The proof of [JP15, Lem. 5.2] is not correct, as it was based on an incorrect proof of [Bur03, Prop. 3.2.19] (Burns' PhD thesis). I realized this when I was asked for a referee report for Burns' thesis. That article is now published as [Bur17], where the corresponding proposition (Prop. 5.21) has been corrected, with help from me and Jesse Peterson. We provide the setup below.

Suppose $P \subset Q$ is an inclusion of semifinite factors acting on a Hilbert space H, and Tr_P and Tr_Q are normal faithful semifinite tracial weights on P and Q respectively. By [Haa79], there is a unique trace preserving operator valued weight $T : Q^+ \to \widehat{P^+}$. Recall that $\mathfrak{p}_{\mathrm{Tr}_Q} = \{x \in Q^+ | \mathrm{Tr}_Q(x) < \infty\}$ (and similarly for P), and $\mathfrak{p}_T = \{x \in Q^+ | T(x) \in P^+\}$.

Lemma ([Bur17, Lem. 5.19], proof due to Jesse Peterson)**.** *There is an* $x \in \mathfrak{p}_T$ *with* $\ker(x) = (0)$.

Immediate consequence of this lemma are the following results:

- [Bur17, Lem. 5.20]: There is a sequence of projections $(p_n) \subset \mathfrak{p}_T$ such that $p_n \nearrow 1$.
- [Bur17, Prop. 5.21]: Given a II_1 subfactor $N \subset M$ of infinite index, there exists an orthonormal M_N-basis, i.e., a subset $\{b\} \subset M$ such that $\sum be_N b^* = 1$, where the $be_N b^*$ are mutually orthogonal projections.

We now use this technique to give a correct proof of [JP15, Lem. 5.2]. We do so in slightly more generality.

Lemma 1. *Suppose* (A, Tr_A) *and* (B, Tr_B) *are semifinite tracial von Neumann algebras with* $A \subseteq B$*, and let* $T : B^+ \to \widehat{A^+}$ *be the unique trace preserving operator valued weight. There is an* $x \in \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ *with* $\ker(x) = (0)$.

Proof. We mimic Jesse Peterson's proof of [Bur17, Lem. 5.19].

Since Tr_B is semi-finite, let $(p_i) \in \mathfrak{m}_{\mathrm{Tr}_B}$ be a sequence of projections with $p_i \nearrow 1$. Since $\mathrm{Tr}_B(p_i) < \infty$, $T(p_i)$ has a spectral resolution

$$T(p_i) = \int_0^\infty \lambda de_\lambda^i.$$

Note that $e_j^i \nearrow 1$ as $j \nearrow \infty$, and $e_j^i p_i e_j^i \in \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ for all $j \geq 0$.

For $i, j \in \mathbb{N}$, pick $\alpha_{i,j} > 0$ such that $y = \sum_{i,j} \alpha_{i,j} e_j^i T(p_i) e_j^i$ converges SOT in A^+, and $x = \sum_{i,j} \alpha_{i,j} e_j^i p_i e_j^i$ converges SOT in $\mathfrak{p}_{\mathrm{Tr}_B} \subset B_+$. This can be done by choosing the $\alpha_{i,j}$ such that

- $\sum_{i,j} \alpha_{i,j} \|e_j^i T(p_i) e_j^i\| < \infty$ so $y \in A_+$,
- $\sum_{i,j} \alpha_{i,j} \|e_j^i p_i e_j^i\| \leq \sum_{i,j} \alpha_{i,j} < \infty$ so $x \in B_+$, and
- $\sum_{i,j} \alpha_{i,j} \mathrm{Tr}_B(e_j^i p_i e_j^i) < \infty$ so $x \in \mathfrak{p}_{\mathrm{Tr}_B}$.

Then by normality of T, we have $T(x) = y$, so $x \in \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$.

We claim that $\ker(x) = (0)$. Let $\xi \in H \setminus \{0\}$. Since $p_i \nearrow 1$, there is an $i \in \mathbb{N}$ such that $p_i \xi \neq 0$. Fixing this i, we have that $p_i e_j^i \to p_i$ SOT as $j \nearrow \infty$, so $p_i e_j^i \xi \to p_i \xi \neq 0$ as $j \nearrow \infty$. Hence there is a $j \in \mathbb{N}$ such that $p_i e_j^i \xi \neq 0$, so $\xi \notin \ker(p_i e_j^i) = \ker(e_j^i p_i e_j^i)$. It follows that $x\xi \neq 0$, and we are finished. □

Corollary 2. *There is a sequence of projections* $(p_n) \subset \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ *with* $p_n \nearrow 1$.

Proof. Let $x \in \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ be as in Lemma 1. Define $p_n = \chi_{[1/n,\infty)}(x)$ using the L^∞-functional calculus. Since $\mathfrak{p}_T$ and $\mathfrak{p}_{\mathrm{Tr}_B}$ are hereditary, $p_n \in \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ for all $n \in \mathbb{N}$, and clearly $p_n \nearrow 1$ since $\ker(x) = 0$. □

Theorem 3. *Suppose ${}_AH_B$ is an $A - B$ bimodule. There exists an orthonormal H_B-basis consisting of bi-bounded vectors in $H^\circ = {}_AH^\circ \cap H_B^\circ$.*

Proof. Using Corollary 2 we can find a sequence of mutually orthogonal projections $(q_n) \subset \mathfrak{p}_T \cap \mathfrak{p}_{\mathrm{Tr}_B}$ such that $\sum_n q_n = 1$, where the sum converges SOT. We claim that for every $n \in \mathbb{N}$ and $\eta \in H_B^\circ$, $q_n\eta \in {}_AH^\circ$. Indeed, for all $a \in A$ and $n \in \mathbb{N}$, $aq_n\eta \in H_B^\circ$, and thus

$$\begin{aligned}\|aq_n\eta\|_H^2 &= \mathrm{Tr}_{(B^{\mathrm{op}})'}(L_{aq_n\eta}L_{aq_n\eta}^*) = \mathrm{Tr}_{(B^{\mathrm{op}})'}(aq_nL_\eta L_\eta^* q_n a) = \mathrm{tr}_A(aT(q_nL_\eta L_\eta^* q_n)a)\\ &\leq \|\eta\|_H^2 \,\mathrm{tr}_A(aT(q_n)a^*) \leq \|\eta\|_H^2 \cdot \|T(q_n)\|_{A_+} \|a\Omega\|_{L^2A}^2.\end{aligned}$$

Thus for every $n \in \mathbb{N}$, each q_nH_B-basis consists of bi-bounded vectors in H°. Thus a disjoint union of orthonormal q_nH_B-bases does the trick. □

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