6. 2-categories

In this section and the next, we will make rigorous the chart from the Quantum Constructions notes:

Formal construction of $k\mathsf{Vec}$ from $(k-1)\mathsf{Vec}$.

$$
\begin{tikzcd}
 & & & & & \mathrm{B}3\mathsf{Vec} \arrow[r, mapsto, "\mathsf{Cauchy_u}"] & 2\mathsf{MultFusCat}? \\
 & & & \mathrm{B}2\mathsf{Vec} \arrow[r, mapsto, "\mathsf{Cauchy_u}"] & \mathsf{MultFusCat} \arrow[r, "\mathsf{Mod}", "\cong"'] & 2\mathsf{Cat_{ss}} \arrow[u, "\mathrm{B}"] \arrow[ur, mapsto, "\Sigma"] & \\
 & \mathrm{B}\mathsf{Vec} \arrow[r, mapsto, "\mathsf{Cauchy_u}"] & \mathsf{Alg}^{\mathsf{sep}}_{\mathsf{fd}} \arrow[r, "\mathsf{Mod}", "\cong"'] & \mathsf{Cat_{ss}} \arrow[u, "\mathrm{B}"] \arrow[ur, mapsto, "\Sigma"] & & & \\
\mathrm{B}\mathbb{C} \arrow[r, mapsto, "\mathsf{Cauchy_u}"] & \mathsf{Vec} \arrow[u, "\mathrm{B}"] \arrow[ur, mapsto, "\Sigma"] & & & & & \\
\mathbb{C} \arrow[u, "\mathrm{B}"] \arrow[ur, mapsto, "\Sigma"] & & & & & &
\end{tikzcd}
$$

Notation:

- B means take the *delooping* [BS10, §5.6], i.e., consider the monoidal k-category as a $(k+1)$-category with one object.
- $\mathsf{Cauchy_u}$ means take a unital higher Cauchy completion [GJF19].
- Σ is the composite $\mathsf{Cauchy_u} \circ \mathrm{B}$, called the *suspension*.
- Mod is the equivalence given by taking the 1- or 2-category of modules for the algebra/multifusion category respectively.

6.1. **2-categories.** In these notes, *n-category* will always mean a fully weak version of n-category. When we wish to strictify in some sense, we will indicate this by adding adjectives.

Definition 6.1.1. A *2-category* $\mathcal{C}$ consists of

- A collection of *objects*, a.k.a. *0-morphisms*; we write $c \in \mathcal{C}$ to denote c is an object of $\mathcal{C}$;
- For each $a, b \in \mathcal{C}$, a hom category $\mathcal{C}(a \to b)$. Objects of $\mathcal{C}(a \to b)$ are called *1-morphisms*. We write ${}_aX_b \in \mathcal{C}(a \to b)$ or $X : a \to b$ to denote that X is a 1-morphism from a to b. Morphisms in $\mathcal{C}(a \to b)$ are called *2-morphisms*. We write $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ or $f : X \Rightarrow Y$ to denote that f is a 2-morphism from X to Y.
- For each $a, b, c \in \mathcal{C}$, a *1-composition* functor $\otimes_b : \mathcal{C}(a \to b) \times \mathcal{C}(b \to c) \to \mathcal{C}(a \to c)$. This functor necessarily satisfies the *exchange relation*

$$(f \otimes \mathrm{id}_Z) \circ (\mathrm{id}_W \otimes g) = (\mathrm{id}_X \otimes g) \circ (f \otimes \mathrm{id}_Y) \qquad \forall\, f \in \mathcal{C}({}_aW_b \to {}_aX_b),\ \forall\, g \in \mathcal{C}({}_bY_c \to {}_bZ_c).$$

- For each ${}_aX_b, {}_bY_c, {}_cZ_d$, an *associator* isomorphism
$$\alpha_{X,Y,Z} : X \otimes_b (Y \otimes_c Z) \Rightarrow (X \otimes_b Y) \otimes_c Z.$$
These associator isomorphisms must be *natural* in each variable and satisfy the obvious *pentagon axiom*.
- For each $c \in \mathcal{C}$, there is a *unit* 1-morphism $1_c \in \mathcal{C}(c \to c)$, along with *unitor* isomorphisms $\rho^c_Y : Y \otimes_c 1_c \Rightarrow Y$ for all $Y \in \mathcal{C}(b \to c)$ for all $b \in \mathcal{C}$, and $\lambda^c_Z : 1_c \otimes_c Z \Rightarrow Z$ for all $Z \in \mathcal{C}(c \to d)$ for all $d \in \mathcal{C}$. Again, these unitors must be *natural* in each variable and satisfy the obvious *triangle axiom*.

A 2-category is called *strict* if all associators and unitors are identity 2-morphisms. A 2-category is called *linear* if all 2-morphism spaces $\mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ are finite dimensional complex vector spaces, and all composition functors are bilinear.

Remark 6.1.2. The collection of 0-morphisms, 1-morphisms, and 2-morphisms with only the source and target data is called a *2D globular set*. We will discuss globular sets in more detail in Definition 6.2.8 below.

Warning 6.1.3. Sometimes in the literature, 2-category means *strict* 2-category, and the fully weak notion is called a *bicategory*.

Example 6.1.4. There is a 2-category of topological spaces, continuous maps, and homotopy classes of homotopies between continuous maps.

Examples 6.1.5. Here are some examples of 2-categories whose objects are algebras.

(1) Alg is the 2-category of complex algebras, bimodules, and intertwiners.
(2) $\mathsf{C^*Alg}$ is the 2-category of C*-algebras, C* Hilbert bimodules, and bounded intertwiners.
(3) vNAlg is the 2-category of von Neumann algebras, Hilbert space bimodules, and bounded intertwiners.

Examples 6.1.6. Here are some examples of strict 2-categories whose objects are categories.

(1) Categories, functors, and natural transformations
(2) Monoidal categories, monoidal functors, and monoidal natural transformations
(3) Braided monoidal categories, braided monoidal categories, and monoidal natural transformations
(4) G-crossed braided monoidal categories, G-crossed braided monoidal functors, and monoidal natural transformations

Exercise 6.1.7. Define your favorite 2-category in explicit detail.

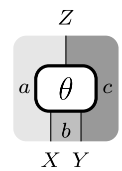
Example 6.1.8. Given a monoidal category $\mathcal{C}$, we get a 2-category $\mathrm{B}\mathcal{C}$ with exactly one object $*$ called the *delooping* of $\mathcal{C}$. We simply define the hom category $\mathrm{B}\mathcal{C}(* \to *) := \mathcal{C}$ with the obvious 1-composition functor, associator, unit, and unitors.

Conversely, given a 2-category $\mathcal{C}$, picking any object $c \in \mathcal{C}$, the *loop space* $\Omega_c\mathcal{C} := \mathcal{C}(c \to c)$ is a monoidal category with the obvious tensor product functor, associator, unit 1_c, and unitors.

Exercise 6.1.9. Suppose $\mathcal{C}$ is a 2-category with one object $*$ and one 1-morphism 1_*. Show that $\mathrm{End}(1_*)$ is a commutative monoid.

Similar to monoidal categories, 2-categories admit a graphical calculus of string diagrams which are dual to pasting diagrams. In a pasting diagram, one represents objects as vertices, 1-morphisms as arrows, and 2-morphisms as 2-cells. In the string diagram calculus, we represent objects by shaded regions, 1-morphisms by (oriented) strands between these regions, and 2-morphisms by coupons.

$$\theta : {}_aX \otimes_b Y_c \Rightarrow {}_aZ_c \qquad \rightsquigarrow \qquad \begin{tikzcd} a \arrow[dr, "X"'] \arrow[rr, bend left, "Z"] & \Uparrow\theta & c \\ & b \arrow[ur, "Y"'] & \end{tikzcd} \qquad \rightsquigarrow$$



As before, we suppress all associators and unitors. 1-composition is denoted by horizontal juxtaposition, and 2-composition is denoted by stacking of diagrams.

Exercise 6.1.10. Formulate the notion of a *rigid* 2-category in which all 1-morphisms are dualizable.

Definition 6.1.11. A linear 2-category $\mathcal{C}$ is called *locally Cauchy complete* if all hom 1-categories are Cauchy complete [[and something about zero objects]]. If in addition $\mathcal{C}$ is rigid, we call $\mathcal{C}$ *pre-semisimple* if all hom 1-categories are semisimple.
Note: This definition of a pre-semisimple 2-category differs slightly from that of presemisimple 2-category in [DR18, Def. 1.2.7]*, but our easier definition will still complete to a semisimple 2-category later on in §6.8.*

Exercise 6.1.12.

(1) Suppose $\mathcal{C}$ is a linear monoidal category. Show how one can canonically endow the Cauchy completion $\mathsf{Cauchy}(\mathcal{C})$ of the underlying linear category with the structure of a monoidal category.
(2) Now suppose $\mathcal{C}$ is a linear 2-category. Show how one can replace all hom 1-categories by their Cauchy completions to obtain a locally Cauchy complete 2-category.

Definition 6.1.13. A 1-morphism ${}_aX_b$ is called *invertible* if there is a 1-morphism ${}_bY_a$ together with 2-isomorphisms $1_a \cong {}_aX \otimes_b Y_a$ and $1_b \cong {}_bY \otimes_a X_b$. Two objects a, b in a 2-category $\mathcal{C}$ are called *equivalent* if there exists an invertible 1-morphism ${}_aX_b \in \mathcal{C}$.

Exercise 6.1.14. Show that if a 1-morphism ${}_aX_b$ is invertible in $\mathcal{C}$, then there is an inverse ${}_bY_a$ such that the isomorphisms $1_a \cong {}_aX \otimes_b Y_a$ and $1_b \cong {}_bY \otimes_a X_b$ also satisfy the zig-zag/snake relations. The 1-morphism ${}_aX_b$ equipped with such an inverse is called an *adjoint equivalence* between $a, b \in \mathcal{C}$.

Definition 6.1.15. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ between 2-categories consists of

- an assignment of an object $F(c)$ to each object $c \in \mathcal{C}$,
- a functor $F_{a,b} : \mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$,
- for all objects $c \in \mathcal{C}$, a *unitor* 2-isomorphism $F_c^1 \in \mathcal{D}(1_{F(c)} \to F(1_c))$, and
- for all 1-morphisms ${}_aX_b, {}_bY_c \in \mathcal{C}$, a *compositor/tensorator* 2-isomorphism $F_{X,Y}^2 \in \mathcal{D}(F(X) \otimes_{F(b)} F(Y) \Rightarrow F(X \otimes_b Y))$

subject to the following axioms:

- (naturality) $F_{X,Y}^2$ is natural in X and Y,

- (associativity) For all ${}_aX_b$, ${}_bY_c$, and ${}_cZ_d$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(X)\otimes(\mathcal{F}(Y)\otimes\mathcal{F}(Z)) \arrow[r, "\mathrm{id}_{\mathcal{F}(X)}\otimes\mathcal{F}^2_{Y,Z}"] \arrow[d, "\alpha^{\mathcal{D}}_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}"] & \mathcal{F}(X)\otimes\mathcal{F}(Y\otimes Z) \arrow[r, "\mathcal{F}^2_{X,Y\otimes Z}"] & \mathcal{F}(X\otimes(Y\otimes Z)) \arrow[d, "\mathcal{F}(\alpha^{\mathcal{C}}_{X,Y,Z})"] \\
(\mathcal{F}(X)\otimes\mathcal{F}(Y))\otimes\mathcal{F}(Z) \arrow[r, "\mathcal{F}^2_{X,Y}\otimes\mathrm{id}_{\mathcal{F}(Z)}"] & \mathcal{F}(X\otimes Y)\otimes\mathcal{F}(Z) \arrow[r, "\mathcal{F}^2_{X\otimes Y,Z}"] & \mathcal{F}((X\otimes Y)\otimes Z)
\end{tikzcd}$$

- (unitality) for all $a,b\in\mathcal{C}$ and ${}_aX_b\in\mathcal{C}(a\to b)$,

$$\begin{tikzcd}
1_{\mathcal{F}(a)}\otimes\mathcal{F}(X) \arrow[r, "\lambda^{\mathcal{F}(a)}_{\mathcal{F}(X)}"] \arrow[d, "\mathcal{F}^1_a\otimes\mathrm{id}_{\mathcal{F}(X)}"] & \mathcal{F}(X) \\
\mathcal{F}(1_a)\otimes\mathcal{F}(X) \arrow[r, "\mathcal{F}^2_{1_a,X}"] & \mathcal{F}(1_a\otimes X) \arrow[u, "\mathcal{F}(\lambda^a_X)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}(X)\otimes 1_{\mathcal{F}(b)} \arrow[r, "\rho^{\mathcal{D}}_{\mathcal{F}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{F}(X)}\otimes\mathcal{F}^1_b"] & \mathcal{F}(X) \\
\mathcal{F}(X)\otimes\mathcal{F}(1_b) \arrow[r, "\mathcal{F}^2_{X,1_b}"] & \mathcal{F}(X\otimes 1_b) \arrow[u, "\mathcal{F}(\rho^b_X)"]
\end{tikzcd}$$

A 2-functor is called:

- *fully faithful* if each functor $F_{a,b}$ is an equivalence, and
- *essentially surjective* if every object $d\in\mathcal{D}$ is equivalent to an object of the form $F(c)$ for some $c\in\mathcal{C}$.
- an *equivalence* if it is fully faithful and essentially surjective (cf. [JY20, Thm. 7.4.1]).

Exercise 6.1.16. A 2-functor is called *strict* if the unitors and tensorators are identities. Show that strict 2-categories and strict 2-functors form a 1-category.

Exercise 6.1.17. For 2-categories $\mathcal{C},\mathcal{D}$ define a *strict* 2-category of 2-functors $\mathrm{Hom}(\mathcal{C}\to\mathcal{D})$.

Exercise 6.1.18. For a monoidal category $\mathcal{C}$, there are three notions of opposite one might take:

- The category $\mathcal{C}^{\mathrm{op}}$ is the arrow opposite.
- The category $\mathcal{C}^{\mathrm{mp}}$ is the *monoidal* opposite, where $a\otimes_{\mathrm{mp}} b := b\otimes a$. The associator is given by $\alpha^{\mathrm{mp}}_{a,b,c} = \alpha^{-1}_{c,b,a}$.
- The category $\mathcal{C}^{\mathrm{mop}}$ is both the arrow and monoidal opposite.

Show that all three of these notions of opposite give monoidal categories. Moreover, show taking two instances of opposite for any of the three above gives back the original category. Finally, show that performing two of these opposites gives the third.

Exercise 6.1.19 ($\star$)**.** Prove that every 2-category is equivalent to a strict 2-category. *Hint: Find a fully faithful 2-functor* $よ : \mathcal{C}\to\mathrm{Hom}(\mathcal{C}^{\mathrm{mp}}\to\mathsf{Cat})$*. (See* [JY20, §8] *for more details.)*

6.2. Simplicial, globular, and Segal sets.

Definition 6.2.1. The *simplicial category* Δ has objects $[n]=\{0\to 1\to\cdots\to n\}$ for $n\in\mathbb{N}=\{0,1,2,\dots\}$ and morphisms the weakly order preserving functions.

Exercise 6.2.2. Prove that Δ has the following presentation by generators and relations:

- Generators: for all n, we have $\delta_i:[n-1]\to[n]$ for $0\le i\le n$ and $\sigma_i:[n+1]\to[n]$ for $0\le i\le n$.

- Relations: $\delta_j\delta_i = \delta_i\delta_{j-1}$, $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}$, and $\sigma_j\delta_i = \begin{cases} \delta_i\sigma_{j-1} & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ \delta_{i-1}\sigma_j & \text{if } i > j+1. \end{cases}$

Hint: Send δ_i to the map which skips i and σ_i to the map which maps i and $i+1$ to i.

Remark 6.2.3. The maps δ_i are called *face maps*, and can be viewed as the inclusion of the $n-1$ simplex into the n-simplex as the face which does not include the vertex i. The maps σ_i are called *degeneracies*

Definition 6.2.4. A *simplicial set* is a *presheaf* on Δ, i.e., a functor $\mathcal{X}_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Set}$. That is, $\mathcal{X}_\bullet$ associates a set $\mathcal{X}_n$ to each $n \in \mathbb{N}$, and maps $d_i : \mathcal{X}_n \to \mathcal{X}_{n-1}$ and $s_j : \mathcal{X}_n \to \mathcal{X}_{n+1}$ which satisfy the opposite relations as in Exercise 6.2.2.

Remark 6.2.5. Given a category $\mathcal{C}$, one can define a simplicial object in $\mathcal{C}$ as a $\mathcal{C}$-valued presheaf on Δ, i.e., a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$.

Exercise 6.2.6. Suppose $\mathcal{X}_\bullet : \Delta^{\mathrm{op}} \to \mathsf{Ab}$ is a simplicial abelian group. For each $n \geq 1$, define $\partial : \mathcal{X}_n \to \mathcal{X}_{n-1}$ by $\partial := \sum_{i=0}^{n} (-1)^i d_i$. Prove that $\partial^2 = 0$. Deduce that $(\mathcal{X}_\bullet, \partial)$ is a chain complex.

Remark 6.2.7. Most definitions of homology start with a space, give some functor to simplicial sets, apply the free functor to get a simplicial abelian group, apply the alternating sum functor to get a chain complex, and then apply the functor $H_n := \ker(\partial)/\operatorname{im}(\partial)$ to get an abelian group.

$$\mathsf{Top} \to \mathsf{Fun}(\Delta^{\mathrm{op}} \to \mathsf{Set}) \xrightarrow{\text{Free}} \mathsf{Fun}(\Delta^{\mathrm{op}} \to \mathsf{Ab}) \xrightarrow{\text{Alt}} \mathsf{Chain} \xrightarrow{H_n} \mathsf{Ab}$$

There are many different definitions one can give for an n-category. Many notions of n-category start with a globular set and impose some composition operations which come with coherence data.

Definition 6.2.8. The *globular category* Γ has objects $[n]$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. Morphisms are given by generators and relations:

- Generators: $\sigma, \tau : [n] \to [n+1]$ for all n.
- Relations: $\sigma \circ \sigma = \tau \circ \sigma$ and $\tau \circ \tau = \sigma \circ \tau$.

Similar to a simplicial set, a globular set $\mathcal{X}_\bullet$ is a presheaf on Γ, i.e., a functor $\mathcal{X}_\bullet : \Gamma^{\mathrm{op}} \to \mathsf{Set}$. That is, $\mathcal{X}$ associates a set $\mathcal{X}_n$ to each $n \in \mathbb{N}$ and *source* and *target* maps $s_n, t_n : \mathcal{X}_{n+1} \to \mathcal{X}_n$ which satisfy $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ t_{n+1} = t_n \circ s_{n+1}$. In other words, in order to have $f \in \mathcal{X}_{n+2}$ with $s(f) = x$ and $t(f) = y$ ($f : x \to y$), we must have $s(x) = s(y)$ and $t(x) = t(y)$.

Other definitions of n-category start with a simplicial object and do not define a unique composite of k-morphisms; instead, one defines a family of k-composites, together with higher coherence data relating the composites. This offers the advantage that it is easier to define these higher categories if one does not have to provide all the data at the start. We will make this procedure more precise for 2-categories by defining the notion of a Segal object in Cat.

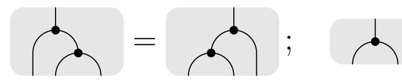
Definition 6.2.9. Suppose $\mathcal{C}$ is a category **TODO:**

TODO: Define a 2-category as a Segal category

6.3. **Algebras, (bi)modules, and intertwiners.** For this section, we fix a 2-category $\mathcal{C}$.

Definition 6.3.1. Suppose $a \in \mathcal{C}$. A pair $({}_aA_a, m : A \otimes_a A \Rightarrow A)$ is called an *algebra* if the following *associativity axiom* is satisfied:

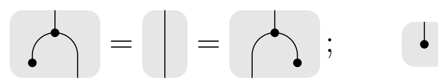
$$\begin{tikzcd} A \otimes (A \otimes A) \arrow[r, "\mathrm{id}_A \otimes m"] \arrow[d, "\alpha_{A,A,A}"] & A \otimes A \arrow[dr, "m"] & \\ & & A \\ (A \otimes A) \otimes A \arrow[r, "m \otimes \mathrm{id}_A"] & A \otimes A \arrow[ur, "m"'] & \end{tikzcd} \quad \rightsquigarrow$$



$= m.$

An algebra $({}_aA_a, m)$ is called *unital* if there is a 2-morphism $i : 1_a \Rightarrow A$ such that the following *unitality axioms* are satisfyied:

$$\begin{tikzcd} 1_a \otimes A \arrow[d, "i \otimes \mathrm{id}_A"] & A \arrow[l, "(\lambda^c_A)^{-1}"'] \arrow[r, "(\rho^c_A)^{-1}"] \arrow[d, "\mathrm{id}_A"] & A \otimes 1_a \arrow[d, "\mathrm{id}_A \otimes i"] \\ A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"'] \end{tikzcd} \quad \rightsquigarrow$$



$= i.$

A unital algebra is called *connected* if $\dim(\mathcal{C}(1_a \Rightarrow A)) = 1$.

There is also a notion of algebra object in a monoidal category $\mathcal{C}$; it is equivalent to an algebra in $\mathrm{B}\mathcal{C}$.

Exercise 6.3.2. Prove that if an algebra (A, m) is unital, then it its unit is unique.

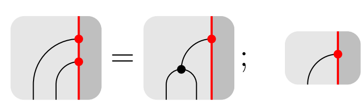
Exercise 6.3.3. Find a complete characterization of (unital) algebras in Set and in Cat.

Exercise 6.3.4. Find a complete characterization of unital algebras in $\mathsf{Vec}(G, \omega)$.

One gets the notion of a module M for an algebra A by taking the axioms for a module and changing the appropriate instance of A to M.

Definition 6.3.5. Suppose $({}_aA_a, m)$ is an algebra in $\mathcal{C}$. A *left A-module* is a pair $({}_aM_b, \lambda : A \otimes_a M \to M)$ for some $b \in \mathcal{C}$ such that the following associativity axiom holds:

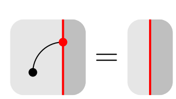
$$\begin{tikzcd} A \otimes (A \otimes M) \arrow[r, "\mathrm{id}_A \otimes \lambda"] \arrow[d, "\alpha_{A,A,M}"] & A \otimes M \arrow[dr, "\lambda"] & \\ & & M \\ (A \otimes A) \otimes M \arrow[r, "m \otimes \mathrm{id}_M"] & A \otimes M \arrow[ur, "\lambda"'] & \end{tikzcd} \quad \rightsquigarrow$$



$= \lambda.$

If $({}_aA_a, m)$ is unital, we call $({}_aM_b, \lambda)$ *unital* if the following unitality axiom is saitsfied:

$$\begin{tikzcd} 1_a \otimes M \arrow[d, "i \otimes \mathrm{id}_M"] & M \arrow[l, "(\lambda^c_A)^{-1}"'] \arrow[d, "\mathrm{id}_M"] \\ A \otimes M \arrow[r, "\lambda"] & M \end{tikzcd} \quad \rightsquigarrow$$

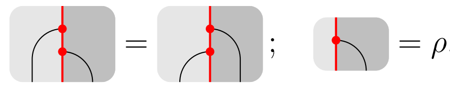


.

We leave the definition of right module to the reader.

Suppose now $({}_aA_a, m_A)$ and $({}_bB_b, m_B)$ are algebras. an $A-B$ *bimodule* is a triple $({}_aM_b, \lambda : A \otimes_a M \to M, \rho : M \otimes_b B \to M)$ such that (M, λ) is a left A-module, (M, ρ) is a right B-module, and the additional associativity axiom holds:

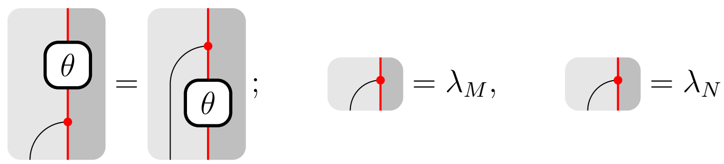
$$\begin{tikzcd} A \otimes (M \otimes A) \arrow[r, "\mathrm{id}_A \otimes \rho"] \arrow[d, "\alpha_{A,M,A}"] & A \otimes M \arrow[dr, "\lambda"] & \\ & & M \\ (A \otimes M) \otimes A \arrow[r, "\lambda \otimes \mathrm{id}_A"] & M \otimes A \arrow[ur, "\rho"] & \end{tikzcd} \quad \rightsquigarrow$$



Remark 6.3.6. As before, there is a notion of a module object in a multitensor category $\mathcal{C}$ for an algebra object; it is a module for the corresponding algebra in $\mathrm{B}\mathcal{C}$.

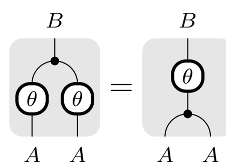
Definition 6.3.7. Suppose $({}_aA_a, m)$ is an algebra. Given two left A-modules $({}_aM_b, \lambda_M)$ and $({}_aN_b, \lambda_N)$, a 2-morphism $\theta \in \mathcal{C}({}_aM_b \Rightarrow {}_aN_b)$ is called a *left A-module map* if the following diagram commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "\lambda_M"] \arrow[d, "\mathrm{id}_A \otimes \theta"] & M \arrow[d, "\theta"] \\ A \otimes N \arrow[r, "\lambda_N"] & N \end{tikzcd} \quad \rightsquigarrow$$

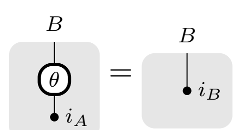


We leave the definition of a right B-module map and an $A-B$ bimodule map to the reader.

Definition 6.3.8. Suppose $({}_aA_a, m_A), ({}_aB_a, m_B)$ are algebras and $\theta : A \Rightarrow B$. We call θ an *algebra map* if



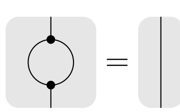
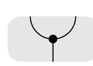
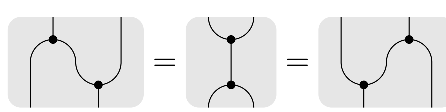
If A, B are unital, we call θ a *unital* algebra map if in addition



Observe that algebra objects in $\Omega_a\mathcal{C}$ and algebra maps form a 1-category.

6.4. **Separable algebras and condensation algebras.**

Definition 6.4.1. An algebra (A, m) (in a 2-category or a tensor category) is called *separable* if the multiplication map splits as an $A-A$ bimodule map, i.e., there is a map $\Delta : A \to A \otimes A$ such that

- (m splits)  where  $= \Delta$
- (as an $A-A$ bimodule) 

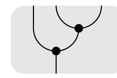
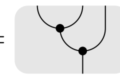
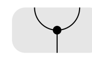
A triple (A, m, Δ) consisting of a separable algebra (A, m) equipped with a separator Δ is called a *condensation algebra* [GJF19].

Remark 6.4.2. The two most natural settings in which to work are condensation algebras and unital separable algebras. Working with separable algebras which are non-unital, but not equipped with a particular splitting, is not a well-behaved notion.

Exercise 6.4.3. Show that a unital algebra in $\mathsf{Vec}_{\mathsf{fd}}$ is separable if and only if it is semisimple.

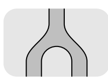
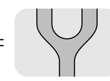
Exercise 6.4.4. Suppose $({}_aA_a, m, \Delta)$ is a condensation algebra. Prove that Δ is *co-associative*, i.e., the following axiom is satisfied:

$$\begin{tikzcd} A\otimes(A\otimes A) \arrow[d, "\alpha_{A,A,A}"] & A\otimes A \arrow[l, "\mathrm{id}_A\otimes\Delta"'] & \\ & & A \arrow[ul, "\Delta"'] \arrow[dl, "\Delta"] \\ (A\otimes A)\otimes A & A\otimes A \arrow[l, "\Delta\otimes\mathrm{id}_A"] & \end{tikzcd} \quad \rightsquigarrow$$

 $=$  ;  $= \Delta.$

Example 6.4.5. Suppose ${}_aX_b \in \mathcal{C}(a \to b)$. A *separable dual* for ${}_aX_b$ is a dual ${}_bX_a^\vee \in \mathcal{C}(b \to a)$ with maps $\mathrm{coev}_X \in \mathcal{C}(1_a \to {}_aX \otimes_b X_a^\vee)$ and $\mathrm{ev}_X \in \mathcal{C}({}_bX^\vee \otimes_a X_b \Rightarrow 1_b)$ such that ev_X admits a right inverse $\epsilon_X \in \mathcal{C}(1_b \to {}_bX^\vee \otimes_a X_b)$.

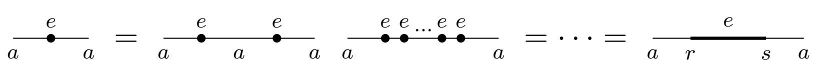
Given a separable dual for ${}_aX_b$, we can canonically endow $X \otimes_b X^\vee$ with the structure of a unital condensation algebra. Indeed, we define

$m :=$  $= \mathrm{id}_X \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}$ $\qquad i :=$  $= \mathrm{coev}_X$ $\qquad \Delta :=$  $= \mathrm{id}_X \otimes \epsilon_X \otimes \mathrm{id}_{X^\vee}$

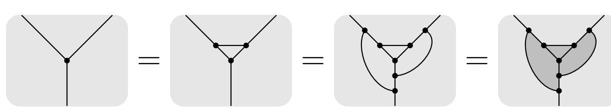
We leave the rest of the straightforward verification to the reader.

Definition 6.4.6. A unital separable algebra ${}_aA_a$ *splits* if it is isomorphic (via a 2-isomorphism) to a unital separable algebra of the form ${}_aX \otimes_b X_a^\vee$ from Example 6.4.5 where ${}_bX_a^\vee$ is a separable dual of ${}_aX_b$.

Remark 6.4.7. A condensation algebra is the 2-categorical analog of an idempotent. An idempotent in a 1-category can replicate freely on a line, and replicating arbitrarily many times leads to the notion of splitting for an idempotent.



Similarly, a condensation algebra can replicate freely in a 2D mesh, and replicating arbitrarily many times leads to the notion of splitting for a separable algebra.



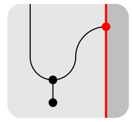
Definition 6.4.8. A locally Cauchy complete 2-category is called *idempotent complete* if every unital separable algebra splits.

In light of Remark 6.4.2, we now define the correct notions of bimodules for unital separable algebras and for condensation algebras.

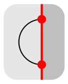
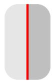
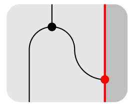
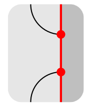
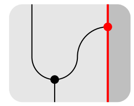
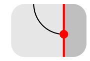
Exercise 6.4.9. Suppose $({}_aA_a, m, i)$ is a unital separable algebra and $({}_aM_b, \lambda)$ is a left A-module. Prove that the following are equivalent:

(1) (M, λ) is unital, and

(2) For any choice of separator $\Delta : A \to A \otimes A$, the map $\delta : M \to A \otimes_a M$ given by

$\delta :=$ 

satisfies

 $=$  and  $=$  $=$  ;  $= \delta.$ (6.4.10)

Repeat the above exercise for right B-modules and $A - B$ bimodules.

Definition 6.4.11. Suppose $({}_aA_a, m, \Delta)$ is a condensation algebra. A *left A condensation module* is a triple $({}_aM_b, \lambda, \delta)$ such that $({}_aM_b, \lambda)$ is a left A-module and $\delta : M \Rightarrow A \otimes_a M$ such that the axioms in (6.4.10) hold. We leave it to the reader to define right B condensation modules and $A - B$ condensation bimodules.

Exercise 6.4.12. Prove that for every $c \in \mathcal{C}$, 1_c is canonically a unital condensation algebra in $\mathcal{C}$. Then prove that for every ${}_aX_b \in \mathcal{C}(a \to b)$, the only $1_a - 1_b$ condensation bimodule structure on X is given by the unitors.
Hint: A condensation bimodule structure for X makes X a unital bimodule by Exercise 6.4.9.

6.5. **Module categories, separability, and semisimplicity.** In this section, $\mathcal{C}$ denotes a multitensor category unless stated otherwise.

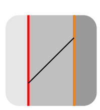
Construction 6.5.1. Given a unital algebra object (A, m, i) in $\mathcal{C}$, we can take the category $\mathsf{Mod}_{\mathcal{C}}(A)$ whose:

- objects are unital right A-module objects (M, ρ) in $\mathcal{C}$, and
- 1-morphisms are right A-module maps.

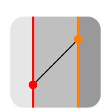
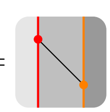
Given $c \in \mathcal{C}$, we get a *free A-module* given by $c \otimes A$ with right action map $\mathrm{id}_c \otimes \rho$. Observe that we get a functor $\mathcal{C} \to \mathsf{Mod}_{\mathcal{C}}(A)$ given by $c \mapsto c \otimes A$ and $(f : c_1 \to c_2) \mapsto f \otimes \mathrm{id}_A$; we call this the *free module functor*.

Exercise 6.5.2. Suppose (A, m) is an algebra object in $\mathcal{C}$. Show that every idempotent right A-module map $e : M_A \to N_A$ splits in $\mathsf{Mod}_{\mathcal{C}}(A)$. Deduce that $\mathsf{Mod}_{\mathcal{C}}(A)$ is Cauchy complete, as is $\mathsf{Bim}_{\mathcal{C}}(A)$, the category of $A - A$ bimodules in $\mathcal{C}$ with $A - A$ bimodule maps.

Definition 6.5.3. Suppose $\mathcal{C}$ is a locally Cauchy complete 2-category and A, B, C are condensation algebras. Given condensation bimodules ${}_AM_B$ and ${}_BN_C$, observe that $M \otimes_b N$ is organically an $A - C$ bimodule, and by Exercise 6.5.2, the category $\mathsf{Bim}_{\mathcal{C}}(A \to C)$ of $A - C$ bimodules in $\mathcal{C}$ is Cauchy complete. We define the *relative tensor product $M \otimes_B N$* by splitting the idempotent

 $:= (\mathrm{id}_M \otimes \lambda_N) \circ (\Delta_M \otimes \mathrm{id}_N)$ (6.5.4)

in $\mathsf{Bim}_{\mathcal{C}}(A \to C)$. Observe that $M \otimes_B N$ is only defined up to unique isomorphism.

Exercise 6.5.5. Verify (6.5.4) is an idempotent. Then show  $=$  .

Exercise 6.5.6. Suppose $\mathcal{C}$ is a locally Cauchy complete 2-category.

(1) Show how to endow $\mathsf{Cauchy}(\mathcal{C})$, the condensation algebras, condensation bimodules, and bimodule maps in $\mathcal{C}$, with the structure of a 2-category.
(2) Show that unital separable algebras, separable (equivalently unital!) bimodules, and intertwiners form a full 2-subcategory $\mathsf{Alg}_{\mathsf{u}}(\mathcal{C})$ of $\mathsf{Cauchy}(\mathcal{C})$.

For $(M, \rho) \in \mathsf{Mod}_{\mathcal{C}}(A)$, observe that $c \otimes M$ also has a right A-module structure with action map $\mathrm{id}_c \otimes \rho$. Thus the category $\mathsf{Mod}_{\mathcal{C}}(A)$ has the structure of a left $\mathcal{C}$-*module category*.

Definition 6.5.7. A left $\mathcal{C}$-module category for a multitensor category $\mathcal{C}$ consists of a linear Cauchy complete category $\mathcal{M}$ together with a left $\mathcal{C}$-action functor $\rhd : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ a family of natural unitor isomorphisms $\lambda_m : 1_{\mathcal{C}} \rhd m \to m$, and a family of natural associator isomorphisms $\alpha_{a,b,m} : a \rhd (b \rhd m) \to (a \otimes b) \rhd m$ which satisfy the pentagon and triangle axioms.

Exercise 6.5.8. Show how to endow $\mathsf{Mod}_{\mathcal{C}}(A)$ with the structure of a left $\mathcal{C}$-module category. *Hint: Use Exercise 6.5.2.*

Exercise 6.5.9. Suppose $\mathcal{C}$ is a multitensor category and $(A, m, \Delta) \in \mathcal{C}$ is a condensation algebra. Prove that $\mathsf{Mod}_{\mathcal{C}}(A)$ is the Cauchy completion of $\mathsf{FreeMod}_{\mathcal{C}}(A)$, whose objects are the right A condensation modules of the form $c \otimes A$ for $c \in \mathcal{C}$.
Hint: Show that a right A-module M is a summand of the free module $M \otimes A$.

Exercise 6.5.10. Prove that the following are equivalent for a unital algebra (A, m, i) in a semisimple tensor category $\mathcal{C}$.

(1) A is separable,
(2) $\mathsf{Mod}_{\mathcal{C}}(A)$ is semisimple, and
(3) $\mathsf{Bim}_{\mathcal{C}}(A)$ is semisimple.

Hint: Use that free modules are projective.

Definition 6.5.11. Suppose $\mathcal{M}, \mathcal{N}$ are two left $\mathcal{C}$-module categories. A $\mathcal{C}$-*module functor* $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is a functor equipped with a family of natural actionator isomorphisms $\mathcal{F}^2_{c,m} : c \rhd \mathcal{F}(m) \to \mathcal{F}(c \rhd m)$ satisfying an associative condition. Given two $\mathcal{C}$-module functors $\mathcal{F}, \mathcal{G} : \mathcal{M} \to \mathcal{N}$, a $\mathcal{C}$-module natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ such that the following compatibility axiom is satisfied with the actionators:

$$\begin{array}{ccc} c \rhd \mathcal{F}(m) & \xrightarrow{\mathcal{F}^2_{c,m}} & \mathcal{G}(c \rhd m) \\ \Big\downarrow{\scriptstyle \mathrm{id}_c \rhd \theta_m} & & \Big\downarrow{\scriptstyle \theta_m} \\ c \rhd \mathcal{G}(m) & \xrightarrow{\mathcal{G}^2_{c,m}} & \mathcal{G}(c \rhd m). \end{array} \tag{6.5.12}$$

Exercise 6.5.13. Show how to endow the left $\mathcal{C}$-modules, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations with the structure of a 2-category. We call this 2-category $\mathsf{Mod}(\mathcal{C})$.

Exercise 6.5.14. Suppose $\mathcal{C}$ is a multitensor category, $A, B \in \mathcal{C}$ are separable unital algebra objects, and $M \in \mathcal{C}$ is an $A{-}B$ bimodule object. Prove that $- \otimes_A M_B : \mathsf{Mod}_{\mathcal{C}}(A) \to \mathsf{Mod}_{\mathcal{C}}(B)$ gives a well-defined $\mathcal{C}$-module functor.

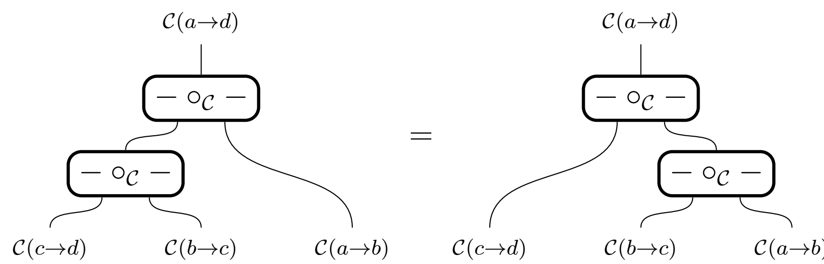
6.6. **Enriched categories and the Barr-Beck/Ostrik theorem for multifusion categories.** For this section, $\mathcal{V}$ denotes a monoidal category. Typically, we will take $\mathcal{V} = \mathsf{Vec}$ in applications, but sometimes we take $\mathcal{V}$ to be sVec or another multifusion category.

Definition 6.6.1 ([Kel05])**.** Given a monoidal category $\mathcal{V}$, a $\mathcal{V}$-*(enriched) category* $\mathcal{C}$ consists of the following data:

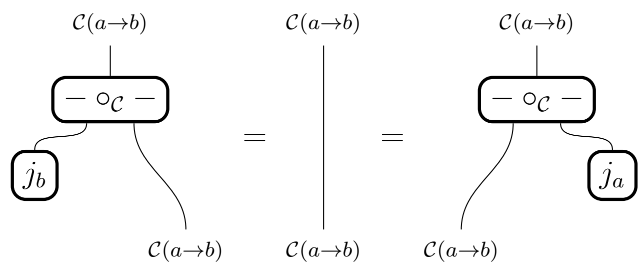
- a collection of objects $c \in \mathcal{C}$,
- for each $a, b \in \mathcal{C}$, a *hom object* $\mathcal{C}(a \to b) \in \mathcal{V}$,
- a unit map $j_c \in \mathcal{V}(1_\mathcal{V} \to \mathcal{C}(c \to c))$ for every $c \in \mathcal{C}$, and
- a composition morphism $- \circ_\mathcal{C} - \in \mathcal{V}(\mathcal{C}(b \to c) \otimes \mathcal{C}(a \to b) \to \mathcal{C}(a \to c))$ for all $a, b, c, \in \mathcal{C}$.

This data must satisfy the following axioms:

- (associativity)



- (identity)



Exercise 6.6.2. Given a $\mathcal{V}$-category $\mathcal{C}$, the *underlying category* $\mathcal{C}^\mathcal{V}$ has the same objects as $\mathcal{C}$, but $\mathcal{C}^\mathcal{V}(a \to b) := \mathcal{V}(1_\mathcal{V} \to \mathcal{C}(a \to b))$. Show how to endow $\mathcal{C}^\mathcal{V}$ with the structure of an ordinary category.

Exercise 6.6.3 ($\star$, [Kel05])**.** Define a notion of $\mathcal{V}$-functor and natural transformation so that $\mathcal{V}$-categories, $\mathcal{V}$-functors, and natural transformations forms a 2-category.

Exercise 6.6.4 ($\star$)**.** Show that taking the underlying category, functor, and natural transformation gives a 2-functor $\mathcal{V}\mathsf{Cat} \to \mathsf{Cat}$.

For the remained of this section, $\mathcal{C}$ denotes a multifusion category.

Exercise 6.6.5. Suppose $\mathcal{M}$ is a finitely semisimple left $\mathcal{C}$-module category.

(1) Prove that for each $m, n \in \mathcal{M}$, the functor $\mathcal{M}(- \triangleright m \to n) : \mathcal{C} \to \mathsf{Vec}$ is representable. *Hint: Show* $\mathcal{M}(- \triangleright m \to n)$ *is exact.*
(2) Let $\widehat{\mathcal{M}}(m \to n)$ denote the *internal hom* representing the functor $\mathcal{M}(- \triangleright m \to n)$. Show that $\widehat{\mathcal{M}}$ is a $\mathcal{C}$-enriched category with the same objects as $\mathcal{M}$.
(3) Deduce that for every $m \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to m)$ has the structure of a unital algebra object in $\mathcal{C}$, and for every $n \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to n)$ is a right $\widehat{\mathcal{M}}(m \to m)$-module.

Remark 6.6.6. In the previous exercise, we saw how to build a $\mathcal{C}$-enriched category $\widehat{\mathcal{M}}$ from a left $\mathcal{C}$-module category. Conversely, given a particularly nice $\mathcal{C}$-enriched category which is *tensored* [Lin81, MPP18], we can build a left $\mathcal{C}$-module category. These two constructions are mutually inverse; indeed the 2-category of these nice tensored $\mathcal{C}$-enriched categories is equivalent to the 2-category of left $\mathcal{C}$-module categories, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations. We refer the reader to [Lin81, MPP18, Del19] for more details.

Exercise 6.6.7. Suppose $\mathcal{M}$ is a finitely semisimple left $\mathcal{C}$-module category.

(1) Find a canonical isomorphism $\widehat{\mathcal{M}}(m_1 \to c \rhd m_2) \cong c \otimes \widehat{\mathcal{M}}(m_1 \to m_2)$.
(2) Use (1) to prove that for any $m \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to -) : \mathcal{M} \to \mathcal{C}$ is a left $\mathcal{C}$-module functor.
(3) Find a canonical isomorphism $\widehat{\mathcal{M}}(c \rhd m_1 \to m_2) \cong \widehat{\mathcal{M}}(m_1 \to m_2) \otimes c^\vee$.

Definition 6.6.8. A *pointing* on a left $\mathcal{C}$-module category $\mathcal{M}$ is a choice of object $m_0 \in \mathcal{M}$ which *generates* $\mathcal{M}$ as a left $\mathcal{C}$-module category. This means that every object of $\mathcal{M}$ is isomorphic to a summand of a direct sum of objects of the form $c \rhd m_0$ where $c \in \mathcal{C}$. Equivalently (exercise!), m_0 generates $\mathcal{M}$ if $\mathcal{M}$ is equivalent to the Cauchy completion of the left $\mathcal{C}$-module subcategory $\mathcal{M}_0 \subset \mathcal{M}$ whose objects are of the form $c \rhd m_0$ for $c \in \mathcal{C}$.

A *pointing* on a $\mathcal{C}$-module functor $\mathcal{F} : (\mathcal{M}, m_0) \to (\mathcal{N}, n_0)$ between two pointed module categories is a distinguished isomorphism $\mathcal{F}_* : n_0 \to \mathcal{F}(m_0)$. A $\mathcal{C}$-module natural transformation $\theta : (\mathcal{F}, \mathcal{F}_*) \Rightarrow (\mathcal{G}, \mathcal{G}_*)$ between pointed $\mathcal{C}$-module functors $(\mathcal{M}, m_0) \to (\mathcal{N}, n_0)$ is *pointed* if the following diagram commutes:

$$\begin{tikzcd} n_0 \arrow[rr, "\mathcal{F}_*"] \arrow[dr, "\mathcal{G}_*"'] & & \mathcal{F}(m_0) \arrow[dl, "\theta_m"] \\ & \mathcal{G}(m_0). & \end{tikzcd}$$

Exercise 6.6.9. Prove that pointed left $\mathcal{C}$-modules, pointed $\mathcal{C}$-module functors, and pointed $\mathcal{C}$-module natural transformations forms a 2-category, which we call $\mathsf{Mod}_*(\mathcal{C})$.

Exercise 6.6.10. Suppose $(\mathcal{F}, \mathcal{F}_*), (\mathcal{G}, \mathcal{G}_*) : (\mathcal{M}, m_0) \to (\mathcal{N}, n_0)$ are two pointed $\mathcal{C}$-module functors. Prove there is at most one pointed $\mathcal{C}$-module natural transformation $\theta : (\mathcal{F}, \mathcal{F}_*) \Rightarrow (\mathcal{G}, \mathcal{G}_*)$, which is necessarily an isomorphism. In this sense, we say $\mathsf{Mod}_*(\mathcal{C})$ is *1-truncated*, i.e., equivalent to a 1-category.

Exercise 6.6.11. Suppose A, B are unital algebras in $\mathcal{C}$ and $\theta : A \to B$ is an algebra map as in Definition 6.3.8.

(1) Show that algebras and algebra maps forms a 1-category.
(2) Show that this 1-category is equivalent to $\mathsf{Mod}_*(\mathcal{C})$.

Definition 6.6.12. Let $\mathsf{Mod}_{\exists *}(\mathcal{C})$ denote the 2-subcategory of $\mathsf{Mod}(\mathcal{C})$ whose objects are modules which admit a pointing. Observe the existence of a pointing is a property, and we do not assume this pointing as extra structure.

Theorem 6.6.13 (Barr-Beck for multifusion categories, [Ost03], [BZBJ18, §4])**.** *Let $\mathcal{C}$ be a multifusion category. The map $A \mapsto \mathsf{Mod}_{\mathcal{C}}(A)$ and ${}_AM_B \mapsto - \otimes_A M_B$ is a 2-equivalence $\mathsf{Alg}_{\mathsf{u}}(\mathcal{C}) \to \mathsf{Mod}_{\exists *}(\mathcal{C})$.*

Remark 6.6.14. We give a straightforward pedestrian proof following [Ost03], but one can use the Barr-Beck Theorem to prove this as well [BZBJ18, §4].

Proof of Theorem 6.6.13. It is straightforward to verify the above map gives a 2-functor.

First, we check that for all unital $A-B$ bimodules ${}_AM_B, {}_AN_B$,

$$\mathrm{Hom}_{A-B}(M \Rightarrow N) \ni \theta \mapsto - \otimes \theta \in \mathsf{Fun}_{\mathcal{C}-}(- \otimes_A M_B \Rightarrow - \otimes_A N_B)$$

is an isomorphism. Indeed, every $\mathcal{C}$-module natural transformation $\zeta : - \otimes_A M_B \Rightarrow - \otimes_A N_B$ is completely determined by ζ_A using (6.5.12) as $\mathsf{Mod}_{\mathcal{C}}(A)$ is the Cauchy completion of $\mathsf{FreeMod}_{\mathcal{C}}(A)$ by Exercise 6.5.9.

Thus to show our 2-functor is fully faithful, we need to prove the hom functors are essentially surjective. Suppose $\mathcal{F} : \mathsf{Mod}_{\mathcal{C}}(A) \to \mathsf{Mod}_{\mathcal{C}}(B)$ is a left $\mathcal{C}$-module functor. Then $\mathcal{F}(A) \in \mathsf{Mod}_{\mathcal{C}}(B)$ carries both a right B-action and a left A-action using the modulator: $\lambda_A := \mathcal{F}^2(m_A) \circ \mathcal{F}^2_{A,A}$. Setting ${}_AM_B := \mathcal{F}(A)$, it is straightforward to check that $\mathcal{F} \cong - \otimes_A M$.

It remains to show the 2-functor is essentially surjective. To do this, we must show that every semisimple left $\mathcal{C}$-module category $\mathcal{M}$ which admits a pointing is equivalent to $\mathsf{Mod}_{\mathcal{C}}(A)$ for some separable unital algebra A. Let $m \in \mathcal{M}$ be a pointing. First, observe that $A := \widehat{\mathcal{M}}(m \to m)$ is a unital algebra object by Exercise 6.6.5. Second, the functor $n \mapsto \widehat{\mathcal{M}}(m \to n)$ takes values in $\mathsf{Mod}_{\mathcal{C}}(A)$ again by Exercise 6.6.5, and is a $\mathcal{C}$-module functor by Exercise 6.6.7. The verification that this $\mathcal{C}$-module functor is an equivalence is provided by the next exercise. Finally, we conclude A is separable by Exercise 6.5.10. ☐

Exercise 6.6.15 ([Ost03, Thm. 3.1])**.** Suppose $(\mathcal{M}, m)$ is a pointed semisimple $\mathcal{C}$-module category and $A := \widehat{\mathcal{M}}(m \to m)$. Follow the steps below to verify that the $\mathcal{C}$-module functor $n \mapsto \widehat{\mathcal{M}}(m \to n)$ is an equivalence.

(1) Prove that since $(\mathcal{M}, m)$ is pointed, the $\mathcal{C}$-module category $\mathcal{M}$ is *indecomposable*, i.e., it does not break up as a direct sum of two non-zero $\mathcal{C}$-module categories.
(2) Prove that $n \neq 0$ in $\mathcal{M}$ implies $\widehat{\mathcal{M}}(m \to n) \neq 0$ in $\mathsf{Mod}_{\mathcal{C}}(A)$.
(3) Prove that $\widehat{\mathcal{M}}(- \to m)$ is faithful.
(4) Prove that $\widehat{\mathcal{M}}(- \to m)$ is surjective on hom spaces of the full subcategory of $\mathcal{M}$ of objects of the form $c \rhd m$ for $c \in \mathcal{C}$. Deduce that $\widehat{\mathcal{M}}(- \to m)$ is surjective on all hom spaces of $\mathcal{M}$.
(5) Prove that $\widehat{\mathcal{M}}(- \to m)$ is essentially surjective on objects.

Definition 6.6.16. Two algebras $({}_aA_a, m_A), ({}_bB_b, m_B)$ are called *Morita equivalent* if there are *Morita equivalence* $A-B$ and $B-A$ bimodules ${}_aM_b$ and ${}_bN_a$ respectively together with an $A-A$ bimodule isomorphism ${}_aA_a \cong {}_aM \otimes_b N_a$ and a $B-B$ bimodule isomorphism ${}_bB_b \cong {}_bN \otimes_a M_b$.

Exercise 6.6.17. Show that two algebra objects $(A, m_A), (B, m_B)$ in a multitensor category $\mathcal{C}$ are Morita equivalent if and only if the categories $\mathsf{Mod}_{\mathcal{C}}(A)$ and $\mathsf{Mod}_{\mathcal{C}}(B)$ are equivalent as left $\mathcal{C}$-module categories.

6.7. **Cauchy completion for linear 2-categories.** In this section, $\mathcal{C}$ denotes a locally Cauchy complete 2-category unless otherwise stated.

Definition 6.7.1 ([GJF19])**.** The *Cauchy completion* $\mathsf{Cauchy}(\mathcal{C})$ is the 2-category of $\mathsf{Cauchy}(\mathcal{C})$ of condensation algebras, condensation bimodules, and intertwiners from Exercise 6.5.6.

Observe there is a canonical inclusion 2-functor $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{Cauchy}(\mathcal{C})$ given by $a \mapsto 1_a$, ${}_aX_b \mapsto {}_aX_b$ with the unique condensation bimodule structure from Exercise 6.4.12, and $(f : {}_aX_b \Rightarrow {}_aY_b) \mapsto f$. We call $\mathcal{C}$ *Cauchy complete* if $\iota_{\mathcal{C}}$ is an equivalence.

Exercise 6.7.2. Write down the coheretors of the 2-functor $\iota_{\mathcal{C}}$.

Exercise 6.7.3 ([DR18, App. A], [CP], $\star\star$)**.** Show that any 2-functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is Cauchy complete factors *uniquely* through $\mathsf{Cauchy}(\mathcal{C})$.

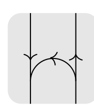
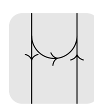
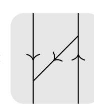
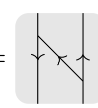
$$\begin{tikzcd} \mathsf{Cauchy}(\mathcal{C}) \arrow[dr, "\exists!\mathsf{Cauchy}(\mathcal{F})"] & \\ \mathcal{C} \arrow[u, hook] \arrow[r, "\mathcal{F}"] & \mathcal{D} \end{tikzcd}$$

Note: the difficulty in this problem is the uniqueness statement. In the case of a linear 1-category, the lift to $\mathsf{Cauchy}(\mathcal{C})$ *was unique up to unique natural isomorphism. That is, there was a whole* category *of lifts, and this category turned out to be a* contractible groupoid*, i.e., a connected groupoid with exactly one isomorphism between any two objects. However, as $\mathcal{C}, \mathcal{D}$ are 2-categories, one obtains a 2-category of lifts, and this 2-category is itself contractible, i.e., equivalent to a point. These notions will be made more precise in §**??** below.*

Exercise 6.7.4. Suppose $\mathcal{C}$ is a rigid locally Cauchy complete 2-category.

(1) Given a condensation algebra $({}_aA_a, m, \Delta)$, show that $({}_aA_a^\vee, m^\vee, \Delta^\vee)$ is a condensation algebra.

(2) Prove that A has the structure of a left $A^\vee$ module, and $A^\vee$ has the structure of a right A-module. Deduce that

$p_{A,A^\vee} :=$  $=$  $=$  $=$ 

is an idempotent. Show that 1-morphism $A^\vee \otimes_A A$ obtained by splitting this idempotent has the structure of a unital condensation algebra.

(3) Show that both $({}_aA_a, m, \Delta)$ and $({}_aA_a, m^\vee, \Delta^\vee)$ are Morita equivalent to the unital condensation algebra $A^\vee \otimes_A A$.

Exercise 6.7.5 ([GJF19, Thms. 3.1.7 and 3.3.3])**.**

(1) Use Exercise 6.7.4 to show that $\mathsf{Cauchy}(\mathcal{C})$ is equivalent to $\mathsf{Cauchy}_{\mathsf{u}}(\mathcal{C})$, the 2-category of unital condensation algebras, condensation bimodules, and intertwiners.

(2) Show that the forgetful 2-functor $\mathsf{Cauchy}_{\mathsf{u}}(\mathcal{C}) \to \mathsf{Alg}_{\mathsf{u}}(\mathcal{C})$ which forgets the separators is an equivalence of 2-categories.

Corollary 6.7.6. $\mathsf{Cauchy}(\mathrm{B}\mathsf{Vec}) \cong \mathsf{Cauchy}_{\mathsf{u}}(\mathrm{B}\mathsf{Vec}) = \mathsf{Alg}^{\mathsf{sep}}_{\mathsf{fd}} = \mathsf{Alg}_{\mathsf{u}}(\mathrm{B}\mathsf{Vec})$.

Proof. Immediate from Exercise 6.7.5. □

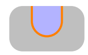
Exercise 6.7.7. Prove that $\mathsf{Alg}^{\mathsf{sep}}_{\mathsf{fd}} \cong \mathsf{Cat}^{\mathsf{fin}}_{\mathsf{ss}}$, the 2-category of finite semisimple categories. We will thus refer to either one of these 2-categories as $2\mathsf{Vec}$.
Hint: Use Theorem 6.6.13 for the fusion category $\mathrm{B}\mathsf{Vec}$*.*

Exercise 6.7.8. Show that $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{Cauchy}(\mathcal{C})$ is fully faithful, i.e., an equivalence on all hom categories. Deduce that $\mathcal{C}$ is Cauchy complete if and only if every condensation algebra in $\mathsf{Cauchy}(\mathcal{C})$ is equivalent to a trivial algebra of the form 1_c for $c \in \mathcal{C}$.

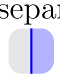
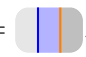
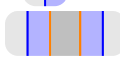
Exercise 6.7.9. Prove that $\mathsf{Cauchy}(\mathcal{C})$ is Cauchy complete.

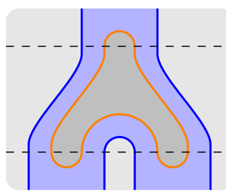
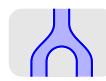
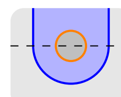
Theorem 6.7.10 (cf. [DR18, [[where?]]])**.** *A locally Cauchy complete 2-category $\mathcal{C}$ is Cauchy complete ($\iota_{\mathcal{C}} : \mathcal{C} \to \mathsf{Cauchy}(\mathcal{C})$ is an equivalence) if and only if every unital separable algebra in $\mathcal{C}$ splits.*

Proof.
$\underline{\Rightarrow}$: Suppose $(A, m) \in \mathcal{C}(a \to a)$ is a unital separable algebra. Since $\mathcal{C} \hookrightarrow \mathsf{Cauchy}(\mathcal{C})$ is an equivalence, it is essentially surjective on objects. This means there is an object $b \in \mathcal{C}$ and an invertible (which implies dualizable) separable bimodule ${}_AN_{1_b} =$  $\in \mathsf{Cauchy}(\mathcal{C})(A \to 1_b)$ together with isomorphisms

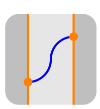
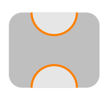
$\varepsilon :=$  $: {}_AN \otimes_{1_b} N^\vee_A \xrightarrow{\cong} {}_AA_A.$ and $\delta :=$  $: {}_{1_b}1_{b1_b} \xrightarrow{\cong} {}_{1_b}N^\vee \otimes_A N_{1_b},$

which satisfy the zigzag conditions. Since ε and δ are invertible, we see that $N^\vee$ is a separable dual for N, and N is a separable dual for $N^\vee$.

Now consider ${}_{1_a}A_A =$  as a $1_a - A$ bimodule, and set $M := {}_{1_a}A \otimes_A N_{1_b} =$ . Then clearly ${}_{1_a}M \otimes_{1_b} M^\vee_{1_a} =$  is isomorphic to ${}_aA \otimes_A A_a$ as algebras via the isomorphism

$u :=$  $\Longrightarrow$  $=$   $=$ .

By Exercise 6.7.11 below, ${}_{1_a}M \otimes_{1_b} M^\vee_{1_a} \cong {}_{1_a}A \otimes_A A_{1_a} \cong {}_{1_a}A_{1_a}$ as algebras in $\mathsf{Cauchy}(\mathcal{C})$. Hence A splits as desired.
$\underline{\Leftarrow}$: Suppose every unital separable algebra $A \in \mathcal{C}(a \to a)$ splits. To show $\mathcal{C}$ is Cauchy complete, by Exercise 6.7.8, it suffices to show that every unital separable algebra ${}_aA_a$ is equivalent to a trivial algebra. Let ${}_aN_b$ be a dualizable 1-morphism with separable dual $({}_bN^\vee_a, \mathrm{ev}_N, \mathrm{coev}_N)$ such that ${}_aA_a \cong {}_aN \otimes_b N^\vee_a$ as algebras. This isomorphism intertwines the $A - A$ bimodule structure on ${}_aA_a$ with the ${}_aN \otimes_b N^\vee_a - {}_aN \otimes_b N^\vee_a$ bimodule structure on ${}_aN \otimes_b N^\vee_a$. Observe that under the above isomorphism, the canonical separability projector $p_{N^\vee,N} \in \mathrm{End}({}_bN^\vee \otimes_a N_b)$ onto ${}_bN^\vee \otimes_A N_b \cong {}_bN^\vee \otimes_{N \otimes_b N^\vee} N_b$ is given as in (6.5.4) by

$p_{N,N^\vee} =$  $=$  $=$  $= \epsilon_N \circ \mathrm{ev}_N,$

where ϵ_N is a right inverse to ev_N as in Exercise 6.4.5. Thus $(\mathrm{ev}_N, \epsilon_N)$ splits $p_{N^\vee,N}$. Since the image of ev_N is 1_b, we see that ${}_AN_b$ is an invertible $A - 1_c$ bimodule in $\mathsf{Cauchy}(\mathcal{C})(A \to 1_b)$, as desired. $\square$

Exercise 6.7.11. Suppose $(A, m) \in \mathcal{C}(a \to a)$ is a unital separable algebra. Prove that the unital separable algebra ${}_{1_a}A \otimes_A A_{1_a}$ constructed as in Exercise 6.4.5 is isomorphic as an algebra to ${}_{1_a}A_{1_a}$.

6.8. **Semisimple 2-categories.** For this section, $\mathcal{C}$ denotes a locally Cauchy complete 2-category.

Definition 6.8.1. Given a finite collection of objects $c_1, \ldots, c_n \in \mathcal{C}$, an object $\boxplus_{i=1}^n c_i \in \mathcal{C}$ together with 1-morphisms $I_j : c_j \to \boxplus_{i=1}^n c_i$ and $P_j : \boxplus_{i=1}^n c_i \to c_j$ is called the *direct sum* of $c_1, \ldots, c_n$ if

($\boxplus$1) $I_j \otimes_{\boxplus_{i=1}^n c_i} P_k$ is isomorphic to 1_j if $j = k$ and the zero 1-morphism if $j \neq k$, and
($\boxplus$2) $\bigoplus_{j=1}^n P_j \otimes_{c_j} I_j$ is isomorphic to $1_{\boxplus_{i=1}^n c_i}$ in $\mathrm{End}_c(\boxplus_{i=1}^n c_i)$.

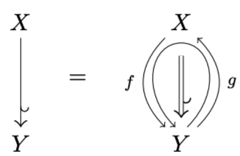
We call $\mathcal{C}$ *additive* if $\mathcal{C}$ admits direct sums. By convention, an additive 2-category must have a zero object (whose identity 1-morphism is a zero 1-morphism), as it is the empty direct sum.

Exercise 6.8.2. Suppose $c_1, c_2 \in \mathcal{C}$ and $I_j : c_j \to c_1 \boxplus c_2$, $P_j : c_1 \boxplus c_2 \to c_j$ for $j = 1, 2$ witness $c_1 \boxplus c_2$ as the direct sum. Show that P_j is both a left and right dual of I_j.
Hint: Try Exercise 6.1.14 first.

Definition 6.8.3. A locally Cauchy complete 2-category is called a *Cauchy complete* 2-category if it is additive and idempotent complete. Observe that a Cauchy complete 2-category is Cauchy complete by Theorem 6.7.10.

For the remainder of this section, $\mathcal{C}$ denotes a pre-semisimple 2-category, i.e., $\mathcal{C}$ is rigid and all hom 1-categories are semisimple.

Definition 6.8.4. A *split surjection* or *condensation* $X \twoheadrightarrow Y$ between objects in an n-category consists of 1-morphisms $f : X \to Y$ and $g : Y \to X$ and another split surjection/condensation $f \circ g \Rrightarrow \mathrm{id}_Y$. At the top level, we require $f \circ g = \mathrm{id}_Y$.



[GJF19, p8, (♠)]

Lemma 6.8.5 (cf. [DR18, Prop. 1.2.4])**.** *The following are equivalent for $c \in \mathcal{C}$, a pre-semisimple 2-category.*

(1) $\mathrm{End}_{\mathcal{C}}(1_c) \cong \mathbb{C}$, *and*
(2) every non-zero 1-morphism $a \to c$ can be augmented to a condensation $a \twoheadrightarrow c$.
(3) ***TODO: equivalent subobject definition***

Such objects $c \in \mathcal{C}$ are called simple.

Proof.
(1) ⇒ (2): Let ${}_aX_c : a \to c$ be non-zero, and consider $\mathrm{ev}_X : {}_cX^\vee \otimes_a X_c \Rightarrow 1_c$ which is also non-zero. Since $\mathrm{End}_{\mathcal{C}}(1_c \oplus {}_cX^\vee \otimes_a X_c)$ is semisimple and 1_c is simple, there is a right inverse $\epsilon_X : 1_c \to {}_cX^\vee \otimes_a X_c$. Thus the data $(X^\vee, \mathrm{ev}_X, \epsilon_X)$ endows ${}_aX_c : a \to c$ with the structure of a condensation.
(2) ⇒ (1): We prove the contrapositive. Suppose $1_c = {}_cE_c \oplus {}_cF_c$ with E, F non-isomorphic and non-zero. (This is always possible when 1_c is not simple in a semisimple multitensor category.) Observe that E, F are idempotent, i.e., $E \otimes_c E \cong E$ and $F \otimes_c F \cong F$ as $\mathrm{End}_{\mathcal{C}}(c)$ is a semisimple multitensor category. We claim that ${}_cE_c : c \to c$ cannot be augmented to a condensation. Indeed, for any other 1-morphism ${}_cX_c : c \to c$ and any 2-morphism $\theta : X \otimes_c E \Rightarrow 1_c$, there does not exist a splitting of θ as the right support of $X \otimes_c E$ is clearly a subobject of E in $\mathrm{End}_{\mathcal{C}}(c)$. $\square$

Definition 6.8.6. A *semisimple* 2-category is a pre-semisimple 2-category that is Cauchy complete. A semisimple 2-category is called *finitely* semisimple if it has only finitely many equivalence classes of simple objects, and every hom 1-category is finitely semisimple.

Exercise 6.8.7. Suppose $\mathcal{F}$ is a multifusion category. Prove that $\mathsf{Mod}(\mathcal{F})$ is a semisimple 2-category.

Exercise 6.8.8. A rigid 2-category is called *connected* if there is a non-zero 1-morphism between any two objects.

(1) Show that every connected finitely semisimple 2-category $\mathcal{C}$ is equivalent to $\mathsf{Mod}(\mathcal{F})$ for some fusion category $\mathcal{F}$.
Hint: Pick your favorite object $c \in \mathcal{C}$, and observe $\Omega_c\mathcal{C} = \mathcal{C}(c \to c)$ is a multifusion category.
(2) Deduce that every finitely semisimple 2-category is equivalent to $\mathsf{Mod}(\mathcal{F})$ for some multifusion category $\mathcal{F}$.

Exercise 6.8.9. The relation 'b admits a condensation from a' denoted $\exists(a \twoheadrightarrow b)$ is an equivalence relation on simple objects in a semisimple 2-category $\mathcal{C}$. The equivalence classes of this relation are called the *components* of $\mathcal{C}$.

Remark 6.8.10. For semisimple n-categories with $n > 2$, having finitely many equivalence classes of objects is no longer the correct finiteness assumption; indeed, this will be in direct conflict with being Cauchy complete, as $\mathsf{Cauchy}(2\mathsf{Vec}) = \mathsf{MultFusCat}$, which has infinitely many simple objects. The correct finiteness assumption is having finitely many *components* of simples.

Exercise 6.8.11. Compute the simple objects and components of the following semisimple 2-categories: $\mathsf{Mod}(\mathsf{Vec}_{\mathsf{fd}}(\mathbb{Z}/2))$, $\mathsf{Mod}(\mathsf{sVec})$.

6.9. **Unitary 2-categories.**

Definition 6.9.1. Suppose $\mathcal{C}$ is a linear 2-category whose hom 1-categories $\mathcal{C}(a \to b)$ are Cauchy complete. A dagger structure on $\mathcal{C}$ consists of an anti-linear map $\dagger : \mathcal{C}({}_aX_b \Rightarrow {}_aY_b) \to \mathcal{C}({}_aY_b \Rightarrow {}_aX_b)$ for all 1-morphisms ${}_aX_b, {}_aY_b \in \mathcal{C}(a \to b)$ for all objects $a, b \in \mathcal{C}$ satisfying the following conditions:

- For all $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$, $f^{\dagger\dagger} = f$.
- For all $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ and $g \in \mathcal{C}({}_aY_b \Rightarrow {}_aZ_b)$, $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.
- For all $f \in \mathcal{C}({}_aW_b \Rightarrow {}_aX_b)$ and $g \in \mathcal{C}({}_bY_c \Rightarrow {}_bZ_c)$, $(f \otimes_b g)^\dagger = f^\dagger \otimes_b g^\dagger$.
- All unitors and associators in $\mathcal{C}$ are *unitary* ($u^\dagger = u^{-1}$).

We call the pair $(\mathcal{C}, \dagger)$ a *$\dagger$-2-category*. We call a $\dagger$-2-category $(\mathcal{C}, \dagger)$ a *pre-unitary 2-category* if $\mathcal{C}$ is rigid and every hom 1-category is a unitary category.

Exercise 6.9.2. Show that a pre-unitary 2-category is pre-semisimple.

Exercise 6.9.3. Formulate the notion of a unitary dual 2-functor on a pre-unitary 2-category.

Exercise 6.9.4 ($\star\star$)**.** Prove that every pre-unitary 2-category is equivalent to a *strict* pre-unitary 2-category.

Definition 6.9.5. An algebra $({}_aA_a, m)$ in a pre-unitary 2-category is called *unitarily separable* if $m^\dagger : A \Rightarrow A \otimes A$ splits m as an $A - A$ bimodule map.

When $({}_aA_a, m)$ is a unitarily separable algebra in a pre-unitary 2-category $\mathcal{C}$, we call a left A-module $({}_aM_b, \lambda)$ *unitarily separable* if $\lambda^\dagger$ splits λ as an A-module map, and $\delta := \lambda^\dagger$ is compatible with $\Delta := m^\dagger$.

Exercise 6.9.6 ([BKLR15, Lem. 3.7], $\star$)**.** Suppose (A, m, i) is a unital algebra in a unitary multitensor category such that m unitarily splits, i.e., $mm^\dagger = \mathrm{id}_A$. Prove that A is unitarily separable, i.e., $m^\dagger$ is an $A - A$ bimodule map.

Question 6.9.7. *Given a separable unital algebra (A, m, i) in a unitary multitensor category, is A isomorphic (via an algebra isomorphism) to a unitarily separable algebra? At the time of writing, this question remains open even for connected separable algebras in unitary fusion categories.*

Exercise 6.9.8. Suppose $({}_aA_a, m)$ is a unitarily separable algebra in a pre-unitary 2-category $\mathcal{C}$, and $({}_aM_b, \lambda)$ is a left A-module such that $\lambda\lambda^\dagger = \mathrm{id}_M$. Prove that ${}_aM_b$ is unitarily separable. *Hint: First do Exercise 6.9.6.*

Exercise 6.9.9. Suppose $({}_aA_a, m)$ is a unitarily separable algebra in a pre-unitary 2-category $\mathcal{C}$, $({}_aM_b, \lambda_M), ({}_aN_b, \lambda_N)$ are unitarily separable left A-modules, and $f \in \mathcal{C}({}_aM_b \Rightarrow {}_aN_b)$ is a left A-module map. Prove that $f^\dagger \in \mathcal{C}({}_aN_b \Rightarrow {}_aM_b)$ is a left A-module map.

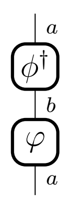
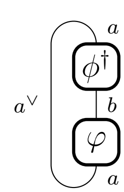
Warning 6.9.10. In contrast to Remark 6.6.6, there is a stark difference between the concepts of being enriched vs. being tensored over $\mathsf{Hilb}_{\mathsf{fd}}$ for a unitary category $\mathcal{C}$. Indeed, in order to be enriched over $\mathsf{Hilb}_{\mathsf{fd}}$, one must endow hom spaces of $\mathcal{C}$ with Hilbert space structures that respect the dagger structure of $\mathcal{C}$. One can describe these choices in the following equivalent ways:

- a *2-Hilbert space structure* on $\mathcal{C}$ [Bae97],
- a *unitary trace* on $\mathcal{C}$ [GMP+18, [[where?]]], or
- a positive real number for every simple object in $\mathcal{C}$.

Observe that there is a canonical choice where each simple object in $\mathcal{C}$ is assigned the positive real number 1, which corresponds to the *isometry inner product*. This extra structure is exactly what is needed to construct the unitary Yoneda embedding $よ : \mathcal{C} \to \mathsf{Fun}^\dagger(\mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb})$ [GMP+18, [[where?]]]; see Exercise 6.9.11 below.

When $\mathcal{C}$ is unitary multitensor, there is a second organic choice corresponding to

While $\mathcal{C}$ has a canonical unitary spherical structure, there are (at least) two organic choices for this 2-Hilbert space structure, corresponding to the *isometry* and the *tracial* inner products. Recall that for $a \in \mathrm{Irr}(\mathcal{C})$ and $b \in \mathcal{C}$, these inner products are given on $\mathcal{C}(a \to b)$ by

$\langle \phi | \varphi \rangle_{\mathrm{Isom}} \cdot \mathrm{id}_a = \phi^\dagger \circ \varphi =$  $\qquad \langle \phi | \varphi \rangle_{\mathrm{tr}} \cdot \mathrm{id}_{1_{\mathcal{C}}} =$  .

However, one can define a natural dagger functor $- \otimes c : \mathsf{Hilb}_{\mathsf{fd}} \to \mathcal{C}$ for any $c \in \mathcal{C}$ without appealing to enriched category theory. Indeed, $\mathsf{Hilb}_{\mathsf{fd}}$ is equivalent to the category whose objects are $\mathbb{C}^n$ and whose morphisms $\mathbb{C}^n \to \mathbb{C}^m$ are $M_{m \times n}(\mathbb{C})$. We define $\mathbb{C}^n \otimes c := \bigoplus_{i=1}^n c$

and for $T : \mathbb{C}^n \to \mathbb{C}^m$, we have $T \otimes \mathrm{id}_c : \bigoplus_{i=1}^n c \to \bigoplus_{i=1}^m c$ is the map whose i, j-th entry is $T_{ij} \cdot \mathrm{id}_c$.

Exercise 6.9.11 ($\star\star$)**.** Suppose $\mathcal{C}$ is a unitary category equipped with a 2-Hilbert space structure.

(1) Construct the unitary Yoneda embedding $よ : \mathcal{C} \to \mathsf{Fun}^\dagger(\mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb})$.
(2) Formulate and prove the unitary Yoneda Lemma.
Note: The object corresponding to a representable dagger functor must be specified up ti unique unitary *isomorphism. For a hint, see* [JP20, [[where?]]].
(3) Suppose $\mathcal{C}$ is a unitary multitensor category and that the 2-Hilbert space structure comes from a unitary pivotal structure. Show how to endow $よ$ with the structure of a $\dagger$ tensor functor.
Hint: Tensor product in $\mathsf{Fun}^\dagger(\mathcal{C}^{\mathrm{op}} \to \mathsf{Hilb})$ *uses a unitary version of the Day convolution product. See* [JP17, [[where?]]] *for a hint.*

Exercise 6.9.12 ($\star\star$)**.** Formulate the notion of left $\mathcal{C}$-module C* category for a unitary multitensor category $\mathcal{C}$. Then formulate and prove the unitary Barr-Beck/Ostrik Theorem for unitary multifusion categories.
Note: The reason for the two star rating is that one should use the unitary Yoneda embedding in order to define hom objects of the $\mathcal{C}$-enriched category $\widehat{\mathcal{M}}$ up to unique unitary isomorphism. Then one must prove that $\widehat{\mathcal{M}}(m \to m)$ is unitarily separable for a fixed pointing $m \in \mathcal{M}$.

Definition 6.9.13. Suppose $\mathcal{C}$ is a pre-unitary 2-category. The *unitary Cauchy completion* $\mathsf{Cauchy}^\dagger(\mathcal{C})$ is the pre-unitary 2-category whose

- objects are unital unitarily separable algebras,
- 1-morphisms are unitarily separable bimodules, and
- 2-morphisms are intertwiners.

By Exercise 6.9.9, the intertwiner endomorphisms in $\mathcal{C}({}_aX_b \Rightarrow {}_aX_b)$ form a unital $\dagger$-closed subalgebra, and is thus a unitary algebra. Hence $\mathsf{Cauchy}^\dagger(\mathcal{C})$ is again a pre-unitary 2-category.

Again, there is a canonical $\dagger$-inclusion $\mathcal{C} \hookrightarrow \mathsf{Cauchy}^\dagger(\mathcal{C})$. We call $\mathcal{C}$ *unitarily Cauchy complete* if this inclusion is a $\dagger$ equivalence. As in Exercise 6.7.8 and Theorem 6.7.10, this happens if and only if every unital unitarily separable algebra in $\mathcal{C}$ unitarily splits.

We will not provide any more detail here, other than to refer the reader to the upcoming article [CPJP].

Definition 6.9.14. A *unitary* 2-category is a pre-unitary 2-category that is unitarily Cauchy complete. A unitary 2-category is called *finite* if it has only finite many unitary equivalence classes of simple objects, and every hom unitary category is finitely semisimple.

Question 6.9.15. *Is every finite unitary 2-category finitely semisimple? Observe that the answer depends on the open Question 6.9.7.*

Exercise 6.9.16. Repeat Exercise 6.8.8 above for finite unitary 2-categories and unitary (multi)fusion categories.

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