Algebras, module categories, and planar algebras

Let (A, μ, i) be an algebra object in a tensor category[1] $\mathcal{C}$. By an argument similar to [KO02, Fig. 4], we have canonical isomorphisms

$$\begin{tikzcd}[column sep=small] & \mathrm{Hom}_{A-\mathsf{Mod}}(A \to A) \arrow[dr, "\cong"] & \\ \mathrm{Hom}_{A-\mathsf{Mod}-A}(A \otimes A \to A) \arrow[ur, "\cong"] \arrow[dr, "\cong"] & & \mathrm{Hom}_{\mathcal{C}}(1 \to A) \\ & \mathrm{Hom}_{\mathsf{Mod}-A}(A \to A) \arrow[ur, "\cong"] & \end{tikzcd} \tag{1}$$

We call (A, μ, i):

- *connected* if the morphism spaces in (1) above are one dimensional.
- *separable* if μ splits as an $A - A$ bimodule map, i.e., there is an $A - A$ bimodule map $s : A \to A \otimes A$ such that $\mu \circ s = \mathrm{id}_A$. In this case, by [HPT16, Props. 2.5 and 2.6][2] and [Ost03, Prop. 3.1.i], $\mathsf{Mod} - A$, $A - \mathsf{Mod}$, and $A - \mathsf{Mod} - A$ are all semisimple categories.

Lemma 1. *Suppose (A, μ, i) is a connected separable algebra object in a tensor category $\mathcal{C}$. There are unique morphisms $\Delta \in \mathrm{Hom}_{A-\mathsf{Mod}-A}(A \to A \otimes A)$ and $\epsilon \in \mathrm{Hom}_{\mathcal{C}}(A \to 1)$ for which $(A, \mu, i, \Delta, \epsilon)$ is a normalized*[3] *special*[4] *Frobenius algebra.*

Proof. Up to scalar, ϵ must be the unique left inverse of $i \in \mathcal{C}(1 \to A)$, which exists by semisimplicity of $\mathcal{C}$. The pairing $\epsilon \circ m$ is non-degenerate by [Ost03, Prop. 3.1.ii]. There is a unique comultiplication Δ making $(A, \mu, i, \Delta, \epsilon)$ a Frobenius algebra by [FRS02, Lem. 3.7], [FS08, Prop. 8]. Finally, A is automatically special (see [GS16, Thm. 2.6]). Indeed, since A is separable, there is a splitting $s \in \mathrm{Hom}_{A-\mathsf{Mod}-A}(A \to A \otimes A) \cong \mathbb{C}$ as in (1). Hence $\Delta = \lambda s$ for some $\lambda \in \mathbb{C}^\times$ as $\Delta \neq 0$. Thus $\mu \circ \Delta = \lambda(\mu \circ s) = \lambda \, \mathrm{id}_A$. Finally, normalize $(A, \mu, i, \Delta, \epsilon)$ by picking an appropriate scaling of ϵ. $\square$

Remark 2. When $\mathcal{C}$ is pivotal and $\dim_{\mathcal{C}}(A) \neq 0$,[5] $(A, \mu, i, \Delta, \epsilon)$ is automatically *symmetric*[6] by [FRS02, Cor. 3.10] (see also [Sch13, Lem. 2.9] and the footnote therein). In this case, by [Sch13, Main Thm. or §4.1], up to scaling, there is a unique way to endow the indecomposable semisimple $\mathcal{C}$-module category $\mathcal{M} := \mathsf{Mod}_{\mathcal{C}}(A)$ with a pivotal trace.[7]

Question 3. *If (A, μ, i) is a connected separable algebra object in a unitary tensor category*[8] *$\mathcal{C}$, is A an irreducible Q-system*[9]*? This question has the following two equivalent formulations:*

[1]Following [EGNO15], a *tensor category* is a semisimple rigid monoidal category with simple unit object.

[2]Use [DMNO13, Prop. 2.7] when $\mathcal{C}$ is fusion.

[3]There are 3 equivalent definitions of normalized: (1) $\epsilon \circ i = \mathrm{id}_1$; (2) $m \circ \Delta = \mathrm{id}_A$; and (3) $\epsilon \circ i = \lambda \, \mathrm{id}_1$ and $\mu \circ \Delta = \lambda \, \mathrm{id}_A$. We use (3) as in [Müg03, Def. 3.13 and Prop. 5.8] and [BKLR15, Def. 3.2 and 3.8].

[4]A Frobnius algebra is called *special* if $\mu \circ \Delta$ is a non-zero scalar multiple of id_A [FRS02, Def. 3.4.i].

[5]The condition $\dim_{\mathcal{C}}(A) \neq 0$ is automatic when $\mathcal{C}$ is a spherical fusion category or a unitary tensor category; see Footnote 8 for the definition of unitary tensor category.

[6]A Frobenius algebra $(A, \mu, i, \Delta, \epsilon)$ in a pivotal category is called *symmetric* if $[(\epsilon \circ \mu) \otimes \mathrm{id}_{A^\vee}] \circ (\mathrm{id}_{A \otimes A} \otimes \mathrm{coev}_A) = [\mathrm{id}_{A^\vee} \otimes (\epsilon \circ \mu)] \circ (\varphi_A \otimes \mathrm{id}_A) \circ (\mathrm{coev}_{A^\vee} \otimes \mathrm{id}_A)$ where $\varphi_A \in \mathcal{C}(A \to A^{\vee\vee})$ is the pivotal structure [Sch13, Def. 2.7].

[7]See [Sch13] for the definition of a pivotal trace for a $\mathcal{C}$-module category. In the unitary setting, we call such a trace *unitary* if it gives us a positive definite inner product in the usual GNS way.

[8]A *unitary tensor category* is a rigid C* tensor category which is Karoubi complete with simple unit object. A unitary tensor category has a canonical spherical structure by [Yam04, Thm. 4.7] and [BDH14, §4].

[9]A C* *Frobenius algebra* in a unitary tensor category is an algebra object (A, μ, i) such that (A, μ, i, μ^*, i^*) is a Frobenius algebra. We call a C* Frobenius algebra *standard* or a *Q-system* if $i^* \circ \mu$ and $\mu^* \circ i$ are standard solutions to the conjugate equations. In this case, $\mu \circ \mu^* = \sqrt{\dim_{\mathcal{C}}(A)} \, \mathrm{id}_A$, $i^* \circ i = \sqrt{\dim_{\mathcal{C}}(A)} \, \mathrm{id}_{1_{\mathcal{C}}}$, and $i^* \circ \mu \circ \mu^* \circ i = \dim_{\mathcal{C}}(A)$. A Q-system is called *irreducible* if the underlying algebra object is connected.

- *Is Δ from Lemma 1 proportional to μ^*?*
- *Is μ^* automatically an $A-A$ bimodule map?*

This question motivates the following definition:

Definition 4. An algebra object (A, μ, i) in a unitary tensor category is called *unitarily separable* [10] if $\mu \circ \mu^* = \lambda \, \mathrm{id}_A$ for some $\lambda > 0$. We further say (A, μ, i) is *normalized* if $\mu \circ \mu^* = \sqrt{d_A} \, \mathrm{id}_A$.

Corollary 5. *A connected normalized unitarily separable algebra (A, μ, i) in a unitary tensor category is an irreducible Q-system.*

Proof. That A is a C* Frobenius algebra from [BKLR15, Lem. 3.7]. That A is standard follows from [Müg03, Rem. 5.6.3]. □

Now assume that $\mathcal{C}$ is a pseudounitary fusion category or a unitary tensor category.[11]

Theorem 6. *There is a canonical bijection*[12] *between equivalence classes of:*

(1) 2-shaded subfactor planar algebras with $\dim(\mathcal{P}_{1,\pm}) = 1$ and principal even part equivalent to $\mathcal{C}$.

(2) indecomposable 2×2 spherical multifusion / unitary multitensor[13] *categories $\mathcal{D}$ such that $1_{\mathcal{D}} \cong 1_1 \oplus 1_2$ and $\mathcal{C} \cong 1_1 \otimes \mathcal{D} \otimes 1_1$, with chosen simple $m \in 1_1 \otimes \mathcal{D} \otimes 1_2$ which generates $\mathcal{D}$.*

(3) pointed[14] *indecomposable $\mathcal{C}$-module C* categories $(\mathcal{M}, m)$ with normalized unitary pivotal trace.*

(4) pointed indecomposable $\mathcal{C}$-module C categories $(\mathcal{M}, m)$.*

(5) connected normalized unitarily separable algebra objects $(A, \mu, i) \in \mathcal{C}$ which generate[15] *$\mathcal{C}$.*

(6) connected normalized separable symmetric Frobenius algbera objects $(A, \mu, i, \Delta, \epsilon) \in \mathcal{C}$ / irreducible Q-systems $(A, \mu, i) \in \mathcal{C}$ which generate $\mathcal{C}$.

Sketch of the proof.

$\underline{(1) \Leftrightarrow (2):}$ This is a rewording of the main result of [Gho11].

$\underline{(2) \Rightarrow (3):}$ Take $\mathcal{M} = 1_1 \otimes \mathcal{D} \otimes 1_2$ and basepoint m.

$\underline{(3) \Rightarrow (4):}$ Forget the trace.

$\underline{(4) \Rightarrow (5):}$ Take $A := \underline{\mathrm{End}}_{\mathcal{C}}(m)$. Then $\mathcal{M} \cong \mathsf{Mod}_{\mathcal{C}}(A)$ as $\mathcal{C}$-module categories [EGNO15, Thm. 7.10.1]. Then A is connected by [EGNO15, Lem. 7.8.12] and separable by [DMNO13, Prop. 2.7]. In the unitary setting, A can be endowed with the structure of an irreducible Q-system such that $\mathcal{M}$ is dagger equivalent to the $\mathcal{C}$-module C* category $\mathsf{Mod}_{\mathcal{C}}(A)$ by [NY17, Thm. A.1].

$\underline{(5) \Rightarrow (6):}$ This is Lemma 1. In the unitary setting, this is Corollary 5.

[10] For a C* Frobenius algebra, this property is called being *special* as in Footnote 4; we reserve this terminology for Frobenius algebras.

[11] We use this hypothesis to ensure that $\dim_{\mathcal{C}}(A) \neq 0$. This is not always true in a spherical semisimple tensor category; there is a counter-example in the free product of the rank 2 Fibonacci and Yang-Lee categories.

[12] One can probably extend this to an equivalence of categories.

[13] Here, *multitensor* means semisimple rigid monoidal, but not necessarily simple unit object. We say 2×2 to indicate that $1_{\mathcal{D}}$ decomposes into two distinct simples: $1_{\mathcal{D}} \cong 1_1 \oplus 1_2$.

[14] A *basepoint* for a semisimple $\mathcal{C}$-module category $\mathcal{M}$ is a distinguished object $m \in \mathcal{M}$. A *pointed* $\mathcal{C}$-module category is a pair $(\mathcal{M}, m)$ where $\mathcal{M}$ is a semisimple $\mathcal{C}$-module category and $m \in \mathcal{M}$ is a simple basepoint.

[15] We say A *generates* $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to a an object obtained from A via direct sums, tensor products, subobjects, and duals.

(6) ⇒ (2) : This is an adaptation of [Müg03, Thm. 5.12].[16] For irreducible Q-systems, we use [Müg03, Prop. 5.5] and [Müg03, Thm. 5.16].[17] □

Remarks 7. Here are some further equivalences:

- The non-unitary proof of (1) ⇔ (4) follows from [Sch13, Main Thm.].
- There is a notion of C*-algebra object in the ind-category of a rigid C*-tensor category from [JP17a]. The main result of [JP17b] gives an equivalence of categories between irreducible Q-systems in $\mathcal{C}$ from (6) in Theorem 6 and connected C*-algebra objects in $\mathcal{C}^\natural$, the fusion category obtained from $\mathcal{C}$ by forgetting the dagger structure.

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