Omega_A, \mathbb{P}_A)$ and $X_B$ on $(\Omega_B, \mathbb{P}_B)$ and a joint probability distribution $\mathbb{P}_{AB}$ on $\Omega_A \times \Omega_B$, we have the following quantities:

- (joint entropy) $H(X, Y) := -\sum_{x,y} p(x, y) \log p(x, y)$
- (conditional entropy) $H(X|Y = y) := -\sum_x p(x|y) \log p(x|y)$, similarly $H(Y|X = x)$

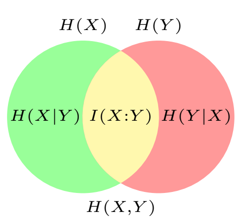
**Definition 1.2.6.** The *mutual information* for $(\Omega_A, \mathbb{P}_A, X_A)$ and $(\Omega_B, \mathbb{P}_B, X_B)$ is given by

$$I(X : Y) := \sum_{y \in Y} \sum_{x \in X} p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right).$$

(Observe that by convention, $x \in X$ means that $p(x) \neq 0$!)

**Exercise 1.2.7.** Prove the following identities:

(1) $H(X) = H(X|Y) + I(X : Y)$
(2) $H(Y) = H(Y|X) + I(X : Y)$
(3) $H(X, Y) = H(X|Y) + H(Y|X) + I(X : Y)$.




(1.2.8)

adapted from [arXiv:1508.02595, p. 12]

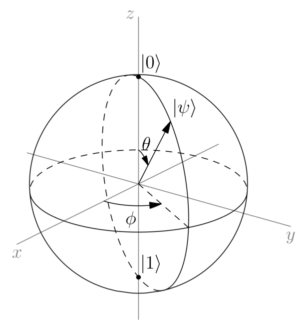
**Exercise 1.2.9.** Prove that $I(X : Y) = 0$ implies $C(X, Y) = 0$. Does the converse hold? *Hint: Observe* $I(X : Y) = 0$ *if and only if* $p(x, y) = p(x)p(y)$.

1.3. **Basics of quantum mechanics.** A *state vector* is a unit (norm 1) vector $|\psi\rangle$ in a Hilbert space $\mathcal{H}$; we denote the inner product by $\langle \cdot | \cdot \rangle$, which is linear on the right. Vectors are denoted by *kets* $|\eta\rangle$ and covectors (linear functionals) are denoted by *bras* $\langle\xi|$. Given a bounded operator $x \in B(H)$, the *adjoint* $x^\dagger \in B(H)$ is defined by $\langle\psi|x\phi\rangle = \langle x^\dagger\psi|\phi\rangle$ for all $\psi, \phi \in \mathcal{H}$ by the Riesz Representation Theorem.

**Principle of superposition:** Given orthogonal state vectors $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{H}$ and $c_1, c_2 \in \mathbb{C}$ with $|c_1|^2 + |c_2|^2 = 1$, $c_1|\psi_1\rangle + c_2|\psi_2\rangle \in \mathcal{H}$ is another state vector.

**Example 1.3.1.** The state vector of a free electron is a unit vector in $\mathbb{C}|0\rangle \oplus \mathbb{C}|1\rangle$, which is also called a *quantum bit* or *qubit*. Here, $|0\rangle$ is *spin up* and $|1\rangle$ is *spin down*. We can represent a qubit *up to a phase* in $U(1) := \{z \in \mathbb{C} | \, |z| = 1\}$ by a point on the surface of the *Bloch sphere*:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$




[arXiv:1508.02595, p. 14]

**Definition 1.3.2.** An *observable* is a self-adjoint (Hermitian) operator $M$ on $\mathcal{H}$. A *measurement* is an eigenvalue $\mu$ of $M$; the set of eigenvalues of $M$ is called the *spectrum* $\operatorname{spec}(M)$ of $M$. Since $\mathcal{H}$ is finite dimensional, by the spectral theorem, we can decompose $M$ canonically as a weighted sum of orthogonal projections:

$$M = \sum_{\mu \in \operatorname{spec}(M)} \mu p_\mu \qquad \sum_{\mu \in \operatorname{spec}(M)} p_\mu = \mathrm{id}_{\mathcal{H}} \qquad E_\mu := p_\mu\mathcal{H} = \{\eta \in H | M\eta = \mu\eta\}.$$

Given an observable $M$, its *avergage/expected value* in state vector $|\psi\rangle$ is

$$\langle M\rangle_\psi := \langle\psi|M|\psi\rangle := \langle\psi|M\psi\rangle.$$

When the system is in state vector $|\psi\rangle$, the probability of observing the measurement $\mu$ for $M$ is $\langle p_\mu\rangle = \langle\psi|p_\mu\psi\rangle$. Observe $\langle M\rangle$ and $\langle p_\mu\rangle$ are independent of the phase of $|\psi\rangle$.

**Definition 1.3.3.** The *Pauli spin matrices* in $M_2(\mathbb{C})$ are given by

$$X := \sigma_X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y := \sigma_Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z := \sigma_Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Observe that $\{I, X, Y, Z\}$ is a basis for $M_2(\mathbb{C})$.

**Example 1.3.4.** If we measure the Pauli $Z$ operator in the qubit state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the probability of measuring 1 is $|\alpha|^2$, while the probability of measuring $-1$ is $|\beta|^2$. If we measure the Pauli $X$ operator in the same state vector, the probability of measuring $\pm 1$ is

$$\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \middle| \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\rangle = \frac{1}{2}\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \middle| \begin{pmatrix} \alpha \pm \beta \\ \pm\alpha + \beta \end{pmatrix} \right\rangle = \frac{1}{2} \pm \operatorname{Re}(\alpha\overline{\beta}).$$

Whereas all random variables on a probability space share the same distribution, measurements do not. Thus we must choose an observable in order to assign a value of uncertainty.

**Definition 1.3.5.** A *Hamiltonian* of a quantum system is an (unbounded) self-adjoint/Hermetian operator $H$ acting on a (dense subset of) a Hilbert space $\mathcal{H}$. (Here, unbounded only applies to systems where $\mathcal{H}$ is infinite dimensional.) Eigenvalues of the Hamiltonian are meant to represent energy levels of the system. The time evolution of states is governed by Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \qquad \rightsquigarrow \qquad |\psi(t)\rangle = e^{-itH/\hbar}|\psi(0)\rangle. \tag{1.3.6}$$

Note that even if $H$ is unbounded, $e^{-itH/\hbar}$ is always unitary. This means that time evolution of the system is reversible. In Heisenberg's formulation, time dependence lies on the observables $M$:

$$\langle\phi(t)|M|\psi(t)\rangle = \langle\phi(0)|\underbrace{e^{itH/\hbar}Me^{-itH/\hbar}}_{:=M(t)}|\psi(0)\rangle$$

Heisenberg's equation of motion for an observable $M$ is

$$i\hbar\frac{\partial}{\partial t}M = [M,H] := MH - HM$$

**Exercise 1.3.7.** Use Equation (1.3.6) to show that time evolution of a mixed state is given by

$$i\hbar\frac{\partial}{\partial t}\rho(t) = -[\rho(t),H]$$

Then show that $\rho(t) = e^{-itH/\hbar}\rho(0)e^{itH/\hbar}$ is a solution of the above differential equation. *Hint: $e^{\pm itH/\hbar}$ commutes with $H$.*

## 1.4. **States.**

**Definition 1.4.1.** A *state* on $B(\mathcal{H})$ is a linear functional $\omega : B(\mathcal{H}) \to \mathbb{C}$ such that $\omega(I) = 1$ and $x \geq 0$ ($\langle\psi|x\psi\rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$) implies $\omega(x) \geq 0$.

**Example 1.4.2.** A state vector $|\psi\rangle$ gives a *vector state* $\omega_\psi(x) := \langle\psi|x\psi\rangle$. Observe that the vector state corresponding to a state vector is independent of the phase.

**Exercise 1.4.3.**

(1) Show that $\omega_{\psi_1} = \omega_{\psi_2}$ if and only if $|\psi_1\rangle = c|\psi_2\rangle$ for some $c \in U(1)$.

(2) Recall that the *trace* on $B(\mathcal{H})$ is given by $\mathrm{Tr}(x) := \sum\langle\eta_i|x\eta_i\rangle$ where $\{|\eta_i\rangle\}$ is any orthonormal basis of $\mathcal{H}$. When $x$ is *normal* ($xx^\dagger = x^\dagger x$), $\mathrm{Tr}(x) = \sum_{\mu\in\mathrm{spec}(x)}\mu\dim(E_\mu)$.

Show that for every state $\omega$ on $B(\mathcal{H})$, there is a unique positive operator $\rho \in B(\mathcal{H})$ called the *density operator*[1] of $\omega$ such that $\omega(x) = \mathrm{Tr}(x\rho)$ for all $x \in B(\mathcal{H})$.

(3) Prove that the density operator $\rho$ of a state $\omega$ satisfies $\mathrm{Tr}(\rho^2) \leq 1$ with equality if and only if $\rho = |\psi\rangle\langle\psi|$ for some state vector $|\psi\rangle \in \mathcal{H}$.

**Vector states and density operators.** A vector state $|\psi\rangle \in H$ is really only physically relevant up to a phase in $U(1)$. The $U(1)$-orbit of $|\psi\rangle$ corresponds to the rank one density operator $|\psi\rangle\langle\psi| \in B(\mathcal{H})$, which is called a *pure state.* A non-trivial convex combination of pure states is called a *mixed state.*

[1]Sometimes, $\rho$ is called the (non-commutative) *Radon-Nikodym derivative* of $\omega$ with respect to Tr.

**Remark 1.4.4.** A mixed state is not in a superposition; rather, a mixed state is what we get by imposing classical probability theory on top of quantum mechanics.

First, a superposition really only makes sense with respect to a certain basis, which are the classical outcomes of a certain observable we would like to measure. For example, suppose Alice wants to measure the spin of an electron in state $|\psi\rangle \in \mathbb{C}^2$ which is in a superposition of $|0\rangle$ and $|1\rangle$. She measures Pauli $Z$; if she observes 1, then the electron is now in state $|0\rangle$, and if she measures -1, the electron is in state $|1\rangle$.

Suppose now $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ so that it is equally likely to measure either outcome, and that Bob measures the electron, but does not tell Alice the result. The electron is either in state $|0\rangle$ or $|1\rangle$, but Alice cannot tell which one; the state of the electron may then be described by the mixed state $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I$.

Indeed, a mixed state $\rho$ can be viewed as a classical probability distribution of a finite collection of pure states:

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| \qquad\qquad 0 \le p_i \le 1,\ \sum_i p_i = 1.$$

Its von Neumann entropy is then equal to the Shannon entropy of the corresponding classical probability distribution $\{p_i\}$:

$$S(\rho) = -\operatorname{Tr}(\rho\log(\rho)) = -\operatorname{Tr}(\sum p_i \log(p_i)|\psi_i\rangle\langle\psi_i|) = -\sum p_i \log(p_i) = H(\{p_i\}).$$

However, observe that the same mixed state can be viewed in multiple ways as a distribution:

$$\rho = \sum p_i |\psi_i\rangle\langle\psi_i| = \sum q_j |\phi_j\rangle\langle\phi_j|.$$

But quantum mechanics cannot distinguish which probability distribution we have!

Some physicists/mathematicians think that in pure quantum mechanics, there are really only pure states; $|\psi\rangle$ describes the state of the universe, and it evolves in time with respect to some Hermitian/self-adjoint Hamiltonian $H$. However, our local information is not enough information to determine this global pure state $|\psi\rangle$. We can thus try and approximate the state based on certain reduced density matrices that we can measure using the principle of maximum entropy below in §1.8.

**Exercise 1.4.5.** An *extreme point* of a convex subset $S$ of a vector space $\mathcal{V}$ is a point $x \in S$ such that

$$x = ty + (1-t)z \text{ for some } 0 \le t \le 1 \qquad \Longrightarrow \qquad y = z = x.$$

Show that the pure states are the extreme points of the convex set of states on $B(\mathcal{H})$.
*Note: the state space is the convex hull of the pure states.*

**Exercise 1.4.6.**

(1) Prove that every mixed state in $B(\mathbb{C}^2)$ can be written as

$$\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma}) = \frac{1}{2}(I + r_x X + r_y Y + r_z Z) = \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

where $X, Y, Z$ are the Pauli matrices and $\vec{r} \in \mathbb{R}^3$ is called the *Bloch vector*.

(2) Deduce that the Bloch sphere is the state space of $M_2(\mathbb{C})$, and the interior corresponds to the mixed states. Which states lie at the intersection of the axes and the surface of the Bloch sphere?

(3) Calculate the eigenvalues of $\rho$ in terms of $\vec{r}$.

**Various notions of states.**

| | |
|---|---|
| state vector | norm 1 $\|\psi\rangle \in \mathcal{H}$ |
| state | positive linear functional $\omega$ on $B(H)$ with $\omega(I) = 1$ <br> or density operator $\rho \geq 0$ with $\mathrm{Tr}(\rho) = 1$ |
| vector state | $B(H) \ni x \mapsto \langle\psi\|x\psi\rangle \in \mathbb{C}$ where $\|\psi\rangle \in \mathcal{H}$ norm 1 |
| pure state | rank 1 projection $\rho = \|\psi\rangle\langle\psi\| \in B(H)$ |
| mixed state | non-trivial convex combination of orthogonal pure states |

One gets a vector state from a pure state by $x \mapsto \mathrm{Tr}(\rho x)$. Thus pure states correspond to state vectors modulo a phase in $U(1)$.

**Expected values of observables for various notions of states.**

| | |
|---|---|
| state vector $\|\psi\rangle$ <br> vector state $\omega_\psi$ <br> pure state $\|\psi\rangle\langle\psi\|$ | $\langle\psi\|M\psi\rangle$ |
| mixed state $\rho$ | $\mathrm{Tr}(M\rho)$ |

**Definition 1.4.7.** The *von Neumann entropy* of a state $\rho$ is given by

$$S(\rho) := -\operatorname{Tr}(\rho\log(\rho)).$$

Here, $\rho\log(\rho)$ is defined via the spectral theorem:

$$\rho = \sum_{\mu\in\operatorname{spec}(\rho)} \mu p_\mu \qquad \Longrightarrow \qquad \rho\log(\rho) = \sum_{\mu\in\operatorname{spec}(\rho)} \mu\log(\mu) p_\mu$$

where by convention, $0\log(0) = 0$.

**Exercise 1.4.8.** Show that von Neumann entropy $S$ is a continuous function from states on $B(\mathcal{H})$ to $[0,\infty)$.
*Hint: One could proceed as follows.*

(1) *For any convergent sequence of normal operators $a_n \to a$ in $B(\mathcal{H})$ and any open neighborhood $U$ of* $\operatorname{spec}(a)$*, eventually* $\operatorname{spec}(a_n) \subset U$.
(2) *For any convergent sequence of normal operators $a_n \to a$ in $B(\mathcal{H})$ and any open neighborhood $U$ of* $\operatorname{spec}(a)$ *and continuous $f : U \to \mathbb{C}$, $f(a_n) \to f(a)$.*

**Exercise 1.4.9.** Prove that the entropy is maximized at a unique point in the state space of $B(\mathcal{H})$. What is this point? For $\mathcal{H} = \mathbb{C}^2$, where does this point lie in the Bloch sphere?
*Hint: Use Exercise 1.2.4*

1.5. **Tensor products.** Given independent observers Alice ($A$) and Bob ($B$) with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, the total Hilbert space is

$$\mathcal{H}_A \otimes \mathcal{H}_B := \operatorname{span}\{|\varphi\otimes\phi\rangle = |\varphi\rangle\otimes|\phi\rangle \,|\, |\varphi\rangle \in \mathcal{H}_A \text{ and } |\phi\rangle \in \mathcal{H}_B\}$$

subject to the relations

$$|c\psi_1 + \psi_2\rangle \otimes |\phi\rangle = c|\psi_1\rangle\otimes|\phi\rangle + |\psi_2\rangle\otimes|\phi\rangle \qquad |\psi\rangle\otimes|c\phi_1+\phi_2\rangle = c|\psi\rangle\otimes|\phi_1\rangle + |\psi\rangle\otimes|\phi_2\rangle,$$

with inner product

$$\langle \varphi_1 \otimes \phi_1 | \varphi_2 \otimes \phi_2 \rangle := \langle \varphi_1 | \varphi_2 \rangle_{\mathcal{H}_A} \cdot \langle \phi_1 | \phi_2 \rangle_{\mathcal{H}_B}.$$

**Notation 1.5.1.** When we have distinguished bases $\mathcal{B}_A$ for $\mathcal{H}_A$ and $\mathcal{B}_B$ for $\mathcal{H}_B$, we write $|\alpha\beta\rangle := |\alpha\rangle \otimes |\beta\rangle$ for $|\alpha\rangle \in \mathcal{B}_A$ and $|\beta\rangle \in \mathcal{B}_B$.

**Exercise 1.5.2.** If $\dim(\mathcal{H}_A) = d_A$ and $\dim(\mathcal{H}_B) = d_B$, then $\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = d_A d_B$.
*Hint: Show that given bases $\mathcal{B}_A$ for $\mathcal{H}_A$ and $\mathcal{B}_B$ for $\mathcal{H}_B$, $\{|\alpha\beta\rangle \,|\, |\alpha\rangle \in \mathcal{B}_A$ and $|\beta\rangle \in \mathcal{B}_B\}$ is a basis for $\mathcal{H}_A \otimes \mathcal{H}_B$.*

> **Tensor product composite:** The Hilbert space of a composite quantum system is the tensor product of all Hilbert spaces of its subsystems.

**Example 1.5.3.** The *computational basis* for an $N$-qubit state is $\{|i_1 \cdots i_N\rangle | i_1, \ldots, i_N \in \{0, 1\}\}$.

**Definition 1.5.4.** Recall that $B(\mathcal{H}_{AB}) \cong B(\mathcal{H}_A) \otimes B(\mathcal{H}_B)$. We have a canonical (non-normalized) *conditional expectation*

$$\mathbb{E}_A : B(\mathcal{H}_{AB}) \cong B(\mathcal{H}_A) \otimes B(\mathcal{H}_B) \to B(\mathcal{H}_A) \qquad\qquad x \otimes y \mapsto \mathrm{Tr}_B(y)x,$$

The conditional expectation of $x \in B(\mathcal{H}_{AB})$ is also determined by the following formula:

$$\langle \phi_1 | \mathbb{E}_A(x) | \phi_2 \rangle = \sum_{\beta \in \mathcal{B}_B} \langle \phi_1 \beta | x | \phi_2 \beta \rangle \qquad\qquad \forall |\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}_A.$$

**Remark 1.5.5.** Sometimes $\mathbb{E}_A$ is called the *partial trace* and denoted by $\mathrm{Tr}_B$ (rather than $\mathrm{id} \otimes \mathrm{Tr}_B$) by a slight abuse of notation. To avoid overloading the notation, we will stick to conditional expectations.

**Definition 1.5.6.** Given a *bipartite* (in tensor product) state vector $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, we get the *reduced density operator* $\rho_A^{\mathrm{red}} := \mathbb{E}_A(\rho_{AB})$ determined by the formula

$$\langle \phi_1 | \rho_A^{\mathrm{red}} \phi_2 \rangle = \sum_{\beta \in \mathcal{B}_B} \langle \phi_1 \beta | \psi_{AB} \rangle \langle \psi_{AB} | = | \phi_2 \beta \rangle \qquad\qquad \forall |\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}_A.$$

Conversely, given a state $\rho_B$, we can construct a state vector $|\psi_{AB}\rangle$ such that $\rho_B = \mathrm{Tr}_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$ via *quantum state purification*. Just pick any ONB $\mathcal{B}_A$ for $\mathcal{H}_A$ and use a spectral decomposition of $\rho_B$:

$$\rho_B = \sum_{j=1}^{d_B} p_j |\beta_j\rangle\langle\beta_j| \qquad \rightsquigarrow \qquad |\psi_{AB}\rangle := \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} \sqrt{p_i} |\alpha_i \beta_j\rangle.$$

This depends on the choice of basis for the auxiliary system $A$.

**Exercise 1.5.7.** Show that any pure state $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has a *Schmidt decomposition*, i.e., there is a $d \leq \min\{d_A, d_B\}$, $p_1, \ldots, p_d > 0$ with $\sum_{i=1}^d p_i = 1$, and orthonormal sets $\{\alpha_1, \ldots, \alpha_d\} \subset \mathcal{H}_A$ and $\{\beta_1, \ldots, \beta_d\} \subset \mathcal{H}_B$ such that

$$|\psi_{AB}\rangle = \sum_{i=1}^{d} \sqrt{p_i} |\alpha_i \beta_i\rangle.$$

*Hint: Use quantum state purification for* $\rho_A^{\text{red}} = \mathbb{E}_A(|\psi_{AB}\rangle\langle\psi_{AB}|)$.
*Note: A high level proof uses the canonical isomorphism* $\mathcal{H}_A \otimes \mathcal{H}_B \cong \operatorname{Hom}(\mathcal{H}_A \to \overline{\mathcal{H}_B})$ *and polar decomposition.*

**Exercise 1.5.8.** Suppose we have a pure bipartite state in a Schmidt decomposition $|\psi_{AB}\rangle = \sum \sqrt{p_i}|\alpha_i\beta_i\rangle$. Compute the reduced density operators $\rho_A^{\text{red}}, \rho_B^{\text{red}}$.

1.6. **Quantum is not classical.**

**Example 1.6.1** (Einstein-Podolsky-Rosen (EPR) Paradox, *spooky action at a distance*)**.** Consider the *singlet state* (all electrons paired) $|EPR\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ in $\mathbb{C}^2 \otimes \mathbb{C}^2$. If $B$ measures $|0\rangle$ ($+1$ for Pauli $Z$), then $A$ has state $|1\rangle$. One might think that this means information has travelled faster than the speed of light, contradicting the theory of special relativity. However, before and after $B$'s measurement, $A$ still has the same reduced density matrix

$$\rho_A^{\text{red}} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I.$$

If $B$ does not tell $A$ they have performed a measurement, then $A$ will not know the state has collapsed. So the information has not really travelled!

It is worth mentioning that if $A$ measures along a different axis than $B$, e.g., $X$ instead of $Z$, there is still a $\frac{1}{2}$ probability for either outcome.

**Example 1.6.2** (Bell's Inequality)**. TODO:**

**Exercise 1.6.3** (Greenberger-Horne-Zeilinger (GHZ) Paradox, *no hidden variables*)**.** Consider the *GHZ state vector* on $\mathbb{C}_A^2 \otimes \mathbb{C}_B^2 \otimes \mathbb{C}_C^2$ given by

$$|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle).$$

(1) Prove that the operators $Z_A \otimes Z_B \otimes I_C$, $I_A \otimes Z_B \otimes Z_C$, and $X_A \otimes X_B \otimes X_C$ pairwise commute. Describe the finite group generated by these operators in terms of tensor products of Pauli operators.
(2) Show that $(X_A^a \otimes Z_B^b \otimes X_Z^c)|GHZ\rangle$ for $a, b, c \in \{0, 1\}$ are an orthonormal basis of eigenvectors for the commuting operators in (1) above. Deduce that $|GHZ\rangle$ spans the intersection of the eigenspaces $E_1$ for the commuting operators in (1) above.
(3) Compute $\langle X_A \otimes X_B \otimes X_C\rangle$, $\langle -Y_A \otimes Y_B \otimes X_C\rangle$, $\langle -Y_A \otimes X_B \otimes Y_C\rangle$, and $\langle -X_A \otimes Y_B \otimes Y_C\rangle$.
(4) Suppose that each of $X_i, Y_j, Z_k$ for $i, j, k \in \{A, B, C\}$ has a universal 'hidden value' in $\{\pm 1\}$ denoted by $v(-)$. Deduce this is impossible by considering the following four equalities:

$$\begin{aligned} v(X_A)v(X_B)v(X_C) &= \langle X_A \otimes X_B \otimes X_C\rangle \\ -v(Y_A)v(Y_B)v(X_C) &= \langle -Y_A \otimes Y_B \otimes X_C\rangle \\ -v(Y_A)v(X_B)v(Y_C) &= \langle -Y_A \otimes X_B \otimes Y_C\rangle \\ -v(X_A)v(Y_B)v(Y_C) &= \langle -X_A \otimes Y_B \otimes Y_C\rangle. \end{aligned}$$

*Hint: Each* $v(-)$ *appears exactly twice on the left, and* $(\pm 1)^2 = 1$.

**Example 1.6.4** (Diecks-Wootters-Zurek, *no cloning*)**.** Suppose $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}$, and fix a state $|\beta\rangle \in \mathcal{H}_B$. There is no unitary $u \in B(\mathcal{H} \otimes \mathcal{H})$ such that

$$u|\alpha\beta\rangle \in U(1)|\alpha\alpha\rangle \qquad \forall\, |\alpha\rangle \in \mathcal{H}_A.$$

Indeed, if $|\alpha\rangle, |\gamma\rangle \in \mathcal{H}_A$, then

$$\langle\gamma|\alpha\rangle_A = \langle\gamma\beta|\alpha\beta\rangle_{AB} = \langle\gamma\gamma|u^\dagger u|\alpha\alpha\rangle_{AB} \in U(1)\langle\gamma\gamma|\alpha\alpha\rangle_{AB} = U(1)\langle\gamma|\alpha\rangle_A^2$$

Taking moduli, we get $|\langle\gamma|\alpha\rangle| = |\langle\gamma|\alpha\rangle|^2$, so $|\langle\gamma|\alpha\rangle| \in \{0,1\}$. But $\alpha, \gamma$ were arbitrary, a contradiction.

1.7. **Quantum entanglement.** A *product* state vector is a state vector $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ for some state vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. A non-product state vector is called an *entangled* state vector. Similarly, a *product* mixed state is a mixed state $\rho_{AB}$ of the form $\rho_A \otimes \rho_B$ where $\rho_A$ is a mixed state on $\mathcal{H}_A$ and $\rho_B$ is a mixed state on $\mathcal{H}_B$. A *separable* mixed state is a convex combination of product mixed states. A non-separable mixed state is called an *entangled* state.

**Separable vs. entangled states.**

| | |
|---|---|
| product state vector | $\|\psi_{AB}\rangle = \|\psi_A\rangle \otimes \|\psi_B\rangle$ |
| product pure state | $\|\psi_{AB}\rangle\langle\psi_{AB}\| = \|\psi_A\rangle\langle\psi_A\| \otimes \|\psi_B\rangle\langle\psi_B\|$ |
| product mixed state | $\rho_{AB} = \rho_A \otimes \rho_B$ |
| separable mixed state | convex combination of product mixed states |
| entangled state | non-separable mixed state |

Product state vectors, pure states, and mixed states are all separable.

**Exercise 1.7.1.** Show that if $\rho_{AB} = \rho_A \otimes \rho_B$ is a product state, then $S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$.

**Remark 1.7.2.** Determining whether a state is separable is called the *separability problem* and is known to be NP-hard.

**Exercise 1.7.3.** The *partial adjoint with respect to $B$* of a state $\rho = \sum p^{ij}_{k\ell}|i\rangle\langle j| \otimes |k\rangle\langle\ell|$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is

$$\rho^{T_B} := (I \otimes T)(\rho) = \sum p^{ij}_{k\ell}|i\rangle\langle j| \otimes |\ell\rangle\langle k| = \sum p^{ij}_{\ell k}|i\rangle\langle j| \otimes |k\rangle\langle\ell|$$

The *positive partial transpose condition* (PPT) is that $\mathrm{spec}(\rho^{T_B}) \subset \mathbb{R}_{>0}$.

(1) Show that PPT for $B$ is equivalent to PPT for $A$.
*Hint:*$\rho^{T_A} = (\rho^{T_B})^T$.
(2) Prove that PPT is necessary for separability.

*Note: In fact, PPT is sufficient for separability in the $2 \times 2$ case, but this is beyond the scope of this course.*

**Definition 1.7.4.** Suppose $\mathcal{H}_A \otimes \mathcal{H}_B$ is a product system in state $\rho_{AB}$ and $M_A, M_B$ are two observables acting locally on the subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The *correlation function* is given by

$$\begin{aligned} C(M_A, M_B) :&= \langle M_A \otimes M_B\rangle - \langle M_A \otimes 1_B\rangle\langle 1_A \otimes M_B\rangle \\ &= \mathrm{Tr}_{AB}((M_A \otimes M_B)(\rho_{AB} - \rho_A^{\mathrm{red}} \otimes \rho_B^{\mathrm{red}})). \end{aligned}$$

**Exercise 1.7.5.** Prove that the following are equivalent.

(1) $\rho_{AB}$ has no correlations, i.e., $C(M_A, M_B) = 0$ for all subsystem observables $M_A, M_B$.

(2) The joint probability distribution on $\mathcal{B}_A \times \mathcal{B}_B$ given by the *projective measurements* (measurement of an observable which is a projection)

$$p_{AB}(\alpha, \beta) := \mathrm{Tr}_{AB}(p_{\alpha\beta}\rho_{AB})$$

has no correlation, where $p_{\alpha\beta} = |\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|$.

(3) $\rho_{AB} = \rho_A^{\text{red}} \otimes \rho_B^{\text{red}}$.

(4) $\rho_{AB}$ is a product state.

**Definition 1.7.6.** The *quantum mutual information* of the bipartite state $\rho_{AB}$ is given by

$$I(A : B) := S(\rho_A^{\text{red}}) + S(\rho_B^{\text{red}}) - S(\rho_{AB}).$$

is a measure of total correlation between $A$ and $B$ which does not depend on an observable.

**Exercise 1.7.7.** Suppose $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ is pure so that $S(\rho_{AB}) = 0$. Use the Schmidt decomposition $|\psi_{AB}\rangle = \sum \sqrt{p_i}|\alpha_i\beta_i\rangle$ where $\sum p_i = 1$ and Exercise 1.5.8 to show that $S(\rho_A^{\text{red}}) = S(\rho_B^{\text{red}}) = H(\{p_i\})$ where $\{p_i\}$ denotes a discrete probability distribution.

**Definition 1.7.8.** The *entanglement* of a bipartite state vector $|\psi_{AB}\rangle$ is given by

$$S(|\psi_{AB}\rangle) = S(\rho_A^{\text{red}}) = S(\rho_B^{\text{red}}).$$

Define the *quantum conditional entropy* by

$$S(A|B) := S(\rho_{AB}) - S(\rho_B^{\text{red}}) \qquad\qquad S(B|A) := S(\rho_{AB}) - S(\rho_A^{\text{red}}).$$

This means there is a similar Venn diagram for quantum mutual information similar to the Venn diagram (1.2.8) for classical mutual information.

**Warning 1.7.9.** While classical conditional entropy can never be negative, quantum conditional entropy can be! Indeed, if $|\psi_{AB}\rangle$ is an entangled state vector, then $S(\rho_{AB}) = 0$, but $S(\rho_A^{\text{red}}) = S(\rho_B^{\text{red}}) > 0$, so $S(A|B) = S(B|A) < 0$.

**Exercise 1.7.10.** Prove that $I(A : B)$ is always non-negative, and equals zero if and only if $\rho_{AB}$ is a product state.

**Exercise 1.7.11.** Use quantum state purification $\rho_{AB} = \mathbb{E}_C(|\psi_{ABC}\rangle\langle\psi_{ABC}|)$ to show that

$$|S(\rho_A^{\text{red}}) - S(\rho_B^{\text{red}})| \leq S(\rho_{AB}).$$

1.8. **Many body correlation.** We are now interested in quantum systems with more than 2 subsystems; for simplicity, we will study those of the form $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. States on this system will be called *tripartite/3-particle* states.

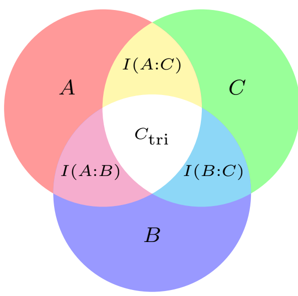
**Definition 1.8.1.** The *total correlation* in the tripartite state $\rho_{ABC}$ is

$$C_T(\rho_{ABC}) := S(\rho_A^{\text{red}}) + S(\rho_B^{\text{red}}) + S(\rho_C^{\text{red}}) - S(\rho_{ABC}).$$

**Example 1.8.2.** For the tripartite state $\rho_{ABC} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)$, observe that

$$\begin{aligned} C_T(\rho_{ABC}) &= 3 \cdot S\left(\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right) - S\left(\frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)\right) \\ &= -6\frac{1}{2}\log\left(\frac{1}{2}\right) + 2\frac{1}{2}\log\left(\frac{1}{2}\right) = 3 - 1 = 2. \\ I(A:B) &= 2 \cdot S\left(\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)\right) - S\left(\frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)\right) = 1 \\ &= I(B:C) = I(A:C) \end{aligned}$$

This means that whereas the mutual information $I(A:B)$ can be viewed as the intersection in a Venn diagram, the corresponding triple intersection $C_{\text{tri}}$ in the following Venn diagram can be negative, and is thus not the correct notion of total correlation between 3 subsystems.




adapted from [arXiv:1508.02595, p.28]

$$\begin{aligned} C_{\text{tri}} :&= S(\rho^{\text{red}}_{AB}) + S(\rho^{\text{red}}_{BC}) + S(\rho^{\text{red}}_{AC}) \\ &\quad - S(\rho^{\text{red}}_{A}) - S(\rho^{\text{red}}_{B}) - S(\rho^{\text{red}}_{C}) - S(\rho_{ABC}) \\ &= C_T(\rho_{ABC}) - I(A:B) - I(B:C) - I(A:C). \end{aligned}$$

Indeed, in the example above, $C_{\text{tri}} < 0$!

**Definition 1.8.3.** For a tripartite state $\rho_{ABC}$, we define its sets of *$k$-reduced density matrics* ($k$-RDMs) for $k = 1, 2, 3$ by

$$\begin{aligned} \text{1-RDMs} &= \{\rho^{\text{red}}_{A}, \rho^{\text{red}}_{B}, \rho^{\text{red}}_{C}\} \\ \text{1-RDMs} &= \{\rho^{\text{red}}_{AB}, \rho^{\text{red}}_{BC}, \rho^{\text{red}}_{AC}\} \\ \text{3-RDMs} &= \{\rho_{ABC}\} \end{aligned}$$

For $k = 1, 2, 3$, we define the sets

$$\begin{aligned} \mathbf{L}_1 &:= \left\{\text{states } \sigma_{ABC} \middle| \sigma^{\text{red}}_{A} = \rho^{\text{red}}_{A},\ \sigma^{\text{red}}_{B} = \rho^{\text{red}}_{B},\ \text{and } \sigma^{\text{red}}_{C} = \rho^{\text{red}}_{C}\right\} \\ \mathbf{L}_2 &:= \left\{\text{states } \sigma_{ABC} \middle| \sigma^{\text{red}}_{AB} = \rho^{\text{red}}_{AB},\ \sigma^{\text{red}}_{BC} = \rho^{\text{red}}_{BC},\ \text{and } \sigma^{\text{red}}_{AC} = \rho^{\text{red}}_{AC}\right\} \\ \mathbf{L}_3 &:= \{\text{states } \sigma_{ABC} | \sigma_{ABC} = \rho_{ABC}\} = \{\rho_{ABC}\} \end{aligned}$$

We can do this in general for an $n$-partite state as well.

Now given a set of density matrices, we can ask if it is a collection of $k$-RDMs for some $N$-partite state.

**Principle of maximum entropy.** For a given set $R$ of $k$-RDMs, the best inference of the $N$-partite state $\rho_{A_1 \cdots A_N}$ is the unique state $\rho^*_k \in \mathbf{L}_k$ with maximal von Neumann entropy.

$$\rho^*_k = \operatorname{argmax}\{S(\sigma) | \sigma \in \mathbf{L}_k(R)\}$$

**Exercise 1.8.4** ($\star$, [MR1979011]). Let $J \subset \mathbb{R}$ be an interval. A function $f : J \to \mathbb{R}$ is called *operator convex* if for all Hermetian/self-adjoint operators $a, b \in M_n(\mathbb{C})$ with $\mathrm{spec}(a), \mathrm{spec}(b) \subset J$, we have $f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$. Show that $f(x) := x\log(x)$ is operator convex.

**Exercise 1.8.5.** Show that $\rho_k^*$ is well-defined, i.e., given a set of $k$-RDMs, there is a unique state in $\mathbf{L}_k$ with maximum entropy.

**Exercise 1.8.6.** Suppose $\rho_{ABC}$ is a tripartite state.

(1) Calculate $\rho_1^*$ and $\rho_3^*$.
(2) Show that the total correlation satisfies $C_T(\rho_{ABC}) = S(\rho_1^*) - S(\rho_3^*)$.

**Exercise 1.8.7.** If $j \leq k$, is $\rho_j^* \in \mathbf{L}_k$?

**Definition 1.8.8.** The *irreducible $k$-partite correlation* of $\rho_{A_1\cdots A_N}$ is

$$C_k := S(\rho_{k-1}^*) - S(\rho_k^*),$$

which measures the $k$-partite correlations that cannot be learned from the $(k-1)$-RDMs. Observe that $C_k \geq 0$, and by Exercise 1.8.6,

$$C_T(\rho_{ABC}) = S(\rho_1^*) - S(\rho_3)^* = S(\rho_1^*) - S(\rho_2)^* + S(\rho_2^*) - S(\rho_3)^* \geq 0.$$

**Exercise 1.8.9.** Compute $C_1, C_2, C_3$ for $|GHZ\rangle$ as in Exercise 1.6.3.

**Remark 1.8.10.** A tripartite state $\rho_{ABC}$ is determined by its 2-RDMs if $C_3 = 0$, or equivalently, $\rho_{ABC} = \rho_2^*$. The only pure tripartite states $|\psi_{ABC}\rangle\langle\psi_{ABC}|$ on $(\mathbb{C}^2)^{\otimes 3}$ with $C_3 \neq 0$ are

$$\alpha|000\rangle + \beta|111\rangle \qquad \alpha, \beta \neq 0,\ |\alpha|^2 + |\beta|^2 = 1.$$

**Definition 1.8.11.** A pure tripartite state vector $|\psi_{ABC}\rangle$ is

- a *product* state vector if it can be written as $|\psi_{ABC}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \otimes |\psi_C\rangle$,
- *entangled* if it is not a product state (eg: $|\psi_A\rangle \otimes |\psi_{BC}\rangle$ for some entangled state $|\psi_{BC}\rangle$), and
- *genuinely entangled* if it cannot be written as a product state with respect to any tensor product decomposition.

**Definition 1.8.12.** Suppose $|\psi_{ABC}\rangle$ is a pure tripartite stat vector, and denote by $|\alpha\rangle = |\alpha_A\rangle \otimes |\alpha_B\rangle \otimes |\alpha_C\rangle$ a product pure tripartite state vector. The *geometric measure of entanglement* is given by

$$E_G(|\psi_{ABC}\rangle) := -\log(\Lambda^2_{\max}(|\psi_{ABC}\rangle) \qquad \text{where} \qquad \Lambda^2_{\max}(|\psi_{ABC}\rangle) := \max_{|\alpha\rangle} |\langle\alpha||\psi_{ABC}\rangle\rangle|.$$

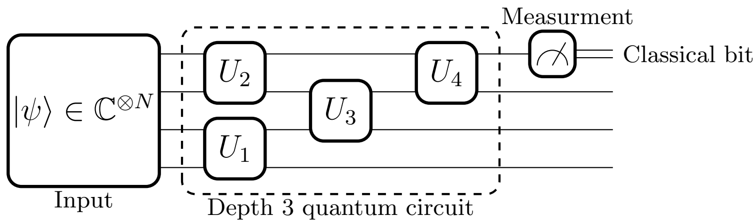
We view $E_G(|\psi_{ABC}\rangle)$ as a measurement of how far $|\psi_{ABC}\rangle$ is from the set of product pure tripartite state vectors.

**Exercise 1.8.13.** Compute $E_G(|GHZ\rangle)$ where $|GHZ\rangle$ is as in Exercise 1.6.3.

1.9. **Quantum computation.** Classical computers manipulate bits to perform calculations. Quantum computers manipulate qubits (state vectors in $\mathbb{C}^2$) or qudits (state vectors in $\mathbb{C}^d$) to perform calculations using *quantum gates*, which are unitary matrices on a tensor product Hilbert space $\mathcal{H} = (\mathbb{C}^d)^{\otimes N}$. We have already seen some examples of quantum gates: the Pauli $X, Y, Z$ operators, which are self-adjoint unitaries.

To perform a quantum calculation, we take our state vector $|\psi\rangle \in \mathcal{H}$ and apply a finite sequence of unitary quantum gates to transform the state. Finally, we perform a measurement on part or all of the total space. It is important to note from the beginning that in contrast with classical computation, this process is *probabilistic*, not deterministic, due to the process of measurement in quantum mechanics.

Here is a cartoon of a quantum computation:




These diagrams exploit the *graphical calculus* for tensor categories; here we work with the tensor C* category of finite dimensional Hilbert spaces. We will use two conventions for these diagrams in these notes. Sometimes these diagrams are written sideways to use less space on a page, where time increases from left to right; sometimes we write these diagrams vertically, where time increases from bottom to top; this is known as the *optimistic convention* (always look up!).

**Some physical realizations of qubits.**

| Name | Physical support | Realized system size |
|---|---|---|
| superconducting qubits | Josephson junction (charge, flux, phase) | 53[a] $\geq$ 2000 DWave[b] |
| photonic | boson sampling | 76[c] |
| cold atom | neutral atoms/ | $72 - 1000$ qubits |
| trapped ion | electrodynamic ion trap | 32 qubits[d] |
| silicon qubits | quantum dots | 2 qubits at 1.5K[e] |
| topological qubits/ anyons | anyon braiding in gapped TPM | $< 1$ qubit |

[a]Google's quantum supremacy [AAB+19]
[b]DWave microchip implements quantum annealing, which is not universal for quantum computation.
[c]Jiuzhang quantum supremacy [ZWD+20]
[d]IonQ [Gib20]. Previously, IBM had 27 trapped ion qubits.
[e]Silicon qubits at higher temperatures could interface better with existing conventional computes [YLH+20]

There are (at least) two sources of error in quantum computation:

- *decoherence*, when information is lost due to interaction with the environment, and
- *accuracy*, our inability to perfectly construct a specific unitary quantum gate.

Both are a manifestation of non-unitary time evolution of the system. We have already seen from the *no cloning theorem* that we cannot just send the same message multiple times, so more robust techniques are necessary. One way to deal with error is via *software*, e.g., error correcting codes, and another method is *hardware*, e.g., *topological quantum computation*.

It is not the case that having $N$ *physical* qubits for a quantum computer gives $N$ *logical* qubits for quantum computation, as many physical qubits are needed for error correction. As topology is robust to deformation, it is believed that topological qubits are manifestly fault tolerant [Kit03] so that one topological qubit would equal one logical qubit.

## 1.10. **Non-unitary time evolution by completely positive maps.**

### 1.10.1. *Completely positive maps.*

**Exercise 1.10.1.** Suppose $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional vector spaces with $\dim(\mathcal{K}) = k$. Construct a unitary isomorphism $\mathcal{H} \otimes \mathcal{K} \cong \mathcal{H}^{\oplus k}$. Deduce that $B(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \cong M_k(\mathcal{B}(\mathcal{H}))$.

**Definition 1.10.2.** Suppose $\mathcal{H}_A$ and $\mathcal{H}_B$ are two quantum systems. A linear map $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is called

- *unital* if $\Phi(I_A) = I_B$,
- *trace preserving* if $\operatorname{Tr}_B(\Phi(x)) = \operatorname{Tr}_A(x)$ for all $x \in \mathcal{B}(\mathcal{H}_A)$,
- *positive* if $x \geq 0$ in $\mathcal{B}(\mathcal{H}_A)$ implies $\Phi(x) \geq 0$, and
- *completely positive* if $(x_{ij}) \geq 0$ in $M_k(\mathcal{B}(\mathcal{H}_A))$ implies $(\Phi(x_{ij})) \geq 0$ in $M_k(\mathcal{B}(\mathcal{H}_B))$ for all $k \in \mathbb{N}$.

A *quantum operation* is a completely positive map such that for all states $\rho$, $\operatorname{Tr}(\Phi(\rho)) \leq 1$. Observe that a quantum operation maps states to states if and only if it is trace preserving. In this case, we call the quantum operation a *quantum channel*. That is, quantum channels are trace preserving completely positive (TPCP) maps.

**Exercise 1.10.3.** Suppose that $E_i : \mathcal{H}_A \to \mathcal{H}_B$ is a family of transformations such that $\sum E_i^\dagger E_i = I_A$. Show that $\Phi(x) := \sum_i E_i x E_i^\dagger$ is completely positive.
*Hint: When $(x_{ij})$ is positive in $M_k(\mathcal{B}(\mathcal{H}_A))$, find a $y$ such that $(\Phi(x_{ij})) = y^\dagger y$.*

**Exercise 1.10.4** (Stinespring Dilation)**.** Suppose $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is completely positive.

(1) Show that $\langle x \otimes \eta | y \otimes \xi \rangle := \langle \eta | \Phi(x^\dagger y) \xi \rangle_H$ on $\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{H}_B$ linearly extends to a well-defined positive sesquilinear form.
(2) Show that for $V$ a vector space with positive sesquilinear form $B(\,\cdot\,|\,\cdot\,)$, $N_B = \{v \in V | B(v|v) = 0\}$ is a subspace of $V$, and $B$ descends to an inner product on $V/N_B$.
(3) Define $\mathcal{K}$ to be completion of $(\mathcal{B}(\mathcal{H}_A) \otimes \mathcal{H}_B)/N_{\langle\cdot|\cdot\rangle}$ in $\|\cdot\|_2$. Find a unital $*$-homormophism $\Psi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{K})$, and an isometry $v \in \mathcal{B}(\mathcal{H}_B \to \mathcal{K})$ such that $\Phi(x) = v^\dagger \Psi(x) v$ for all $x \in \mathcal{B}(\mathcal{H}_A)$.

*Note: For this problem, we may replace $\mathcal{B}(\mathcal{H}_A)$ with any* C*-*algebra, and $\mathcal{H}_B$ need not be finite dimensional.*

**Exercise 1.10.5** (Kraus operators)**.** Suppose $\Phi : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ is completely positive.
[[maybe quantum operation or TPCP?]]

(1) Prove there are a family of transformations $T_i : \mathcal{H}_A \to \mathcal{H}_B$ with $\sum E_i^\dagger E_i = I_A$ such that $\Phi(x) = \sum_i E_i x E_i^\dagger$ for all $x \in \mathcal{B}(\mathcal{H}_A)$.
*Note: Such a family* $(E_i)$ *is called a system of* Kraus operators *for* $\Phi$.

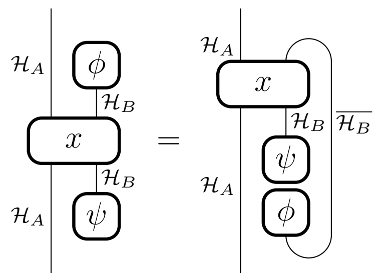
(2) Suppose $(E_i)_{i=1}^N$ and $(F_i)_{i=1}^N$ are systems of Kraus operators for $\Phi$. Prove that there is a unitary $(u_{ij}) \in M_N(\mathcal{B}(\mathcal{H}_B))$ such that $F_i = \sum_{j=1}^N u_{ij} E_j$.

1.10.2. *Non-unitary time evolution.* Suppose we have a quantum system $\mathcal{H}_S$, which is unavoidably coupled with the environment $\mathcal{H}_E$, so that the total space is $\mathcal{H}_S \otimes \mathcal{H}_E$. Suppose the system is in state vector $|\psi_{SE}\rangle$ at time $t_0$, and set $\rho_{SE}(t_0) := |\psi_{SE}\rangle\langle\psi_{SE}|$. Recall that in the Hamiltonian formulation of quantum mechanics, the state of the system at time $t > t_0$ is given by

$$\rho_{SE}(t) = U_{SE}(t)\rho_{SE}U_{SE}(t)^\dagger.$$

The time evolution of the quantum system $S$ is given by the RDM $\rho_S^{\mathrm{red}}(t) = \mathbb{E}_S(\rho_{SE}(t))$.

**Notation 1.10.6.** For two state vectors $\phi, \psi \in \mathcal{H}_B$ and $x \in \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B)$, we write

$\langle\phi|x|\psi\rangle_A := (I_A \otimes \phi)x(I_A \otimes \psi) =$  $= \mathbb{E}_A(x(I_A \otimes |\psi\rangle\langle\phi|))$.



In the diagram, we identify $|\psi\rangle$ with a linear map $\mathbb{C} \to \mathcal{H}_B$ and $\langle\phi|$ with a linear map $\mathcal{H}_B \to \mathbb{C}$. In the middle equality, we use the graphical calculus to rotate $\phi$ around the right hand side. We will justify this step later.

**Exercise 1.10.7.** Prove that the canonical trace preserving conditional expectation $\mathbb{E}_A = \mathrm{id}_A \otimes \mathrm{Tr}_B : \mathcal{B}(\mathcal{H}_A) \otimes \mathcal{B}(\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_A)$ is given by $\mathbb{E}_A(x) = \sum_\beta \langle\beta|x|\beta\rangle_A$ for any orthonormal basis $\{\beta\}$ of $\mathcal{H}_B$.

Suppose now that $\rho_{SE}(t_0) = \rho_S(t_0) \otimes |0_E\rangle\langle 0_E|$ is a product state and $\{|i\rangle\}$ is some orthonormal basis of $\mathcal{H}_E$. In this case,

$$\begin{aligned}
\rho_S^{\mathrm{red}}(t) = \mathbb{E}_S(\rho_{SE}(t)) &= \mathbb{E}_S(U_{SE}(t)(\rho_S(t_0) \otimes |0_E\rangle\langle 0_E|)U_{SE}(t)^\dagger) \\
&= \sum_i \langle i|U_{SE}(t)(\rho_S(t_0) \otimes |0_E\rangle\langle 0_E|)U_{SE}(t)^\dagger|i\rangle_S \\
&= \sum_i \underbrace{\langle i|U_{SE}(t)|0_E\rangle_S}_{=:E_i} \rho_S(t_0)\langle 0_E|U_{SE}(t)^\dagger|i\rangle_S \\
&= \sum_i E_i(t)\rho_S(t_0)E_i(t)^\dagger
\end{aligned}$$

where $E_i(t) := \langle i|U_{SE}(t)|0_E\rangle_S$. the reader is encouraged to perform the above calculation diagrammatically.

**Exercise 1.10.8.** Verify that $\sum_i E_i(t)^\dagger E_i(t) = I_S$ so that $\Phi_t(x) := \sum_i E_i(t)xE_i(t)^\dagger$ is completely positive by Exercise 1.10.3.

These Kraus operators $\{E_i\}$ are the *noise* or *error* operators when $\Phi$ represents the error-producing quantum operation of the environment. Hence we see that the time evolution of $\rho_S(t)$ under interaction with the environment is no longer given by conjugation by a unitary operator, but instead by the family $\{\Phi_t\}$ of completely positive maps $\rho_S^{\mathrm{red}}(t) = \Phi_t(\rho_S(t_0))$.

1.11. **Error correction.** We have seen in the previous section that errors are introduced in quantum computation due to non-unitary evolution of the system $\mathcal{H}$. Our presentation of error correction codes follows [KL97, §3], but we use different notation.

**Definition 1.11.1.** For $k \le n$, an $(n,k)$-*quantum code* is a triple $(\mathcal{Q}, \mathcal{H}, u)$ where

- $\mathcal{Q}$ is a $k$ dimensional Hilbert space,
- $\mathcal{H}$ is an $n$-dimensional Hilbert space called the *coding space*, and
- $u : \mathcal{Q} \to \mathcal{H}$ is an injective partial isometry called the *encoding operator*. Its adjoint $u^\dagger$ is called the *decoding operator*.

For such a triple, we define the associated *code* $\mathcal{C} := u\mathcal{Q} \subset \mathcal{H}$, whose state vectors are called *code words*.

**Definition 1.11.2.** Suppose we have a quantum code $(\mathcal{Q}, \mathcal{H}, u, \mathcal{C} = u\mathcal{Q})$, and suppose $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is an 'error-producing' quantum operation. A *recovery operator* is a quantum operation $\Theta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, and we call $(\mathcal{C}, \Theta)$ a *quantum error correcting code*. The *error* of $(\mathcal{C}, \Phi, \Theta)$ is given by

$$\mathrm{Err}(\mathcal{C}, \Phi, \Theta) := 1 - F(\mathcal{C}, \Phi, \Theta) \qquad \text{where} \qquad F(\mathcal{C}, \Phi, \Theta) := \min_{\substack{\text{state vectors} \\ |\psi\rangle \in \mathcal{C}}} \langle \psi | \Theta(\Phi(|\psi\rangle\langle\psi|)) | \psi \rangle$$

is called the *fidelity*.

**Exercise 1.11.3.** Why may we define the fidelity in terms of a min and not an inf?

**Exercise 1.11.4.** Suppose $\{E_i\}, \{R_j\}$ are systems of Kraus operators for $\Phi, \Theta$ respectively. Show that the fidelity is given by

$$F(\mathcal{C}, \Phi, \Theta) := \min_{\substack{\text{state vectors} \\ |\psi\rangle \in \mathcal{C}}} \sum_{i,j} |\langle \psi | R_j E_i | \psi \rangle|^2$$

and the error is given by

$$\mathrm{Err}(\mathcal{C}, \Phi, \Theta) = \max_{\substack{\text{state vectors} \\ |\psi\rangle \in \mathcal{C}}} \sum_{i,j} \|(R_j E_i - \langle \psi | R_j E_i | \psi \rangle) |\psi\rangle\|^2.$$

**Example 1.11.5** (Bit flip code, [ZCZW19, §3.2.1])**.** Suppose we have a qubit $|\psi\rangle \in \mathbb{C}^2$ together with a noisy quantum operation $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ which flips $|0\rangle \leftrightarrow |1\rangle$ with probability $p$, e.g.,

$$\Phi(\rho) := (1-p)\rho + p(X\rho X).$$

Observe that $\Phi$ is completely positive by Exercise 1.10.3, since the Kraus operators here are $\{\sqrt{1-p}I, \sqrt{p}X\}$. If the qubit is in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$, then applying $\Phi$ gives us the mixed state

$$\rho = (1-p)|\psi\rangle\langle\psi| + p(|X\psi\rangle\langle X\psi|).$$

The only reasonable choice for the error correction code is to set $\mathcal{C} = \mathbb{C}^2$ and $\Theta = \mathrm{id}$. The error is then given by

$$\begin{aligned}\mathrm{Err}_1(\alpha,\beta) &= 1 - \langle\psi|\rho|\psi\rangle = 1 - ((1-p)\langle\psi|\psi\rangle^2 + p\langle\psi|X|\psi\rangle^2) = p(1-\langle\psi|X|\psi\rangle^2)\\ &= p(1 - 4\,\mathrm{Re}(\alpha\overline{\beta})^2).\end{aligned}$$

We thus see that the maximum the error can be is $p$.

However, if we amplify the space $\mathbb{C}^2$ and use an error correcting code, we can decrease our error by an order of magnitude. Setting $\mathcal{A} = (\mathbb{C}^2)^{\otimes 2}$, we amplify the total space to $\mathcal{H}\otimes\mathcal{A}$, and we encode our basis elements by $|0\rangle \mapsto |000\rangle$ and $|1\rangle \mapsto |111\rangle$. We assume that each of the 3 qubits we have prepared are independently subject to bit flip noise with probability $p$. After applying $\Phi^{\otimes 3}$ to $|\psi\rangle\langle\psi|$, we get the mixed state

$$\begin{aligned}\Phi^{\otimes 3}(|\psi\rangle\langle\psi|) &= (1-p)^3|\psi\rangle\langle\psi| + p(1-p)^2\sum_{i=1}^{3} X_i|\psi\rangle\langle\psi|X_i\\ &\qquad + p^2(1-p)\sum_{i=1}^{3}\widehat{X_i}|\psi\rangle\langle\psi|\widehat{X_i} + p^3 X_1X_2X_3|\psi\rangle\langle\psi|X_1X_2X_3,\end{aligned}\tag{1.11.6}$$

where we use the shorthand $\widehat{X_i} = X_jX_k$ for distinct $i,j,k\in\{1,2,3\}$. Observe $\Phi^{\otimes 3}$ is again visibly completely positive by Exercise 1.10.3. (What are the Kraus operators $E_i$ for $\Phi^{\otimes 3}$?)

Our quantum code is now $\mathcal{C} = \mathrm{span}_{\mathbb{C}}\{|000\rangle, |111\rangle\}$, and our recover operator is given by

$$\Theta(\rho) := \sum_{j=0}^{3} R_j\rho R_j^\dagger := P_0\rho P_0 + \sum_{j=1}^{3} X_jP_j\rho P_jX_j \qquad\qquad R_0 := P_0,\ R_j := X_jP_j$$

where the $P_j$ are the orthogonal projections

$$\begin{aligned}P_0 &= |000\rangle\langle 000| + |111\rangle\langle 111|\\ P_1 &= |100\rangle\langle 100| + |011\rangle\langle 011|\\ P_2 &= |010\rangle\langle 010| + |101\rangle\langle 101|\\ P_3 &= |001\rangle\langle 001| + |110\rangle\langle 110|.\end{aligned}$$

**Exercise 1.11.7.** Verify that $\sum_j R_j^\dagger R_j = I$. Check that the map $\rho\mapsto P_0\rho P_0$ acts as the identity on $|\phi\rangle\langle\psi|$ for $|\phi\rangle,|\psi\rangle\in\{|111\rangle\langle 111|\}$. Then check that the map $\rho\mapsto X_jP_j\rho P_jX_j$:

- $\underline{j=1}$: corrects $|100\rangle$ to $|000\rangle$ and $|011\rangle$ to $|111\rangle$ on operators of the form $|\phi\rangle\langle\psi|$ for $|\phi\rangle,|\psi\rangle\in\{|100\rangle\langle 011|\}$.
- $\underline{j=2}$: corrects $|010\rangle$ to $|000\rangle$ and $|101\rangle$ to $|111\rangle$ on operators of the form $|\phi\rangle\langle\psi|$ for $|\phi\rangle,|\psi\rangle\in\{|010\rangle\langle 101|\}$.
- $\underline{j=3}$: corrects $|001\rangle$ to $|000\rangle$ and $|110\rangle$ to $|111\rangle$ on operators of the form $|\phi\rangle\langle\psi|$ for $|\phi\rangle,|\psi\rangle\in\{|001\rangle\langle 110|\}$.

For $|\psi\rangle = \alpha|000\rangle + \beta|111\rangle$, we write $\Phi^{\otimes 3}(|\psi\rangle\langle\psi|)$ from (1.11.6) as $\rho_0+\rho_1+\rho_2+\rho_3$ where

$$\begin{aligned}
\rho_0 &= (1-p)^3|\psi\rangle\langle\psi| + p^3X_1X_2X_3|\psi\rangle\langle\psi|X_1X_2X_3\\
\rho_1 &= p(1-p)^2X_1|\psi\rangle\langle\psi|X_1 + p^2(1-p)X_2X_3|\psi\rangle\langle\psi|X_2X_3\\
\rho_2 &= p(1-p)^2X_2|\psi\rangle\langle\psi|X_1 + p^2(1-p)X_1X_3|\psi\rangle\langle\psi|X_1X_3\\
\rho_3 &= p(1-p)^2X_3|\psi\rangle\langle\psi|X_1 + p^2(1-p)X_1X_2|\psi\rangle\langle\psi|X_1X_2.
\end{aligned}$$

Observe that $R_j\rho_kR_j^\dagger$ vanishes if $j\neq k$. We calculate $\Theta(\Phi^{\otimes 3}(|\psi\rangle\langle\psi|)) = \sum_{j=0}^3 R_j\rho_jR_j^\dagger$ where

$$\begin{aligned}
R_0\rho_0R_0^\dagger &= (1-p)^3|\psi\rangle\langle\psi| + p^3X_1X_2X_3|\psi\rangle\langle\psi|X_1X_2X_3\\
R_1\rho_1R_1^\dagger &= R_2\rho_2R_2^\dagger = R_3\rho_3R_3^\dagger = p(1-p)^2|\psi\rangle\langle\psi| + p^2(1-p)X_1X_2X_3|\psi\rangle\langle\psi|X_1X_2X_3.
\end{aligned}$$

Finally, we compute the error as

$$\begin{aligned}
\mathrm{Err}_3(\alpha,\beta) &= 1-\langle\psi|\Theta(\Phi^{\otimes 3}(|\psi\rangle\langle\psi|))|\psi\rangle = 1-\sum_{j=0}^{3}\langle\psi|R_j\rho_jR_j|\psi\rangle\\
&= 1-(1-p)^3+3p(1-p)^2-(p^3+3p^2(1-p))\langle\psi|X_1X_2X_3|\psi\rangle^2\\
&= p^2(3-2p)(1-\langle\psi|X_1X_2X_3|\psi\rangle^2) = p^2(3-2p)(1-4\,\mathrm{Re}(\alpha\overline{\beta})^2)\le p^2(3-2p).
\end{aligned}$$

Observe that when $p<1/2$, $\mathrm{Err}_3<\mathrm{Err}_1$. To achieve arbitrarily good precision, we can encode $|0\rangle$ as $|0\cdots 0\rangle\in(\mathbb{C}^2)^{\otimes 2r-1}$.

**Remark 1.11.8.** Although we will not discuss it here, one should read about the phase flip code and Shor's code.

**Definition 1.11.9.** Ideally, we would like to choose our code $(\mathcal{Q},\mathcal{H},u,\mathcal{C}=u\mathcal{Q})$ and our recovery operator $\Theta$ to minimize the error. When $\Theta$ satisfies $\mathrm{Err}(\mathcal{C},\Phi,\Theta)=0$, we say that $(\mathcal{C},\Theta)$ is a $\Phi$-*correcting code*.

**Exercise 1.11.10** ($\star$ [KL97, Thm. 3.2],[Got10, Thm. 2 and 3],[Pre, §7.2])**.** Prove that the following are equivalent for a quantum code subspace $\mathcal{C}\subset\mathcal{H}$ and error producing quantum operation $\Phi(x)=\sum_i E_i^\dagger xE_i$ with $\sum_i E_i^\dagger E_i = I_{\mathcal{H}}$.

(QEC1) $\mathcal{C}$ corrects the error set $\{E_i\}$,

(QEC2) There is an orthonormal basis $\{\kappa_k\}$ of $\mathcal{C}$ such that

$$\langle\kappa_k|E_i^\dagger E_j|\kappa_\ell\rangle_{\mathcal{H}} = c_{ij}\delta_{k=\ell}$$

where $c_{ij}$ is a constant independent of $k,\ell$,

(QEC3) Denoting the orthogonal projection onto $\mathcal{C}$ by $p_{\mathcal{C}}$,

$$p_{\mathcal{C}}E_i^\dagger E_jp_{\mathcal{C}} = c_{ij}p_{\mathcal{C}}$$

where $c_{ij}$ is a constant, and

(QEC4) For all state vectors $|\kappa\rangle,|\kappa'\rangle\in\mathcal{C}$ and all $E\in\mathrm{span}\{E_i\}\subset\mathcal{B}(\mathcal{H})$,

$$\langle\kappa|E^\dagger E|\kappa\rangle_{\mathcal{H}} = \langle\kappa'|E^\dagger E|\kappa'\rangle_{\mathcal{H}}.$$

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