Jones’ basic construction in finite dimensions

The exercises marked (∗) below are more advanced and can be skipped on first read through. For this handout, \( A \subset B \) will always denote a unital inclusion of finite dimensional complex multimatrix algebras.

1 Conditional expectations

Pick faithful tracial states \( \text{tr}_B \) on \( B \) and \( \text{tr}_A \) on \( A \).

**Definition 1.** A *conditional expectation* \( E : B \rightarrow A \) is a linear map such that

- \( (A - A \text{ bilinear}) \ E(axb) = aE(x)b \) for all \( x \in B \) and \( a, b \in A \).
- \( (\text{unital}) \ E(1) = 1 \), and
- \( (\text{projection}) \ E^2 = E \), i.e., for all \( x \in B \), \( E(E(x)) = E(x) \).

A conditional expectation is called:

- *trace preserving* if \( \text{tr}_A(E(x)) = \text{tr}_B(x) \) for all \( x \in B \).
- *faithful* if \( E(x^*x) = 0 \) implies \( x = 0 \).

**Exercise 2.** Prove that \( E|_A = \text{id}_A \). Deduce that if \( E : B \rightarrow A \) is trace preserving, then \( \text{tr}_B |_A = \text{tr}_A \).

**Exercise 3.** Show that if \( E,F : B \rightarrow A \) are two conditional expectations such that for all \( a \in A \) and \( b \in B \), \( \text{tr}_A(aE(b)) = \text{tr}_A(aF(b)) \), then \( E = F \). Deduce that there is at most one trace preserving conditional expectation \( B \rightarrow A \).

*Hint:* Show that for all \( b \in B \) and \( a \in A \), \( \langle E(b)\Omega,a\Omega \rangle = \langle F(b)\Omega,a\Omega \rangle \) in \( L^2(A,\text{tr}_A) \).

**Exercise 4.** Suppose \( E \) is trace preserving. Show that \( E(x^*) = E(x)^* \) for all \( x \in B \).

*Hint:* First prove that \( \text{tr}_B(x^*) = \text{tr}_B(x) \) for all \( x \in B \). Then show \( \langle E(x^*)\Omega,a\Omega \rangle = \langle E(x)^*\Omega,a\Omega \rangle \) for all \( x \in B \) and \( a \in A \).

**Exercise 5.** Suppose \( E \) is trace preserving. Show that for any \( x \in B \), \( E(x^*x) \geq 0 \).

*Hint:* Compute \( \langle E(x^*x)a\Omega,a\Omega \rangle \) in \( L^2(A,\text{tr}_A) \).

**Exercise 6.** Consider the subspace \( \text{A}\Omega \subset L^2(B,\text{tr}_B) \). Let \( e_A \in B(L^2(B,\text{tr}_B)) \) be the orthogonal projection onto \( \text{A}\Omega \). Define \( E : B \rightarrow A \) by \( E(b) = a \) where \( a \in A \) is the unique element such that \( e_A(b\Omega) = a\Omega \). Prove that \( E \) is a faithful conditional expectation. Prove that \( E \) is trace preserving if and only if \( \text{tr}_B |_A = \text{tr}_A \).

**Exercise 7.** Continue the notation of Exercise 6.

1. Show that for all \( b \in B \), \( E(b)e_A = e_Abe_A \).
2. Show that for all \( b \in B \), we have \( b \in A \) if and only if \( e_Ab = be_A \).

**Exercise 8.** Compute the unique trace preserving conditional expectation for the following unital inclusions:

1. The inclusion \( M_k(\mathbb{C}) \hookrightarrow M_{nk}(\mathbb{C}) \) with the unique normalized traces.
The connected inclusion \( A = M_n(\mathbb{C}) \oplus M_k(\mathbb{C}) \hookrightarrow M_{n+k}(\mathbb{C}) = B \) with trace vector on \( A \) given by \( \frac{1}{n-k}, \frac{1}{n-k} \) and the unique normalized trace on \( B \).

Note: First verify that \( n_A \lambda_A = 1 \) where \( n_A \) denotes the dimension row vector of \( A \).

The connected inclusion \( A = \mathbb{C} \oplus \mathbb{C} \hookrightarrow M_2(\mathbb{C}) \oplus \mathbb{C} = B \) with Bratteli diagram and trace vectors for \( A \) and \( B \) given by

\[
\Lambda := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \lambda_a := (\phi^{-2}, \phi^{-1}), \quad \lambda_b := (\phi^{-2}, \phi^{-3})
\]

where \( \phi := \frac{1+\sqrt{5}}{2} \) (so \( \phi^2 = 1 + \phi \)).

Note: First verify that \( n_A \lambda_A = 1 = n_B \lambda_B \) where \( n_A, n_B \) denotes the dimension row vector of \( A, B \) respectively.

## 2 The basic construction

Suppose \( A \subset B \) is a unital inclusion of multimatrix algebras and \( \text{tr}_B \) is a faithful normal trace on \( B \). Define \( \text{tr}_A = \text{tr}_B \mid_A \), and let \( e_A \in B(L^2(B, \text{tr}_B)) \) be the orthogonal projection with range \( L^2(A, \text{tr}_A) = A\Omega \) as in Exercise 6. Let \( E : B \to A \) be the canonical trace-preserving conditional expectation, which is defined by \( E(b)\Omega := e_A(b\Omega) \) for all \( b \in B \).

**Definition 9.** The **basic construction** of \( A \subset B \) is the unital \( \ast \)-subalgebra \( \langle B, e_A \rangle \subset B(L^2(B, \text{tr}_B)) \) generated by \( B \) and \( e_A \).

**Exercise 10.** Prove that \( \langle B, e_A \rangle = B + Be_A B = \text{span} \{a + be_{AC}a, b, c \in B \} \subset B(L^2(B, \text{tr}_B)) \).

Define \( J : L^2(B, \text{tr}_B) \to L^2(B, \text{tr}_B) \) by \( Jb\Omega := b^*\Omega \).

**Exercise 11.** Prove that for all \( a, b \in B \), \( \langle Ja\Omega, b\Omega \rangle = \langle Jb\Omega, a\Omega \rangle \).

**Exercise 12.** Use Exercise 4 to show that \( Je_A = e_A J \) on \( L^2(B, \text{tr}_B) \).

**Exercise 13.** Recall \( A' = \{ x \in B(L^2(B, \text{tr}_B)) \mid xa = ax \text{ for all } a \in A \} \). Show that \( JA'J = (JAJ)' \).

**Exercise 14.** Show that \( \langle M, e_A \rangle = JA'J \).

Using this last exercise, we see that the basic construction algebra naturally arises as the missing algebra in the following picture.

\[
\begin{array}{ccc}
?? & \overset{A'}{\longrightarrow} \\
B & \longrightarrow & L^2(B, \text{tr}_B) & \longrightarrow & JB = B' \\
& \langle B, e_A \rangle & \leftarrow & JBJ = B' & \langle B, e_A \rangle \\
\end{array}
\]

## 3 The inclusion \( B \subset \langle B, e_A \rangle \)

We now compute the Bratteli diagram for the inclusion \( B \subset \langle B, e_A \rangle \) in terms of the Bratteli diagram for \( A \subset B \).

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4 Traces on the basic construction

5 Loop algebras

6 Pimsner-Popa bases

References