Categorified Morita equivalence

These notes were compiled from the following sources:

- Rieffel's "Morita Equivalence for C^*-algebras and W^*-algebras"
- Mueger's "From subfactors to categories and topology I" (arXiv:math/0111204)
- Ostrik's "Module categories, weak Hopf algebras and modular invariants" (arXiv:math/0111139)
- Etingof, Nikshych, and Ostrik's "Fusion categories and homotopy theory" (arXiv:math/0909.3140)
- Noah Snyder's talk on 2/20/10 at the Subfactor Tahoe Retreat

Morita equivalence of rings

Let R, S be rings.

Definition 1. R, S are Morita equivalent if there are bimodules ${}_SP_R$ and ${}_RQ_S$ and isomorphims

$$ {}_SP \underset{R}{\otimes} Q_S \cong {}_SS_S \text{ and } {}_RQ \underset{S}{\otimes} P_R \cong {}_RR_R. $$

Theorem 2. *R, S are Morita equivalent if and only if there is an additive equivalence of categories ${}_R\mathsf{Mod} \cong {}_S\mathsf{Mod}$.*

Sketch of Proof. If R, S are Morita equivalent, then we have additive functors

$$ {}_R\mathsf{Mod} \to {}_S\mathsf{Mod} \text{ by } {}_RM \mapsto {}_SP \underset{R}{\otimes} M \text{ and} $$
$$ {}_S\mathsf{Mod} \to {}_R\mathsf{Mod} \text{ by } {}_SN \mapsto {}_RQ \underset{S}{\otimes} N $$

and the two ways of composing them are clearly naturally isomorphic to identity functors.

Suppose there is an equivalence of categories ${}_R\mathsf{Mod} \cong {}_S\mathsf{Mod}$ where $F\colon {}_R\mathsf{Mod} \to {}_S\mathsf{Mod}$ is one of the functors of the equivalence. Then F is right exact, and by a theorem of Homological algebra, there is a bimodule ${}_SP_R$ such that F is naturally isomorphic to ${}_SP \underset{R}{\otimes} -$. $\square$

Remark 3. Keeping the picture

$$ R \overset{P}{\underset{Q}{=\!=}} S $$

in mind, we can hide half the diagram as follows: If R and S are Morita equivalent via bimodules ${}_SP_R$ and ${}_RQ_S$, then $S \cong \mathrm{End}_R(P_R)$ and $Q \cong \mathrm{Hom}(P_R, R_R)$.

Example 4.

(1) Let R be a ring. Then R is Morita equivalent to $M_n(R)$ for all $n \in \mathbb{N}$ via the bimodule R^n where isomorphims are given by inner and outer products.

(2) Suppose $N \subset M$ is a II_1-subfactor. Recall that the trace tr on M is unique. Let $e \in B(L^2(M, \mathrm{tr}))$ be the projection onto $L^2(N, \mathrm{tr}|_N)$. Then $E = e|_M$ is the unique ultraweakly continuous trace-preserving conditional expectation $E\colon M \to N$ such that $E(1) = 1$ and $E(amb) = aE(m)b$ for all $a, b \in N$ and $m \in M$. In fact, E is a Banach space projection of norm 1. The basic construction of $N \subset M$ is the von Neumann algebra $M_1 = \langle M, e\rangle \subset B(L^2(M, \mathrm{tr}))$. If $N \subset M$ is finite index, then

$$M_1 = MeM \cong M \underset{N}{\otimes} M \cong \mathrm{End}_N(M_N).$$

Hence N is Morita equivalent to M_1 as rings via ${}_N M_{M_1}$.

Strong Morita equivalence of C^*-algebras

Let A, B be C^*-algebras.

Definition 5. A left pre-C^* B-module ${}_B X$ is a complex vector space X which is a left B-module together with a positive, antisymmetric B-valued sesquilinear form

$$\langle -, - \rangle_B \colon X \times X \to B,$$

i.e.,

(i) $\langle bx + y, z\rangle_B = b\langle x, z\rangle_B + \langle y, z\rangle_B$

(ii) $\langle x, y\rangle_B^* = \langle y, x\rangle_B$, and

(iii) $\langle x, x\rangle_B \geq 0$

for all $b \in B$ and $x, y, z \in X$.

A right pre-C^* A module is defined similarly, except (i) above is changed to

$$\langle x|y + za\rangle_A = \langle x|y\rangle_A + \langle x|z\rangle_A a.$$

A pre-C^* $B - A$-bimodule is both a left pre-C^* B-module and a right pre-C^* A-module such that

$$\langle x, y\rangle_B z = x\langle y|z\rangle_A \text{ for all } x, y, z \in X.$$

A left C^* B-module is full if $\mathrm{span}(\mathrm{im}(\langle -, -\rangle_B))$ is dense in B. If B is a von Neumann algebra, we require density in the ultraweak topology. Similarly we define full right pre-C^*-modules and full pre C^*-bimodules.

Remark 6. $\langle -, -\rangle_B$ is B-linear in the first variable and conjugate linear in the second. $\langle -|-\rangle_A$ is A-linear in the second variable and conjugate linear in the first.

Remark 7. Given a left pre-C^* B-module ${}_BX$, we can define a seminorm by

$$\|x\|_B^2 = \|\langle x, x\rangle_B\|.$$

Similarly for right pre-C^* A-modules. If ${}_BX_A$ is a pre-C^* $B - A$ bimodule, then $\| - \|_A = \| - \|_B$.

Definition 8. A left pre-C^* B-module ${}_BX$ is called a left C^* B-module if $\langle -, -\rangle_B$ is definite and X is complete for $\| - \|_B$. Similarly, we define right C^*-modules and C^*-bimodules. Fullness is defined accordingly.

Remark 9. If A, B are von Neumann algebras, we require the maps

$$\begin{aligned} B &\to A \text{ by } b \mapsto \langle x|by\rangle_A \text{ and} \\ A &\to B \text{ by } a \mapsto \langle ax, y\rangle_B \end{aligned}$$

to be normal, i.e., ultraweakly continuous.

Definition 10. A, B are strongly Morita equivalent if there are full C^*-bimodules ${}_BX_A$ and ${}_AY_B$ such that

$${}_BX \underset{A}{\otimes} Y_B \cong {}_BB_B \text{ and } {}_AY \underset{B}{\otimes} X_A \cong {}_AA_A$$

(isomorphic as C^*-bimodules). Note that this tensor product requires completion after defining

$$\begin{aligned} \langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_B &= \langle\langle x_2, x_1\rangle_B y_1, y_2\rangle_B \text{ and} \\ \langle x_1 \otimes y_1 | x_2 \otimes y_2\rangle_B &= \langle x_1\langle y_1|y_2\rangle_B|x_2\rangle_B, \end{aligned}$$

and similarly for $Y \otimes_A X$.

Definition 11. Suppose ${}_BX_A$ is a C^*-bimodule. The contragredient or dual C^*-bimodule is given by

$${}_A\overline{X}_B = \overline{{}_BX_A} = \left\{\overline{x}\middle|x \in X\right\} \text{ and } a \cdot \overline{x} \cdot b = \overline{b^*xa^*} \text{ for all } a \in A, b \in B.$$

Definition 12. Given $x, y \in X_A$, a right C^*-module, we can define the "A-finite rank" operator

$$|x\rangle\langle y| \colon X_A \to X_A \text{ by } |x\rangle\langle y|z = x\langle y|z\rangle.$$

Then $|x\rangle\langle y|^* = |y\rangle\langle x|$ (A-finite rank operators are adjointable), and we let $\mathrm{End}_A^0(X_A)$ denote the norm closure of the span of A-finite rank operators. Then $\mathrm{End}_A^0(X_A)$ is a C^*-algebra. We let $\mathrm{End}_A(X_A)$ be the von Neumann algebra of all adjointable operators.

Remark 13. If A, B are Morita equivalent via X, Y, once more we can hide half of the diagram

$$A \underset{Y}{\overset{X}{=\!=}} B$$

by noting that $B \cong \mathrm{End}_A^0(X_A)$ and $Y \cong \overline{X}$ (if A, B are von Neumann algebras, then $B \cong \mathrm{End}_A(X_A)$).

Definition 14. If A is a C^*-algebra, let ${}_A\mathsf{Hilb}$ is the category of left A-Hilbert modules (HIlbert spaces with a left nondegenerate $*$-representation of A). If A is a von Neumann algebra, we require these representations to be normal (ultraweakly continuous).

Definition 15. Suppose ${}_BX_A$ is a C^*-bimodule and ${}_AH \in_A \mathsf{Hilb}$. We form the relative tensor product ${}_BX \otimes_A H$ as the completion of the algebraic tensor product (over A) completed in the norm induced by the inner product defined by the linear extension of

$$\langle x_1 \otimes \xi_1, x_2 \otimes \xi_2 \rangle = \langle \langle x_2 | x_1 \rangle_A \xi_1, \xi_2 \rangle_H.$$

Similarly, we may define $K \otimes_B X_A$ for $K_B \in \mathsf{Hilb_B}$ where the inner product is given by

$$\langle \eta_1 \otimes x_1, \eta_2 \otimes x_2 \rangle = \langle \eta_1, \eta_2 \langle x_2, x_1 \rangle_B \rangle_K.$$

Theorem 16. *Suppose A, B are Morita equivalent via ${}_BX_A$. Then ${}_A\mathsf{Hilb} \cong_\mathsf{B} \mathsf{Hilb}$ via*

$${}_AH \mapsto {}_BX \underset{A}{\otimes} H.$$

Example 17. Suppose $N \subset M$ is a finite index II_1-subfactor. Recall there is an anti-linear unitary $J\colon L^2(M, \mathrm{tr}) \to L^2(M, \mathrm{tr})$ given by the extension of $x\Omega \mapsto x^*\Omega$ where Ω is the image of 1 in $L^2(M, \mathrm{tr})$. Then $Je = eJ$ and $M_1 = JN'J$. For $x \in M_1$, we define the right action $\xi x = Jx^*J\xi$ for $\xi \in L^2(M, \mathrm{tr})$. One easily computes for $m \in M$ that

$$m\Omega(xey) = Jy^*ex^*Jm\Omega = E(mx)y\Omega.$$

On ${}_NM\Omega_{M_1}$, define (obviously surjective) maps

$$\langle x\Omega, y\Omega \rangle_N = E(xy^*) \text{ and } \langle x\Omega | y\Omega \rangle_{M_1} = x^*ey.$$

Then ${}_NM\Omega_{M_1}$ is a C^*-bimodule which gives a Morita equivalence between N and M_1 as

$$\langle x\Omega, y\Omega \rangle_N z\Omega = E(xy^*)z\Omega = x\Omega(y^*ez) = x\Omega\langle y\Omega | z\Omega \rangle_{M_1}.$$

Categorification of rings to tensor catgories

Categorification is the process of replacing sets, functions, and equations with categories, functors, and natural isomorphims such that when one decategorifies, usually by taking the Grothendieck group K_0, we get back the sets, functions, and equations.

Remark 18. We require our categories to be skeletally small, i.e., the collection of isomorphism classes of objects in our category is a set.

Definition 19. The Grothendieck group of a category C is $K_0(\mathsf{C}) = \{[x] | x \in \mathsf{C}\}$, the set of isomorphism classes of objects.

Remark 20. If C has more structure, then so does $K_0(\mathsf{C})$. For example,

(1) If C is abelian, then $K_0(\mathsf{C})$ is an abelian group with $[x] + [y] = [x \oplus y]$

(2) If C is also a tensor category, then $K_0(\mathsf{C})$ is a ring with $[x] \cdot [y] = [x \otimes y]$.

(3) If C is also braided (a chosen collection of isomorphisms $B_{X,Y}$ for each $X, Y \in \mathsf{C}$ which satisfy the (Yang-Baxter) braid relation), then $K_0(\mathsf{C})$ is a commutative ring.

(4) If C is also rigid (C has left and right duals) and pivotal (C has "nice" duals, i.e., there are natural isomorphisms ${}^*X \cong X^*$ and $X^{**} \cong X$ for all $X \in \mathsf{C}$), then $K_0(\mathsf{C})$ is a $*$-ring.

(5) If C is semisimple (every object is a finite direct sum of simple objects) and also has finitely many isomorphism classes of simple objects, then $K_0(\mathsf{C})$ is finitely generated.

Definition 21. A fusion category over a field k is a rigid, semisimple tensor category enriched over k-vector spaces with only finitely many isomorphism classes of simple objects such that $\mathrm{End}(1) \cong k$.

Remark 22. For simplicity, we will assume our fusion categories are pivotal. In general, this is an open problem.

Examples 23. The following are fusion categories:

(1) ${}_{\mathsf{f.d.}}\mathsf{Vect}_k$, the category of finite dimensional k-vector spaces.

(2) $\mathsf{Rep}(G)$, the category of representations of a finite group or finite dimensional quantum group.

(3) The subcategory in ${}_N\mathsf{Mod}_N$ (respectively ${}_M\mathsf{Mod}_M$) generated by taking summands of (finite) tensor products of ${}_N M_N$ over N (respectively ${}_M M \otimes_N M_M$ over M) if $N \subset M$ is a finite index, finite depth II_1-subfactor.

Categorified Morita Equivalence of Fusion Categories

All categories mentioned from here on will be abelian, semisimple, enriched over k-vector spaces (k algebraically closed), and have finite dimensional Hom spaces. Let C, D be fusion categories.

Definition 24. A left module category over a tensor category C is a category ${}_{\mathsf{C}}\mathsf{M}$ and an exact bifunctor $\otimes\colon \mathsf{C} \times \mathsf{M} \to \mathsf{M}$ satisfying some unit and associativity axioms up to an associator.

Remark 25. Heuristically, one should think of this functor as a categorification of a ring action on a module, e.g., $\lambda\colon A \otimes X \to X$. The associativity of the action means the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes X & \xrightarrow{m \otimes \mathrm{id}_X} & A \otimes X \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes \lambda} & & \downarrow{\scriptstyle \lambda} \\ A \otimes X & \xrightarrow{\lambda} & X \end{array}$$

where $m\colon A \otimes A \to A$ is the multiplication map. This means we have to have some type of associator isomorphisms in the categorified version.

Remark 26. Given a right C-module category M_{C} and a left C-module category ${}_{\mathsf{C}}\mathsf{N}$ we can form an abelian category

$$\mathsf{M} \underset{\mathsf{C}}{\boxtimes} \mathsf{N} \cong \mathsf{Fun}_{\mathsf{C}}(\mathsf{M}^{\mathsf{op}}, \mathsf{N}).$$

If M and N are bimodule categories, then so is $\mathsf{M} \underset{\mathsf{C}}{\boxtimes} \mathsf{N}$.

Definition 27. A bimodule category ${}_{\mathsf{D}}\mathsf{M}_{\mathsf{C}}$ is invertible if one of the following equivalent conditions holds:

(1) $\mathsf{M} \underset{\mathsf{C}}{\boxtimes} \mathsf{M}^{\mathsf{op}} \cong {}_{\mathsf{D}}\mathsf{D}_{\mathsf{D}}$ as $\mathsf{D}-\mathsf{D}$-bimodule categories,

(2) $\mathsf{M}^{\mathsf{op}} \underset{\mathsf{D}}{\boxtimes} \mathsf{M} \cong {}_{\mathsf{C}}\mathsf{C}_{\mathsf{C}}$ as $\mathsf{C}-\mathsf{C}$-bimodule categories,

(3) The functor $\mathsf{D} \to \mathsf{Fun}_{-\mathsf{C}}(\mathsf{M}, \mathsf{M})$ given by $X \mapsto (M \mapsto X \otimes M)$ is an equivalence, and

(4) The functor $\mathsf{C} \to \mathsf{Fun}_{\mathsf{D}-}(\mathsf{M}, \mathsf{M})$ given by $Y \mapsto (M \mapsto M \otimes Y)$ is an equivalence.

Definition 28. Let C and D be fusion categories. C, D are Morita equivalent if there is an invertible bimodule category ${}_{\mathsf{D}}\mathsf{M}_{\mathsf{C}}$.

Definition 29. If M is a left C-module category, the dual fusion category of C with respect to M is $\mathsf{C}^*_{\mathsf{M}} = \mathsf{Fun}_{\mathsf{C}-}(\mathsf{M}, \mathsf{M})$.

Remark 30. Note that ${}_{\mathsf{C}}\mathsf{M}$ is a right $\mathsf{C}^*_{\mathsf{M}}$-module category, so C is Morita equivalent to $\mathsf{C}^*_{\mathsf{M}}$.

Example 31. In the subfactor case, $\mathsf{C}^*_{\mathsf{M}} = {}_M\mathsf{Mod}_M$.

Algegra objects from module categories

Remark 32. A complex algebra is a complex vector space A with a map $m\colon A \otimes A \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes m} & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A. \end{array}$$

Definition 33. An algebra object in a fusion category C is an object $A \in \mathsf{C}$ and a map $m\colon A \otimes A \to A$ satisfying the unit and associativity axioms up to the associator.

An algebra object $A \in \mathsf{C}$ is called a Frobenius algebra object if it comes with a map $\mathrm{tr}\colon A \to 1$ satisfying a certain nondegeneracy axiom (the categorified "bilinear form" $A \otimes A^* \to$ has a biadjoint).

Examples 34.

(1) Let G be a finite group. Let C be the category of G-graded vector spaces, i.e., vector spaces V which are the direct sum of vector spaces V_g for each $g \in G$:

$$V = \bigoplus_{g \in G} V_g.$$

C is a tensor category where $\otimes$ is given by

$$(V \otimes W)_g = \bigoplus_{hk=g} V_h \otimes_{\mathbb{C}} W_k.$$

The group algebra $\mathbb{C}G$ is an algebra object in this category.

(2) ${}_N M_N \in {}_N\mathsf{Mod}_N$ is a Frobenius algebra object.

Exercise 35. Show that the multiplications induce the algebra object structures in the above examples.

Remark 36. Given an algebra object $A \in \mathsf{C}$, we can make a left module category as follows: set M equal to the category of *right* A-module objects, i.e., those objects $X \in \mathsf{C}$ with a map $\rho \colon X \otimes A \to X$ satisfying the associativity axiom up to an associator:

$$\begin{array}{ccc} X \otimes A \otimes A & \xrightarrow{\mathrm{id}_X \otimes m} & X \otimes A \\ \Big\downarrow{\scriptstyle \mathrm{id}_A \otimes \rho} & & \Big\downarrow{\scriptstyle \rho} \\ X \otimes A & \xrightarrow{\quad \rho \quad} & X \end{array}$$

Note that if X is a right A-module object and $Y \in \mathsf{C}$, then $Y \otimes X$ is also a right A-module object with the map $\mathrm{id}_Y \otimes \rho$.

Inner hom

Osterik's internal Hom construction gives algebra objects from a module category. Heuristically, internal Hom is a way of creating objects in a category in a natural way from two given objects. In the category of vector spaces, $\mathrm{Hom}(X, Y)$ is a complex vector space.

Definition 37. Given $M_1, M_2 \in \mathsf{M}$, the contravariant functor

$$h_{M_1,M_2} \colon \mathsf{C} \to {}_{\mathsf{f.d.}}\mathsf{Vect}_k \text{ by } X \mapsto \mathrm{Hom}(X \otimes M_1, M_2)$$

is exact. By abstract nonsense and the Yoneda Lemma, there is a unique object $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathsf{C}$ up to unique isomorphism representing h_{M_1,M_2}, and $\underline{\mathrm{Hom}}(-,-)$ is a bifunctor.

Examples 38.

(1) If $\mathsf{C} = \mathsf{M} = {}_{\mathsf{f.d.}}\mathsf{Vect}_k$, then $\underline{\mathrm{Hom}}(X, Y) = Y \otimes_k X^*$.

(2) Let G be a finite group. The category $\mathsf{Rep}(G)$ of finite dimensional complex representations of G thought of as $G - \{e\}$-bimodules where $\{e\}$ is the trivial group is a module category over G-graded vector spaces, and $\underline{\mathrm{Hom}}(X, Y) = Y \otimes_{\mathbb{C}} X^*$.

(3) Let $N \subset M$ be a finite depth, finite index II_1-subfactor. Let C be the subcategory of ${}_N\mathsf{Mod}_N$ generated by taking summands of (finite) tensor products of $X =_N M_N$ over N. Let M be be subcategory of ${}_N\mathsf{Mod}_M$ generated by taking summands of $X =_N M_M$, $X \otimes X^* \otimes X, \ldots$ Then $\underline{\mathrm{Hom}}(Y, Z) = Y \otimes_M Z^*$.

Lemma 39. *Let M be a left C-module category. Given $X \in \mathsf{C}$ and $M_1, M_2 \in \mathsf{M}$, there are natural isomorphisms*

(1) $\underline{\mathrm{Hom}}(X \otimes M_1, M_2) \cong \underline{\mathrm{Hom}}(M_1, M_2) \otimes X^*$ *and*

(2) $\underline{\mathrm{Hom}}(M_1, X \otimes M_2) \cong X \otimes \underline{\mathrm{Hom}}(M_1, M_2)$.

There is also a canonical multiplication map

$$\underline{\mathrm{Hom}}(M_2, M_3) \otimes \underline{\mathrm{Hom}}(M_1, M_2) \to \underline{\mathrm{Hom}}(M_1, M_3)$$

which is natural in each variable.

Fact 40. *Given a module category M over C and an object $M \in \mathsf{M}$, $\underline{\mathrm{Hom}}(M, M)$ is an algebra object in C.*

Remarks 41.

(1) In the subfactor setting, we want $M \in \mathsf{M}$ to be a simple object.

(2) Just as ${}_N M_M$ is the preferred object in the module category ${}_N\mathsf{Mod}_M$, if we have an algebra object $A \in \mathsf{C}$, the preferred object in the left module category of right A-module objects is A as a right A module.

Subfactors from algebra objects

Theorem 42. *If $X \in {}_N\mathsf{Mod}_N$ is a simple Frobenius algebra object, then X comes from a factor P where $N \subset P$. Moreover, any unitary tensor category with simple 1 can be realized as a category of bimodules over a factor N (see Yamagami). This means every algebra object can be realized as a subfactor.*

Remark 43. The index of the subfactor coming from an algebra object in a tensor category is the Frobenius-Perron dimension of the object, not the square of the dimension.