1 Dmitri Nikshych

Work in a field k of characteristic D, algebraically closed (e.g. $k = \mathbb{C}$).

Categories are k-linear, semi-simple. $\mathrm{Hom}(X, Y)$ is a k vector space, and composition of morphisms is a bilinear map:

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z).$$

All objects are direct sums of simple objects. An object is simple if it has no sub-objects.

Definition 1.1. A fusion category $\mathcal{C}$ is a category satisfying the above conditions, along with a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a simple unit object 1 with associativity and unit constraints.
Associator isomorphisms: $a_{X,Y,Z} \colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$
Units: $r_X \colon X \otimes 1 \to X$ and $\ell_X \colon 1 \otimes X \to X$.
These satisfy the pentagon and triangle axioms.

Every object has left and right duals. Duals are defined axiomatically rather than constructively. A left dual of an object V is an object V^* along with two maps:
evaluation $\mathrm{ev}_V \colon V^* \otimes V \to 1$
coevaluation $\mathrm{coev}_V \colon 1 \to V \otimes V^*$
These maps should satisfy the zig-zag relation:

$$V \xrightarrow{r_{V^{-1}}} 1 \otimes V \xrightarrow{\mathrm{coev}} (V \otimes V^*) \otimes V \longrightarrow V \otimes (V^* \otimes V) \longrightarrow V$$

should be the identity map. In fact, if V^* exists, it is unique up to isomorphism.

Finally, there should be finitely many isomorphism classes of simple objects.

Example 1.2. Let $\mathcal{C} = \mathsf{Vec}$, the category of finite dimensional vector spaces. Then $(V \otimes U) \otimes W$ is canonically isomorphic to $V \otimes (U \otimes W)$, but these objects are not equal.

MacLane's strictness:

We have an isomorphism for each object $V \cong k^n$, and linear transformations are matrices. Then $k^n \otimes k^m = k^{mn}$.

For duals: $\mathrm{ev}_V \colon V^* \otimes V \to k$ is the map $\langle \phi, v \rangle = \phi(v)$, and $\mathrm{coev}_V \colon k \to V \otimes V^*$ is $1 \mapsto \sum_i v \otimes v_i^*$. Then $v = \sum_i v_i^*(v) v_i$.

Definition 1.3. Let $\mathcal{C}_1, \mathcal{C}_2$ be fusion categories. A tensor functor is a functor $F \colon \mathcal{C}_1 \to \mathcal{C}_2$ which respects $\otimes$:

$$\mu_{X,Y} \colon F(X \otimes Y) \cong F(X) \otimes F(Y).$$

You should also have an isomorphism $u \colon F(1_{\mathcal{C}_1}) \to 1_{\mathcal{C}_2}$. These isomorphisms should be consistent. They should satisfy compatibility requirements. They should play well with the associators from $\mathcal{C}_1$ and $\mathcal{C}_2$.

Example 1.4. Let G be a finite group. Then $\mathsf{Rep}(G)$, the category of finite dimensional representations of G over k is a fusion category. Associativity is the usual one for vector spaces $g(v \otimes w) = g(v) \otimes g(w)$. $1 = k$ is the trivial representation.

Example 1.5. Let H be a semi-simple finite dimensional Hopf algebra. Then $\mathsf{Rep}(H)$ is also a fusion category. The comultiplication $\Delta\colon H \to H \otimes H$ makes $V \otimes W$ a representation.

Example 1.6. Let G be a finite group. Then G-graded factor spaces Vec_G forms a fusion category. $V = \bigoplus_{g \in G} V_g$.

$$(V \otimes W)_g = \bigoplus_{xy=g} V_x \otimes W_y.$$

The simple 1-dimensional objects are the δ_g. The associativity constraint is the obvious one.

Instead of the obvious associativity constraint, we can get another fusion category Vec_G^ω by using a 3-cocycle $\omega \in Z^3(G, k^\times)$.

$$\omega(g,h,k)\colon (\delta_g \otimes \delta_h) \otimes \delta_k \longrightarrow \delta_g \otimes (\delta_h \otimes \delta_k).$$

The pentagon axiom means that $\omega\colon G \times G \times G \to k$ satisfies the cocycle condition:

$$\omega(xy,z,w)\omega(x,y,zw) = \omega(x,yz,w)\omega(x,y,z)\omega(y,z,w).$$

Definition 1.7. Vec_G^ω is the most difficult example of a <u>pointed</u> fusion category, i.e., each simple object is invertible.

Definition 1.8. Let $X_1, \dots, X_n$ be the simple objects of $\mathcal{C}$. We define structure constants $N_{i,j}^k$ by

$$X_i \otimes X_j \cong \bigoplus N_{i,j}^k X_k$$

and an involution $i \mapsto i^*$ of $\{1, \dots, n\}$ by taking duals. One gets the <u>Grothendieck ring</u> $K_0(\mathcal{C})$ by taking the ring generated as above.

There is a unique homomorphism $\mathrm{FPdim}\colon K_0(\mathcal{C}) \to \mathbb{R}$ such that $\mathrm{FPdim}(X) > 0$ for all objects of $\mathcal{C}$, $X \neq 0$. Here every $X \in \mathcal{C}$ determines $N^X\colon K_0(\mathcal{C}) \to K_0(\mathcal{C})$ by $V \mapsto X \otimes V$. Since $K_0(\mathcal{C})$ is free of rank n, N^X gives an $n \times n$ matrix with non-negative entries. If it's not nilpotent, then it has a strictly positive eigenvalue. It is not nilpotent because of rigidity ($X^* \otimes X \neq 0$). Hence $\mathrm{FPdim}(X)$ is an algebraic integer.

Define $\mathrm{FPdim}(\mathcal{C}) = \sum_{i=1}^n \mathrm{FPdim}(X_i)^2$ (analog of size of group being the sum of squares of sizes of irreducible representations).

Example 1.9. $\mathrm{FPdim}(\mathsf{Rep}(G)) = |G| = \mathrm{FPdim}(\mathsf{Vec}_G^\omega)$.

Proposition 1.10 (Ocneanu Rigidity)**.** *Fusion categories are not deformable, i.e., there are not continuous families of fusion categories.*

- *There are only finitely many fusion categories with a given ring for* $K_0(\mathcal{C})$.

- *There are only finitely many fusion categories $\mathcal{C}$ with $\mathrm{FPdim}(\mathcal{C}) < N$, for some $N > 0$.*
- *There are only finitely many functors between two fusion categories.*
- *There are only finitely many module categories of a given fusion categories.*

Definition 1.11. A $\mathcal{C}$-module category $\mathcal{M}$ is a category along with a bifunctor $\otimes \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ which satisfies an associativity constraint

$$\mu_{X,Y,M} \colon (X \otimes Y) \otimes M \longrightarrow X \otimes (Y \otimes M)$$

(these are isomorphisms that satisfy an axiom that involves 3 objects of $\mathcal{C}$).

Examples 1.12.

(1) $\mathcal{C}$ is a regular module over itself.

(2) If $L < G$ is a subgroup, then $\mathcal{M} = \mathsf{Rep}(L)$ is a module category over $\mathsf{Rep}(G)$ by restriction.

(3) Let φ be a 2-cocycle on L. Then projective representations $\mathsf{Rep}_\varphi(L)$ gives another module category over $\mathsf{Rep}(G)$. In this case, the module category is indecomposable.

Question 1.13. *Given a fusion category, describe all module categories over it.*

Example 1.14. Given a subfactor $N \subset M$, take for $\mathcal{C}$ the $N - N$ bimodules, and for $\mathcal{M}$ the $N - M$ bimodules.

Definition 1.15. Let A be an algebra in $\mathcal{C}$. Let $\mathcal{M}$ be the category of right A-modules in $\mathcal{C}$, i.e., there is a map

$$\rho \colon A \times V \to V$$

which satisfies some axioms. Let $X \in \mathcal{C}$. Then

$$A \otimes (V \otimes X) \cong (A \otimes V) \otimes X \xrightarrow{\rho} V \otimes X.$$

Then A-modules form a $\mathcal{C}$-module category.

Theorem 1.16 (Ostrik)**.** *Every module category appears in this way.*

Definition 1.17. A-bimodules in $\mathcal{C}$ can be tensored with each other. C_A^*, the set of such bimodules, is a fusion category which is Morita equivalent to $\mathcal{C}$.

Proposition 1.18 (Mueger)**.** *Morita equivalence is an equivalence relation.*

Example 1.19. $\mathsf{Rep}(G) \sim \mathsf{Vec}_G$ is a Morita equivalence.

Morita equivalence is a good approach to classification.

2 Noah Snyder

Translating between subfactors and tensor categories

Question 2.1. *Why do $\otimes$-categories appear in studying von Neumann algebras?*

We can look at some collection of (bifinite) $N - N$ bimodules given a von Neumann algebra N, and these form a tensor category. So when someone says fusion category, you can think of a finite collection of $N - N$ bimodules, closed under $\otimes$.

Example 2.2. Vec_G can be realized as bimodules over R, the hyperfinite II_1-factor.

There are a couple things lost in translation:

- (Unitarity) The von Neumann algebra side has a $*$-structure. Hom-spaces are Hilbert spaces instead of vector spaces. There are more examples on the fusion category side.
- By passing to this category, you are losing information about N. Which von Neumann algebra is N?

Remark 2.3. Semi-simplicity is automatic (from analysis). However, in general, there can be infinitely many isomorphism classes of simple objects. The category is a fusion category if and only if the subfactor is finite depth.

What about starting with a subfactor? Let $N \subseteq M$ be a finite index subfactor. Then M is an $N - N$ bimodule, and also and algebra.

Example 2.4. $\mathbb{C}[G] = \bigoplus_{g\in G} \delta_g \in \mathsf{Vec}_G$. Need to check that multiplication is a map in this category. This algebra object gives the crossed-product subfactor $N \subseteq N \rtimes G$.

Definition 2.5. The standard invariant of a subfactor $N \subseteq M$ is

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathsf{Mod}-A"] & A - \mathsf{Mod} - A \arrow[l, shift left, "A-\mathsf{Mod}"] \end{tikzcd}$$

The extra $\otimes$ exists as Ostrik's internal hom, denoted Hom. In the 2-category language, we get a unitary 2-cateroy, where the extra $\otimes$ is just composition.

In the subfactor world:

$$\begin{tikzcd} N - \mathsf{Mod} - N \arrow[r, shift left, "N-\mathsf{Mod}-M"] & M - \mathsf{Mod} - M \arrow[l, shift left, "M-\mathsf{Mod}-N"] \end{tikzcd}$$

$N - \mathsf{Mod} - N$ is the even part.
$M - \mathsf{Mod} - M$ is the dual even part.
The bimodule categories $N - \mathsf{Mod} - M$ and $M - \mathsf{Mod} - N$ are the odd parts.

Instead of an algebra $A \in \mathcal{C}$, can think of a tensor category $\mathcal{C}$, a module category $\mathcal{M} = \mathsf{Mod} - A$, and an object $X \in \mathcal{M}$ (where $A = X \otimes X^* = \underline{\mathrm{Hom}}(X, X)$. In fact, $X = A$ thought of in $\mathsf{Mod} - A$). Then we can look at the alternating $\otimes$-powers.

Or, we can look at two tensor categories $\mathcal{C}, \mathcal{D}$ and a module category between them $\mathcal{M}$.

Definition 2.6. Given $X \in \mathcal{C}$, we can look at the fusion graph for $- \otimes X$. It has vertices the simple objects in $\mathcal{C}$, and $N_{Y,X}^Z$ edges from Y to Z

In the subfactor case, we get the 4-partite Ocneanu graph.

Here is a table:

Subfactors	Tensor categories
Q-system	s.s. $A \in \mathcal{C}$, tensor category $N - \mathsf{Mod} - N$
$N \subseteq N \rtimes G$	$\mathbb{C}[G] = \bigoplus_{g \in G} \delta_G$
finite depth	fusion category
N a factor	1 is simple
irreducible	$\dim(\mathrm{Hom}(1, A)) = 1$
index	$\mathrm{FPdim}(A)$ (note $A \simeq X \otimes X^*$)
global index	global dimension $\mathrm{FPdim}(\mathcal{C}) = \sum_V (\mathrm{FPdim}(V))^2$
depth	how many X's in $X \otimes X^* \otimes X \otimes \cdots$ until everything appears
connection	$6j$-symbols

The standard invariant also appears as a planar algebra. We look at the diagrammatic calculus. When we tune planar algebras to the subfactor setting, we pick our particular object X, and we only allow one strand type. The odd and even part give a checkerboard shading. This allows for looking at special structure, like the rotation.

The standard invariant also appears as a connection on the 4-partite Ocnenu graph. This amounts to looking at the $6j$-symbols. Since 2 of the objects are X, X^*, the number only depends on 4 numbers, and it gives a number for each loop on the 4-partite graph. "This is a discrete version of a connection on a manifold." There are unitary and tetrahedral conditions, which is called biunitarity, and then there's a condition called flatness, which is hard.

3 Monday afternoon

Question 3.1 (Fusion categories of small global dimension)**.** *Find all fusion categories $\mathcal{C}$ with $\mathrm{FPdim}(\mathcal{C}) < N$ (up to equivalence).*

If $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{N}$, then this is reasonably well-understood for small number of divisors of N.

- p^n

- $p^a q^b$
- pqr

This should get us up to 60 without a problem.

For example, suppose $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}[\sqrt{2}]$. Then it is of the form $(1+\sqrt{2})^k p_1^{a_1} \cdots p_n^{a_n}$

Question 3.2. *What if* $\dim(\mathcal{C}) = (1+\sqrt{2})^k$*?*

Proposition 3.3 (Yamagami)**.** *A unitary fusion category is spherical*

So one could first classify the modular tensor categories which could be the quantum double $Z(\mathcal{C})$. But note that $\mathrm{FPdim}(Z(\mathcal{C})) = \mathrm{FPdim}(\mathcal{C})^2$.

Proposition 3.4. *If $\mathcal{C} \subset \mathcal{D}$ is a tensor subcategory, and $\mathcal{C}, \mathcal{D}$ are fusion categories, then* $\mathrm{FPdim}(\mathcal{D}) / \mathrm{FPdim}(\mathcal{C})$ *is an algebraic integer.*

Question 3.5 (Open questions in classifying small fusion categories)**.**

(1) What are all the group extensions of even parts of ADE subfactors?

(2) Can you classify all fusion categories $\mathcal{C}$ where $\mathrm{FPdim}(\mathcal{C})$ *is a unit?*

(3) How about modular categories up to dimension N? After which, how do you get fusion categories up to $\sqrt{N}$?

(4) For which (n, N) do we already know fusion categories with rank $\leq N$ and $\mathrm{FPdim}(\mathcal{C}) \leq N$*? (What about with no $\otimes$-subcategories?) (What about generated by a self-dual element?)*

(5) Can we really classify all weakly integral categories up to $N = 60$?

(6) What can we say about near-group categories ($G \cup \{\rho\}$ where $\rho^2 = n\rho \oplus \bigoplus_{g \in G} g$)? (How far do Masaki's results get us?)

4 Michael Müger

Definition 4.1 ('63, MacLane)**.** If $\mathcal{C}$ is a tensor category, then a symmetry is an isomorphism

$$c_{X,Y} \colon X \otimes Y \to Y \otimes X$$

which is natural with respect to X, Y, and $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all X, Y. There is also a hexagon axiom with the associators that must be satisfied. It allows us to express $c_{X \otimes Y, Z}$ in terms of $c_{Y,Z}$ and $c_{X,Z}$ and the associators.

Definition 4.2 ('86-'93, Joyal-Street)**.** A braiding is the same as above, but where we drop the condition $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$, and there is another hexagon identity that relates to the maps $c_{X, Y \otimes Z}$.

If you have a symmetric tensor category, and an object X and an $n \in \mathbb{N}$, get a map

$$S_n \to \operatorname{End}(X^{\otimes n})$$

If you have a braided tensor category, and an object X and an $n \in \mathbb{N}$, get a map

$$B_n \to \operatorname{End}(X^{\otimes n})$$

Definition 4.3. The braid category is the free braided tensor category on 1 object. Hence the objects is $\mathbb{Z}_{\geq 0}$, $\otimes = +$, and $\operatorname{Hom}(n, m) = B_n$ if $m = n$ and empty otherwise.

You can make this category rigid by allowing bent curves, and get the category of tangles.

Note that given a braided tensor category, can define $\widetilde{c_{X,Y}} = c_{Y,X}^{-1}$. Of course $\widetilde{\widetilde{c}} = c$, and $\widetilde{c} = c$ if and only if $\mathcal{C}$ is symmetric.

Definition 4.4. X, Y commute if $c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y}$.

Definition 4.5. If $\mathcal{C}$ is braided and $\mathcal{D} \subset \mathcal{C}$, then the centralizer of $\mathcal{D}$ is

$$C_{\mathcal{C}}(\mathcal{D}) = \text{ full subcategory with objects } \left\{ X \in \mathcal{C} \middle| c_{Y,X} \circ c_{X,Y} = \mathrm{id}_{X \otimes Y} \text{ for all } Y \in \mathcal{D} \right\}.$$

The symmetric center of $\mathcal{C}$ is

$$Z_2(\mathcal{C}) = C_{\mathcal{C}}(\mathcal{C}).$$

We call $X \in \mathcal{C}$ transparent if $X \in Z_2(\mathcal{C})$. "Can pull line associated to X through any other." It is obvious that $Z_2(\mathcal{C})$ is symmetric, and $\mathcal{C}$ is symmetric if and only if $\mathcal{C} = Z_2(\mathcal{C})$.

Question 4.6. *What are the maximally non-commutative braided tensor categories?*

There are other center constructions:

$$cat \xrightarrow{Z_0} \otimes - cat \xrightarrow{Z_1} braided - \otimes - cat \xrightarrow{Z_2} sym - \otimes - cat$$

Definition 4.7. If $\mathcal{C}$ is a category, then $Z_0(\mathcal{C}) = \operatorname{End}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C} \to \mathcal{C})$. An action of a tensor category $\mathcal{C}$ on $\mathcal{M}$ is just a tensor functor $\mathcal{C} \to Z_0(\mathcal{M})$.

Definition 4.8. The braided center $Z_1(\mathcal{C})$ of a strict tensor category is defined as follows: (also called the Drinfel'd center) Given $X \in \mathcal{C}$, a half braiding for X is a natural family

$$e_X \colon X \otimes Y \to Y \otimes X$$

such that

$$e_X(Y \otimes Z) = (\mathrm{id}_Y \otimes e_X(Z)) \circ (e_X(Y) \otimes \mathrm{id}_Z)$$

(need associators if $\mathcal{C}$ is not strict). Also,

$$e_{X \otimes Y}(Z) = (e_X(Z) \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes e_Y(Z))$$

The braided center $Z_1(\mathcal{C})$ is the category whose objects are pairs (X, e_X), and

$$\begin{aligned}\mathrm{Hom}((X, e_X), (Y, e_Y)) &= \left\{ s\colon X \to Y \middle| s \text{ is compatible with the half braidings} \right\} \\ &= \left\{ s\colon X \to Y \middle| (\mathrm{id}_X \otimes s) \circ e_X(Z) = e_Y(Z) \circ (s \otimes \mathrm{id}_Z) \right\}.\end{aligned}$$

The trivial object is $(1, \mathrm{id})$, and $c_{(X,e_X),(Y,e_Y)} = e_X(Y)$.

Example 4.9. If H is a finite dimensional Hopf algebra, then $Z_1(H - \mathsf{Mod}) \cong D(H) - \mathsf{Mod}$, where $D(H)$ is the Drinfel'd double of H.

This process can be done for sets, monoids,

We should hope that the center of a center is "trivial." More on this later.

Definition 4.10. Let $\mathcal{C}$ be a braided fusion category over k. $\mathcal{C}$ is called pre-modular if $\mathcal{C}$ is spherical (ribbon structure). $\mathcal{C}$ is called non-degenerate if $Z_2(\mathcal{C})$ is trivial, i.e., equivalent to Vec_k. $\mathcal{C}$ is modular if $\mathcal{C}$ is pre-modular and non-degenerate.

Remark 4.11. If $\mathcal{C}$ is spherical, we have traces, and the left trace is the right trace. So for every pair of objects, we can define the S-matrix

$$S_{X,Y} = \mathrm{tr}_{X \otimes Y}(c_{Y,X} \circ c_{X,Y}) \in \mathrm{End}(1) \cong k.$$

Turaev defined modular as semi-simple, braided, fusion category, and $\{S_{X,Y}\}_{X,Y \in \mathrm{Irr}(\mathcal{C})}$ is invertible.

Fact 4.12 (Rehren '90, Müger '04, more)**.** *$\mathcal{C}$ is semi-simple braided fusion category. Then S is invertible if and only if $Z_2(\mathcal{C})$ is trivial.*

Examples 4.13.

(1) $D(G) = D(\mathbb{C}[G])$ is modular.

(2) If H is a finite dimensional semi-simple, co-semi-simple Hopf algebra, then $D(H)-\mathsf{Mod}$ is modular.

(3) Quantum groups at roots of unity.

Proposition 4.14. *$\mathcal{C}$ a spherical fusion category over k, algebraically closed, and $\mathrm{FPdim}(\mathcal{C}) \neq 0$, then $Z_1(\mathcal{C})$ is modular, and $\mathrm{FPdim}(Z_1(\mathcal{C})) = (\mathrm{FPdim}(\mathcal{C}))^2$.*

Some important parts of the proof. If $\mathcal{C}$ is a fusion category, then $Z_1(\mathcal{C})$ is non-degenerate. It is easy to see if $\mathcal{C}$ is spherical, then $Z_1(\mathcal{C})$ is spherical. $Z_1(\mathcal{C})$ is semi-simple. We know $\mathrm{End}_{Z_1(\mathcal{C})}((X, e_X)) \subset \mathrm{End}_{\mathcal{C}}(X)$, and there is a conditional expectation which gives the semi-simplicity. $Z_1(\mathcal{C})$ is Morita-equivalent to $\mathcal{C} \boxtimes \mathcal{C}^{\mathsf{op}}$. We find the Longo-Rehren Q-system, $Q = \bigoplus X_i \boxtimes X_i^{\mathsf{op}}$. □

Fact 4.15. *It is not the case that every modular tensor category is the center of a fusion category. Use the Gauss sums:*

$$\Omega^{\pm}(\mathcal{C}) = \sum_{i \in I} d(X_i)^2 \theta_i^{\pm 1}.$$

In general, $\Omega^+(\mathcal{C})\Omega^-(\mathcal{C}) = \dim(\mathcal{C})$. It is always the case that $\Omega^{\pm}(Z_1(\mathcal{C})) = \dim(\mathcal{C})$, but it is not always the case that $\Omega^+(\mathcal{C}) = \Omega^-(\mathcal{C})$.

Algebras and module categories in BTCs

If $\mathcal{C}$ is symmetric and $A \in \mathcal{C}$ is a commutative algebra, then ${}_A\mathcal{C}$ is symmetric, and $F_A \colon X \mapsto A \otimes X$ is symmetric.

If $\mathcal{C}$ is braided and $A \in \mathcal{C}$ is commutative, then ${}_A\mathcal{C}$ is a tensor category, F_A is monodial. There is a braiding on ${}_A\mathcal{C}$ making F_A braided if and only if $A \in Z_2(\mathcal{C})$.

F_A always factors through $Z_1({}_A\mathcal{C})$. $F_A \colon \mathcal{C} \to Z_1({}_A\mathcal{C}) \to_A \mathcal{C}$.

(A, μ) an algebra. An A-module X is dyslexic if $\mu \circ c_{X,A} \circ c_{A,X} = \mu$. We set ${}_A\mathcal{C}^o$ to be the full subcategory of ${}_A\mathcal{C}$ of dyslexic A-modules. This category is braided.

If $\mathcal{C}$ is braided and $A \in \mathcal{C}$ is a commutative algebra, then $F_A \colon \mathcal{C} \to Z_1({}_A\mathcal{C})$ is faithful. When $\mathcal{C}$ is modular or non-degenerate, F_A is full, so $\mathcal{C} \subset Z_1({}_A\mathcal{C})$.

If $\mathcal{C}$ is modular, and $\mathcal{D} \subset \mathcal{C}$ is a full modular subcategory, then $\mathcal{C} \cong \mathcal{D} \boxtimes C_{\mathcal{C}}(\mathcal{D})$, so $Z_1({}_A\mathcal{C}) \cong \mathcal{C} \boxtimes \widetilde{{}_A\mathcal{C}^o}$.

Theorem 4.16. *If $\mathcal{C}$ admits a commutative algebra such that ${}_A\mathcal{C}^o$ is trivial, then $\mathcal{C} \cong Z_1({}_A\mathcal{C})$.*

Conversely, $Z_1(\mathcal{C})$ contains a commutative, separable algebra A such that ${}_AZ_1(\mathcal{C})^o$ is trivial.

Fact 4.17. *$Z_1(\mathcal{C}_1) \cong Z_1(\mathcal{C}_2)$ if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ are Morita equivalent.*

Definition 4.18. $\mathcal{C}_1, \mathcal{C}_2$ non-degenerate braided fusion categories. Then $\mathcal{C}_1, \mathcal{C}_2$ are Witt-equivalent if there are fusion categories $\mathcal{D}_1, \mathcal{D}_2$ such that

$$\mathcal{C}_1 \boxtimes Z_1(\mathcal{D}_1) \cong \mathcal{C}_2 \boxtimes Z_1(\mathcal{D}_2)$$

as braided tensor categories.

Witt-equivalence is an equivalence relation.

We can define $W_{non-deg}$ to be the set (actually countable!) of braided non-degenerate fusion categories modulo Witt-equivalence. In fact it is a group, with multiplication $[\mathcal{C}_1] \cdot [\mathcal{C}_2] = [\mathcal{C}_1 \boxtimes \mathcal{C}_2]$, and $[\mathcal{C}]^{-1} = [\widetilde{\mathcal{C}}]$, where the braiding is flipped, and $1_W = [\mathsf{Vec}]$.

$$[\mathcal{C}] \cdot [\widetilde{\mathcal{C}}] = [\mathcal{C} \boxtimes \widetilde{\mathcal{C}}] = [Z_1(\mathcal{C})] = [\mathsf{Vec}] = 1_W.$$

Conjecture 4.19. *$W_{non-deg}$ is generated by the Witt-classes of the quantum groups at roots of unity.*

If $A \in \mathcal{C}$ is a commutative, separable algebra, then $[\mathcal{C}] = [{}_A\mathcal{C}^o]$.

If $\mathcal{A}$ is a conformal quantum field theory, then finite extensions of $\mathcal{A} \subset \mathcal{B}$ are classified by commutative separable algebras $A \in \mathsf{Rep}(\mathcal{A})$ such that $\mathsf{Rep}(\mathcal{B}) \cong_A (\mathsf{Rep}(\mathcal{A}))^o$.

5 Richard Ng

Definition 5.1. Given a fusion category over $\mathbb{C}$, can define notions of pivotal structure $j \colon \mathrm{id}_{\mathcal{C}} \to (-)^{**}$, and pivotal trace.

$$\mathrm{ptr}_\ell(f) \colon 1 \longrightarrow V \otimes V^* \xrightarrow{jf \otimes \mathrm{id}} V^{**} \otimes V^* \longrightarrow 1$$

$$\mathrm{ptr}_r(f) \colon 1 \longrightarrow V \otimes V^* \xrightarrow{\mathrm{id} \otimes fj^{-1}} V^{**} \otimes V^* \longrightarrow 1$$

$\mathcal{C}$ is spherical if $\mathrm{ptr}_r = \mathrm{ptr}_\ell$. If $\mathcal{C}$ is spherical, then $\mathrm{ptr}(\mathrm{id}_X) = d(X)$ for all X.

$\mathcal{C}$ is a strict pivotal category if $(V \otimes W)^* = W^* \otimes V^*$, $(-)^{**} = \mathrm{id}_\mathcal{C}$, and $j\colon \mathrm{id}_\mathcal{C} \to (-)^{**}$ is the identity.

Theorem 5.2. *Every pivotal category is equivalent to a strict pivotal category.*

We will assume $\mathcal{C}$ is a braided, spherical fusion category over $\mathcal{C}$. Let the braiding be given by

$$c_{V,W}\colon V \otimes W \to W \otimes V.$$

Now look at the trace of $c_{W,V} \circ c_{V,W}$, looks like a Hopf link, labeled by V, W.

Definition 5.3. The S-matrix is given by

$$S_{i,j} = \mathrm{tr}(c_{j,i^*} c_{i^*,j})$$

which is a Hopf link, where the left edge of each loop is labelled by i, j.

$\mathcal{C}$ is modular if S is invertible.

Remark 5.4. Note that we get a ribbon structure if $\mathcal{C}$ is modular. This gives a diagonal matrix called the T matrix, where the i,i-th entry is the θ_i which is the scalar corresponding to a twist in a swing from i to i (handedness?). Note that θ_i is not 0, and in fact T has finite order, so θ_i is a root of unity.

Fact 5.5. *The modular group $SL(2,\mathbb{Z})$ is generated by two elements:*

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A presentation is given by $\langle s, t | s^2 = I \text{ and } (st)^3 = s^2 \rangle$.

We get a projective representation $\rho_\mathcal{C}\colon SL(2,\mathbb{Z}) \to PGL(n+1,\mathbb{C})$ by $s \mapsto S$ and $t \mapsto T$.

We denote the Frobenius-Schur exponent as $\mathrm{ord}(T) = N$. Then

(1) $\rho_\mathcal{C}$ factors through $SL(2,\mathbb{Z}/N)$, and N is the level of the congruence subgroup.

(2) $S_{i,j} \in \mathcal{O}(\mathbb{Q}_N)$ (cyclotomic integers)

In fact, $\rho_\mathcal{C}$ can be lifted to an ordinary representation. There is a map $\rho\colon SL(2,\mathbb{Z}) \to GL(n+1,\mathbb{C})$.

Theorem 5.6. *Consider t, s in the image of ρ.*

(1) $\ker(\rho)$ is a congruence subgroup of level $m = \mathrm{ord}(t)$. The map ρ factors through $SL(2,\mathbb{Z}/m)$.

(2) Have renormalized s. $s \in \mathbb{Q}_m$, the cyclotomic field of degree m.

(3) Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$. Then $\sigma(s) = G_\sigma s$ where G_σ is a signed permutation matrix $\hat{\sigma} \in {}^{\mathcal{C}}sym_{n+1}$.

In fact, $\sigma = \sigma_a$, raising the root of unity to the a-th power, where $(a, n) = 1$.

(4) $G_{\sigma_a} = t^a st^b st^a s^{-1}$ *where* $ab \equiv 1 \mod n$*.*

Definition 5.7. Suppose $\mathcal{C}$ is a strict, spherical fusion category over $\mathcal{C}$. Let $V \in \mathcal{C}$ and $n \in \mathbb{N}$. Then $\mathcal{C}(1, V^{\otimes n})$ is a finite dimensional vector space. Define a map

$$E_V^{(n)}\colon \mathcal{C}(1, V^{\otimes n}) \to \mathcal{C}(1, V^{\otimes n})$$

by rotation! Put all the string down, then rotate the left-most strand over the top. Then $\nu_n(V) = \mathrm{tr}(E_V^{(n)})$.

Note that $E^n = I$, so $\nu_n(V) = \mathrm{tr}(E) \in \mathcal{O}(\mathbb{Q}_n)$.

The Frobenius-Schur indicators are numerical invariants of a spherical fusion category.

Examples 5.8.

(1) If $\mathcal{C} = \mathsf{Rep}(G)$, and $j\colon V \to V^{**}$ is the pivotal structure, then $\nu_n(V)$ is the classical indicator.
$$\nu_n(\mathbb{C}[G]) = \#\left\{x \in G \middle| x^n = e\right\}.$$

(2) If $\mathcal{C} = \mathsf{Rep}(H)$, where H is a semi-simple Hopf algebra, j the pivotal structure, and $S^2 = 1$, then $\nu_n(V)$ is the indicator of Linchenko-Montgomery.

(3) In rational conformal field theory, given a modular tensor category, $\nu_2(k)$ is given by Bantay's formula.

If $\mathcal{C}$ is a spherical fusion category, then $Z(\mathcal{C})$ is modular, so it has a T-matrix. Then $\mathrm{ord}(T) = N$, so for each $V \in \mathcal{C}$,

$$\nu_N(V) = d(V) \in \mathcal{O}(\mathbb{Q}_N).$$

Then the $S_{i,j}$, which are based on $N_{i,j}^k$, the θ_i, and the d_j are in $\mathcal{O}(\mathbb{Q}_N)$.

In fact, $\mathrm{ord}(T)$ is the smallest N such that $\nu_N(V) = d(V)$ for all $V \in \mathcal{C}$.

If $\mathcal{C}$ is a spherical fusion category, then $\nu_2(V) = 0, \pm 1$ for all simple $V \in \mathcal{C}$, and $\nu_1(V) = \delta_{0,V}$. FOr a modular tensor category,

$$\nu_n(V_k) = \frac{1}{\dim(\mathcal{C})} \sum_{i,j} N_{i,j}^k d_i d_j \frac{\theta_i^n}{\theta_j^n}.$$

Theorem 5.9 (A version of Cauchy's theorem)**.** *If* $\mathcal{C}$ *is an integral fusion category, then* $\mathrm{FPdim}(\mathcal{C})$ *and* $\exp(\mathcal{C})$ *have the same prime factors, where* $\exp(\mathcal{C}) = \mathrm{ord}(c^2)$ *(order of the braiding) of* $Z(\mathcal{C})$*.*

6 Tuesday afternoon

Question 6.1 (Wang's conjecture)**.** *For any fixed* $n \in \mathbb{N}$*, there are finitely many modular categories of rank* n *(rank is number of simple objects).*

Status:

For $n \leq 4$, it is true, and there is a complete classification. (For $n = 4$, the only problem is $\overline{fib} \boxtimes fib$, where the first has objects of dimension $1, \overline{\tau}$, where $\overline{\tau}\tau = -1$.

For $n = 5$, and there is some $X \not\cong X^*$, it is known.

For $n = 6$ and integral, classified.

ENO: Wang's conjecture holds if $\mathrm{FPdim}(\mathcal{C}) \in \mathbb{Z}$.

Question 6.2. *Is there a function of* $\mathrm{rank}(\mathcal{C})$ *which bounds* $\mathrm{rank}(Z(\mathcal{C}))$*?*

Question 6.3. *It is known that* $\exp(W_{torr}) = 32$*, and* $\mathrm{ord}([so(2n+1)_{2n+1}]) = 32$*. Are these distinct?*

Remark 6.4. To remember conventions, just remember $so(3)$ at level 3 is $su(2)$ at level 6.

Question 6.5. *When is the near group category $A \cup \{Y\}$ modular, where A is a finite abelian group?*

Question 6.6. *Is there a computational approach to $Z(\mathcal{C})$?*

Question 6.7. *What is the range of $\nu_n(V)$ for $V \in \mathcal{C}$?*

7 Scott Morrison

Constructing exotic objects

Start with a fusion ring (either for a fusion category or for a subfactor), or perhaps $X \in \mathrm{Irr}(\mathcal{C})$ and the multiplicities of $- \otimes X$ and $- \otimes X^*$.

Want general purpose methods to find $\mathcal{C}$.

Theorem 7.1. *Every non-degenerate planar algebra embeds in its graph planar algebra.*

Definition 7.2. $\mathcal{G}_n = \mathcal{G}(\Gamma)_n$ is $\mathbb{C}$-valued functions on the loops of length n on Γ.

The graph planar algebra is a linear algebra gadget which only depends on the combinatorial data of the graph and Frobenius-Perron dimensions.

How do you find the embedding?

(1) Identify canonical elements of the planar algebra and some relations it must satisfy, and then think of those relations as equations in the graph planar algebra, and try to solve them. For example, if Γ is $(n-1)$-supertransitive, and excess 1, then there is a low weight rotational eigenvector S which satisfies $S^2 = (r-1)S + rf^{(n)}$. Often solving these equations gives a discrete set of solutions.

(2) $P_0 = \mathbb{C}$, but $\mathcal{G}_0 = \mathbb{C}^{\#\text{vertices}}$. If S is in the image of the embedding map, then $\mathrm{tr}(S^n)$ is inside the image of P_0, which is the constants. This gives a huge collection of polynomial constraints. For example, this rigidly determines the embedding of the 2221 subfactor planar algebra into its graph planar algebra.

(3) A biunitary connection gives you the flat subalgebra of $\mathcal{G}$. In particular, if you have a subfactor planar algebra to start with, it defines a biunitary connection whose flat subalgebra coincides with that of the embedding. In general, we start with a biunitary connection, and find flat elements with respect to it.

One solves the flatness equations and the low-weight equations which are all linear.

Next, we look at the subalgebra Q generated by S in $\mathcal{G}$. This subalgebra is always spherical, unitary, and non-trivial! We need to know:

(1) Is $Q_0 \cong \mathbb{C}$? If so, we have some fusion category/subfactor planar algebra.

(2) Is it what we were looking for? Is the principal graph of Q equal to Γ?

The challenge is to show all closed networks in S are multiples of the empty diagram. Need evaluation algorithms, e.g., D_{2n}, jellyfish algorithm.

8 Wednesday afternoon

All known fusion categories/subfactors

8.1 "Ab initio"

- $\mathsf{Rep}(U_q(\mathfrak{g}))$, e.g., sl_2, sl_3, etc.

  (also via BMW, Hecke, TL algebras)

- Loop groups

- Cuntz algebras, e.g., 2^n1 and 3^n when there are solutions.

  Note that 3^4 is at index $3+\sqrt{5}$ gives 4^32 by equivariantization, and one (in a family?) between $A_3 * A_4$ and $A_3 \boxtimes A_4$.

  more generally, adding planar generators to a category with infinitely many objects

- Quantum subgroups of $su(n)$, $n = 2, 3, 4, 5$? (commutative algebra objects)

- Bisch-Haagerup subfactors $R^K \subset R \rtimes H$ (quotient of $\mathsf{Rep}(K) * \mathsf{Vec}(H)$)

- Group theoretical, e.g., $R^G \subset R^H$, $\mathsf{Rep}(G)$, Vec_G^ω

- Asaeda-Haagerup

- Extended Haagerup

- Flat elements of graph planar algebras

- $\mathsf{Rep}(H)$, H a quasi-Hopf algebra

8.2 New from old

- Deligne tensor product $\boxtimes$
- Free product (not fusion)
- Subcategories
- Morita-equivalence (intermediate subfactors)
- reduced subfactor construction, which is changing your favorite object
- Orbifold
- Coset
- De-equivariantization (simple current extension is the abelian case), e.g., D_{2n} from A_n.
- Centers
- Conformal inclusions (commutative algebras in $\mathsf{Rep}(U_q(G))$
- G-extensions ($\mathcal{D}$ is G-graded with $\mathcal{D}_e = \mathcal{C}$)

  braided G-crossed categories
- short exact sequences
- Galois actions (gives non-unitary)
- adjoints of forgetful functors, e.g., symmetrization of braided categories, braidification of tensor categories
- $\mathcal{C}^T$, T-modules in $\mathcal{C}$ where T is a semi-simple Hopf monad.

9 Open problem session, Wednesday afternoon

9.1 Fusion categories

Problem 9.1. Is there a conceptual construction of the even half of 4442? The even half of 4442 looks like a copy of the even part of affine E_6 fusion category (it is also $\mathsf{Rep}(A_4)$) and another copy of affine E_6, but as a module category. It looks like $\mathsf{Rep}(A_4)$, graded by the fibonacci category.

Problem 9.2. Is there an extension theory for fusion categories extended by fusion rings? (E.g., near group categories)

Problem 9.3 (Snyder)**.** Is there a sense in which a randomly chosen fusion graph doesn't have cylotomic dimensions?

Problem 9.4 (Snyder)**.** Look at all spoke graphs with $N > 0$ arms. Are there finitely many N-tuples $(\ell_1, \dots, \ell_N)$ such that the spoke graph with N arms of lengths $\ell_1, \dots, \ell_N$ has cyclotomic norm squared?

Problem 9.5 (Davydov)**.** Can we find all fusion categories with a given smallest simple object (which is not invertible)?

Problem 9.6 (Wenzl)**.** How many fusion categories have the same given fusion rules?

Can solve for $su(n)$. It seems you should be able to do this for all quantum groups at roots of unity. (need braided assumption) have to look at nice examples, or it is intractable.

Problem 9.7 (Wang)**.** Is there an effective version of Ocneanu rigidity? Is there a sub-exponential bound on the number of unitary fusion categories with respect to N, the sum of all the fusion multiplicities $N_{i,j}^k$?

Problem 9.8 (Etingof-Snyder)**.** Can you classify all algebras in fusion categories with small Frobenius-Perron dimension (e.g., less than $3 + \sqrt{3}$)?

Problem 9.9 (Snyder)**.** Is there a way to find the Frobenius-Schur exponent of $\mathcal{C}$ without computing $Z(\mathcal{C})$?

Ng: This is known for quasi-Hopf algebras.

Problem 9.10 (Gelaki)**.** Describe functors between group-theoretical categories.

What is known for Verlinde categories? (quantum groups at roots of unity)

Problem 9.11 (Peters)**.** Compute the center of the even half of the Asaeda-Haagerup and Extended Haagerup subfactors.

Problem 9.12 (Gelaki)**.** Suppose $\mathcal{C}$ is a finite tensor category over $\mathcal{C}$ with prime Frobenius-Perron dimension. Is $\mathcal{C}$ fusion?

(Hence, it would be of the form $\mathsf{Vec}(\mathbb{Z}/p, \omega)$. This would be an extension of a result in Hopf algebras.)

Problem 9.13 (Rowell, Property F conjecture)**.** Given a braided, weakly integral fusion category $\mathcal{C}$, is the image of the braid group finite?

Is braided and weakly integral fusion equivalent to finite image of the braid group?

(Interesting because you could not construct a universal quantum computer from it. If a fusion category has property F, then can't use braiding alone for a universal quantum computer.)

Problem 9.14. Is there a physical model which gives infinite image for the braid group?

Problem 9.15. Is any integral fusion category unitarizable?

Problem 9.16. Is every integral fusion category weakly group theoretical?

Problem 9.17. Does pseudo-unitary imply unitarizable?

(Wang: Physicists are really interested in this question. Given a conformal field theory which is not unitary, there is a negative dimension.)

Problem 9.18. Are all fusion categories pivotal? Are all fusion categories spherical? Does it depend on the ground field (k v.s. $\mathbb{C}$)?

Problem 9.19. Is $\mathrm{FPdim}(\mathcal{C})/\mathrm{FPdim}(X)$ an algebraic integer for every $X \in \mathrm{Irr}(\mathcal{C})$?

Problem 9.20 (Gelaki)**.** How much from modular representations of finite groups can be carried to finite tensor categories?

Problem 9.21 (Jordan)**.** Classify module categories and Brauer-Picard groups for known examples.

Problem 9.22 (Rowell)**.** What values can $\dim(X)$ take in $[2,3]$ for $X \in \mathcal{C}$, a braided fusion category?

($(\sqrt{3}+\sqrt{7})/2, (1+\sqrt{13})/2$ do not appear in the braided case.)

Problem 9.23 (Snyder)**.** What are all the $\mathcal{C}$ generated by X with $\mathrm{FPdim}(X) \leq 2$, and X not self-dual? Are they group theoretical if $\dim(X) = 2$?

(can do X is self-dual and unitary (this is the subfactor case), Snyder: I think I can do it if $X \otimes X^* \cong X^* \otimes X$, Rowell: enough if X is self-dual and the Grotheneick ring is commutative)

Problem 9.24. For unitary theories, can choose a gauge so the F matrices formed by $6j$ symbols are unitary, and the braiding matrices are diagonal with respect to a certain basis. Can this happen for some non-unitary fusion category?

(Is the unitarity of the F matrices equivalent to unitarity? Snyder: if F matrices are unitary, and dimensions are positive, and maybe something about θ's, then it is unitary)

Problem 9.25 (Morrison)**.** Are there any numbers which are positive real numbers which are cyclotomic integers and are the largest amongst its Galois conjugates which are not realized as dimensions of objects in a fusion category?

Problem 9.26. What can you say about all fusion categories $\mathcal{C}$ for which

$$\#\left\{\dim(X)\middle|X \in \mathcal{C}\right\} = 2?$$

Problem 9.27 (Jones, fusion category version of supertransitivity)**.** In a fusion category, is there an upper bound on the N such that $X^{\otimes N}$ is a simple object (where $\dim(X) > 1$)?

(Haagerup: If the fish exist, then no!)

9.2 Subfactors

Problem 9.28 (Jones)**.** Is there an upper bound on the supertransitivity of a subfactor planar algebra?

(The supertransitivity with respect to an object X with $\dim(X) > 2$ is the largest N such that $\mathrm{Hom}(1, X^{\otimes n})$ is Temperley-Lieb.)

Supertransitivity is the analog of transitivity of group actions. Note that the group case was solved by the classification of finite simple groups.

Problem 9.29 (Snyder)**.** Find a non-number theoretic argument to rule out the rest of the Haagerup family vine.

E.g., is there a diagram that evaluates in two different ways?

Problem 9.30. There are accumulation points from below for $\mathrm{FPdim}(X)$ for an object in a fusion category or $[M\colon N]$ for finite depth subfactors. Are there any accumulation points from above?

(Note that there are no accumulation points at all for $\mathrm{FPdim}(\mathcal{C})$ for a fusion category $\mathcal{C}$ by Ocneanu rigidity.)

Problem 9.31 (Morrison-Peters)**.** Is there a polymer theory of principal graphs? What graphs can appear as subgraphs of principal graphs?

(need a bound on the rank at each depth)

10 David Jordan

G-extensions of fusion categories (results of ENO)

Definition 10.1. A fusion category $\mathcal{D}$ is a G-graded extension of a fusion category $\mathcal{C} = \mathcal{D}_e$ if

$$\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$$

with $\mathcal{D}_e = \mathcal{C}$ and $\mathcal{D}_g \otimes \mathcal{D}_h \subseteq \mathcal{D}_{gh}$. This gives a grading on $K_0(\mathcal{D})$.

Examples 10.2.

(1) Vec_G^ω is a G-graded extension of Vec.

(2) Vec_G^ω is also a G/N-graded extension of $\mathsf{Vec}_N^{\omega|_N}$, where N is normal.

(3) Tambara-Yamagami categories. Fix a finite group G. Want $K_0(TY) = \mathbb{Z}[G] \oplus \mathbb{Z}[X]$. Here $X \otimes X = \bigoplus_{g \in G} g$. This is the simplest near-group category. This is weakly integral. This is a $\mathbb{Z}/2$-extension of Vec_G^ω (and in fact, we will show ω is trivial).

Their initial classification was quite intensive, and relied on computing all the $6j$-symbols. This was the motivation for ENO.

We want to study $\mathcal{D}$ which are G-extensions of $\mathcal{C}$.

Remark 10.3. Each $\mathcal{D}_g$ is a $\mathcal{C}$-$\mathcal{C}$ bimodule category. In fact, $\mathcal{C} \boxtimes_{\mathcal{C}} \mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{C} \cong \mathcal{D}_g$. $\mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{D}_{g^{-1}} \cong \mathcal{D}_e = \mathcal{C}$, so each $\mathcal{D}_g$ is invertible. This is an analog to the graded-ring case, in which each graded piece is an invertible bimodule.

Definition 10.4. The Brauer-Picard groupoid $\mathrm{BrPic}_3(\mathcal{C})$ (using the subscript instead of number of underlines) is as follows: The set of objects is $\{*\}$, the 1-mnorphisms is the set of invertible $\mathcal{C} - \mathcal{C}$ bimodule categories, the 2-morphisms are bimodule equivalences, and the 3-morphisms are natural isomorphisms.

In this 3-category, everything is invertible. Hence we can think of it as a topological space. The algebraic space $|\mathrm{BrPic}_3(\mathcal{C})|$ is the classifying space of $\mathrm{BrPic}_3(\mathcal{C})$.

Theorem 10.5 (ENO)**.** *There is an equivalence*

$$\frac{\{G - \textit{extensions } \mathcal{D} \textit{ of } \mathcal{C}\}}{\sim} \cong [BG, |\mathrm{BrPic}_3(\mathcal{C})]$$

where $BG = K(G, 1)$, which is uniquely determined by $\pi_1(BG) = G$ and $\pi_k(BG) = 0$ for all $k \neq 1$.

Theorem 10.6 (ENO)**.** *Let $\pi_k := \pi_k(|\mathrm{BrPic}_3(\mathcal{C})|)$. Then $\pi_0 = \{*\}$, $\pi_1 = \mathrm{BrPic}(\mathcal{C})$ is the Brauer-Picard group of $\mathcal{C}$, which is the equivalence classes of invertible bimodules, π_2 is the invertible objects in $Z(\mathcal{C})$, $\pi_3 = \mathbb{C}^\times$, and $\pi_k = 0$ for all $k \geq 4$.*

Remark 10.7. Can compute invertible objects in $Z(\mathcal{C})$ without actually computing $Z(\mathcal{C})$.

This gives an algorithm for understanding G-extensions.

Corollary 10.8. *Extensions $\mathcal{D}$ of $\mathcal{C}$ by G are built as follows:*

(1) Give a homomorphism $\rho\colon G \to \mathrm{BrPic}(\mathcal{C})$, for which a certain obstruction $\mathcal{O}_3(\rho) \in H^3(G, \pi_2)$ must vanish.

(2) If so, write $\mathcal{O}_3(\rho) = \partial M$, where $M \in H^2(G, \pi_2)$.

(3) For this M, there is another obstruction $\mathcal{O}_4(\rho, M) \in H^4(G, \mathbb{C}^\times)$ which must vanish. If so, choose α such that $\alpha = \mathcal{O}_4$, where $\alpha \in H^3(G, \mathbb{C}^\times)$.

Roughly, ρ corresponds to choosing the bimodule decomposition $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$, M corresponds to the isomorphisms $\mathcal{D}_g \boxtimes_{\mathcal{C}} \mathcal{D}_h \cong \mathcal{D}_{gh}$, and α corresponds to the associativity data.

Example 10.9. $\mathcal{C} = \mathsf{Vec}$, $|cD = \bigoplus_{g \in G} \mathsf{Vec}$ Then π_2 is the invertible objects in $Z(\mathsf{Vec}) = \mathsf{Vec}$, which is $\mathbb{C}$. Hence π_2 is trivial. Hence $\mathcal{O}_3(\rho) = 0$ automatically. Now $\mathcal{O}_4 \in H^4(G, \mathbb{C}^\times)$ must vanish. Then we choose $\alpha \in H^3(G, \mathbb{C}^\times)$ and this gives us Vec_G^α.

We now outline the ENO treatment of the Tambara-Yamagami classification.

Theorem 10.10 (Tambara-Yamagami)**.** *Categorifications of the TY ring $K_0(TY)$ are given by*

(1) $\omega = 0$ and A is abelian.

(2) Choose a non-degenerate bilinear, symmetric bicharacter $\chi\colon A \times A \to \mathbb{C}^\times$.

(3) Choice of sign ± 1.

Proof.

(0) Assume the order of G is odd for simplicity in the Tambara-Yamagami theory.

(1) First, $\mathcal{C} = \mathsf{Vec}_G^\omega$ for some G, ω. We will rule out the possibility of a nonzero ω. Now the functor $g \mapsto g \otimes X$, where X is the new object, defines a fiber functor, so $\omega = 0$.

Now Vec_G comes with a corresponding bimodule M (the X part). If we take the dual $(\mathsf{Vec}_G)^*_M = \mathsf{Rep}(G) = \mathsf{Vec}_G$, so $G = A$ is abelian.

(2) We have a map $\rho\colon \mathbb{Z}/2 \to \mathrm{BrPic}(\mathsf{Vec}_A)$, which is braided auto-equivalences of $Z(\mathsf{Vec}_A)$ (by ENO). Now $Z(\mathsf{Vec}_A) = (\mathsf{Vec}_{A\times A^\times}, G)$ where $A^\times$ is the group of characters and $G_{a,b}\colon a \otimes b \to b \otimes a$ via the bicharacter $\langle a, b\rangle \in \mathbb{C}^\times$. Hence the braided auto-equivalences is $\mathcal{O}(A \oplus A^\times)$ (the split orthogonal group). Now we choose $\rho(\varepsilon) \in \mathcal{O}(A \oplus A^\times)$. We write $\rho(\varepsilon)$ as

$$\rho(\varepsilon) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where $\alpha\colon A \to A$, $\beta\colon A \to A^\times$, etc. and the transpose looks like

$$\rho(\varepsilon)^T = \begin{pmatrix} \delta^T & \beta^T \\ \gamma^T & \alpha^T \end{pmatrix}.$$

Now we have $\alpha^T = \delta$, $\beta^T = \beta, \gamma^T = \gamma$, and we can reduce to the case $\alpha = \delta = 0$. Now we have

$$\rho(\varepsilon) = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

must square to itself, so $\beta = \gamma^{-1}$, and $\beta = \beta^*$ is the data in (2) (the non-degenerate, bilinear, symmetric bicharacter).

(3) Since $|A|$ is odd, $\mathcal{O}_3(\rho(\varepsilon) \in H^2(\mathbb{Z}/2, A \oplus A^\times) = 0$, so $M \in H^2(\mathbb{Z}/2, \mathbb{C}^\times) = 0$. Then $\mathcal{O}_4 \in H^4(\mathbb{Z}/2, \mathbb{C}^\times) = 0$, and $\alpha \in H^3(\mathbb{Z}/2, \mathbb{C}^\times) = \mathbb{Z}/2$, which is the choice of sign. □

Remark 10.11 (Snyder)**.** Start with self-dual subfactor planar algebra, try to erase shading, but this comes down to a choice of bicharacter.

Can use the same machinery to classify fusion categories of dimension pq^2. ENO have a Burnside type theorem which classifies these up to Morita equivalence.

Theorem 10.12 (Jordan-Larsen)**.** *If $\dim(\mathcal{C}) = pq^2$, then either:*

(1) $\mathcal{C}$ is group theoretical,

(2) $p = 2$ is a Tambara-Yamagami category,

(3) p is odd, $p|(q+1)$, and $\mathcal{C}$ is a category $\mathcal{C}(p,q,\{\zeta_1,\zeta_2\},\xi)$ where $\zeta_i \in \mathbb{F}_{q^2}$ is root of unity, $\zeta_1 \neq \zeta_2$, $\zeta_1\zeta_2 = 1$, $\zeta_i^p = 1$, and $\xi \in H^3(\mathbb{Z}/p, \mathbb{C}^\times) = \mathbb{Z}/p$.

Sketch of proof. First, show $\mathcal{C}$ is $\mathbb{Z}/p$ graded. The same tricks as in TY show that $\mathcal{O}_3, \mathcal{O}_4$ are 0. Then $\rho(\varepsilon) \in \mathcal{O}((\mathbb{Z}/q)^4)$, and ζ_1, ζ_2 are eigenvalues of one block of $\rho(\varepsilon)$. □

TODO: replace BrPic_3 **with** $\underline{\underline{\underline{\mathbf{BrPic}}}}$

11 Sonia Natale

Exact sequences of (tensor) fusion categories.

For finite groups, if we have a short exact sequence

$$1 \to G' \to G \to G'' \to 1,$$

we can recover G from G', G'' and cohomological data.

An exact sequence of semi-simple Hopf algebras is a sequence

$$k \to H' \xrightarrow{\iota} H \xrightarrow{\pi} H'' \to k.$$

such that $\pi \circ \iota = \varepsilon 1$. Then $H' = H^{co\pi} = \left\{h \in H \middle| (\mathrm{id} \otimes \pi)\delta(h) = h \otimes 1\right\}$. ($H'' \cong H/H(H')^+$ where $(H')^+ = \ker(\varepsilon_{H'})$)

Definition 11.1. $H' \subseteq H$ is a normal Hopf subalgebra if $h_{(1)}aS(h_{(2)}) \in H'$ for all $h \in H, a \in H'$.

In this case, get a short exact sequence as above, and there exists a good section $H'' \to H$ which makes it possible to recover H from H', H'' and some cohomological data in the structure of a bicrossed product.

This talk is about joint work with Alain Bruegières, which extends the notion of short exact sequences of Hopf algebras to the fusion category setting.

Definition 11.2. GIven a finite group action $\rho\colon G \to \mathrm{Aut}_\otimes(\mathcal{C})$, then the equivariantization $\mathcal{C}^G$ is a fusion category, which is endowed with a natural forgetful functor $\mathcal{C}^G \to \mathcal{C}$, by $(X, u^g) \mapsto X$. Have isomorphisms $u^g\colon \rho^g(X) \to X$. In this setting, get

$$\mathsf{Rep}(G) \to \mathcal{C}^G \to \mathcal{C}$$

Definition 11.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a tensor functor with $\mathcal{C}, \mathcal{D}$ fusion, which is strong monodical and exact.

- F is dominant if for all $Y \in \mathcal{D}$, there is an $X \in \mathcal{C}$ such that Y embeds in $F(X)$. Equivalently, F is surjective.
- F is called normal if for all simple $X \in \mathrm{Irr}(\mathcal{C})$ such that $\mathrm{Hom}_{\mathcal{D}}(1, F(X)) \neq 0$ we have $F(X) \cong \bigoplus^n 1$.

Example 11.4. If $G' \subseteq G$ is a subgroup, get a map $\mathsf{Rep}(G) \to \mathsf{Rep}(G')$ by restriction, which is always dominant. It is normal if and only if G' is a normal subgroup.

Definition 11.5. A sequence $\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$ of tensor functors between fusion categories is called an exact sequence of fusion categories if

(1) F is dominant and normal,

(2) f is a full embedding, and

(3) the essential image of f is $\ker(F)$, where $\ker(F) \subseteq \mathcal{C}$ is the full subcategory whose objects are those $X \in \mathcal{C}$ such that $F(X) \cong \bigoplus^n 1$, i.e., $F(X) \in \langle 1 \rangle$.

Definition 11.6. Suppose $\mathcal{C}' \to \mathcal{C} \xrightarrow{F} \mathcal{C}''$ is exact. Then F induces by restriction a fiber functor $\omega_F \colon \mathcal{C}' \to \mathsf{Vec}_k$, by $X \mapsto \mathrm{Hom}(1, F(X))$. Then there exists a (co-semi simple) Hopf algebra H such that $\mathcal{C}' \cong \mathrm{comod} - H$, where $H = \mathrm{coned}(\omega_F)$. H is called the induced Hopf algebra of the exact sequence.

(We will work with the case $k = \mathbb{C}$ so that H is co-semis impel and semi simple.)

Examples 11.7.

(1) Given

$$1 \to G' \to G \to G'' \to 1$$

a short exact sequence of finite groups, this gives rise to an exact sequence of fusion categories

$$\mathsf{Rep}(G'') \to \mathsf{Rep}(G) \xrightarrow{\mathrm{res}^G_{G'}} \mathsf{Rep}(G').$$

The induced Hopf algebra is $H = k^{G''}$.

(2) If $\rho \colon G \to \mathrm{Aut}_\otimes(\mathcal{C})$, then

$$\mathsf{Rep}(G) \to \mathcal{C}^G \xrightarrow{U} \mathcal{C}$$

is exact.

Proposition 11.8. *If* $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ *is exact, then* $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')\,\mathrm{FPdim}(\mathcal{C}'')$.

Moreover, if $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ *is a sequence such that* $\mathcal{C}' \subseteq \ker(F)$, *then the sequence is exact if and only if* $\mathrm{FPdim}(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}')\,\mathrm{FPdim}(\mathcal{C}'')$.

Definition 11.9 (Hopf monads, Bruguières, Virelizier, Lack)**.** $\mathcal{C}$ a fusion category. a monad on $\mathcal{C}$ is an algebra $T \in \mathrm{End}(\mathcal{C})$. In particular, have a multiplication $\mu \colon T \otimes T \to T$ and a unit $\eta \colon \mathrm{id}_{\mathcal{C}} \to T$.

If T is a monad, then we have the category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$, where objects are (X, r) where $X \in \mathcal{C}$ and $r \colon T(X) \to X$ is a morphism in $\mathcal{C}$ such that $rT(r) = r\mu$ (commutative diagram)

$$\begin{tikzcd} T^2X \arrow[r, "T(r)"] \arrow[d, "\mu"] & T(X) \arrow[d, "r"] \\ TX \arrow[r, "r"'] & X \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "="'] & TX \arrow[d, "r"] \\ & X \end{tikzcd}$$

Now we have a forgetful functor $U\colon \mathcal{C}^T \to \mathcal{C}$, which has a left adjoint L. $U(X,r) = X$, and $L(X) = (TX, \mu_X)$, where $T = UL$.

Definition 11.10. If we have $\mathcal{C}, \mathcal{D}$ fusion categories, and $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{C} \to \mathcal{D}$ are functors. Suppose G is left-adjoint to F, $FG = T \in \mathrm{End}(\mathcal{C})$ is a monad. Then F is monadic if $\kappa\colon \mathcal{D} \to \mathcal{C}^T$ is an equivalence.

(Question from Jordan: when is the right adjoint equal to the left adjoint? Maybe: If G is left-adjoint to F, and F is left-adjoin tho R, then $R(X) = G(X^*)^*$.)

Fact 11.11. *$\mathcal{C}, \mathcal{D}$ fusion categories, F a tensor functor. Then F has adjoints and it is monadic.*

Definition 11.12. Let T be a monad on $\mathcal{C}$. T is a bimonad if $\mathcal{C}^T$ is a monodical category and the forgetful functor $U\colon \mathcal{C}^T \to \mathcal{C}$ is strong monodical. Equivalently, T has a comonoidal structure, which is a natural transformation (not isomorphism) $T(X \otimes Y) \xrightarrow{T_2(X,Y)} T(X) \otimes T(Y)$ which is natural in X, Y, and $T_0\colon T(1) \to 1$ satisfying some conditions.

If T is a bimonad, then it is called a Hopf monad if $\mathcal{C}^T$ is rigid.

A Hopf monad $T\colon \mathcal{C} \to \mathcal{C}$ is called normal if $T(1) \in \langle 1 \rangle$.

Examples 11.13.

(1) If H is a Hopf algebra over k, then $T_H = H \otimes -$ defines a Hopf monad on Vec_k, and $(\mathsf{Vec}_k)^{T_H} \cong \mathsf{Rep}(H)$.

(2) If T is normal, then $T|_{\langle 1 \rangle}$ is a Hopf monad on $\langle 1 \rangle = \mathsf{Vec}_k$ by restriction, so $T_{\langle 1 \rangle} \cong H \otimes -$, where H is the induced Hopf algebra of T.

Theorem 11.14. *Suppose we have tensor categories $\mathcal{C}'', \mathcal{C}'$ over k where $\mathcal{C}'$ is finite (e.g., they are fusion categories). Then the following are equivalent:*

(1) Extensions $\mathcal{C}' \to \mathcal{C} \to \mathcal{C}''$ of $\mathcal{C}''$ by $\mathcal{C}'$.

(2) Normal, faithful k-linear exact Hopf monads on $\mathcal{C}"'$ with induced Hopf algebra H endowed with an equivalence $\mathcal{C}' \cong comod - H$.

Proof. If T is as in (2), then we get a short exact sequence by setting $\mathcal{C} = (\mathcal{C}'')^T$, and $F = U\colon (\mathcal{C}'')^T \to \mathcal{C}''$ is the forgetful functor. This is normal and dominant since T is normal and T is faithful.

Conversely, given a short exact sequence as in (1), then the left adjoint G of F gives a monadic adjunction, and we get an equivalences $\mathcal{C} \cong (\mathcal{C}'')^T$. □

Example 11.15. This extends equivariantization under a finite group action. Suppose $\rho\colon G \to \mathrm{Aut}_\otimes(\mathcal{C})$. Then we get a short exact sequence $\mathsf{Rep}(G) \to \mathcal{C}^G \xrightarrow{U} \mathcal{C}$. This is a special case. Set $T = \bigoplus_{g \in G} \rho^g \in \mathrm{End}(\mathcal{C})$, and $\mu\colon T^2 \to T$ by

$$T^2(X) = \bigoplus_{g,h \in H} \rho^g \rho^h(X) \xrightarrow{\rho^2_{g,h}} T(X) = \bigoplus_{g \in G} \rho^g(X)$$

where $\rho^2_{g,h}$ is given by the monodical structure of ρ. Then T is exact, normal, faithful Hopf monad on $\mathcal{C}$, and $\mathcal{C}^G \cong \mathcal{C}^T$.

Theorem 11.16. *$F\colon \mathcal{C} \to \mathcal{D}$ dominant,* $\mathrm{FPind}(F) = 2$*, where*

$$\mathrm{FPind}(F) = \mathrm{FPdim}(\mathcal{C}) / \mathrm{FPdim}(\mathcal{D}).$$

Then F is normal and $\mathsf{Rep}(\mathbb{Z}/2) \to \mathcal{C} \to \mathcal{D}$ is exact (analog of subgroup of index 2 is normal).

Theorem 11.17. *Suppose $F\colon \mathcal{C} \to \mathcal{D}$ dominant where $\mathcal{C}, \mathcal{D}$ are weakly integral. Suppose* $\mathrm{FPind}(F) = p$*, where p is the smallest prime dividing* $\mathrm{FPdim}(\mathcal{C})$*. Then F is normal and $\mathsf{Rep}(\mathbb{Z}/p) \to \mathcal{C} \to \mathcal{D}$ is exact.*

Remark 11.18. In both theorems above, both short exact sequences are equivariantizations.

12 Thursday afternoon

Question 12.1. *Finite monodromy for modular extensions.*

Question 12.2. *Extensions of $\mathcal{C}$ by $\mathcal{D}$ without $\mathcal{D}$ being $\mathsf{Rep}(H)$.*

Question 12.3. *Izumi's question about TY for self-dual Hopf algebras.*

Question 12.4. *Integral modular categories*

Question 12.5. *Understanding $Z(\mathcal{C})$ for near group $\mathcal{C}$.*

Question 12.6. *Frobenius-Schur exponent and planar algebras*

Question 12.7. *Compute center of Asaeda-Haagerup or Extended Haagerup via class equation, etc.*

Question 12.8. *Fusion categories of small global dimension*

13 Zhenghan Wang

IBM claims that we will probably have quantum computers in about 15 years. (Not Wang's answer though.)

A unitary modular category is the same as many topological quantum computers.

Any generic unitary modular category is a universal quantum computer, i.e., whatever you claim you can do with a quantum computer, you can do it with the unitary modular category with at most a polynomial time slowdown.

Unitary modular categories can be realized by real physical systems (with boundaries).

(Bisch: That's like saying every self-adjoint operator is a physical observable, which is not true... Wang: This is better, and explanation will come)

There are two extremes: the Drinfel'd centers (doubles) and quantum groups (Chern-Simons theories). The first have central charge 0, and the second (conjecturally) generate the Witt group, and they are maximally chiral. The second group seems pretty real, but it is hard to understand them in terms of Hamiltonians. But we understand the first one completely, but we cannot construct them easily. The quantum double of $\mathbb{Z}/2$ is realized in superconductors.

Peter Shor: quantum algorithm for factoring numbers.

Alexei Kitaev: need to do topology

Start with $\mathbb{Z}/2$, and take the double $D(\mathbb{Z}/2)$, called toric code, $\mathbb{Z}/2$-gauge theory, $\mathbb{Z}/2$-spin liquid, etc.

Kitaev model/Levin-Wen model - Hamiltonian formulation of $D(\mathbb{Z}/2)$/ Turaev-Viro TQFTs.

Definition 13.1. A quantum system is a pair $(\mathcal{L}, H)$ of a local Hilbert space and a Hermitian operator called a Hamiltonian. (In fact, $\mathcal{L}$ is finite dimensional!)

The $6j$-numbers are all algebraic numbers, so they are computable. Hence this Hamiltonian is computable. To remember H, you just need to remember an orthonormal basis and some numbers. Fix a certain tensor product decomposition of $\mathcal{L}$ (gives a sense of locality).

Kitaev: On $\mathbb{T}^2$, we give a cell-ulation (or choose a square lattice on the torus). Write down a local Hilbert space and a Hermitian operator. Define $\mathcal{L} = \bigotimes \mathbb{C}^2$ over all edges in the lattice (this is electrons, one spin up, one spin down, or the group algebra $\mathbb{C}[\mathbb{Z}/2]$, which is just $|0\rangle$ and $|1\rangle$).

(Bisch: in operator algebras, would associate matrix algebras. Wang: You can associate any finite dimensional C^*-algebras, but then you have to choose a basis.)

The Hamiltonian which measures the energy is given by

$$H = -\sum_{\text{verticces}} A_v - \sum_{\text{faces}} B_p$$

Take a vertex v. There are 4 edges touching it, so $A_v = \bigotimes_{4 \text{ edges}} \sigma^z \otimes \bigotimes_{\text{rest}} \text{id}$ where σ^z is the Pauli matrix

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Given a face p, the $B_p = \bigotimes_{4 \text{ edges on } \partial p} \sigma^X \otimes \bigotimes_{\text{rest}} \text{id}$, where

$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is very difficult to detect a 3-body interaction ($\sim 1\%$ of the 2-body interaction). In this system, we have a 4-body interaction. For this Hamiltonian:

(1) All terms A_v, B_p commute with each other. The only trouble is when a vertex is touching a face, but then they share 2 edges. This means they can all be simultaneously diagonalized.

The spectrum of H is the set of eigenvalues $\{\lambda_0 < \lambda_1 < \cdots\}$. The eigenspace E_{λ_0} is the ground-state manifold. The eigenspace E_{λ_1} is the first excited state.

(2) The ground state manifold on $\mathbb{T}^2$ is isomorphic to $\mathbb{C}^4$.

(3) The elementary excitations E_{λ_1} is isomorphic to $D(\mathbb{Z}/2)$.

This Hamiltonian is rigorously solvable (not exactly solvable, but this means something else). The dimension of E_{λ_0} is always equal to the rank of the TQFT. In fact, $\mathbb{C}^4 = \mathbb{C}[H_1(\Sigma, \mathbb{Z}/2)]$ you always get the group algebra of the $\mathbb{Z}/2$-homology of the surface. We can renormalize to $\lambda_0 = 0$ by shifting. What's the next possible eigenvalue? It is $\lambda_1 = 2$.

$$|\psi\rangle \in E_{\lambda_0} \iff \text{for all } v, p,\ A_v|\psi\rangle = |\psi\rangle \text{ and } B_p|\psi\rangle = |\psi\rangle.$$

This leads to a constraint: elementary excitation can only be created in pairs (conservation of physical charge). There are particles $1, e, m, em$ from some rules on the torus, and looking at the braiding rules returns $D(\mathbb{Z}/2)$.

The above machinery works for any unitary modular category.

Theorem 13.2. *For any unitary fusion category $\mathcal{C}$, there exists a quantum schema to realize the Drinfel'd center $Z(\mathcal{C})$.*

Here a schema is just a procedure to write something down. Given $\mathcal{C}$ and an oriented surface Y with a cellulation Δ of Y, can write down a quantum system $(\mathcal{L}_{Y,\Delta,\mathcal{C}}, H_{Y,\Delta,\mathcal{C}})$ so that the ground state of H is canonically isomorphic to the Turaev-Viro Hilbert space associated to Y, and the fusion rules give the quantum double of $\mathcal{C}$.

(Bisch: What about higher eigenvaues? Wang: The ground state is stable. We want to deal with low-energy physics, not high-energy physics which is unstable. This relates to renormalization)

(1) The ground state manifold of $(\mathcal{L}, H) \cong V^{RT}_{Z(\mathcal{C})}(Y) \cong V^{TV}_{\mathcal{C}}(Y)$

(2) The elementary excitation is $D(\mathcal{C}) \cong Z(\mathcal{C})$.

Given a subfactor, take the even part to get all this data. The odd part gives a module category. How does this appear? Rather than a torus, draw a disk, so we have a lattice with a boundary. The boundary conditions are given by module categories. What is the condition to give that the module category is indecomposable? These are the stable ones. Decomposable ones will break down quickly.

If $\mathcal{C}$ is a group ($\mathsf{Rep}(G)$?) then we get a trivalent graph labelled by group elements. The mapping class group changes this graph to another graph, and they are related. If we have two triangulations of the same surface with the same number of vertices, we can always move one to the other by the diagonal flip. If we dualize this rule, we get the F-rule.

Mass-gap: Given an H on a finite dimensional $\mathcal{L}$, the quantity $\lambda_1^\Delta - \lambda_0^\Delta > 0$. We want the limit as Δ gets more fine. Is the difference bounded below?

"$su(2)_2$" Ising. $1, \sigma, \psi$ are the simples, with rules $\sigma^2 = 1 \oplus \psi$, $\psi^2 = 1$, and $\sigma\psi = \psi\sigma = \sigma$. σ is non-abelian anyon. $\theta_\psi = -1$, and $\psi^* = \psi$ is a Majorana fermion. This is probably already discovered in condensed matter.

Fractional quantum Hall effect. At $N_e/\#\text{Flux}$ is $1/3$, this is the modular category of $\mathbb{Z}/3$. $5/2$ corresponds to Ising model.

14 Reports on small groups

14.1 Computing center of even part of Asaeda-Haagerup

Found some stuff. Some information about what possible dimensions you can have. The global dimension is $\mathrm{FPdim}(AH_{even}) = 8\sqrt{17}(4+\sqrt{17})$, and the right hand side is the fundamental unit, so not many numbers divide it. Another restriction is that conjecturally, the dimensions of objects in the center should be sums of 1's and sums of $(4+\sqrt{17})$'s. This gives a limited list (of length 13) on possible dimensions of objects. This also gives a bound on the rank of 270-something, but they guess the rank is in the 20's.

Theorem 14.1 (???, Osterik?)**.** *The list of dimensions of summands of the induction of* 1 *in the center,* $I(1)$ *in* $Z(\mathcal{C})$*, can be read off just from the Grothendeick group of* $\mathcal{C}$ *itself via formal codegrees. They are* $\mathrm{FPdim}(\mathcal{C})$*/formal codegrees.*

This allows them writing down 5+ dimensions of objects in the center. There are some heuristics to believe why the rank is somewhere in the 20s and 30s. They know 5 non-trivial objects, and 3 of the dimensions.

Idea: look at inductions of other objects, and try Frobenius reciprocity, and this will give a better bound on the rank.

Induction is dominant (restriction is surjective).

14.2 Classifying integral modular tensor categories of dimension $p^a q^b$

When are these categories group theoretical?

Theorem 14.2 (Etingof, Rowell, Witherspoon)**.** *If* $\mathcal{C}$*is group theoretical, braided fusion, then* $\mathcal{C}$ *has property* F*.*

If $\dim(\mathcal{C}) = p^n, pq, pqr$, and $\mathcal{C}$ is fusion, then $\mathcal{C}$ is group theoretical.

If $\dim(\mathcal{C}) = pq^2, pq^3$ and $\mathcal{C}$ modular, then $\mathcal{C}$ is group theoretical.

For p^2q^2, there are non-group theoretical where $\mathrm{FPdim}(\mathcal{C}) = 36$. There are 2 of them of ranks 8 and 10. The one of rank 8 is "like $\mathsf{Rep}(D(S_3))$," and the other is $su(3)_3$.

The idea is to classify by the pointed sub-categories. In each case, there are pointed non-trivial subcategories.

For pq^4, it is group theoretical unless $|\mathcal{C}_{pt}| = q^2$, and for p^2q^2, it is group theoretical unless $|\mathcal{C}_{pt}| = p$ or q.

The conjecture that the two of rank 8 and 10 are the only two non group-theoretical categories with $\mathrm{FPdim}(\mathcal{C}) = 36$.

There is a theorem of Müger that tells us when a category is group-theoretical.

14.3 Frobenius Schur exponents

Try calculating a simple case of this. Try calculating $\mathrm{ord}(T)$ for A_n. Since A_n is modular, can calculate $\mathrm{ord}(T)$ from the $f^{(n)}$. In general, the following formula is wrong, since A_n at $q = \exp(2\pi i/(n+1))$ is not modular!

$$\mathrm{ord}(T) = \begin{cases} 2(n+1) & \text{if } n \text{ is odd} \\ 4(n+1) & \text{if } n \text{ is even} \end{cases}$$

At index greater than 4, only contribution to the FS-indicator are the new rotational eigenvectors of P_n are $n = \mathrm{ord}(T)$. For Haagerup, $n = 39$, and there are roughly 45 million new linearly independent things.

15 Problems for friday

Question 15.1.

(1) Low rank modular tensor category classification

(2) Construct the connection on the graph of the missing AH subfactor

(3) Formal codegrees: what is the smallest possible?

(4) Find small dimensional objects in braided subfactors

(5) More on combinatorics of Z(AH)