# More detail on [Pen20, Lem. 3.26]

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We provide more detail on the proof of [Pen20, Lem. 3.26]. We thank Giovanni Ferrer and Brett Hungar for their careful reading of the manuscript, leading to this note.

Let $(\mathcal{C}, \varphi)$ be a (semisimple) pivotal tensor category. The following definition is based on André Henriques' notion of *modular distortion* for bimodules over a $\mathrm{II}_1$ factor [BCE+19, BCE+20].

**Definition 1.** We say an object $c \in \mathcal{C}$ has *constant distortion* $\delta \in \mathbb{C}^\times$ if

$$\operatorname{tr}_L(f) = \delta \cdot \operatorname{tr}_R(f) \qquad \forall\, f \in \operatorname{End}_{\mathcal{C}}(c). \tag{1}$$

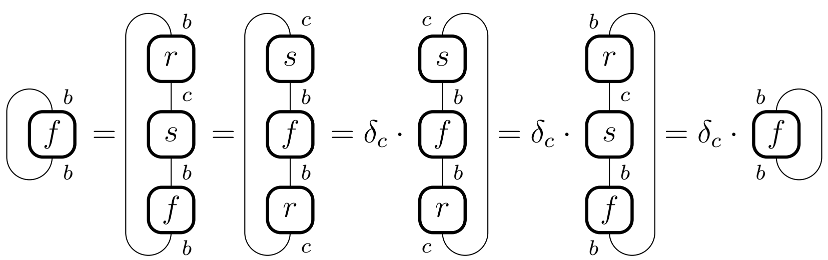
Observe that every simple object has constant distortion.

**Lemma 2.**

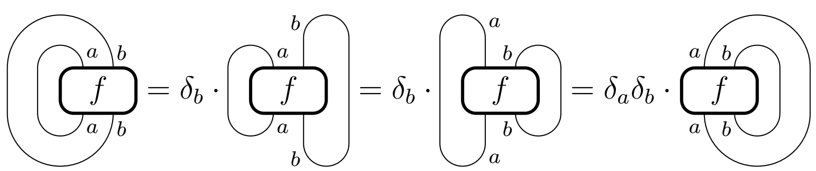
*(1) If $c \in \mathcal{C}$ has constant distortion $\delta_c$, then so does every subobject $b \subseteq c$.*

*(2) If $a, b \in \mathcal{C}$ have constant distortion $\delta_a, \delta_b$ respectively, then $a \otimes b$ has constant distortion $\delta_a \delta_b$.*

*Proof.* To prove (1), let $r : c \to b$ and $s : b \to c$ such that $r \circ s = \mathrm{id}_b$. Then for all $f \in \operatorname{End}_{\mathcal{C}}(b)$, suppressing the pivotal structure $\varphi$,



To prove (2), we observe that for all $f \in \operatorname{End}_{\mathcal{C}}(a \otimes b)$,



□

By Lemma 2 above, we get a (most likely non-faithful) $\mathbb{C}^\times$-grading on $\mathcal{C}$ given by $\mathcal{C} = \bigoplus_{z \in \mathbb{C}^\times} \mathcal{C}_z$ where $\mathcal{C}_z$ is the semisimple subcategory of $\mathcal{C}$ whose objects have constant distortion $z$. Observe $\mathcal{C}_w \otimes \mathcal{C}_z \subseteq \mathcal{C}_{wz}$, and if $c \in \mathcal{C}_z$, then $c^\vee \in \mathcal{C}_{z^{-1}}$.

Denote by $G$ the subgroup of $\mathbb{C}^\times$ such that $\mathcal{C}_z \neq 0$ so that $\mathcal{C}$ is faithfully graded by $G$. By [Pen20, Rem. 3.17], there is a surjective group homomorphism from the universal grading group $U_{\mathcal{C}} \twoheadrightarrow G$. Composing with the inclusion map $G \hookrightarrow \mathbb{C}^\times$ gives a group homomorphism $U_{\mathcal{C}} \to \mathbb{C}^\times$. In summary, we have the following proposition.

**Proposition 3.** *Let $(\mathcal{C}, \varphi)$ be a pivotal (semisimple) tensor category. The map $\delta : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}^\times$ given by $c \mapsto \delta_c := \dim_L(c)/\dim_r(c)$ gives a group homomorphim from the universal grading group $U_{\mathcal{C}}$ to $\mathbb{C}^\times$. In particular, $\mathcal{C}_e$ is spherical.*

Now suppose $(\mathcal{C},\varphi)$ is a pivotal (semisimple) multitensor category with unit decomposition $1=\bigoplus_{i=1}^r 1_i$. We write $\mathcal{C}_{ij}=1_i\otimes\mathcal{C}\otimes 1_j$ and $c_{ij}=1_i\otimes c\otimes 1_j$ for $c\in\mathcal{C}$.

**Definition 4.** We say $c\in\mathcal{C}_{ij}$ has constant distortion $\delta$ if (1) holds under the identification $\operatorname{End}_{\mathcal{C}}(1_i)\cong\mathbb{C}$ and $\operatorname{End}_{\mathcal{C}}(1_j)\cong\mathbb{C}$ by mapping the identities to $1_{\mathbb{C}}$.

We omit the proof of the following lemma, which is similar to the proof of Lemma 2.

**Lemma 5.**

*(1) If $c\in\mathcal{C}_{ij}$ has constant distortion $\delta_c$, then so does every subobject $b\subseteq c$.*
*(2) If $a\in\mathcal{C}_{ij}$ and $b\in\mathcal{C}_{jk}$ have constant distortion $\delta_a$ and $\delta_b$ respectively, then $a\otimes b\in\mathcal{C}_{ik}$ has constant distortion $\delta_a\delta_b$.*

Similar to above, we get a (most likely non-faithful) $\mathcal{G}_r\times\mathbb{C}^\times$-grading on $\mathcal{C}$ given by $\mathcal{C}=\bigoplus(\mathcal{C}_{ij})_z$, where $\mathcal{G}_r$ is the groupoid with $r$ objects and a unique isomorphism between any two objects, and $(\mathcal{C}_{ij})_z$ is the semsimple subcategory of $\mathcal{C}_{ij}$ whose objects have constant distortion $z$. Observe that $(\mathcal{C}_{ij})_w\otimes(\mathcal{C}_{jk})_z\subseteq(\mathcal{C}_{ik})_{wz}$, and if $c\in(\mathcal{C}_{ij})_z$, then $c^\vee\in(\mathcal{C}_{ji})_{z^{-1}}$.

Denote by $\mathcal{G}$ the subgroupoid of $\mathcal{G}_r\times\mathbb{C}^\times$ such that $\mathcal{C}$ is faithfully graded by $\mathcal{G}$. Observe that the map

$$(\mathcal{C}_{ij})_z\ni c\longmapsto z\in\mathbb{C}^\times$$

descends to a well-defined groupoid homomorphism $\mathcal{G}\to\mathbb{C}^\times$. By [Pen20, Rem. 3.17], there is a surjective groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}}\twoheadrightarrow\mathcal{G}$. Composing these two homomorphisms gives a groupoid homomorphism $\mathcal{U}_{\mathcal{C}}\to\mathbb{C}^\times$. In summary, we have the following proposition.

**Proposition 6.** *Let $(\mathcal{C},\varphi)$ be a pivotal (semisimple) multitensor category. The map $\delta:\operatorname{Irr}(\mathcal{C})\to\mathbb{C}^\times$ given by $c\mapsto\delta_c:=\dim_L(c)/\dim_R(c)$ gives a groupoid homomorphism from the universal grading groupoid $\mathcal{U}_{\mathcal{C}}$ to $\mathbb{C}^\times$. In particular, for any idempotent $e\in\mathcal{U}_{\mathcal{C}}$, $\mathcal{C}_e$ is spherical.*

## References


[BCE+19] Marcel Bischoff, Ian Charlesworth, Samuel Evington, Luca Giorgetti, David Penneys, and André Henriques, *Modular distortion for* $\mathrm{II}_1$ *factor bimodules*, Subfactors and Applications (Organized by Dietmar Bisch, Terry Gannon, Vaughan Jones, and Yasuyuki Kawahigashi, eds.), Oberwolfach Rep., vol. 49, 2019, DOI:10.4171/OWR/2019/49, pp. 42–46.

[BCE+20] Marcel Bischoff, Ian Charlesworth, Samuel Evington, Luca Giorgetti, and David Penneys, *Distortion for multifactor bimodules and representations of multifusion categories*, 2020, arXiv:2010.01067.

[Pen20] David Penneys, *Unitary dual functors for unitary multitensor categories*, High. Struct. **4** (2020), no. 2, 22–56, MR4133163 arXiv:1808.00323.