

# Markov sequences of von Neumann algebras

## 1 Markov sequences

**Definition 1.** A *Markov sequence* consists of a sequence  $(M_n, \text{tr}_n)_{n \geq 0}$  of finite dimensional von Neumann algebras with faithful normal tracial states such that  $\text{tr}_{n+1}|_{M_n} = \text{tr}_n$  for all  $n \geq 0$ , and a sequence of *Jones projections*  $e_n \in M_{n+1}$  for all  $n \geq 1$  such that:

- the projections  $(e_n)$  satisfy the Temperley-Lieb-Jones relations:
  - (1)  $e_i^2 = e_i = e_i^*$  for all  $i$ ,
  - (2)  $e_i e_j = e_j e_i$  for  $|i - j| > 1$ , and
  - (3) there is a fixed constant  $d > 0$  such that  $e_i e_{i \pm 1} e_i = d^{-2} e_i$  for all  $i$ .
- for all  $x \in M_n$ ,  $e_n x e_n = E_n(x) e_n$  where  $E_n : M_n \rightarrow M_{n-1}$  is the canonical trace-preserving conditional expectation.
- (pull down) For  $n \geq 1$ ,  $M_{n+1} e_n = M_n e_n$ .

**Exercise 2.** Prove that the pull down condition is equivalent to  $M_n e_n M_n$  is a 2-sided ideal in  $M_{n+1}$  for all  $n \geq 1$ .

## 2 Elementary properties of Markov sequences

Markov sequence satisfies the following elementary properties for  $n \geq 1$ .

- (A) The map  $M_n \ni y \mapsto y e_n \in M_{n+1}$  is injective.
- (B)  $E_{n+1}(e_n) = d^{-2}$ .
- (C) For all  $x \in M_{n+1}$ ,  $d^2 E_{n+1}(x e_n)$  is the unique element  $y \in M_n$  such that  $x e_n = y e_n$  [PP86, Lem. 1.2].
- (D) The traces  $\text{tr}_{n+1}$  satisfy the following *Markov property* with respect to  $M_n$  and  $e_n$ : for all  $x \in M_n$ ,  $\text{tr}_{n+1}(x e_n) = d^{-2} \text{tr}_n(x)$ .
- (E)  $e_n M_{n+1} e_n = M_{n-1} e_n$ .
- (F)  $X_{n+1} := M_n e_n M_n$  is a 2-sided ideal of  $M_{n+1}$ , and  $M_{n+1}$  splits as a direct sum of von Neumann algebras  $X_{n+1} \oplus Y_{n+1}$ . (In [GdlHJ89, Thm. 4.1.4 and Thm. 4.6.3],  $Y_{n+1}$  is the so-called ‘new stuff’.) By convention, we define  $Y_0 = M_0$  and  $Y_1 = M_0$ , so that  $X_0 = (0)$  and  $X_1 = (0)$ .
- (G) The map  $a e_n b \mapsto a p_n b$  gives a  $*$ -isomorphism from  $X_{n+1} = M_n e_n M_n$  to  $\langle M_n, p_n \rangle = M_n p_n M_n$ , the Jones basic construction of  $M_{n-1} \subseteq M_n$  acting on  $L^2(M_n, \text{tr}_n)$ .
- (H) Under the isomorphism  $X_{n+1} \cong M_n p_n M_n$ , the canonical non-normalized trace  $\text{Tr}_{n+1}$  on the Jones basic construction algebra  $M_n p_n M_n$  satisfying  $\text{Tr}_{n+1}(a p_n b) = \text{tr}_n(ab)$  for  $a, b \in M_n$  equals  $d^2 \text{tr}_{n+1}|_{X_{n+1}}$ .
- (I) If  $y \in Y_{n+1}$  and  $x \in X_n$ , then  $yx = 0$  in  $M_{n+1}$ . Hence  $E_{n+1}(Y_{n+1}) \subseteq Y_n$ . (“The new stuff comes only from the old new stuff” [GdlHJ89].)

(J) If  $Y_n = (0)$ , then  $Y_k = (0)$  for all  $k \geq n$ .

**Exercise 3.** Prove Elementary Properties (A) – (J) above for  $n \geq 1$ .

**Exercise 4.** Show that for  $n \geq 1$ , the Bratteli diagram for the inclusion  $M_n \subset M_{n+1}$  consists of the reflection of the Bratteli diagram for the inclusion  $M_{n-1} \subset M_n$ , together with possibly some new edges and vertices corresponding to simple summands of  $Y_{n+1}$ . Moreover, show that the new vertices at level  $n + 1$  only connect to the vertices that were new at level  $n$ .

**Definition 5.** The *principal graph* of the Markov sequence  $(M_n, \text{tr}_n)$  with Jones projections  $(e_n)$  consists of the *new* vertices at every level  $n$  of the Bratteli diagram, together with all the edges connecting them.

The sequence is said to have *finite depth* if the principal graph is finite.

**Exercise 6.** Show that the Markov sequence has finite depth if and only if there is an  $n \in \mathbb{N}$  such that  $Y_n = (0)$  as in Elementary Property (J).

**Exercise 7.** Suppose our Markov sequence has finite depth. Let  $n \in \mathbb{N}$  such that  $Y_n = (0)$  as in the previous exercise. Show that for all  $k \geq n$ , the Bratteli diagram for the inclusion  $M_{k-1} \subset M_k$  can be canonically identified with the principal graph.

**Exercise 8.** Show that the Temperley-Lieb algebras of modulus  $d \geq 2$  with the usual Jones projections form a Markov sequence.

### 3 Example from unitary multifusion categories

Let  $\tilde{\mathcal{C}}$  be a  $3 \times 3$  unitary multifusion category where  $1_{\tilde{\mathcal{C}}} = 1_1 \oplus 1_2 \oplus 1_3$ , and let  $\mathcal{C}$  be the (non-unital)  $2 \times 2$  unitary multifusion subcategory with  $1_{\mathcal{C}} = 1_1 \oplus 1_2$ . We get a unitary *right*  $\mathcal{C}$ -module category by considering  $\mathcal{M} = \mathcal{C}_{31} \oplus \mathcal{C}_{32}$ .

$$\mathcal{M} = (\mathcal{C}_{31} \quad \mathcal{C}_{32}) \quad \mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} \subset \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} \end{pmatrix} = \tilde{\mathcal{C}}$$

Fix a simple object  $x \in \mathcal{C}_{12}$ .

**Notation 9.** For  $n \in \mathbb{N}$ , we define the  $n$ -fold *alternating* tensor product of  $x$  by

$$x^{\text{alt}\otimes n} := \underbrace{x \otimes \bar{x} \otimes x \otimes \cdots \otimes x^?}_{n \text{ tensorands}},$$

where  $x^? = \bar{x}$  if  $n$  is even and  $x$  if  $n$  is odd. Similarly, we write

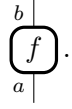
$$\bar{x}^{\text{alt}\otimes n} := \underbrace{\bar{x} \otimes x \otimes \bar{x} \otimes \cdots \otimes \bar{x}^?}_{n \text{ tensorands}},$$

where  $\bar{x}^? = x$  if  $n$  is even and  $\bar{x}$  if  $n$  is odd.

Since  $\mathcal{C}$  is indecomposable as a multifusion category, we see that every object of  $\mathcal{C}_{11}$  or  $\mathcal{C}_{12}$  is isomorphic to a direct summand of  $x^{\text{alt}\otimes n}$  for some  $n \in \mathbb{N}$ . In this sense, we say  $x$  *generates*  $\mathcal{C}$ .

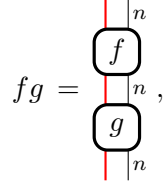
Now fix any simple  $m \in \mathcal{C}_{31} \subset \mathcal{M}$ . Since  $\tilde{\mathcal{C}}$  is indecomposable as a multifusion category, every object of  $\mathcal{M}$  is isomorphic to a direct summand of  $m \otimes x^{\text{alt}\otimes n}$  for some  $n \in \mathbb{N}$ .

**Notation 10.** We read our string diagrams bottom to top. We denote a morphism  $f : a \rightarrow b$  by a coupon



Composition of morphisms corresponds to vertical stacking of coupons, and tensor product of morphisms corresponds to horizontal juxtaposition of coupons.

**Definition 11.** Let  $A_n := \text{End}_{\mathcal{C}}(m \otimes x^{\text{alt}\otimes n})$  equipped with the multiplication



where the red strand stands for  $m$ , the label  $n$  stands for the object  $x^{\text{alt}\otimes n}$ , and the label  $\bar{n}$  stands for the object  $\bar{x}^{\text{alt}\otimes n}$ . This is a finite dimensional von Neumann algebra. It comes with a faithful tracial state  $\text{tr}_n : A_n \rightarrow \mathbb{C}$  given by

$$\text{tr}_n(f) := \frac{1}{\dim_{\mathcal{C}}(m) \dim_{\mathcal{C}}(x)^n} \cdot \text{tr}_{\mathcal{C}}(f) = \frac{1}{\dim_{\mathcal{C}}(m)} \cdot \frac{1}{d^n} \cdot \left( \bar{m} \left( f \right) \bar{n} \right) \quad (1)$$

where  $d := \dim(x) = \dim(\bar{x})$ . Further down, a black strand with label 1 stands for either  $x$  or  $\bar{x}$ .

We have inclusions  $A_n \rightarrow A_{n+1}$  compatible with the traces given by



and trace preserving conditional expectations  $E_n : A_n \rightarrow A_{n-1}$  given by

$$\left( f \right) \mapsto \frac{1}{d} \cdot \left( \left( f \right) \right) \quad (3)$$

The Jones projection for the inclusion  $A_{n-1} \subset A_n$  is given by

$$e_n = \frac{1}{d} \cdot \left( \left| \begin{array}{c} \cup \\ n-1 \\ \cap \end{array} \right| \right) \in A_{n+1}. \quad (4)$$

**Exercise 12.** Explain why the diagram in (1) can be viewed as a scalar in  $\mathbb{C}$ .

*Hint: Recall  $m \in \mathcal{C}_{31}$  is simple.*

**Exercise 13.** Show that for all  $n \geq 1$ , prove that

(1) For all  $a, b \in A_{n-1}$  and  $x \in A_n$ ,  $E_n(axb) = aE_n(x)b$ .

(2)  $\text{tr}_n = \text{tr}_{n-1} \circ E_n$ .

(3)  $\text{tr}_n|_{A_{n-1}} = \text{tr}_{n-1}$ , where we identify  $A_{n-1}$  with its image in  $A_n$  under the inclusion map (2).

**Exercise 14.** Prove that the sequence  $(A_n, \text{tr}_n)$  with Jones projections  $(e_n)$  is Markov.

**Exercise 15.** Prove that the Markov sequence  $(A_n, \text{tr}_n)$  with Jones projections  $(e_n)$  has finite depth.

## References

- [GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F.R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989, x+288 pp. ISBN: 0-387-96979-9, [MR999799](#).
- [PP86] Mihai Pimsner and Sorin Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 57–106, [MR860811](#).