Finite dimensional complex multimatrix algebras

Exercises and sections marked $(*)$ below are more advanced and can be skipped on first read through. Exercises marked $(**)$ are very difficult relative to the exposition!

1 Basic facts about $M_n(\mathbb{C})$

Exercise 1. Show that if $a \in M_n(\mathbb{C})$ commutes with all $b \in M_n(\mathbb{C})$, then $a = \lambda 1$ for some $\lambda \in \mathbb{C}$.

Exercise 2. Prove that $M_n(\mathbb{C})$ has no non-trivial 2-sided ideals.

Exercise 3. Use Exercise 2 to show that any (not necessarily unital) $*$-algebra map out of $M_n(\mathbb{C})$ into another complex $*$-algebra is either injective or the zero map.

The matrix algebra $M_n(\mathbb{C})$ acts on the inner product (Hilbert) space $\mathbb{C}^n$ with inner product given by $\langle \eta, \xi \rangle := \sum_{j=1}^{n} \eta_j \overline{\xi_j}$.

Definition 4. An element $a \in M_n(\mathbb{C})$ is called *positive*, denoted $a \geq 0$, if for every $\xi \in \mathbb{C}^n$, $\langle a\xi, \xi \rangle \geq 0$.

Exercise 5. Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

(1) $a \geq 0$.

(2) a is normal ($aa^* = a^*a$) and all eigenvalues of a are non-negative.

(3) There is a $b \in M_n(\mathbb{C})$ such that $b^*b = a$.

(4) There is a $b \in M_{n\times k}(\mathbb{C})$ for some $k \in \mathbb{N}$ such that $b^*b = a$.

2 Finite dimensional complex multimatrix algebras

In this section, A will always denote a finite dimensional complex $*$-algebra.

Definition 6. A linear functional $\varphi : A \to \mathbb{C}$ is called:

- a *trace* or *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a *state* if φ is positive and $\varphi(1) = 1$.
- *faithful* if φ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

Definition 7. A finite dimensional complex $*$-algebra A is called a *multimatrix algebra* if it is $*$-isomorphic to a $*$-algebra of the form

$$M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

The row vector $n_A := (n_1, \ldots, n_k)$ is called the *dimension row vector* of A. For $1 \leq i \leq k$, we denote by $p_i \in A$ the minimal central projection corresponding to the summand $M_{n_i}(\mathbb{C})$, so that $p_iAp_i \cong M_{n_i}(\mathbb{C})$.

Exercise 8. Prove that $M_n(\mathbb{C})$ has a unique trace such that $\text{tr}(1) = 1$. In this case, prove that tr is positive (so tr is a state) and faithful.

Exercise 9 $(*)$**.** Prove that for any state φ on $M_n(\mathbb{C})$, there exists $d \in M_n(\mathbb{C})$ with $d \geq 0$ and $\text{tr}(d) = 1$ such that $\varphi(a) = \text{tr}(da)$ for all $a \in M_n(\mathbb{C})$. Prove that φ is a faithful if and only if d is also invertible.
The matrix d is called the density matrix *of φ with respect to* tr*.*

Exercise 10. Suppose tr is a trace on a multimatrix algebra. Show that:

(1) tr is positive if and only if $\text{tr}(p) \geq 0$ for all projections $p \in A$ ($p = p^* = p^2$).

(2) tr is positive and faithful if and only if $\text{tr}(p) > 0$ for all projections $p \in A$.

Exercise 11. Find a bijective correspondence between faithful tracial states on a finite dimensional complex multimatrix algebra with dimension row vector $n_A = (n_1, \ldots, n_k)$ and column vectors $\lambda \in (0,1)^j$ such that $n_A\lambda = 1$. Under this correspondence, what does the entry λ_i signify?

3 Finite dimensional operator algebras $(*)$

Let H denote a finite dimensional inner product (Hilbert) space. Denote by $B(H)$ the unital $*$-algebra of linear operators on H, where $*$ is the adjoint operation.

Exercise 12. Show that a choice of orthonormal basis of H gives a unitary linear map $u : H \to \mathbb{C}^n$ ($uu^* = \text{id}_{\mathbb{C}^n}$ and $u^*u = \text{id}_H$) such that $x \mapsto uxu^*$ is a unital $*$-algebra isomorphism $B(H) \to M_n(\mathbb{C})$, where the $*$ on the latter is conjugate transpose.

Definition 13. Suppose H is a finite dimensional inner product (Hilbert) space, and denote by $B(H)$ the linear operators on H. For a subset $S \subset B(H)$, the *commutant* of S is $S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}$

Exercise 14. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$.

Exercise 15. Show that if $S \subset B(H)$, then $S' = S'''$.

Exercise 16 $(**)$**.** Show that if $A \subset B(H)$ is a unital $*$-subalgebra, then $A = A''$.
Hint: See [Jon10, Thm. 3.2.1].

Exercise 17 $(**)$**.**

(1) Show that a finite dimensional von Neumann algebra is a multimatrix algebra.

(2) Show that a finite dimensional C*-algebra is a multimatrix algebra.

4 The GNS construction

Suppose A is a multimatrix algebra and φ is a faithful state.

Exercise 18. Show that $\langle a, b \rangle := \varphi(b^*a)$ defines a positive definite inner product on A (thought of as a $\mathbb{C}$-vector space).

Definition 19. We define $L^2(A, \varphi)$ to be A as an inner product (Hilbert) space with the inner product from Exercise 18. We denote the image of $1 \in A$ in $L^2(A, \varphi)$ by Ω, so $a\Omega$ is the image of $a \in A$.

Exercise 20. Prove that if $a \in A$, the map given by $b\Omega \mapsto ab\Omega$ defines a left multiplication operator $\lambda_a \in B(L^2(A,\varphi))$. Prove that the adjoint of this operator is λ_{a^*} given by $b\Omega \mapsto a^*b\Omega$.

Exercise 21. Prove that if $a \in A$, the map given by $b\Omega \mapsto ba\Omega$ defines a right multiplication operator $\rho_a \in B(L^2(A,\varphi))$. Prove that the adjoint of this operator is ρ_{a^*} given by $b\Omega \mapsto ba^*\Omega$.

Exercise 22. Suppose φ is a faithful state on $M_n(\mathbb{C})$. Prove that every linear operator on $L^2(M_n(\mathbb{C}),\varphi)$ can be written as a left multiplication operator followed by a right multiplication operator. Deduce that the commutant of the *left* $M_n(\mathbb{C})$ action on $L^2(M_n(\mathbb{C}),\varphi)$ is the *right* $M_n(\mathbb{C})$ action.

Exercise 23. Suppose φ is a faithful state on A. Prove that the commutant of the *left* A acting on $L^2(A,\varphi)$ is the *right* action of A on $L^2(A,\varphi)$.

Exercise 24 $(*)$**.** Show that a finite dimensional complex $*$-algebra is a multimatrix algebra if and only if it has a faithful state.
Hint: For the forward direction, use Exercise 11. For the reverse direction, if A has a faithful state, then the image of A inside the linear operators on $L^2(A,\varphi)$ is a unital $*$-subalgebra, and thus a finite dimensional von Neumann algebra by Exercise 16. The result now follows by (1) of Exercise 17.

5 Inclusions of multimatrix algebras

Definition 25. Consider a multimatrix algebra B and a $*$-subalgebra $A \subset B$ such that A is also a multimatrix algebra (so A is unital). We call the inclusion $A \subset B$ *unital* if the unit of A is also the unit of B.

Exercise 26. Give examples of unital and non-unital inclusions of multimatrix algebras.

Exercise 27 $(*)$**.** Show that $M_k(\mathbb{C})$ isomorphic to a unital $*$-subalgebra of $M_n(\mathbb{C})$ if and only if $k \mid n$. Then show that up to unitary conjugation in $M_n(\mathbb{C})$, the isomorphism above is given by

$$M_k(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix} \in M_n(\mathbb{C})$$

where x is repeated on the diagonal j times where $jk = n$.

Consider a unital inclusion of multimatrix algebras $A \subset B$. Suppose B has dimension row vector $n_B = (n_1, \ldots, n_\ell)$ and A has dimension row vector $m_A = (m_1, \ldots, m_k)$. Denote the minimal central projections of A by $p_1, \ldots, p_k$ and the minimal central projections of B by $q_1, \ldots, q_\ell$. Consider for $1 \leq i \leq k$ and $1 \leq j \leq \ell$ the $*$-homomorphism $\varphi_{ij} : M_{m_i}(\mathbb{C}) \to M_{n_j}(\mathbb{C})$ given by

$$\begin{aligned} M_{m_i}(\mathbb{C}) \hookrightarrow A \hookrightarrow B \twoheadrightarrow M_{n_j}(\mathbb{C}). \\ x \mapsto p_i x = p_i x \mapsto p_i q_j x. \end{aligned}$$

That is, $\varphi_{ij}(x) := p_i q_j x \in B$. Note that φ_{ij} need not be unital, but note that by Exercise 3, $\varphi_{i,j}$ is either injective or zero.

Exercise 28. Show that if we consider φ_{ij} as a map $p_i A \to p_i q_j B p_i q_j$, then φ_{ij} is a unital $*$-homomorphism.

By Exercises 27 and 28 there is a non-negative integer $\Lambda_{ij} \in \mathbb{N}_{\geq 0}$ such that up to unitary conjugation in B, $\varphi_{ij}(x)$ consists of Λ_{ij} copies of x along the diagonal of $p_i q_j B p_i q_j$. Let $\Lambda = (\Lambda_{i,j}) \in M_{k\times\ell}(\mathbb{C})$

Exercise 29. Show that since $A \subset B$ is a unital inclusion of multimatrix algebras ($1_B \in A$), we must have $m_A\Lambda = n_B$.

Definition 30. The *Bratteli diagram* of the inclusion $A \subset B$ is the bipartite graph Γ with:

- k even vertices labelled by the integers $m_1, \ldots, m_k$,
- ℓ odd vertices labelled by the integers $n_1, \ldots, n_\ell$, and
- Λ_{ij} edges from the i-th even vertex to the j-th odd vertex.

That is, Γ is the bipartite graph with adjacency matrix Λ whose even and odd vertices are labelled by the entries of the dimension row vectors of A and B respectively.

Exercise 31 ($*$)**.** Let B be a multimatrix algebra. Prove that up to unitary conjugation in B, any unital $*$-subalgebra $A \subset B$ is completely determined by its Bratteli diagram.

Exercise 32. Suppose λ_A and λ_B are trace column vectors for A and B satisfying $m_A\lambda_A = n_B\lambda_B$ respectively as in Exercise 11. Assume the entries of λ_A and λ_B are all strictly positive. Denote by tr_A and tr_B the corresponding faithful tracial states on A and B. Prove that $\mathrm{tr}_B|_A = \mathrm{tr}_A$ if and only if $\Lambda\lambda_B = \lambda_A$.

6 Connected inclusions

We continue the notation of the previous section for an inclusion $A \subset B$ with dimension row vectors $m_A = (m_1, \ldots, m_k)$ and $n_B = (n_1, \ldots, n_\ell)$ respectively.

Definition 33. The inclusion $A \subset B$ is called *connected* if the graph Γ is connected.

Exercise 34 ($*$)**.** Prove that Γ is connected if and only if $Z(A) \cap Z(B) = \mathbb{C}$.

Exercise 35 ($*$)**.** Show that if $A \subset B$ is connected, there is a unique $d > 0$ and unique trace vector λ_B such that $m_B\lambda_B = 1$ and $\Lambda^T\Lambda\lambda_B = d^2\lambda_B$. Then deduce:

(1) If $\lambda_A := \Lambda\lambda_B$, then $\Lambda^T\lambda_A = d^2\lambda_B$.

(2) $\begin{pmatrix} 0 & \Lambda \\ \lambda^T & 0 \end{pmatrix} \begin{pmatrix} \lambda_B \\ d\lambda_A \end{pmatrix} = d \begin{pmatrix} \lambda_B \\ d\lambda_A \end{pmatrix}$.

Hint: Use the Frobenius-Perron Theorem.

Definition 36. If $A \subset B$ is connected, the scalar d from Exercise 35 is called the *Frobenius Perron eigenvalue*. The trace vector λ_B is called a *Frobenius Perron eigenvector*.

References

[Jon10] Vaughan F. R. Jones, *Von Neumann algebras*, 2010, http://math.berkeley.edu/~vfr/MATH20909/VonNeumann2009.pdf.