Markov sequences of von Neumann algebras

1 Markov sequences

Definition 1. A Markov sequence consists of a sequence \((M_n, tr_n)_{n\geq 0}\) of finite dimensional von Neumann algebras with faithful normal tracial states such that \(tr_{n+1}|_{M_n} = tr_n\) for all \(n \geq 0\), and a sequence of Jones projections \(e_n \in M_{n+1}\) for all \(n \geq 1\) such that:

- the projections \((e_n)\) satisfy the Temperley-Lieb-Jones relations:
  1. \(e_i^2 = e_i = e_i^*\) for all \(i\),
  2. \(e_ie_j = e_je_i\) for \(|i - j| < 1\), and
  3. there is a fixed constant \(d > 0\) such that \(e_i e_{i \pm 1} e_i = d^{-2} e_i\) for all \(i\).

- for all \(x \in M_n, e_n xe_n = E_n(x)e_n\) where \(E_n : M_n \to M_{n-1}\) is the canonical faithful trace-preserving conditional expectation.

- (pull down) for all \(n \geq 1\), \(M_{n+1}e_n = M_ne_n\).

Exercise 2. Prove that the pull down condition is equivalent to \(M_ne_nM_n\) is a 2-sided ideal in \(M_{n+1}\) for all \(n \geq 1\).

2 Elementary properties of Markov sequences

Markov sequence satisfies the following elementary properties for \(n \geq 1\).

(A) The map \(M_n \ni y \mapsto ye_n \in M_{n+1}\) is injective.

(B) For all \(x \in M_{n+1}\), \(d^2 E_{n+1}(xe_n)\) is the unique element \(y \in M_n\) such that \(xe_n = ye_n\) [PP86, Lem. 1.2].

(C) The traces \(tr_{n+1}\) satisfy the following Markov property with respect to \(M_n\) and \(e_n\): for all \(x \in M_n\), \(tr_{n+1}(xe_n) = d^{-2} tr_n(x)\).

(D) \(e_nM_{n+1}e_n = M_{n-1}e_n\).

(E) \(X_{n+1} = M_ne_nM_n\) is a 2-sided ideal of \(M_{n+1}\), and \(M_{n+1}\) splits as a direct sum of von Neumann algebras \(X_{n+1} \oplus Y_{n+1}\). (In [GdlHJ89, Thm. 4.1.4 and Thm. 4.6.3], \(Y_{n+1}\) is the so-called ‘new stuff’.) By convention, we define \(Y_0 = M_0\) and \(Y_1 = M_0\), so that \(X_0 = (0)\) and \(X_1 = (0)\).

(F) The map \(ae_n b \mapsto ap_n b\) gives a *-isomorphism from \(X_{n+1} = M_ne_nM_n\) to \(\langle M_n, p_n \rangle = M_np_nM_n\), the Jones basic construction of \(M_{n-1} \subseteq M_n\) acting on \(L^2(M_n, tr_n)\).

(G) Under the isomorphism \(X_{n+1} \cong M_np_nM_n\), the canonical non-normalized trace \(Tr_{n+1}\) on the Jones basic construction algebra \(M_np_nM_n\) satisfying \(Tr_{n+1}(ap_n b) = tr_n(ab)\) for \(a, b \in M_n\) equals \(d^2 tr_{n+1}|_{X_{n+1}}\).

(H) If \(y \in Y_{n+1}\) and \(x \in X_n\), then \(yx = 0\) in \(M_{n+1}\). Hence \(E_{n+1}(Y_{n+1}) \subseteq Y_n\). (“The new stuff comes only from the old new stuff” [GdlHJ89].)
(1) If $Y_n = (0)$, then $Y_k = (0)$ for all $k \geq n$.

**Exercise 3.** Prove Elementary Properties (A) – (I) above for $n \geq 1$.

**Exercise 4.** Show that for $n \geq 1$, the Bratteli diagram for the inclusion $M_n \subset M_{n+1}$ consists of the reflection of the Bratteli diagram for the inclusion $M_{n-1} \subset M_n$, together with possibly some new edges and vertices corresponding to simple summands of $Y_{n+1}$. Moreover, show that the new vertices at level $n + 1$ only connect to the vertices that were new at level $n$.

**Definition 5.** The principal graph of the Markov sequence $(M_n, \text{tr}_n)$ with Jones projections $(e_n)$ consists of the new vertices at every level $n$ of the Bratteli diagram, together with all the edges connecting them.

The sequence is said to have finite depth if the principal graph is finite.

**Exercise 6.** Show that the Markov sequence has finite depth if and only if there is an $n \in \mathbb{N}$ such that $Y_n = (0)$ as in Elementary Property (I).

**Exercise 7.** Suppose our Markov sequence has finite depth. Let $n \in \mathbb{N}$ such that $Y_n = (0)$ as in the previous exercise. Show that for all $k \geq n$, the Bratteli diagram for the inclusion $M_{k-1} \subset M_k$ can be canonically identified with the principal graph.

**Exercise 8.** Show that the Temperley-Lieb algebras of modulus $d \geq 2$ with the usual Jones projections form a Markov sequence.

### 3 Example from unitary multifusion categories

Let $\tilde{C}$ be a $3 \times 3$ unitary multifusion category where $1_{\tilde{C}} = 1_1 \oplus 1_2 \oplus 1_3$, and let $C$ be the (non-unital) $2 \times 2$ unitary multifusion subcategory with $1_C = 1_1 \oplus 1_2$. We get a unitary right $C$-module category by considering $\mathcal{M} = C_{31} \oplus C_{32}$.

$$\mathcal{M} = \begin{pmatrix} C_{31} & C_{32} \end{pmatrix} \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \subset \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} = \tilde{C}$$

Fix a simple object $x \in C_{12}$.

**Notation 9.** For $n \in \mathbb{N}$, we define the $n$-fold alternating tensor product of $x$ by

$$x^{\text{alt} \otimes n} := \underbrace{x \otimes x \otimes \cdots \otimes x}_{n \text{ tensorands}},$$

where $x^? = x$ if $n$ is even and $x$ if $n$ is odd. Similarly, we write

$$\overline{x}^{\text{alt} \otimes n} := \underbrace{\overline{x} \otimes \overline{x} \otimes \cdots \otimes \overline{x}}_{n \text{ tensorands}},$$

where $\overline{x}^? = x$ if $n$ is even and $\overline{x}$ if $n$ is odd.

Since $C$ is indecomposable as a multifusion category, we see that every object of $C_{11}$ or $C_{12}$ is isomorphic to a direct summand of $x^{\text{alt} \otimes n}$ for some $n \in \mathbb{N}$. In this sense, we say $x$ generates $C$.

Now fix any simple $m \in C_{31} \subset \mathcal{M}$. Since $\tilde{C}$ is indecomposable as a multifusion category, every object of $\mathcal{M}$ is isomorphic to a direct summand of $m \otimes x^{\text{alt} \otimes n}$ for some $n \in \mathbb{N}$.
**Notation 10.** We read our string diagrams bottom to top. We denote a morphism $f : a \rightarrow b$ by a coupon

```
\begin{array}{c}
  \text{a} \\
  \text{f} \\
  \text{b}
\end{array}
```

Composition of morphisms corresponds to vertical stacking of coupons, and tensor product of morphisms corresponds to horizontal juxtaposition of coupons.

**Definition 11.** Let $A_n := \text{End}_C(m \otimes x^{\text{alt} \otimes n})$ equipped with the multiplication

```
\begin{array}{c}
  f \\
  g
\end{array}
```

$fg =

```
\begin{array}{c}
  f \\
  g \\
  n
\end{array}
```

where the red strand stands for $m$, the label $n$ stands for the object $x^{\text{alt} \otimes n}$, and the label $\overline{n}$ stands for the object $\overline{x}^{\text{alt} \otimes n}$. This is a finite dimensional von Neumann algebra. It comes with a faithful tracial state $\text{tr}_n : A_n \rightarrow \mathbb{C}$ given by

$$
\text{tr}_n(f) := \frac{1}{\text{dim}_C(m) \text{dim}_C(x)^n} \cdot \text{tr}_C(f) = \frac{1}{\text{dim}_C(m)} \cdot \frac{1}{d^n} \cdot \left( \overline{m} \begin{array}{c} \overline{f} \end{array} \overline{n} \right)
$$

where $d := \text{dim}(x) = \text{dim}(\overline{x})$. Further down, a black strand with label 1 stands for either $x$ or $\overline{x}$.

We have inclusions $A_n \rightarrow A_{n+1}$ compatible with the traces given by

```
\begin{array}{c}
  f \\
  n
\end{array}
```

$\rightarrow

```
\begin{array}{c}
  f \\
  \overline{n}
\end{array}
```

(2)

and trace preserving conditional expectations $E_n : A_n \rightarrow A_{n-1}$ given by

```
\begin{array}{c}
  f \\
  n
\end{array}
```

$\rightarrow

```
\begin{array}{c}
  1 \\
  n
\end{array}
```

(3)

where $\overline{n} := \text{dim}(x) = \text{dim}(\overline{x})$. Further down, a black strand with label 1 stands for either $x$ or $\overline{x}$.

The Jones projection for the inclusion $A_{n-1} \subset A_n$ is given by

$$
e_n = \frac{1}{d} \cdot \left( \bigcup_{n-1}^{1} \right) \in A_{n+1}.
$$

(4)

**Exercise 12.** Explain why the diagram in (1) can be viewed as a scalar in $\mathbb{C}$.

*Hint: Recall $m \in C_{31}$ is simple.*

**Exercise 13.** Show that for all $n \geq 1$, prove that

(1) For all $a, b \in A_{n-1}$ and $x \in A_n$, $E_n(axb) = aE_n(x)b$. 


(2) $\text{tr}_n = \text{tr}_{n-1} \circ E_n$.

(3) $\text{tr}_n |_{A_{n-1}} = \text{tr}_{n-1}$, where we identify $A_{n-1}$ with its image in $A_n$ under the inclusion map (2).

**Exercise 14.** Prove that the sequence $(A_n, \text{tr}_n)$ with Jones projections $(e_n)$ is Markov.

**Exercise 15.** Prove that the Markov sequence $(A_n, \text{tr}_n)$ with Jones projections $(e_n)$ has finite depth.

**References**
