Finite dimensional complex multimatrix algebras

Exercises and sections marked (•) below are more advanced and can be skipped on first read through. Exercises marked (••) are very difficult relative to the exposition!

1 Basic facts about $M_n(\mathbb{C})$

Exercise 1. Show that if $a \in M_n(\mathbb{C})$ commutes with all $b \in M_n(\mathbb{C})$, then $a = \lambda 1$ for some $\lambda \in \mathbb{C}$.

Exercise 2. Prove that $M_n(\mathbb{C})$ has no non-trivial 2-sided ideals.

Exercise 3. Use Exercise 2 to show that any (not necessarily unital) $*$-algebra map out of $M_n(\mathbb{C})$ into another complex $*$-algebra is either injective or the zero map.

The matrix algebra $M_n(\mathbb{C})$ acts on the inner product (Hilbert) space $\mathbb{C}^n$ with inner product given by $\langle \eta, \xi \rangle := \sum_{j=1}^n \eta_j \overline{\xi_j}$.

Definition 4. An element $a \in M_n(\mathbb{C})$ is called positive, denoted $a \geq 0$, if for every $\xi \in \mathbb{C}^n$, $\langle a\xi, \xi \rangle \geq 0$.

Exercise 5. Show that the following are equivalent for $a \in M_n(\mathbb{C})$.

1. $a \geq 0$.
2. $a$ is normal ($aa^* = a^*a$) and all eigenvalues of $a$ are non-negative.
3. There is a $b \in M_n(\mathbb{C})$ such that $b^*b = a$.
4. There is a $b \in M_{n \times k}(\mathbb{C})$ for some $k \in \mathbb{N}$ such that $b^*b = a$.

2 Finite dimensional complex multimatrix algebras

In this section, $A$ will always denote a finite dimensional complex $*$-algebra.

Definition 6. A linear functional $\varphi : A \to \mathbb{C}$ is called:

- a trace or tracial if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- positive if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a state if $\varphi$ is positive and $\varphi(1) = 1$.
- faithful if $\varphi$ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

Definition 7. A finite dimensional complex $*$-algebra $A$ is called a multimatrix algebra if it is $*$-isomorphic to a $*$-algebra of the form $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.

The row vector $n_A := (n_1, \ldots, n_k)$ is called the dimension row vector of $A$. For $1 \leq i \leq k$, we denote by $p_i \in A$ the minimal central projection corresponding to the summand $M_{n_i}(\mathbb{C})$, so that $p_i Ap_i \cong M_{n_i}(\mathbb{C})$. 

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Exercise 8. Prove that $M_n(\mathbb{C})$ has a unique trace such that $\text{tr}(1) = 1$. In this case, prove that $\text{tr}$ is positive (so $\text{tr}$ is a state) and faithful.

Exercise 9 (*). Prove that for any state $\varphi$ on $M_n(\mathbb{C})$, there exists $d \in M_n(\mathbb{C})$ with $d \geq 0$ and $\text{tr}(d) = 1$ such that $\varphi(a) = \text{tr}(da)$ for all $a \in M_n(\mathbb{C})$. Prove that $\varphi$ is a faithful if and only if $d$ is also invertible.

The matrix $d$ is called the density matrix of $\varphi$ with respect to $\text{tr}$.

Exercise 10. Suppose $\text{tr}$ is a trace on a multimatrix algebra. Show that:

1. $\text{tr}$ is positive if and only if $\text{tr}(p) \geq 0$ for all projections $p \in A$ ($p = p^* = p^2$).
2. $\text{tr}$ is positive and faithful if and only if $\text{tr}(p) > 0$ for all projections $p \in A$.

Exercise 11. Find a bijective correspondence between faithful tracial states on a finite dimensional complex multimatrix algebra with dimension row vector $n_A = (n_1, \ldots, n_k)$ and column vectors $\lambda \in (0,1)^j$ such that $n_A \lambda = 1$. Under this correspondence, what does the entry $\lambda_i$ signify?

3 Finite dimensional operator algebras (*)

Let $H$ denote a finite dimensional inner product (Hilbert) space. Denote by $B(H)$ the unital $*$-algebra of linear operators on $H$, where $*$ is the adjoint operation.

Exercise 12. Show that a choice of orthonormal basis of $H$ gives a unitary linear map $u : H \to \mathbb{C}^n$ ($uu^* = \text{id}_{\mathbb{C}^n}$ and $u^*u = \text{id}_H$) such that $x \mapsto uxu^*$ is a unital $*$-algebra isomorphism $B(H) \to M_n(\mathbb{C})$, where the $*$ on the latter is conjugate transpose.

Definition 13. Suppose $H$ is a finite dimensional inner product (Hilbert) space, and denote by $B(H)$ the linear operators on $H$. For a subset $S \subset B(H)$, the commutant of $S$ is $S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}$

Exercise 14. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$.

Exercise 15. Show that if $S \subset B(H)$, then $S' = S''$.

Exercise 16 (**). Show that if $A \subset B(H)$ is a unital $*$-subalgebra, then $A = A''$.

Hint: See [Jon10, Thm. 3.2.1].

Exercise 17 (**).

1. Show that a finite dimensional von Neumann algebra is a multimatrix algebra.
2. Show that a finite dimensional C*-algebra is a multimatrix algebra.

4 The GNS construction

Suppose $A$ is a multimatrix algebra and $\varphi$ is a faithful state.

Exercise 18. Show that $\langle a, b \rangle := \varphi(b^*a)$ defines a positive definite inner product on $A$ (thought of as a $\mathbb{C}$-vector space).

Definition 19. We define $L^2(A, \varphi)$ to be $A$ as an inner product (Hilbert) space with the inner product from Exercise 18. We denote the image of $1 \in A$ in $L^2(A, \varphi)$ by $\Omega$, so $a\Omega$ is the image of $a \in A$. 
Exercise 20. Prove that if \( a \in A \), the map given by \( b\Omega \mapsto ab\Omega \) defines a left multiplication operator \( \lambda_a \in B(L^2(A, \varphi)) \). Prove that the adjoint of this operator is \( \lambda_{a^*} \) given by \( b\Omega \mapsto a^*b\Omega \).

Exercise 21. Prove that if \( a \in A \), the map given by \( b\Omega \mapsto ba\Omega \) defines a right multiplication operator \( \rho_a \in B(L^2(A, \varphi)) \). Prove that the adjoint of this operator is \( \rho_{a^*} \) given by \( b\Omega \mapsto ba^*\Omega \).

Exercise 22. Suppose \( \varphi \) is a faithful state on \( M_n(\mathbb{C}) \). Prove that every linear operator on \( L^2(M_n(\mathbb{C}), \varphi) \) can be written as a left multiplication operator followed by a right multiplication operator. Deduce that the commutant of the left \( M_n(\mathbb{C}) \) action on \( L^2(M_n(\mathbb{C}), \varphi) \) is the right \( M_n(\mathbb{C}) \) action.

Exercise 23. Suppose \( \varphi \) is a faithful state on \( A \). Prove that the commutant of the left \( A \) acting on \( L^2(A, \varphi) \) is the right action of \( A \) on \( L^2(A, \varphi) \).

Exercise 24 (*). Show that a finite dimensional complex \( * \)-algebra is a multimatrix algebra if and only if it has a faithful state.

Hint: For the forward direction, use Exercise 11. For the reverse direction, if \( A \) has a faithful state, then the image of \( A \) inside the linear operators on \( L^2(A, \varphi) \) is a unital \( * \)-subalgebra, and thus a finite dimensional von Neumann algebra by Exercise 16. The result now follows by (1) of Exercise 17.

5 Inclusions of multimatrix algebras

Definition 25. Consider a multimatrix algebra \( B \) and a \( * \)-subalgebra \( A \subset B \) such that \( A \) is also a multimatrix algebra (so \( A \) is unital). We call the inclusion \( A \subset B \) unital if the unit of \( A \) is also the unit of \( B \).


Exercise 27 (*). Show that \( M_k(\mathbb{C}) \) isomorphic to a unital \( * \)-subalgebra of \( M_n(\mathbb{C}) \) if and only if \( k \mid n \). Then show that up to unitary conjugation in \( M_n(\mathbb{C}) \), the isomorphism above is given by

\[
M_k(\mathbb{C}) \ni x \mapsto \begin{pmatrix} x & \cdots & x \\ \cdots & \cdots & \cdots \\ x & \cdots & x \end{pmatrix} \in M_n(\mathbb{C})
\]

where \( x \) is repeated on the diagonal \( j \) times where \( jk = n \).

Consider a unital inclusion of multimatrix algebras \( A \subset B \). Suppose \( B \) has dimension row vector \( n_B = (n_1, \ldots, n_\ell) \) and \( A \) has dimension row vector \( m_A = (m_1, \ldots, m_k) \). Denote the minimal central projections of \( A \) by \( p_1, \ldots, p_k \) and the minimal central projections of \( B \) by \( q_1, \ldots, q_\ell \). Consider for \( 1 \leq i \leq k \) and \( 1 \leq j \leq \ell \) the \( * \)-homomorphism \( \varphi_{ij} : M_{m_i}(\mathbb{C}) \to M_{n_j}(\mathbb{C}) \) given by

\[
M_{m_i}(\mathbb{C}) \ni A \mapsto B \ni M_{n_j}(\mathbb{C}).
\]

\[
x \mapsto p_ix = p_iBp_jq_jx.
\]

That is, \( \varphi_{ij}(x) := p_iq_jx \in B \). Note that \( \varphi_{ij} \) need not be unital, but note that by Exercise 3, \( \varphi_{i,j} \) is either injective or zero.

Exercise 28. Show that if we consider \( \varphi_{ij} \) as a map \( p_iA \to p_iq_jBp_jq_j \), then \( \varphi_{ij} \) is a unital \( * \)-homomorphism.
By Exercises 27 and 28 there is a non-negative integer $\Lambda_{ij} \in \mathbb{N}_{\geq 0}$ such that up to unitary conjugation in $B$, $\varphi_{ij}(x)$ consists of $\Lambda_{ij}$ copies of $x$ along the diagonal of $p_i q_i B p_i q_i$. Let $\Lambda = (\Lambda_{ij}) \in M_{k \times \ell}(\mathbb{C})$

**Exercise 29.** Show that since $A \subset B$ is a unital inclusion of multimatrix algebras ($1_B \in A$), we must have $m_A \Lambda = n_B$.

**Definition 30.** The *Bratteli diagram* of the inclusion $A \subset B$ is the bipartite graph $\Gamma$ with:

- $k$ even vertices labelled by the integers $m_1, \ldots, m_k$,
- $\ell$ odd vertices labelled by the integers $n_1, \ldots, n_\ell$, and
- $\Lambda_{ij}$ edges from the $i$-th even vertex to the $j$-th odd vertex.

That is, $\Gamma$ is the bipartite graph with adjacency matrix $\Lambda$ whose even and odd vertices are labelled by the entries of the dimension row vectors of $A$ and $B$ respectively.

**Exercise 31 (★).** Let $B$ be a multimatrix algebra. Prove that up to unitary conjugation in $B$, any unital ∗-subalgebra $A \subset B$ is completely determined by its Bratteli diagram.

**Exercise 32.** Suppose $\lambda_A$ and $\lambda_B$ are trace column vectors for $A$ and $B$ satisfying $m_A \lambda_A = n_B \lambda_B$ respectively as in Exercise 11. Assume the entries of $\lambda_A$ and $\lambda_B$ are all strictly positive. Denote by $\text{tr}_A$ and $\text{tr}_B$ the corresponding faithful tracial states on $A$ and $B$. Prove that $\text{tr}_B |_A = \text{tr}_A$ if and only if $\Lambda \lambda_B = \lambda_A$.

6 Connected inclusions

We continue the notation of the previous section for an inclusion $A \subset B$ with dimension row vectors $m_A = (m_1, \ldots, m_k)$ and $n_B = (n_1, \ldots, n_\ell)$ respectively.

**Definition 33.** The inclusion $A \subset B$ is called *connected* if the graph $\Gamma$ is connected.

**Exercise 34 (★).** Prove that $\Gamma$ is connected if and only if $Z(A) \cap Z(B) = \mathbb{C}$.

**Exercise 35 (★).** Show that if $A \subset B$ is connected, there is a unique $d > 0$ and unique trace vector $\lambda_B$ such that $m_B \lambda_B = 1$ and $\Lambda^T \lambda_B = d^2 \lambda_B$. Then deduce:

1. If $\lambda_A := \Lambda \lambda_B$, then $\Lambda^T \lambda_A = d^2 \lambda_B$.

2. \[
\begin{pmatrix}
0 & \Lambda \\
\Lambda^T & 0
\end{pmatrix}
\begin{pmatrix}
\lambda_B \\
0
\end{pmatrix}
= d \begin{pmatrix}
\lambda_B \\
0
\end{pmatrix}.
\]

*Hint: Use the Frobenius-Perron Theorem.*

**Definition 36.** If $A \subset B$ is connected, the scalar $d$ from Exercise 35 is called the *Frobenius Perron eigenvalue*. The trace vector $\lambda_B$ is called a *Frobenius Perron eigenvector*.

References