Markov sequences of von Neumann algebras

1 Markov sequences

Definition 1. A *Markov sequence* consists of a sequence $(M_n, \mathrm{tr}_n)_{n\geq 0}$ of finite dimensional von Neumann algebras with faithful normal tracial states such that $\mathrm{tr}_{n+1}|_{M_n} = \mathrm{tr}_n$ for all $n \geq 0$, and a sequence of *Jones projections* $e_n \in M_{n+1}$ for all $n \geq 1$ such that:

- the projections (e_n) satisfy the Temperley-Lieb-Jones relations:
  1. $e_i^2 = e_i = e_i^*$ for all i,
  2. $e_i e_j = e_j e_i$ for $|i-j| > 1$, and
  3. there is a fixed constant $d > 0$ such that $e_i e_{i\pm 1} e_i = d^{-2} e_i$ for all i.
- for all $x \in M_n$, $e_n x e_n = E_n(x) e_n$ where $E_n : M_n \to M_{n-1}$ is the canonical faithful trace-preserving conditional expectation.
- for all $n \geq 1$, $E_{n+1}(e_n) = d^{-2}$.
- (pull down) for all $n \geq 1$, $M_{n+1} e_n = M_n e_n$.

Exercise 2. Prove that the pull down condition is equivalent to $M_n e_n M_n$ is a 2-sided ideal in M_{n+1} for all $n \geq 1$.

2 Elementary properties of Markov sequences

Markov sequence satisfies the following elementary properties for $n \geq 1$.

(A) The map $M_n \ni y \mapsto y e_n \in M_{n+1}$ is injective.

(B) For all $x \in M_{n+1}$, $d^2 E_{n+1}(x e_n)$ is the unique element $y \in M_n$ such that $x e_n = y e_n$ [PP86, Lem. 1.2].

(C) The traces tr_{n+1} satisfy the following *Markov property* with respect to M_n and e_n: for all $x \in M_n$, $\mathrm{tr}_{n+1}(x e_n) = d^{-2}\,\mathrm{tr}_n(x)$.

(D) $e_n M_{n+1} e_n = M_{n-1} e_n$.

(E) $X_{n+1} := M_n e_n M_n$ is a 2-sided ideal of M_{n+1}, and M_{n+1} splits as a direct sum of von Neumann algebras $X_{n+1} \oplus Y_{n+1}$. (In [GdlHJ89, Thm. 4.1.4 and Thm. 4.6.3], Y_{n+1} is the so-called 'new stuff'.) By convention, we define $Y_0 = M_0$ and $Y_1 = M_0$, so that $X_0 = (0)$ and $X_1 = (0)$.

(F) The map $a e_n b \mapsto a p_n b$ gives a $*$-isomorphism from $X_{n+1} = M_n e_n M_n$ to $\langle M_n, p_n \rangle = M_n p_n M_n$, the Jones basic construction of $M_{n-1} \subseteq M_n$ acting on $L^2(M_n, \mathrm{tr}_n)$.

(G) Under the isomorphism $X_{n+1} \cong M_n p_n M_n$, the canonical non-normalized trace Tr_{n+1} on the Jones basic construction algebra $M_n p_n M_n$ satisfying $\mathrm{Tr}_{n+1}(a p_n b) = \mathrm{tr}_n(ab)$ for $a, b \in M_n$ equals $d^2\,\mathrm{tr}_{n+1}|_{X_{n+1}}$.

(H) If $y \in Y_{n+1}$ and $x \in X_n$, then $yx = 0$ in M_{n+1}. Hence $E_{n+1}(Y_{n+1}) \subseteq Y_n$. ("The new stuff comes only from the old new stuff" [GdlHJ89].)

(I) If $Y_n = (0)$, then $Y_k = (0)$ for all $k \geq n$.

Exercise 3. Prove Elementary Properties (A) – (I) above for $n \geq 1$.

Exercise 4. Show that for $n \geq 1$, the Bratteli diagram for the inclusion $M_n \subset M_{n+1}$ consists of the reflection of the Bratteli diagram for the inclusion $M_{n-1} \subset M_n$, together with possibly some new edges and vertices corresponding to simple summands of Y_{n+1}. Moreover, show that the new vertices at level $n+1$ only connect to the vertices that were new at level n.

Definition 5. The *principal graph* of the Markov sequence (M_n, tr_n) with Jones projections (e_n) consists of the *new* vertices at every level n of the Bratteli diagram, together with all the edges connecting them.

The sequence is said to have *finite depth* if the principal graph is finite.

Exercise 6. Show that the Markov sequence has finite depth if and only if there is an $n \in \mathbb{N}$ such that $Y_n = (0)$ as in Elementary Property (I).

Exercise 7. Suppose our Markov sequence has finite depth. Let $n \in \mathbb{N}$ such that $Y_n = (0)$ as in the previous exercise. Show that for all $k \geq n$, the Bratteli diagram for the inclusion $M_{k-1} \subset M_k$ can be canonically identified with the principal graph.

Exercise 8. Show that the Temperley-Lieb algebras of modulus $d \geq 2$ with the usual Jones projections form a Markov sequence.

3 Example from unitary multifusion categories

Let $\widetilde{\mathcal{C}}$ be a 3×3 unitary multifusion category where $1_{\widetilde{\mathcal{C}}} = 1_1 \oplus 1_2 \oplus 1_3$, and let $\mathcal{C}$ be the (non-unital) 2×2 unitary multifusion subcategory with $1_{\mathcal{C}} = 1_1 \oplus 1_2$. We get a unitary *right* $\mathcal{C}$-module category by considering $\mathcal{M} = \mathcal{C}_{31} \oplus \mathcal{C}_{32}$.

$$\mathcal{M} = \begin{pmatrix} \mathcal{C}_{31} & \mathcal{C}_{32} \end{pmatrix} \qquad\qquad \mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} \subset \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} \\ \mathcal{C}_{21} & \mathcal{C}_{22} & \mathcal{C}_{23} \\ \mathcal{C}_{31} & \mathcal{C}_{32} & \mathcal{C}_{33} \end{pmatrix} = \widetilde{\mathcal{C}}$$

Fix a simple object $x \in \mathcal{C}_{12}$.

Notation 9. For $n \in \mathbb{N}$, we define the n-fold *alternating* tensor product of x by

$$x^{\mathrm{alt}\otimes n} := \underbrace{x \otimes \overline{x} \otimes x \otimes \cdots \otimes x^?}_{n \text{ tensorands}},$$

where $x^? = \overline{x}$ if n is even and x if n is odd. Similarly, we write

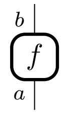
$$\overline{x}^{\mathrm{alt}\otimes n} := \underbrace{\overline{x} \otimes x \otimes \overline{x} \otimes \cdots \otimes \overline{x}^?}_{n \text{ tensorands}},$$

where $\overline{x}^? = x$ if n is even and $\overline{x}$ if n is odd.

Since $\mathcal{C}$ is indecomposable as a multifusion category, we see that every object of $\mathcal{C}_{11}$ or $\mathcal{C}_{12}$ is isomorphic to a direct summand of $x^{\mathrm{alt}\otimes n}$ for some $n \in \mathbb{N}$. In this sense, we say x *generates* $\mathcal{C}$.

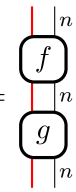
Now fix any simple $m \in \mathcal{C}_{31} \subset \mathcal{M}$. Since $\widetilde{\mathcal{C}}$ is indecomposable as a multifusion category, every object of $\mathcal{M}$ is isomorphic to a direct summand of $m \otimes x^{\mathrm{alt}\otimes n}$ for some $n \in \mathbb{N}$.

Notation 10. We read our string diagrams bottom to top. We denote a morphism $f : a \to b$ by a coupon



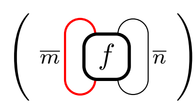
Composition of morphisms corresponds to vertical stacking of coupons, and tensor product of morphisms corresponds to horizontal juxtaposition of coupons.

Definition 11. Let $A_n := \mathrm{End}_{\mathcal{C}}(m \otimes x^{\mathrm{alt}\otimes n})$ equipped with the multiplication

$$fg =$$ 

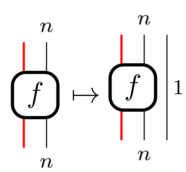
where the red strand stands for m, the label n stands for the object $x^{\mathrm{alt}\otimes n}$, and the label $\overline{n}$ stands for the object $\overline{x}^{\mathrm{alt}\otimes n}$. This is a finite dimensional von Neumann algebra. It comes with a faithful tracial state $\mathrm{tr}_n : A_n \to \mathbb{C}$ given by

$$\mathrm{tr}_n(f) := \frac{1}{\dim_{\mathcal{C}}(m)\dim_{\mathcal{C}}(x)^n} \cdot \mathrm{tr}_{\mathcal{C}}(f) = \frac{1}{\dim_{\mathcal{C}}(m)} \cdot \frac{1}{d^n} \cdot \left(\quad \right) \tag{1}$$



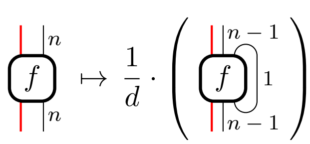
where $d := \dim(x) = \dim(\overline{x})$. Further down, a black strand with label 1 stands for either x or $\overline{x}$.

We have inclusions $A_n \to A_{n+1}$ compatible with the traces given by



(2)

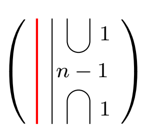
and trace preserving conditional expectations $E_n : A_n \to A_{n-1}$ given by



(3)

The Jones projection for the inclusion $A_{n-1} \subset A_n$ is given by

$$e_n = \frac{1}{d} \cdot \left(\quad \right) \in A_{n+1}. \tag{4}$$



Exercise 12. Explain why the diagram in (1) can be viewed as a scalar in $\mathbb{C}$.
Hint: Recall $m \in \mathcal{C}_{31}$ is simple.

Exercise 13. Show that for all $n \geq 1$, prove that

(1) For all $a, b \in A_{n-1}$ and $x \in A_n$, $E_n(axb) = aE_n(x)b$.

(2) $\mathrm{tr}_n = \mathrm{tr}_{n-1} \circ E_n$.

(3) $\mathrm{tr}_n |_{A_{n-1}} = \mathrm{tr}_{n-1}$, where we identify A_{n-1} with its image in A_n under the inclusion map (2).

Exercise 14. Prove that the sequence (A_n, tr_n) with Jones projections (e_n) is Markov.

Exercise 15. Prove that the Markov sequence (A_n, tr_n) with Jones projections (e_n) has finite depth.

References

[GdlHJ89] Frederick M. Goodman, Pierre de la Harpe, and Vaughan F.R. Jones, *Coxeter graphs and towers of algebras*, Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989, x+288 pp. ISBN: 0-387-96979-9, MR999799.

[PP86] Mihai Pimsner and Sorin Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 57–106, MR860811.