Jones' basic construction in finite dimensions

The exercises marked $(*)$ below are more advanced and can be skipped on first read through. For this handout, $A \subset B$ will always denote a unital inclusion of finite dimensional complex multimatrix algebras.

1 Conditional expectations

Pick faithful tracial states tr_B on B and tr_A on A.

Definition 1. A *conditional expectation* $E : B \to A$ is a linear map such that

- ($A - A$ bilinear) $E(axb) = aE(x)b$ for all $x \in B$ and $a, b \in A$.
- (unital) $E(1) = 1$, and
- (projection) $E^2 = E$, i.e., for all $x \in B$, $E(E(x)) = E(x)$.

A conditional expectation is called:

- *trace preserving* if $\operatorname{tr}_A(E(x)) = \operatorname{tr}_B(x)$ for all $x \in B$.
- *faithful* if $E(x^*x) = 0$ implies $x = 0$.

Exercise 2. Prove that $E|_A = \operatorname{id}_A$. Deduce that if $E : B \to A$ is trace preserving, then $\operatorname{tr}_B|_A = \operatorname{tr}_A$.

Exercise 3. Show that if $E, F : B \to A$ are two conditional expectations such that for all $a \in A$ and $b \in B$, $\operatorname{tr}_A(aE(b)) = \operatorname{tr}_A(aF(b))$, then $E = F$. Deduce that there is at most *one* trace preserving conditional expectation $B \to A$.
Hint: Show that for all $b \in B$ and $a \in A$, $\langle E(b)\Omega, a\Omega\rangle = \langle F(b)\Omega, a\Omega\rangle$ in $L^2(A, \operatorname{tr}_A)$.

Exercise 4. Suppose E is trace preserving. Show that $E(x^*) = E(x)^*$ for all $x \in B$.
Hint: First prove that $\operatorname{tr}_B(x^*) = \overline{\operatorname{tr}_B(x)}$ for all $x \in B$. Then show $\langle E(x)^*\Omega, a\Omega\rangle = \langle E(x^*)\Omega, a\Omega\rangle$ for all $x \in B$ and $a \in A$.

Exercise 5. Suppose E is trace preserving. Show that for any $x \in B$, $E(x^*x) \geq 0$.
Hint: Compute $\langle E(x^*x)a\Omega, a\Omega\rangle$ in $L^2(A, \operatorname{tr}_A)$.

Exercise 6. Consider the subspace $A\Omega \subset L^2(B, \operatorname{tr}_B)$. Let $e_A \in B(L^2(B, \operatorname{tr}_B))$ be the orthogonal projection onto $A\Omega$. Define $E : B \to A$ by $E(b) = a$ where $a \in A$ is the unique element such that $e_A(b\Omega) = a\Omega$. Prove that E is a faithful conditional expectation. Prove that E is trace preserving if and only if $\operatorname{tr}_B|_A = \operatorname{tr}_A$.

Exercise 7. Continue the notation of Exercise 6.

(1) Show that for all $b \in B$, $E(b)e_A = e_A b e_A$.

(2) Show that for all $b \in B$, we have $b \in A$ if and only if $e_A b = b e_A$.

Exercise 8. Compute the unique trace preserving conditional expectation for the following unital inclusions:

(1) The inclusion $M_k(\mathbb{C}) \hookrightarrow M_{nk}(\mathbb{C})$ with the unique normalized traces.

(2) The connected inclusion $A = M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \hookrightarrow M_{n+k}(\mathbb{C}) = B$ with trace vector on A given by $(\frac{1}{n+k}, \frac{1}{n+k})$ and the unique normalized trace on B.
Note: First verify that $n_A \lambda_A = 1$ where n_A denotes the dimension row vector of A.

(3) The connected inclusion $A = \mathbb{C} \oplus \mathbb{C} \hookrightarrow M_2(\mathbb{C}) \oplus \mathbb{C} = B$ with Bratteli diagram and trace vectors for A and B given by

$$\Lambda := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \lambda_a := (\phi^{-2}, \phi^{-1}) \qquad \lambda_b := (\phi^{-2}, \phi^{-3})$$

where $\phi := \frac{1+\sqrt{5}}{2}$ (so $\phi^2 = 1 + \phi$).
Note: First verify that $n_A \lambda_A = 1 = n_B \lambda_B$ where n_A, n_B denotes the dimension row vector of A, B respectively.

2 The basic construction

Suppose $A \subset B$ is a unital inclusion of multimatrix algebras and tr_B is a faithful normal trace on B. Define $\mathrm{tr}_A = \mathrm{tr}_B|_A$, and let $e_A \in B(L^2(B, \mathrm{tr}_B))$ be the orthogonal projection with range $L^2(A, \mathrm{tr}_A) = A\Omega$ as in Exercise 6. Let $E : B \to A$ be the canonical trace-preserving conditional expectation, which is defined by $E(b)\Omega := e_A(b\Omega)$ for all $b \in B$.

Definition 9. The *basic construction* of $A \subset B$ is the unital $*$-subalgebra $\langle B, e_A \rangle \subset B(L^2(B, \mathrm{tr}_B))$ generated by B and e_A.

Exercise 10. Prove that $\langle B, e_A \rangle = B + Be_AB = \mathrm{span}\,\{a + be_Ac | a, b, c \in B\} \subset B(L^2(B, \mathrm{tr}_B))$.

Define $J : L^2(B, \mathrm{tr}_B) \to L^2(B, \mathrm{tr}_B)$ by $Jb\Omega := b^*\Omega$.

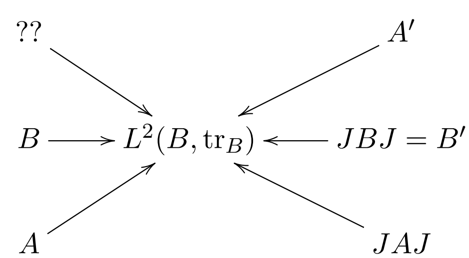
Exercise 11. Prove that for all $a, b \in B$, $\langle Ja\Omega, b\Omega \rangle = \langle Jb\Omega, a\Omega \rangle$.

Exercise 12. Use Exercise 4 to show that $Je_A = e_AJ$ on $L^2(B, \mathrm{tr}_B)$.

Exercise 13. Recall $A' = \left\{x \in B(L^2(B, \mathrm{tr}_B)) \middle| xa = ax \text{ for all } a \in A\right\}$. Show that $JA'J = (JAJ)'$.

Exercise 14. Show that $\langle M, e_A \rangle = JA'J$.

Using this last exercise, we see that the basic construction algebra naturally arises as the missing algebra in the following picture.



3 The inclusion $B \subset \langle B, e_A \rangle$

We now compute the Bratteli diagram for the inclusion $B \subset \langle B, e_A \rangle$ in terms of the Bratteli diagram for $A \subset B$.

4 Traces on the basic construction

5 Loop algebras

6 Pimsner-Popa bases

References