Mason II: notes online at NCGOA website.

1. Axioms for VOA
2. Examples + existence thus.
3. Representations with modular data, invariants, forms.

Axioms: \( k \) commutative ring, \( V \) a \( k \)-module, \( \infty \) bilinear products \( \otimes \rightarrow V \) \( u(n)v = u_{n+v}, n \in \mathbb{Z} \).

1. \( u_n u_{n'} v = 0 \) for \( n > 0 \).
2. \( \exists 1_G V \), vacuum vector \( u(-1)1 = u \), \( u(n)1 = 0 \) \( (n \geq 0) \).
3. Jacobi (Borcherds) identity: \( [s, r, t] = 1 \)
   
   [no \( a_n \)'s are associative, commutative, kind of weird...]

\[ \sum_{n=0}^{\infty} \left( \frac{2}{n} \right) (u(n+s)v)(r+t-n)w \]

\[ = \sum_{n=0}^{\infty} \left( \frac{2}{n} \right) (u(n+s-t)v)(r+t-n)w - (-1)^t u(n-s+v)(r+t)w \]

Category of vertex \( k \)-alg's. Morphism \( V \rightarrow W \) must preserve \( (n) \) and vacuum \( \left[ (L_0)x \right] \left[ \Delta L_0 \right] = \Delta [u(x)] \)

\[ \Delta (1) = 1_{\otimes} \]

Ex: \( k \)-alg. \( k \)-commutative assoc alg \( \Rightarrow 1 \). \( u(-1) \) \( v = u \), \( 1 \) \( = \).

Modes: \( u \in V \) \( u(n) : V \rightarrow V \) \( \in \mathbb{Z} \)-linear endomorphism.

1st mode of \( u \).

\[ \sum_{n \in \mathbb{Z}} \ (u(n)z^{-n}) \text{ vector operator} \]

formal generating. for. for these alg's.

\[ \otimes End(V)[[z, z^{-1}]] \]

\[ \sum_{n \in \mathbb{Z}} \ (u(n)v z^{-n}) \text{ only finitely many negative } z \]'s.
\[ F(V) = \left\{ \sum_{n \in \mathbb{N}} a(n) z^{-n} \mid a(n) \in \text{End}(V) \text{ } \right\} \]

- \( \forall u \in V, \ Y(u, z) \in F(V) \).
- \( Y : V \rightarrow F(V) \) by \( u \mapsto Y(u, z) \). \text{ b:m map}
  "state-field correspondence in QFT"
- \( Y(u, z) \ 1 = \sum_{n} a(n) z^{-n-1} = u + O(z) \)
  "creation axiom"
- "translation covariance"

Introduce \( D : V \rightarrow \mathfrak{v} V \) by \( u \mapsto u(z-2) \ 1 \).

\[ [D, u(n)] = D u(n) - u(n) D = -n u(n-1) \quad (\text{J.I.)} \]

operators on \( V \)

in terms of vertex operators: \( \left[ \partial, Y(u, z) \right] = \sum_{n} [D, u(n)] z^{-n-1} = \frac{2}{z} Y(u, z) \) formal partial derivative.

- "locality" (perhaps most important!) for \( t \gg 0 \):

1. \( \sum_{n \geq 0} \frac{\lambda t} {n!} \left( \sum_{s \geq 1} (u_{n+s} + c) v_{s+t} - 2(1) v_{3s+t} + O(n+n+1) s \right) = 0. \)

2. \( (z_{-1} - \eta_{z_{-1}} \ Y(u_{z_{-1}}) \ Y(u_{z_{-1}}) = C \eta_{z_{-1}} \ Y(u_{z_{-1}}) Y(u_{z_{-1}}) = 0. \)

\[ [Y(u_{z_{-1}}), Y(u_{z_{-1}})] = 0 \implies u_{n}(v(n) = v(n) u(n) \text{  then} \]

- not true in general, need to mult. by \( z_{-1} \ 3z \) (large \( t \)).
- its smallest \( t \) str. \( u(t) v = 0. \)
- unlike \( Y(u_{z_{-1}}) \ 2 \neq Y(u_{z_{-1}}). \)

\[ [u_{n+1}, u_{s}] = [u_{n}, u_{s+1}] = 0, \]

\[ t = \omega \ 2 : [u_{n+2}, v_{s}] = -2[u_{n+1}, v_{s+1}] + [u_{n}, v_{s+2}] = 0. \]
Thm: \((V, Y, 1, D)\) is a \(\text{VerAlg}\). Let \(D: V \to V, \ Y: V \to F(V)\) s.t. \(\forall v\in V, \ Y_1(v, 1) = 1, \ Y_2(v, 1) = v + O(\alpha^2) + \alpha v, \ Y_1(v, 2) = 0\). Then:

\[ Y(v, 1^2) = \text{Id} \] (follows from \(Y_1\))

Then: \((V, Y, 1, D)\) is a \text{VerAlg}.

- Std. \(\exists\) a VerAlg, affine but \(\text{Kac-Moody}\)
- Look at subset of \(V\) which generates \(V\)

Thm: Assume given \((V, Y, 1, D)\) s.t. above hold true. Let \(V \subseteq V\).

If \(V = \text{Span} \langle \xi(a_1) \xi(a_2) \ldots \xi(a_k) 1 \rangle\),

Then get a ! extension of \(Y, D\) so \((V, Y, 1, D)\) is a \text{VerAlg} as before.

- Usually want \(a_i\) to be a finite set.
- Get \(\text{non-trivial}\) ex's which are singly generated,
  - e.g. Heisenberg + Virasoro alg's.

\text{Heisenberg VerAlg:} \ CCR \space \text{Relation} \space \frac{d}{dx} a_{\alpha} = i \alpha \cdot a_{\alpha}\).

Start with \(x_1, \ldots, x_n \ldots\) \(\in\) variables, \(\in \left[ \frac{d}{dx}, i x_1 \right] = \text{id}.

\[ V = C[x_1] \in \mathbb{C} \]

\[ Y(x_1, z) = \sum_{N} x_1^N z^{-N-1} + \mathcal{D} x_1 z^{-N-1} \] is a quantum field.

\[ \text{Commutation} \rightarrow Y(x_1, z) \approx Y(x_1, z) \]: call \(a_{\alpha} = x_1 a_{\alpha}, \ a_{\alpha} = a_{\alpha}\).

\[ Y(x_1, 1^2) = \sum_{N} a_{\alpha} a_{\alpha} z^{-N-1} \] \(a_{\alpha}, a_{\alpha} = m_{N, \alpha, \alpha} \text{id} \). 

\[ D = \sum_{N} x_1 x_1 \sum_{N} D_1 Y_2(1, 2) f(\alpha) 1^2 = i \alpha \cdot Y(x_1, 1^2). \]
- Heisenberg not familiar.

- Each $\mathfrak{g}$ gives an affine Lie alg.
- WZW models in rational case are a direct generalization.

**Virasoro:**

\[ \mathfrak{vir} = \bigoplus \mathfrak{g}_{\mathfrak{ln}} \oplus \mathfrak{ck} \]

- Virscco is called witt alg.
- $\mathfrak{g}_{\mathfrak{ln}}$ is the Lie algebra of $\mathfrak{sl}(2)$.
- $\mathfrak{sl}(2)$ is the Lie algebra of $S^1$.
- Usually we need reps, so we get reps of central extension.

**Exercise:** $\mathfrak{vir}$ module $M$. $K$ acts as a scalar called central charge.

Consider $L(\varphi) = \sum L_{-n} \varphi^{n+1}$ on $M$. If $L(\varphi) \in \mathcal{F}(\varphi)$.

- $\varphi$ is a quantum field, then $L(\varphi) \in \mathcal{F}(\varphi)$. (log calculation)
- $\varphi$ is a quantum field, then $L(\varphi) \in \mathcal{F}(\varphi)$. (no nil structure though...)

**Def:** $W(A)$ as $V$: $(V, Y, T, D)$ as before, but

- (at central charge) $Y(w, z) = \sum L_{-n} \varphi^{n+1}$
- $L(\varphi)$ is a simple operator on $V$.
- $\mathfrak{sp}(\mathfrak{g}) \subseteq \mathfrak{z}$, bdd below.

- $W(A)$ is a semi-simple operator on $V$. $H = \sum \lambda \varphi^{n+1}$.

- Eigenvalues are $\pm \text{dim} \varphi$.

**Confined theory:** $V = \bigoplus V_n$, where $V_n = \mathcal{F}(\mathfrak{g}|_{\mathfrak{g}_{\mathfrak{ln}}})$.

- $H$ is the Hamiltonian, $\mathfrak{g}$ is energy.

- Assume $H$ has a minimal positive energy.

- Nicest theories have $V_0 = \mathbb{C}1$ spanned by vacuum.
Important part is decomposition of $V = \bigoplus \nu \nu_n$.

lots of quantum fields which give $V$ at various central charges.

Hence $\nu^\alpha_0: (\nu, \nu, \nu, \nu) = \nu^\nu >$.

yields $\nu^\alpha_0$: $\nu^\alpha_0(x, \nu, \nu, \nu) = \nu^\nu <$.

Central charge $c = 1$. $V$ is an induced module $\text{Ind}_{\mathfrak{sl}_2}^{GL_2}$.

$\nu^\nu \nu^\alpha_0: \nu^\nu = \bigoplus \nu \nu_n \oplus \mathfrak{sl}_2$. $L(\nu) = \sum_{n=1}^\infty \frac{\nu_{\nu_n - 2}}{\nu_{\nu_n - 2}} = \nu(\nu_1, \nu_2)$$

$V = \langle \nu_1, \nu_2 | \nu_1, \nu_2 \rangle$

- take e 1 dim rep. $\text{Ind}_{\mathfrak{sl}_2}^{GL_2}$: Knapp by $c$.
- get vOA generated by $\nu$

$V(\nu) = \langle \nu_1, \nu_2 | \nu_1, \nu_2 \rangle = \langle \nu_1, \nu_2 \rangle$ $\nu = \nu_1, \nu_2$.

[Note: $L(\nu) = \nu$, affine form of this and vertex module.]

will show $\nu_n$'s are fractional + $\nu = \nu^\nu \nu_n$. tomorrow.

**Facts:**

- In $\nu^\nu \nu^\alpha_0$ $\nu = \nu_1, \nu_2$, spin 2 particle,
  like $\nu$ graviton.
- $\nu^\nu \nu_n$ (has weight $k$), $\nu^\nu_n$ $\nu$ mode $\nu^\nu_{\nu_n}$. $\nu^\nu_n$: $V_\nu \rightarrow V_{\nu + k}$.
- $\nu_{\nu_n + k} = \nu_{\nu_n+k-1} \nu_{\nu_n}$, $\nu_{\nu_n} = k \nu$, $\nu_{\nu_n} = p \nu$ true t.t.
- Zero mode of $\nu^\nu \nu_n V_\nu (v)$: $\nu^\nu_{\nu_n} \nu = \nu^\nu_{\nu_n+1} \nu_1 \rightarrow \nu^\nu_{\nu_n+1} \nu_1$

$\sum \nu^\nu \nu_n T_{\nu^\nu \nu_n} (v)^q \wedge q$ formal $q$-expansion

$\sum \nu^\nu \nu_n T_{\nu^\nu \nu_n} (v)^q \wedge q$ formal $q$-expansion

- Have associated $V \rightarrow$ $q$-expansions $\mathcal{B}(v, q)$
- Correlation fct.
- For a good vOA, these are modular forms. Correlation fcts.
- Need to understand rep theory of $V$. 

Genus 1
Think about formal series \( \sum_{n \geq 0} \frac{\omega(n)}{n!} q^n = \sum_{\nu \in \mathbb{Z}^2} Z_v(\nu) q^\nu \).

**Heisenberg VOA:** try \( v = 1 \). \( \mathfrak{o}(1) = \text{Id} \). 
\[ Z_v(1, \nu) = \mathcal{F}(\nu) \sum_{n \geq 0} \frac{\omega(n)}{n!} q^n \]

\( V = \text{Fock space} \ L^2 \otimes \mathbb{C} \oplus \Lambda \) with \( \deg(x_i) = i \). \( V = \oplus \nu \in \mathbb{Z}^2 \text{C} \nu \). 
\[ \dim(V_\nu) = p(\nu) \text{ partition of } \nu. \]

\[ \Rightarrow Z_v(1, \nu) = \mathcal{F}(\nu) \mathcal{P}(\nu) \]

\[ \mathcal{P}(\nu) = \sum_{\lambda \in \mathcal{P}(\nu)} \prod_{m \in \lambda} ^{\lambda(m)} q^m \]

\[ \mathcal{F}(\nu) = \frac{1}{\eta^2(\nu)} \]

\[ \eta^2(\nu) = \frac{1}{\eta(\nu)} \text{ weight } \frac{1}{2}. \]

\[ \mathcal{C} = \mathcal{C}(\nu) = \mathcal{F}(\nu) \mathcal{P}(\nu) \text{ modular form} \]

- Heisenberg is not rational CFT.

\[ Z_v(v, \nu) = \mathcal{C}(\nu) \mathcal{P}(\nu) \]

**Rig of "quasimodular" forms** \( \mathcal{H}(P, Q, R) \)

\[ P = \prod_{\nu \in \mathbb{Z}^2} \frac{\eta^2(\nu)}{\eta(\nu)} \]

\[ Q = \prod_{\nu \in \mathbb{Z}^2} \frac{\eta^2(\nu)}{\eta(\nu)} \]

\[ R = \prod_{\nu \in \mathbb{Z}^2} \frac{\eta^2(\nu)}{\eta(\nu)} \]

where \( \mathcal{H}(P, Q, R) \) is algebraic rational under \( \mathbb{C} \).

**Theorem:** if \( v \) Heisenberg VOA, \( Z_v(v, \nu) \) if quasimodular for \( \nu \in \mathbb{Z}^2 \).

Conversely, each such \( f \) appears in this way.

\[ V_{\text{virasoro}} \text{ VOA: } V(\nu) = \mathcal{C}(\nu) \mathcal{P}(\nu) \]

\[ \text{Fibonacci basis form induced null construction.} \]

\[ Z_{\text{Vir}^0}(1, \nu) = \prod_{n \geq 0} \frac{1}{(1-q^n)^{-2}} \]

\[ \text{NOT quasimodular form, } \frac{1}{\mathcal{C}(\nu)} \mathcal{P}(\nu) \]

\[ \text{Genus } 1 \]

\[ \nu \to \nu + 2 \]

\[ \text{Not Virasoro} \]
Vertex Algs's

\[ \text{VOA}_C \xrightarrow{\psi} V', \text{ same central charge. Morphisms not always injective.} \]
\[ \ker \phi \leq V \text{ ideal.} \]
\[ \ker \phi \supseteq \mathcal{O}_V \phi \ni v. \]
\[ \text{VOA of CFT type} \quad \mathcal{O}_V \phi \ni f \text{ positive energy}. \]
\[ \text{- proper ideal} \subseteq \mathcal{O}_V \phi \ni f \text{ maximal proper ideal}. J. \]
\[ \text{Facts for Heisenberg} \quad J = (0). \]
\[ \text{For } V|_c, \text{ have nonzero max proper ideal, with } c \text{ in discrete series.} \]
\[ \text{Thm: } V|_c \phi \text{ max proper ideal } J. J = (0) \Leftrightarrow c \text{ in discrete series.} \]

- If \( J = (0) \), let \( L(c) = V|_c / J \text{ simple VOA.} \)
- Its partition function is a modular form.

\[ Z(L(c), q) = \text{ modular form of weight } 0 \text{ on } \Gamma_0(N). \]
\[ \Gamma_0(N) = \{ (a,b) \mid a \equiv b \equiv 0 \pmod{N} \} \]
\[ \text{If } V|_c \text{ not regular, } V|_c / J \text{ is regular for } c \text{ discrete series.} \]

**Def:** A VOA \( V \) is regular if \( V \text{-mod is semi-simple.} \)
- If semi-simple, \( V \) only finitely many co-classes of simples.

**Def:** V-Module: \( M \) a space, normal parity of VOA where it needs sense.
\[ \gamma_M : V \rightarrow \mathcal{E}(M) \quad u \rightarrow \gamma_M(u,v) = \sum v_i(v) t^{-i}, \theta_M(1,t) = \gamma_M \]
+ Jacobi id. for \( V|_M. \)
Simple module $N$ over $V$, 
$M = \bigoplus N_{n+3} \text{ spaces of } L_0$

$M_{(a)} = M_{x} \otimes M_{y} \otimes \cdots \ 	ext{ called conformal weight of } M$.

Fact: If $V$ rational, then $L_0 \in \mathbb{Q}$.

- In general, can get any $L_0$, e.g., for Heisenberg.

$V \ (a,b) \in \mathcal{B}(2,2), \ \mathcal{Z}(a, b) \frac{2}{x(t)^2 + y(t)^2} = \sum_{n \geq 0} q^{n}$

Singularity at $0$ or most a pole.

- Infinite number under congruence subgroup above.
- Only a finite # of $\theta$ expansions that can be obtained this way.
- Put in “cusps” to compactify modular curve, only finitely many.
- Space of $\theta$-expansions in (but is famous) is $\text{SL}(2, \mathbb{Z})$.
- Called a $\eta$-valued modular form.

$\text{SL}(2, \mathbb{Z}) \ \rac{\text{acts}}{\text{on}} \ \langle \mathbb{Z} \ (a, b) \ | \ (a, b) \in \mathcal{B}(2,2) \rangle$

$\ \rac{\text{acts}}{\text{on}} \ \langle \mathbb{Z} \ (a, b) \ | \ (a, b) \in \mathcal{B}(2,2) \rangle$

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Congruence factors through a finite quotient.

- If $V$ a rational VOA, $L_0$ $2$-cofinite
  $\Rightarrow$ finitely many $\theta$ classes. $\mu_n, \ldots, \mu_n$.
- Look at $\langle Z_{\mu_1} (2), \ldots, Z_{\mu_n} (2) \rangle \hookrightarrow \text{SL}(2, \mathbb{Z})$

$\text{as a module over } \mathbb{Z}$

$\text{Congruence factors through a finite quotient...}$

- If $V$ of CFT type, $V < V_{\text{CFT}}$ (self-dual $\Lambda$ cut of reps)
  module category is rigid + $?? \Rightarrow$ action factors

Congruence don’t hold 2-cofinite...
Suppose $A$ is a commutative algebra in a braided category. Why does $A$-mod have an overbraiding?

We have a functor $e \rightarrow A$-mod

$X \rightarrow A \otimes X$

The objects that have an overbraiding. (Exercise - for $D_{2n}$, this means $R$ has an overbraiding...)

Two things to check -

- this is a map of $A$-modules.
- the obvious inverse is an inverse.

A better way to say this is that there's a functor

$C \rightarrow \mathcal{Z}(A$-mod).

Q: What if the ambient $C$ isn't braided, and instead $A$ lifts to the centre? Presumably the above only works for $X \in C$ which lift.
\[ \text{Obj: } \square \]

\[ \text{Hom}(\square \to \square) = 2^nN \text{ if some shading} \]

\[ \text{Hom}(n \to m) = \begin{array}{c}
\text{Diagram}
\end{array} \]

\[ A\text{-mod- } A \]

\[ A\text{-mod- } B \]

\[ B\text{-mod- } A \]

\[ B\text{-mod- } A \]
Gammon II: judges not CFT

① InfraF CFT
② use CFT to motivate VOA's and conformal nets.
③ mathematical approaches to CFT

InfraF CFT in physics:

Step 1: Experimental measures something - scattering cross-sections.
- compare experimental results to theory. Compute amplitudes.
- evolve distribution \( \langle \mathcal{O}(t) \rangle \)

Step 2: too damn hard. Look at \( \langle \mathcal{O}(t) \rangle \) in limit \( t \to \infty \).
- entries of scattering matrix (S-matrix)

Step 2: LSZ reduction formulae which describe asymptotic amplitudes as
\[
\int \Phi_i(x_i, t_i) \Phi_j(x_j, t_j) dx_i dx_j \ldots
\]
- correlation functions/Green's functions.
- quantum fields.

Step 3: That's impossible. So we Taylor expand in coupling constant.
- Feynman integrals as effective descriptions of terms.
- Taylor expansion

Simplest case is 2d CFT written - CFT as math.

States in classical mechanics: particle \( x(t) \), \( \dot{x}(t) \) state
- observables are ftd. of state, e.g. energy \( \frac{1}{2} m \dot{x}^2 + V(x) \).

Classical field theory - field on spacetime
- sector of bundle on spacetime. E.g.
- fields are as important as particles.

E.g. \( \Phi(x, t) \), \( \dot{\Phi}(x, t) \)

Quantum field theory: algebraic span of states = \( \mathcal{H} \) (CS*)
Observables are six operators on $S$

energy $E$, \( \hat{p} \) momentum $\vec{p}$ and \( \hat{J} \) \( \phi \)-current by \( \phi \)-derivatives...

\( \mathcal{F} \) states are generated by fields acting on vacuum \( \phi \).

- operator-induced distribution on space-time.

- correlation follows \( \langle 0 | T [ \phi_i(x_1, t_1), \ldots, \phi_n(x_\eta, t_\eta)] | 0 \rangle \)

(One-point convergence)

- quantum fields are, \( \phi \), from \( \mathcal{F} \).
- modes \( \omega_n \) are fields smeared out in an appropriate way.

Let \( S = V \), space-time is a tube -semi-infinite cylinder.

Feynman's integral: \( \phi \), particle is potential. \( V(x) \) - Schrödinger eq...

\[ K(x'', t''; x', t') \] = amplitude of particle starting at \( (x', t') \) and ending at \( (x'', t'') \)

Solves Schrödinger equation.

\[ Y(x''; t'') = \int K(x'', t''; x', t') \phi(x', t') d^3x \]

kernel which solves the problem.

\[ N \; K(\cdots) = \int \phi_{x'} \phi_{x''} e^{i \frac{\pi}{\hbar} S(x',t')} \; Dx' \]

\( \phi \) to \( 2 \times 10^6 \) paths, look at errors, they are "paths"

\( \rightarrow \) new \( S = S_{\text{free}} + S_{\text{interaction}} \)

\( S_{\text{int}} = V \) potential.

Expand as Taylor in \( \hbar \).

\[ N \; K(\cdots) = \int e^{S_{\text{free}} + \frac{\pi}{\hbar} \sum \left( \frac{\phi_i^2}{2} \right) S^2} D\phi \]

reduce \( S \) to stuff happening at distance \( \phi \).
- Look at Feynman diagrams - only thing that matters is "vertices" on diagram.
- only role of potentials is localization - action is a vertex operator
- particles arise from Poisson bracket. - build up space by using commutators.

Stry to show: $\phi(x,t) \equiv \sum \phi_i(x,t)$ is some model.
- need a conformal structure to get a start.

$\omega$ \( \rightarrow \phi \equiv +oo \) - tells us where we are in the moduli space.
- only messy part is what happens at the vertices.

\[ O \leftrightarrow Y \leftrightarrow \bigcirc \] sphere or 3 punctures.
- each term is a correlation feat. in CFT.
- CFT and string theory highly related...

2d CFT: next time.

- Canonical: automorphism of CFT and VOA - modular.
- Bosonic VOA \( \rightarrow \) CFT

- Euclidean version (rather than Minkowski) Osterwalder-Schrader (1970s)
- apply to $\mathbb{R}^4$ Riemann Sphere.
- Quantum field (QFT) is an operator distribution on space-time.
- $\mathfrak{f}(\mathbb{R}^4)$ smooth dense subspace of $L^2$. $\mathfrak{f}(L) : \mathbb{S} \rightarrow \mathbb{S}$
I. Inner product: \( G_{\alpha \beta}(z_1, \ldots, z_n) = \langle 0 | T(\phi_{\alpha}(z_1) \cdots \phi_{\beta}(z_n)) | 0 \rangle \).

Think of \( z = t + ix \) as parameterized \( \mathbb{R}^\infty \). These \( G_{\alpha \beta} \)'s are contr. func. on conformal space \( \mathbb{C} \cup \{ \infty \} \).

Answers: 1. Locality: \( G_{\alpha \beta}(z_1, \ldots, z_i) = \frac{c_{\alpha \beta}}{|z_i-z_j|^4} \) \( \forall i \neq j \in \mathbb{N} \).

2. Positivity: \( \langle 0 | \phi_{\alpha}(z_1) \phi_{\beta}(z_2) | 0 \rangle \) is always \( \langle 0 | \phi_{\alpha}(z_2) \phi_{\beta}(z_1) | 0 \rangle \)

up to singularities when \( z_1 = z_2 \). No dense domain/continuousness.

In fact \( \langle \phi_{\alpha}(z), \phi_{\beta}(z) \rangle = 0 \)

(II) Positive definite covariance: \( E_2 = SL(2, \mathbb{R}) \times \mathbb{C} \). 3-dim representation.

Formal weights \( h; \bar{h}; \tau \Lambda \) i.e. real \#s.

\( G_{\Lambda}(\frac{z}{\Lambda}) = \frac{(\frac{z}{\Lambda})}{(\frac{z}{\Lambda})}{\Lambda}^{-2} \) \( G_{\bar{\Lambda}}(\frac{z}{\bar{\Lambda}}) \) \( \Lambda \in E_2 \), \( \bar{\Lambda} = \bar{\tau}(\frac{z}{\Lambda}) \) conformal action.

Also Möbius transformations.

\( G_{\alpha \beta}(z_1, z_2) = G_{\gamma \delta}(z_1, \tau^{-1} z_2) \delta_{\gamma \alpha} \delta_{\delta \beta} (\bar{z}_1 - \bar{z}_2)^{-h_1 - h_2} \) \( \bar{z}_1 - \bar{z}_2 \)

Unitarity - involution: \( \Lambda \in \Gamma \rightarrow \bar{\Lambda} \)

\( G_{\alpha \beta}(\bar{z}) = G_{\alpha \beta}(z) \)

inner prod. on wave packets: \( f \in S(\mathbb{C}^\infty) \int f(z) \phi_{\alpha}(z) \phi_{\beta}(z) \overline{c}^n \)

\( \langle \phi_{\alpha}(z), \phi_{\beta}(z) \rangle = \int G_{\alpha \beta}(z_1, \ldots, z_n) \phi_{\alpha}(z_1 \cdots \phi_{\beta}(z_n) \overline{c}^n \)

"Reflection positivity"

Condition is this is positive \( \overline{c}^n \) def inner prod.

(Re)construction: Thus, can build from this data.

Fock space \( S = \text{span of} \{ |0\rangle \} \)

H = completion.

Then, \( \phi_{\alpha}(z) : \mathbb{C} \rightarrow S \)

\( \phi_{\alpha}(z) \phi_{\beta}(z) = G_{\alpha \beta}(z) \phi_{\alpha}(z) \phi_{\beta}(z) \)
$L = 0 >$ is the vacuum vector, zero pt. for $[\Phi, 0] \in \mathcal{F}$. 

- get unitary rep. of $\mathfrak{e}_2$.
- get an energy operator - Hamiltonian $\geq 0$ positive spectrum.
- analytic continuation $\mathcal{G}_1 \cdots \mathcal{G}_n (z_1, \dotsc, z_n) = \phi (T(\Phi (z_1), \cdots, \Phi (z_n)))$.
- it analytically extends to $\mathcal{H}_1 \cdots \mathcal{H}_n (z_1, z_1', \dotsc, 2_1, 2_1')$.
- separately holomorphic on $2_1, \dotsc, 2_n$ and $2_1', \dotsc, 2_n'$.
- respect to $\mathcal{G}_i = z_i$, get $\phi$ back.

- more curious like state-field correspondence.

- Conformal symmetry on $\mathfrak{e}_2$, conf. symmetries are Möbius transformations.
- conserved current, conserved charge. $T(\mathcal{G})$, $\bar{T}(\mathcal{G})$ currents

- classical symmetry broken, but gently.

- by Vir $\cong \overline{\text{Vir}}$ 2 copies, one adjoint of other in unitary case.
- in a rational CFT, can extend to $A \otimes A$ vort's.
- sets of $z_1$ and $z_1'$ are sets of $z$. Chiral algebra.
- $H = \oplus_1^\infty H_1 \otimes H_2$ not needed. reps of $\text{Vir} \overline{\text{Vir}}$

- want this to be a finite sum $\oplus_1^\infty H_1 \otimes H_2$

- $H_1 \cdots \mathcal{G}_n (z_1, 2_1, z_1', 2_1') = \sum H(z_1, \dotsc, 2_n) \mathcal{G}(2_1, \dotsc, 2_n')$

- Correlation. for.

- super T-dual reps at mapping class groups.
- chiral blocks for $\overline{\text{Vir}}$ action of $SU(2, \mathbb{Z})$, $M_{\mathbb{C}}$ of $T^2$.

- .....

- Conformal: full CFT - correlation facts are more important than quantum fields
  - chiral blocks ($\text{Vir}$)

- Heavy-Kostler visions (1960s) !

- local quantum physics - Kanai-Iwasaki (2007)?

- to each region $\Omega$ in spectrum $K$ a Mackiwski $\mathcal{A}(\Omega)$ observables.

- $\mathcal{A}(\Omega) \subset B(\mathcal{H})$ some $H \in \mathcal{A}(\Omega)$ is hyperfinite. III.$\infty$.

- if $\Omega_1$, $\Omega_2$, are spacelike separated, $\mathcal{E}(\Omega_1) \not\subset \mathcal{A}(\Omega_2)$.
Note 2. $\text{diff}(S')$'s "light cones" no project to chiral halves.

If local conformal net on $S'$, projective rep of $\text{diff}(S')$,

- fields of $\text{diff}(S')$ is subalgebra of $S'$, completely no WZW alg.

The central extension (since it's a proj rep) is $\text{Vir}(c)$.

Can talk about a character of $\text{Rep}(A')$, which is $\frac{2\pi}{\text{dim}(V_{\lambda})}$.

Complete rationality: each $\lambda_i$ converges to vacuum $\otimes_i \lambda_i$. Exercise.

\[ A(I_2 + I_3) \cong A(I_2 + I_4) \] finite index

Thm (Kawahigashi-Longo-Müger) If rep $\lambda_1$, category of reps

$\text{Rep}(A)$ is MTC.

Feng Xu (2022) completely rational conformal net $A'$, from $\text{grp} G$,

then $A_G$ equivariantization is completely rational.

Conjecture: sufficiently nice $\Rightarrow$ sufficiently nice $\Rightarrow$ VOAs

completely rational $\Rightarrow$ rational? VOAs

subfactor (finite index, finite depth) $\Rightarrow$ fusion category

where $\text{FRC}$ has weak braiding, $\text{NC}$ $\Rightarrow$ braided SFs.

\[ \text{FRC} \Rightarrow \text{FRC/Weak.} \]

$\text{from } G \Rightarrow \text{SF}$

Lessons for $S'$ from VOAs:

Zhu's algebra $\Rightarrow$ chiral block correlation function.
Assigning \( \mathfrak{h} \) to each prime/each \( \mathfrak{h} \) space \( \mathfrak{h} \) (as \( \mathcal{V} \) \( \mathfrak{v} \))

- goes up to \( \text{Hom}(V, \mathfrak{h}) \) large kernel.
- is a fort. on \( A = V/\text{null} \) (indepened of \( \mathfrak{h} \))
- from \( \text{Chevalley} \) for a neutral VOA, \( \mathfrak{h} = 0 \) \( \text{Hom}(\mathfrak{h}, \mathfrak{h}) \)

- non-vanishing (Gao/Horn)
- logarithmic ???

Lessons for VOA's from SF:

- Hasegawa, not braked. take double, find character \( \mathfrak{h} \)'s; \( C = 8 \).
- new construction of VOA's. -sperey of \( C = 1 \) doubles (Duch)
- constructing SF is \( \mathcal{V}^* \) at MTC, take \( 2x/2 \) to get MTC's.
- Subfactor \( \to \) full CFT
- \( \Rightarrow \) \( \mathfrak{h} \)-induction \( A(\mathfrak{h}) \) \( \mathfrak{B} \) (CFT)
  - no analog for VOA's.
Lecture I: Basic objects of CFT: vertex algebras, reps + primaries

1) alg. geometry - C alg. curves w/ singularities at most 2x pts or nodes
2) holomorphic approach - using Riemann surfaces
   - to Riemann surface y^2 = x^3, three classes of ops.
3) op. alg. - vertex alg. L-G, Diff S^1 comes from spin or boson's.

"Closed string theory" (closed) "open string theory" (open)
open problem.

3 sets of lecture notes: dynamics, website.

Plan:
1) Singular ops.
2) Quantization
3) CFT on Riemann surfaces

Hilbert transform on $S^1$: $L^2(S^1)$

$H^1(S^1) \ni f = \sum a_n e^{in\theta} \mapsto a_0 = 0 \forall n < 0. L^2$ 2-valued of
holo. fts in $D$.

Proj $L^2(S^1) \to H^1(S^1)$. Szegö proj.

$f = \left( \begin{array}{c} a_0 \frac{a_1}{a_0} \frac{a_2}{a_0} \cdots \frac{a_n}{a_0} \end{array} \right)$ sum of squares of coeff. by Parseval's identity $\sum_{n=1}^{\infty} |a_n|^2$.

$T(f) = P_m(f)$ Toeplitz op. is Fredholm if $f \not\equiv 0$ on $S^1$.

(Use $f^* : S^1 \to C$ (continuous))

Commutator $[T(f), P] = (-P) m(f) P - (P) m(f)(-P)

H = c(2P-1) Hilbert transform $[H, m(f)]$ op. with smooth kernel

Trace class $\text{Tr}(T) < \infty. \text{Tr}(T^*) < \infty.$

If $f \in L^1$, Cauchy $\Rightarrow F(t) = F(\text{e}^{i\theta}) = \frac{1}{2\pi i} \int \frac{f(\text{e}^{it})}{i} \, dt = \frac{1}{2\pi i} \int \frac{f(t)}{t-i} \, dt.$

Take limit as $r \to 1$. Want to get singular!...
Define truncated Hilb. trans : $H_\pi f(x) = \frac{1}{2\pi} \int \frac{f(x+e^{-i\omega})}{1-e^{-i\omega}} \, d\omega$  

Check on e^{i\omega} for phys, $H_\pi f = \mathcal{H} f + \text{wtf} \quad \mathcal{H} f = -u^{-1} \mathcal{H} (u f) \quad u = m(2)$.  

If $\|H\| < M$, get $x \leq M + \mathcal{H}$.

Check $1-e^{i\omega}$, $u^{-1}$. Compute Fourier Trans. $\mathcal{P}^{-1} X \{ e^{i\omega} \}$.  

Show Fourier coefficients are uniformly bounded, the $\|f\|_2$ + d + 1. (Hint by PIP.)

Compute: [e.g.,] $\frac{9}{c^3} - \frac{9}{c^3}$, singularity disappears. $c \rightarrow \infty$ if $g = \infty$.

If $H \mathcal{D} f(\xi) \neq 0$ for $f(\xi) = f(\xi(\xi))$. Then $V^{-1}HV - H \rightarrow H(\xi) \frac{e^{i\xi}}{\xi - \xi(\xi)} - \frac{e^{-i\xi}}{\xi - \xi(\xi)}$.

$L^p$ continuity of $H$ on $L^p$.

Classically, this is the M. Riesz theorem, the Poisson integral.  

$f(x) = \sum_\nu e^{i\nu x}$ \quad $P(x) \rightarrow f(x) \sin \nu \pi$ \quad $V(\nu) = \sum \nu c \left( 1 + e^{i\nu x} \right) \frac{1 - \nu^2}{1 - 2\nu x + x^2}$.

Note: $\|H\| = \text{tr} |H|$. (Define $f \rightarrow H$ if $f \rightarrow 0$ as $\nu \rightarrow \infty$.

But $H \rightarrow H \mathcal{L}$ = truncated Hilbert integral of Poisson transforms.

- just need to show $H$ is $L^p$.

Cottlori $(Hf)^2 = f^2 + 2H(\mathcal{L}Hf)$ \quad Cont $L^p \rightarrow Cont L^p$. \quad Cont $L^p \rightarrow Cont L^q$. \quad \text{rest}, \quad \text{interpolate.}$  

(Hebermay 3 in then)

Application: Conformal welding $\mathbb{C} \rightarrow \mathbb{C}$.

Find $f: \mathbb{D} \rightarrow \mathbb{C}$ conformal.  

$g: \mathbb{D} \rightarrow \mathbb{C}$ \quad $f(\zeta) = f(\xi)$.  

$f(\zeta) \rightarrow \zeta$ \quad $\mathbb{D} \rightarrow \bar{\mathbb{D}}$.

$f(\zeta) = e^{4\pi i \zeta} \quad \mathbb{D} \rightarrow \mathbb{D}$.

$f(\zeta) = e^{4\pi i \zeta} \quad \mathbb{D} \rightarrow \mathbb{D}$.

$f(\zeta) = e^{4\pi i \zeta} \quad \mathbb{D} \rightarrow \mathbb{D}$.
Hilbert transform on a closed curve:

\[ H \mathcal{E} = \frac{1}{\pi} \int \frac{f(t)}{2(t-x)} \, dt \]

- Scale length so it's 2π. Then \( H \mathcal{E} \) has smooth kernel.

Then \( H = i(2E-1) \mathcal{E} = \mathcal{E} \), but \( \mathcal{E} \neq E^* \).

Kernan-Stein: \( P = E(1+\mathcal{E}^{-1})^* \), proj onto \( E(\mathcal{E}) = e^\mathcal{E} \).

\[ EP = P, \quad PE = E \quad \text{so} \quad PE^*P = P(1+\mathcal{E}^{-1}) = E \]

Cayley maps: \( \mathbb{R} \to S \), \( \mathcal{E}(x) = \frac{x}{\|x\|} \quad \text{on} \quad S \), \( \mathcal{O} \)

- Poisson is \( e^{-|z|^2} \) in Farrow, all is similar as before.

Hilbert transform on \( L^2 \) (since \( R \to \mathbb{C} \) transforms; Betti's transform)

Riesz transform \( R^k \) is mult. by \( e^{ikx} \) in Farrow on \( L^2(\mathbb{R}) \).

\[ R^k(\mathcal{E}) = \frac{k}{\pi} \hat{f}(k) \] "manifestly unitary"

\[ H^k(z) = \frac{k}{2\pi i} \log z \quad \text{for} \quad \mathbb{C}^+ \setminus \mathbb{R} \]

- Poisson mult. \( e^{-|z|^2} \)

\( L^p \): Calderon-Zygmund (\( C = R \times R \), \( C(\mathcal{E}) = C(R) \circ C(R) \))

\[ H \mathcal{E} = H \]

\[ u = f(z) \quad \text{on} \quad \mathbb{C} \]

\[ T \mathcal{E} \mathcal{E} = \frac{1}{\pi} \int \frac{f(x)}{(x-t)^2} \, dx \]

\[ T \mathcal{E} = \mathcal{E} \]

Wasserman II:

- Betti's transform, mult. by \( \hat{S}_n \) in Farrow on \( C^2(\mathbb{R}) \).

\[ \text{Sinc} \quad \frac{\sin \pi x}{\pi x} \quad T = \mathbb{R}^2 \quad \text{flkn} = \frac{1}{\pi} \int \frac{f(x)}{\left( \frac{x-t}{\pi} \right)^2} \, dx \quad \text{div} \quad \text{in} \quad \mathbb{R}^2 \]

\[ T \left( \frac{d}{dx} \right) = \frac{2}{\pi} T \frac{d}{dx} \quad \frac{2g}{\pi} = R \]

- Like Hilbert, \( T \) has commuting with conformal change of coord. \( \gamma \).

\[ \gamma \mathcal{E} \mathcal{E} \mathcal{E} \mathcal{E} \quad T(\mathcal{E}) \mathcal{E} \mathcal{E} \] has smooth kernel.
Define $\mathcal{A}^2(L^2)$. Be careful: space of $L^2$ holomorphic sections on $\mathbb{C}^2$.

$\mathcal{A}^2(L^2)$ acts on $\mathcal{A}^2(L^2)$ (matrix valued) via operator dual $(K^N, L^N)$ called Grassmann operator.

$K^2$ s.a. are op. of trace class. Fredholm exists.

If $K^2$ and $L^2$ arise from $V = g^2$ of $\det(1 - K^2) = \frac{1}{\text{dim}}(2\log g^2 + \frac{1}{2} + \log g^2 + \frac{1}{2} + \log (g_+ - g_-))$.

Turns out to be $= 0$ to $\text{det}_{K^2}(L^2)$ of $\mathcal{A}^2(L^2)$ (reps of Diff(S^1) for C-measure).

Beltrami eq.: require $\text{det}^2g^2$ on $\mathbb{C}^2$.

Pull back by $D \to L^2$ (after not conformal), $\text{det} = \text{det}^2g^2$.

$\text{det} = \text{Beltrami coeff}$. Well $< 0$.

$f: \mathbb{D} \to \mathbb{D}$, $f: \mathbb{C} \to \mathbb{C}$, $f: S^1 \to S^1$.

Take Beltrami eq. $\frac{f^*}{f} = -\alpha$.

Arrange for metric on $D$ to reflect to a smooth metric on $C^2$.

For reflections $\mathbb{C}$ are like $S^1$ of $D$ and $\mathbb{C}$ are like $\mathbb{R}^2$ of $\mathbb{R}^2$.

$\frac{df}{2i} = \alpha \frac{dz}{2i}$.

$y = -z$, $h = \frac{df}{dz} = \frac{1}{2}$.

$h = (1 - T)^{-1} T$.

Proof: Quaternion.

Vector alg of a single real fermion. (wave formula + lower than)

* Helium sequence

* Trace-class sequence

* Oscillator rep

* Dirac on $\mathbb{R}^3$. Even rep of $\text{Diff} \times \mathbb{R}^3$. Even rep of $\text{Diff} \times \mathbb{R}^3$.

* Fermionic on $\mathbb{R}^3$.

* Fermionic sequence.

Wave alg of single real fermion $\psi \in (\mathbb{C}^2)^{\mathbb{Z}_2}$.

energy op. $[\mathbf{u}_x, \psi_\beta] = -i \chi$.

$\Lambda = \mathbb{Z}_2$, $\psi_\beta(2x) = \psi_\beta(x)$.
\[ U(\bar{z}) = \sum_{\nu} \psi_{\nu} \bar{z}^{-\nu - \frac{1}{2}} \quad \text{(by school of Sato)} \]

- If pos. energy rep. inv., get by vacuum R. it's done.

\[ \psi_{\nu} \cdot \psi^{*}_{\nu} \quad \text{vector alg.} \quad \psi_{\nu} \rightarrow \psi_{\nu}^* \quad \text{Fock space.} \]

Define \[ L_{\pm} = \frac{1}{2} \sum_{\nu} \psi_{\nu}^{*} \psi_{\nu} \pm \frac{1}{2} \sum_{\nu} \left( \frac{1}{\nu^2} - 1 \right) \psi_{\nu}^{*} \psi_{\nu} \quad \text{with Virasoro alg.,} \]

\[ \{ L_{n}, L_{m} \} = (m-n) L_{m+n} + \frac{1}{12} (m^3 - m) \delta_{m+n,0} \quad \text{(cos \hbar)} \]

\[ \{ \psi_{n}, \psi_{m} \} = -i \delta_{m+n,0} \quad n \in \mathbb{N}. \]

Virasoro acts lived. on Fock, Fock \( \mathcal{F} \). (gave explicitly using)

\[ \mathcal{F}(\omega) = \sum_{n \in \mathbb{Z}} \psi^{*}_{n} \psi_{n} \quad \text{field \( A(\omega) \) is annihilate. At annih. point of \( \omega \).} \]

\[ A^{-} = \text{neg. powers of } A \] counting annih. + creation annih.

Ordering. \[ \mathcal{A}(\omega) \mathcal{B}(\omega) = \mathcal{A}^{-}(\omega) \mathcal{B}(\omega) \pm \mathcal{B}(\omega) \mathcal{A}^{-}(\omega). \]

\[ \mathcal{A}_{1} \mathcal{A}_{2} \mathcal{A}_{3} \cdots \mathcal{A}_{n} = \mathcal{A}_{1} \left( \mathcal{A}_{2} \left( \cdots \mathcal{A}_{n} \right) \cdots \right) \quad \text{and so on.} \]

\[ \mathcal{A}_{1} \cdots \mathcal{A}_{n} = \sum_{\sigma} \mathcal{A}_{\sigma_{1}}^{\dagger} \mathcal{A}_{\sigma_{2}}^{\dagger} \cdots \mathcal{A}_{\sigma_{n}}^{\dagger} \quad \text{for any}\quad \sigma \in S_{n}. \]

So since \[ \psi_{\nu_{1}} \cdots \psi_{\nu_{n}} \rightarrow \left( \bar{z}^{\nu_{1}} \right) \psi_{\nu_{1}} \cdots \left( \bar{z}^{\nu_{n}} \right) \psi_{\nu_{n}} \]

\[ \alpha \rightarrow V(\alpha, z) \quad \text{(Wick's conv.)} \]

\[ V(\alpha, z) \overset{z \rightarrow \infty}{\rightarrow} \alpha. \]

- Immediate that \[ \left[ \mathcal{L}_{-1}, \mathcal{V}(\omega, z) \right] = \frac{d}{dz} \mathcal{V}(\omega, z) + \mathcal{L}_{0} \mathcal{V}(\omega, z) \quad \text{if } \mathcal{L}_{0} = \omega \mathcal{A} \]

Check locality + associativity. Both immediate from Wick's formula, use Taylor's formula.

\[ V(\alpha, z) V(b, w) = \pm V(\beta, w) V(\alpha, z) = V(V(\beta, z-w) b, w) \]

Taylor: \[ f(z - \omega) = f(\omega) + z \frac{d}{d\omega} f(\omega) \]

Wick: \[ \mathcal{A}_{1} \cdots \mathcal{A}_{n} \mathcal{B}_{1} \cdots \mathcal{B}_{n} = \sum \pm (\mathcal{A}_{i} \mathcal{B}_{i}) \cdots (\mathcal{A}_{i} \mathcal{B}_{i}) \cdot \mathcal{A}_{i} \cdots \mathcal{A}_{i} \mathcal{B}_{i} \cdots \mathcal{B}_{i} \]

\[ (A(\bar{z}) B(\bar{w})) (\alpha, \beta) \overset{\alpha \rightarrow \beta}{\rightarrow} \frac{1}{2\pi i} \ln \frac{\alpha - \beta}{\bar{z} - \bar{w}} \]

\[ (A(\bar{z}) B(\bar{w})) (\alpha, \beta) \overset{\alpha \rightarrow \beta}{\rightarrow} \frac{1}{2\pi i} \ln \frac{\alpha - \beta}{\bar{z} - \bar{w}} \]
Def: A new gp category has finite gp \( G \), if simple, 
\( gh = gh' \Rightarrow h = h' \Rightarrow g \circ g' = g \circ g' \Rightarrow g = g' \). 
\( f^2 = f \circ f \Rightarrow f = \Phi f \Rightarrow f = \Phi f \). 

\[ f = \Phi f \]

Ex. Fibonacci, euler pt. \( \Phi = SU(2)_2 \), \( f^2 = 10 \Phi f \). 

Ex. Ising model \( f^2 = 1 + \alpha \). 

Ex. \( \text{Rep}(S_3) \) \( f^2 = 1 + 2 f + \alpha \). 

Ex. euler pt. \( E_6 \) \( f^2 \equiv \Phi f \Rightarrow f^2 = 1 + 2 f + \alpha \). 

Tambun: Koyagami '98. \( m = 0 \). Then Gabelon, classified by 
\((G, \circ, \times, E)\) where \( E = \pm 1 \), \( \circ \) nondeg, symmetric bicharacter. \( \Phi \Phi \Phi \Phi \Phi \Phi \). 

Ostrik '03 If \( G = \mathbb{Z}/3 \Rightarrow m = 1 \), only 1 unitary case (non-zero unitary) 

Emersaf-Gelaki-Ostrik '04 \( G = \mathbb{Z}/m \), \( m = n - 1 \). Then \( n = 3^{a - 1} \), and 
\( \hat{Z} \) \( \cong \mathbb{F}_{3^a} \) canonically. Classified by a subset of \( H^2(F_{3^a}, \mathbb{F}_{3^a}) \) \( w \in \mathbb{F}_{3^a} \). 

So \( w(1, x^*) = 1 \).
Theorem: Assume $n \geq 0$, $r = \# G$, $d = \dim(f) = \frac{m+n}{2}$.

1. When $d = 0$, then either $\dim(M) \leq G = \mathbb{Z}_n$ (EG0), or
2. $n = 2^{2k-1}$, $m = 2^k$, $G$ extra special $2p$. 
   \[ [G,G] = \mathbb{Z}(G) = \mathbb{Z}_2, \quad \# G = 3. \]
   \[ \mathbb{G}_{28}, \mathbb{G}_{28}. \]

2. when $d \geq 1$, $G$ abelian, then $m = 0$, $d = 2^{2k}$.

Theorem: Assume $l = 1$, $G$ essential, $G = \mathbb{Z}_l$.

1. $l = 2$, $d = 2^k$, no such category.
2. $l = 2$, no such category.
3. $d = 2^k$, $G$ a category.

We will focus on case $l = 1$. We construct algs to map $\varepsilon_i$, construct $C_0$.

Definition: Curto alg $C_0 = C^*(S_{i_1}, ..., S_n)$ s.t.
\[ S_{i_1}^3 = S_{i_2}^3 = S_{i_3}^3 = 1. \]

Suppose $(C, \otimes) \in \text{End}(M)$, $M$ type II factor.

1. $\otimes$ is $G$-equivariant.
2. $C \otimes M$.
3. $\text{Hom}(f, G) = \mathbb{C}G \otimes M$, $T(\varepsilon) = G(\varepsilon \otimes T)$, $\forall \varepsilon \in G$.
4. $[\gamma, \varepsilon](x) = \sum_s x_s \otimes \gamma_s(x_s)$, $\varepsilon \in G, x_s \in C$, $\gamma \\ \in G_\otimes C_M$.

We will define $\otimes$ to be equivalent.

Definition: $d(f) = |\mathbb{Z} : \text{ker}(f)|^{1/2}$.

$\rho$ is $G$-equivariant s.t. $A \rho|_G = f$.

Hom $G \to C^*(S_{1}, S_2)$
\[ T \mapsto T \otimes I. \]

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\[ T \mapsto T \otimes I. \]

$G = \mathbb{Z}_2$, $\dim = 2$.

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$G = \mathbb{Z}_2$, $\dim = 2$.
\[ \text{All diagrams} \quad f \circ g = \overline{f(g)} \quad \overline{g} \circ f = \overline{f(g)} \overline{g} \quad f \circ \overline{g} = \overline{f(g)} g \quad \overline{f} \circ \overline{g} = \overline{f(g)} \overline{g} \]

Thus these eq's are \( \gamma = 1 \circ \gamma \).

When \( f = \overline{f}, \quad \overline{g} = \overline{g} \), \( r_f, r_g \) are real \( \gamma - r_g \) is pseudoreal.

2. War up:

Assume \( E \subset \text{End}(X) \) Tannaka-Kamagami; i.e. \( \text{End}(E) = \mathbb{C} \).

\[ \operatorname{deg} f = \gamma f, \quad \operatorname{deg} \circ f = \gamma f \text{ for all } f \]

Fix \( g \). One lift from eq. class. May assume \( \operatorname{deg} f = f \) exactly.

\[ (f \circ \overline{g}) = \operatorname{deg} \circ \overline{g} \text{ or } \operatorname{deg} \circ \overline{g} = f \]

Since \( (f, g) = 0 \) \( \Rightarrow \) \( \operatorname{deg} f = \gamma f \text{ or } \operatorname{deg} g = \gamma g \)

- association is always trivial.

Fix \( f = \overline{g} \in (f, f^2) \).

\[ \operatorname{deg} (f) \circ \operatorname{deg} (f^2) = (\operatorname{deg} f, \operatorname{deg} f^2) = (\operatorname{deg} f, f^2) \]

Thus \( \gamma f \), satisfying comm. alg. rels. \( \gamma f \).

\[ \gamma f \operatorname{deg} = \gamma f, \quad \gamma f \gamma f = 1 \]

All info of \( E \) contained in \( \operatorname{deg} f \), restricted to \( \mathbb{C} \).

\[ \operatorname{deg} (f) = \gamma f \text{ want to show } f(E) = \operatorname{deg} (f) \text{ along } \gamma \text{ continuity.} \]

\[ (f^2, f^2) = \overline{\gamma} f^2 \text{ gives proj. rep. of } G \text{ on } (f^2, f^2). \]

Consider \( \gamma (f) \).

\[ \gamma f \circ \gamma (f) = \overline{\gamma} f \text{, may assume } \gamma f = f. \]

\[ \Gamma f \text{ the unitary rep of } G \text{ on } (f^2, f^2) \text{ orthogonal.} \]

The eq's are true in \( \mathbb{C} \).

\[ \gamma (f) = \sum_{(g)} f_{\gamma}(f) g \quad \text{ for } \gamma (f) \text{ in } \Gamma f. \]

\[ \text{of class } 2. \]
Claim: \( \mathbf{L}_h(\mathbf{u}_g) = \mathbf{X}_h(\mathbf{g}) \mathbf{u}_g \)
and \( h \to \mathbf{X}_h \in \mathcal{G} \) is a gp. homomorphism.

\[ f \circ \mathbf{L}_h(\mathbf{u}_g) = \mathbf{L}_h(\mathbf{L}_g(\mathbf{u}_g)) = \mathbf{L}_h(\mathbf{e}(\mathbf{g})) = \mathbf{e}_h \mathbf{L}_h(\mathbf{u}_g) = \mathbf{L}_h(\mathbf{u}_g) \]

Now \( c = \mathbf{S}_c \mathbf{L}_h(\mathbf{u}_g) \mathbf{S}_c^{-1} = \mathbf{L}_h(\mathbf{S}_c \mathbf{u}_g \mathbf{S}_c^{-1}) = \mathbf{X}_h(\mathbf{g}) \mathbf{u}_g \mathbf{S}_h^{-1} \).

Claim: \( \mathbf{P}(\mathbf{u}_g) \mathbf{S}_h = \mathbf{S}_h \mathbf{u}_g \mathbf{S}_{h^{-1}} \)

\[ \mathbf{C} = \mathbf{X}_h \mathbf{S}_h \mathbf{P}(\mathbf{u}_g) \mathbf{S}_{h^{-1}} \]

we know \( \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \mathbf{P} \mathbf{e}_h = \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \)

\[ \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \sum \mathbf{S}_h \mathbf{P} \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \]

\[ \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \sum \mathbf{S}_h \mathbf{P} \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \]

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\[ \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \sum \mathbf{S}_h \mathbf{P} \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \]

\[ \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \sum \mathbf{S}_h \mathbf{P} \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \]

\[ \mathbf{S}_h \mathbf{P}(\mathbf{S}_h) = \sum \mathbf{S}_h \mathbf{P} \mathbf{e}_h \mathbf{P} \mathbf{e}_h \mathbf{S}_h^{-1} \]
\( \langle h, g \rangle = K_h(g) \) symmetric, nondegenerate characters, \( \varepsilon = \pm 1 \).

\( \text{Log}(S_h) = S_h \gamma \); \( f(S_g) = \text{Log}(\gamma) (\text{Log}(S_g) S_h) \). \( \gamma^* \). \( \text{Log} = \sum \langle h, g \rangle S_h S_g \).

Reconstruction comes for free! Use formulas to define \( \text{Aut}(\mathfrak{g}_N) \) get \( g \in \text{End}(\mathfrak{g}_N) \). Check satisfy correct rels.

\( \Rightarrow \langle h, g \rangle \in \text{End}(\mathfrak{g}_N) \).

To get \( N \in \text{End}(\mathfrak{g}_N) \), look at graded gauge action, \( T \in N \), do GNS construction.

*General case.* \( C \in \text{End}(\mathfrak{h}) \) where \( C \) is reductive, \( m = 0 \).

Thus may assume by previous argument that \( \text{Log}g = g, \text{Log}(S_h) = S_h \).

\( \text{Log}(f, g) = \text{Log}(g) S_h \). \( \text{Characters} : K_h(g) = K_h(g) \).

\( \text{In general don't have nondegeneracy! Sometimes is degenerate} \).

*Look at* \( K = (g^{p_2}) \). Hiib space \( \langle v, w \rangle = v^*w \) inner prod.

*Choose \( \mathcal{E} \in S_2^{\mathfrak{m}}, \text{ONS}. \)

\( v^2 = 0 \varepsilon \gamma = m \gamma \), so \( p^2 = 0 \varepsilon \gamma \gamma \).

\( \text{Choose} \ (p_1^2, p^2) = \bigoplus \mathcal{S}_2^{(2)} \mathcal{S}_2^{(2)} \oplus \text{Span} \mathcal{E} \). \( \varepsilon \gamma \gamma \).

\( \gamma^* \in (p_1^2, p^2) \).

\( \gamma^* = \bigoplus X_{X} S_h S_g + \gamma \gamma \), where \( \varepsilon \gamma \gamma \) using \( m = 0 \) in \( \mathfrak{g}(K) \).

**Frobenius reciprocity:** two natural maps.

\( J^i : K \to \mathbb{C}^n \)

\( J^i : K \to \mathbb{C}^n \) \( \mathbb{C}^n \) antisymmetric, \( J^i J^j - E \).

\( J^1 = \pi (T, T^*); J^2 = Z \).

\( J^1 = \pi (T, T^*); J^2 = Z \).

\( J^1 = \pi (T, T^*); J^2 = Z \).

\( f(S_0) = \left( \sum X_{X} S_h S_g + \sum T, T^* \right) f(S_0) \).

\( f(S_0) = \left( \sum X_{X} S_h S_g + \sum T, T^* \right) f(S_0) \).

\( f(S_g) = \left[ \sum X_{X} S_h S_g + \sum T, T^* \right] (\gamma^*) \).

\( f(S_0) = \left( \sum X_{X} S_h S_g + \sum T, T^* \right) f(S_0) \).

\( f(S_g) = \left[ \sum X_{X} S_h S_g + \sum T, T^* \right] (\gamma^*) \).
\[ f(\mathbf{g}) \mathbf{g}(\mathbf{h}) \mathbf{g}^{-1} = \mathbf{g}^* \text{Tr} (\mathbf{g} \mathbf{h}) \]

\[ = \left( \frac{1}{d} \sum \mathbf{S}_n + \frac{1}{d} \sum J_0 (T_k) T_k \right) \left( \frac{1}{d} \sum \mathbf{X}_n (\mathbf{g}) \mathbf{S}_n + \frac{1}{d} \sum \mathbf{Y}_n (\mathbf{g}) J_0 (T_k) T_k \right) \]

\[ = \frac{1}{d} \sum \mathbf{X}_n (\mathbf{g}) + \frac{1}{d} \sum J_0 (T_k) \left( \langle \mathbf{Y}_n (\mathbf{g}) T_k \rangle \langle T_k \rangle \right) \langle \mathbf{Y}_n (\mathbf{g}) \rangle \]

\[ = \frac{1}{d} \sum \langle \mathbf{Y}_n (\mathbf{g}) T_k \rangle \langle T_k \rangle = \langle \mathbf{Y}_n (\mathbf{g}) \rangle \]

\[ = \frac{1}{d} \sum \mathbf{X}_n (\mathbf{g}) + \frac{1}{d} \text{Tr}(\mathbf{g}) \quad \text{since} \quad d = n + md. \]

\[ \sum \left( (n+md) \text{Tr}(\mathbf{g}) \right) = n \sum \mathbf{X}_n (\mathbf{g}) + n \text{Tr}(\mathbf{g}) \]

\[ \Rightarrow \text{Q abelian, and m=0.} \]

For \( d \not\in \mathbb{Q} \), all characters are all 1, completely degenerate.

\[ m=2^m \Rightarrow \text{factor something...} \]

What goes wrong for \( f(\mathbf{w}) \in M \)?

\[ \log f = f \Rightarrow \mathcal{M} \mathcal{M}^G = f(\mathbf{w}), \quad \mathcal{M} \mathcal{M} = f(\mathbf{w}) \]

\[ \Rightarrow \text{M intermediate.} \]

In case 1, since \( d > 0 \),

\[ \mathcal{M} \mathcal{M}^G = f(\mathbf{w}) \Rightarrow \mathcal{M} \mathcal{M} = f(\mathbf{w}) \]

\[ \Rightarrow \mathcal{M} \mathcal{M} = f(\mathbf{w}) \]

In case 2, \( \mathcal{M} \mathcal{M} = f(\mathbf{w}) \]

\[ \text{square much more complicated!} \]
Teacher E: Functional field theories + Ring Spectra

by Stephen Stolz, many contributors.

Goal: construct spaces of field theories. (d=FT) of

d-dim geometric field theories

topological, Euclidean, conformal, quantum

-need higher categories + classify spaces.

Thoughts: 5 stands for discrete (only know any left of table Chain)*

10-TFT \cong H^0 \text{ ordinary C-cohomology}
10-TFT \cong H^0 \text{ C*}
10-TFT \cong H^0 \text{ super symmetries}.

Cay: 1211-FT = spectral topological modular forms.

Chiral CFT: mathematics described by a vertex algebra or chiral conformal nets

Segal CFT: 2-dim conformal functional field theory.

\text{Fun}(\mathcal{C} \text{-Bord}_d, \text{Hilb}) \text{ over } \mathbb{C},

\text{obj}(\mathcal{C} \text{-Bord}_d) \text{ are closed oriented conformal 1-mfd's. Let S}

\text{man (i: } i^2 \text{) are conformal surfaces w/ a split into 0-out conf}

conf welding - can glue along a diffeomorphism

- triple of small collars on S', need for smooth, Riemann, etc.

- look at collars, can see WRT.

Converse: cobordisms have 3 equipped by collars so

- can see how to glue...

- work up to conformal zonometry, color structure w/ orientation.

- very hard to construct.

General: A flavor of a functional field theory is specified by

a dimension d, geometry C,T

started by

spectral 

evolution of space
Ex's for G: Conformal state, T smooth state, Tor smooth state, Tom smooth state. E Endiemi, C Lorentzian.
also raise the concern of subfields to study, call it C. (looking)

To get a (C-G) category $\text{Fun}_G(d, \text{Bord}, d, \text{vect})$, another choice.

Picture for TFT's: C.

Chris Schommer-Pries:

1. TFT = Commutative Frobenius Alg's (frob Alg)
   - Present terms (L-vec, $\mathbb{F}_2$) (folklore)

2. $d$-TFT = semi-simple Frobenius Alg's.
   - Exp: $d$-vecc = (Alg, $\text{Bnd}$, Invariants)

3. $d$-TFT = MTC
   - Exp: $\text{Frob}$

3. $d$-TFT = Fusion Cat's.
   - $(d$-vec, $\otimes) = \text{Tensor-cat}$.

What's osse MTC for Segal CFT?

"Twisted" anomalies Segal CFT $\mathcal{Z}$

2. Relatedaremos
   - $d$-Bond $\rightarrow$ $d$-Bond $\rightarrow$ Linear Cat's

"$d$-TFT model twisted by Chern Simons..." (Examples?)
   - Equivalent to a weak CFT "modular functor"

We could $d$-TFT model twisted thing from $d$-vec.

Lecture 7: $d$-GFT = $\text{Fun}_G(d$, $\text{Bord}, d$, $\text{vect}, \otimes)$, a $d$-category.

Get! Define $d$-space, co-simplicial space ("spectrum"), ring spectrum.

C category, $\text{Bord}$ classifying space, "planar algebra line".

$\text{Bord} = \text{B}$ in $\text{Gen}$, as $\text{Gen}$, $\text{Gen}$ spaces.
\[ \text{Example: } E = \mathbb{R}_+ \times \mathbb{N}_0. \text{ Then } B(E) = \mathbb{S}^1 \text{, the circle.} \]

Note that \( B(\mathbb{R}_+, \mathbb{S}^1) \). 

**Definition** An *oo-loop space* (= infinite loop spectrum) is a seq. of pointed spaces

\[ E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \ldots \rightarrow E_k \rightarrow \ldots \]

**Example:** Claim that \( S^1 \) is an oo-loop space (= so is \( \mathbb{Z} \)).

**Prop:** Any topological ab. gr. is an oo-loop space.

**Fact:** \( \text{Bat} \) is also a top. ab. gr.

- can form \( B(\text{Bat}) = \mathbb{S} \text{A} \), another top. ab. gr.

\[ \mathbb{S} \text{A} = \{ \mathbb{R}^2 \} \]

\( \mathbb{S} \text{A} \) is configurations of \( n \) pts labeled by \( A \).

- not cyclic bar equiv!
- map background = "inclusion map!"
- possibly some centers "well-behaved"

**Exercise:** \( \mathbb{S}^n \langle \mathbb{N}_0 \rangle = \text{Conf}(\mathbb{R}^n \setminus \{0\}) = \mathbb{K} \) homotopy type un

Fundamental thin of algebra. map is take the 

\[ \text{map } \{ \} \text{ to } \{ \} \text{ in } \mathbb{N} \]
\[ CP^o = K(\mathbb{Z}, 2) = B^2(\mathbb{Z}) : S^v = B(\mathbb{Z}) \]

"decomposition differences homotopy gaps by 1."

**Def:** \( H^A = \langle A, B^A, \partial A \rangle \): Eilenberg-MacLane spectrum for \( A \); connected.

because \( H^A (X; A) \simeq [X, B^A] \). The \( |H^A| \) is the \( A \).

**Def:** \( T_A (E) = T_A (B^A, A) \) minus of \( E \).

**Def:** \( E^k (X) = [X, E_k] \) the "E-cohomology"

- If you plug in a point, might not get zero.
- Cohomology they use this name.
- This is represented by \( \Sigma_k \) spectrum.
- In [cohomology theory, this is suspension \( \Sigma_k \).]

A little bit of hope: only on these spaces as well! the \( E_k \)-spectra are \( E_k \)-cohomoogy.

**Comparison:** \( \text{O-GFT} = \text{Fun}^\otimes (\text{O-vect}_\otimes, \text{O-vect}_\otimes) \)

- \( S^0 \) is the symmetric, cocomplete.
- \( S_\infty \) is the symmetric monoidal at an \( S \).
- \( \text{O-vect}_\otimes \) by \( e_{\text{O-vect}} \).

\[ 1 \text{-vect}_\otimes = (\text{O-vect}_\otimes)^S \therefore \text{O-vect}_\otimes = \mathcal{L}_\infty (1 \text{-vect}_\otimes) = \mathcal{C} \]

Claim: \( \text{O-GFT}_H = H_\mathcal{C} (C, +) \)

- In general, we require \( (d\text{-vect}_\otimes, \otimes) \). Then \( (\text{O-GFT}_H, \otimes) \) is a symmetric monoidal category. Next the (co)homology spectrum is none necessary.

- Now \( H_\mathcal{C} (C, +) = \text{Eilenberg-Maclane} \).

*Proof:* (for theory; see notes on top, below.

**3.** \( 1 \text{-vect}_\otimes \) is itself (categorical, \( \mathcal{K}^\infty \)\

**Techniques:** 1) \( \text{Top-abstract}(\mathcal{C}, \otimes) \) by Kontsevich

2) \( \text{Top-abstract}(\mathcal{C}, \otimes) \), top. case. by Kontsevich.

\[ \text{Top-abstract}(\mathcal{C}, \otimes) \]

\[ \text{Top-abstract}(\mathcal{C}, \otimes) \]

\[ \text{Top-abstract}(\mathcal{C}, \otimes) \]
Apply $F(x) = V \otimes x \otimes V$.

Note that proj @ gives trace-class ops. For Hilb $\otimes$ does not exist at all.

2. OHS ops.

$F(x) = e^{-tA}$ continuous semigp, generator $A$ "Laplacian".

If $(\cdot, \cdot)$ psd def inner prod, then $e^{-tA}$ trace class, be.

$F(x) = e^{VxV^*}$ trace class ops.

Note: \( \lim_{t \to 0} F(x) = \lim_{t \to -\infty} e^{-tA} = 1 \).

But \( \lim_{t \to 0} F_{\text{trace}}(x) = \text{trace class, not trace class.} \)

- Complete space
- $1:1$ b/c homology many, split.
- Spin vector/Hilb spaces.
- For $111$-BFT, $F(x) = 2i$ graded.
- Spin irrep is target.
- $F(x) = e^{-tD_0}$ Doob. Direct op.

To have a symmetric spectrum, consider an $O$ plus antiholomorphic.

Then $1 \to D_0$ TFT, say we get some axiom for $111$-BFT.

Have map $\text{Spec} \rightarrow \text{Spec}$. 

```
A \rightarrow (A, BA, \ldots, \partial^n A, \ldots)  
Ab \rightarrow \text{Spec} 
Conf(\mathbb{R}^n, t) \rightarrow \text{Spec}
```

$\text{Spec} = \text{Spec} \otimes \mathbb{R}$.

$\text{Spec} \otimes \mathbb{R}$.

$\text{Spec}$ is $(\mathbb{R}, \text{Spec})$.

Sane, but $\text{Spec} \otimes \mathbb{R}$ not her.

```
K \otimes \mathbb{R} \rightarrow \text{Spec} 
```

(Or make $\otimes$, add $\otimes$).

Have map $\text{Spec} \rightarrow \text{Spec}$ by $(E_0, E_1, \ldots) \rightarrow (E_0, E_1, \ldots)$.

$\text{Spec} \rightarrow \text{Ab}$ by completion.

Can get a spectrum from a sym. monoidal category.

Why does this make sense philosophically?

Want to extend further $\text{Spec} \rightarrow \text{Spec}$. Let $K$ to $\text{Spec}$.

Add $K$ to it.

- Can extend to $\text{Spec}$. $K$ to $\text{Spec}$.

```
\text{Def: } \text{SpecFT} := \text{Spec} \otimes (\text{SpecFT})
```

$\text{SpecFT}$ to $\text{Spec}$. 

Can add $\text{SpecFT}$ to $\text{Spec}$. 

$\text{SpecFT}$ to $\text{Spec}$.
Rem: $K$, constructed by Graeme Segal, using $T$-spaces as a model for concretely spectra (think of topoloqy). $T$-spaces $\cong$ spectra.

Modern Philosophy: Why does $K_i$ exist? (to make spectra use $T$-spaces)

A category $\mathbf{C}$

$$
\mathbf{C} = \mathbf{C}(\cdot, \cdot)
$$

A $T$-space $T = T_{(\cdot, \cdot)}$ a $T$-space from $T$.

Both compositions are $+$.

What is $\text{Sym} \mathbf{C}^{-1}$?

Ex. $\text{Sym} \mathbf{C}^{-1} = \mathbf{C}$

Q. What is $\text{Sym} \mathbf{C}^{-1}$?

For trivial $\mathbf{C}$, you must be symmetric!

Recall from free theories:

Lem 1: $T \cong T = \text{Fun}^R(\text{1-Bend}^T, \text{1-Bend}^T)$

In discrete case, take $K$ to be $\text{Rate}$; $\text{R}$

Lem 2: this category is equivalent to $\text{Proj}^R$. Map is $\text{End}(K^{R}) = \text{End}(\text{R})$

Indeed, $\text{Fun}^R(\text{1-Bend}^T, (C(\cdot), \otimes)) \cong \text{cat}$

Lem 3: $\text{End}(\text{R}) = \text{Fun}^R(T, (C(\cdot), \otimes))$.

Not only is $\text{End}(\text{R})$ equivalent to $T$-space, but it is fully dualizable.

Lem 4: $\text{End}(\text{R})$ has internal monoids.

Lem 5: $\text{End}(\text{R})$ has the presentation as a Sym Cat.

Lem 6: $\text{End}(\text{R})$ has $\text{Horse}$ theory.

Copr: $\text{Fun}^R(\text{1-Bend}^T, (C(\cdot), \otimes)) \cong \text{Fun}^R(\text{1-Bend}^T, (C(\cdot), \otimes))$

For 1-manifold,
A factorization alg. on a nfdl $M$ (spacetime) is
an assignment

$$\begin{align*}
\text{on } M, \text{Div open} & \longrightarrow A(U) \text{ obj-3sp.} \\
& A(U) \text{ a sheaf, cplx } H^*(A(U)) \text{ physical observables.}
\end{align*}$$

$$\begin{align*}
\text{on } U \subset V \subset CV & \rightarrow (A(U) \otimes \cdots \otimes A(U)) \rightarrow A(V)
\end{align*}$$

Properties: compatible w/ composition.

- Every property to be a genuine factorization alg.
  - $U \subset V \rightarrow CV \quad A(U) \otimes \cdots \otimes A(U) \rightarrow A(U \cup U)$
    - So in $H^*$
  - "Cohom property" adjusted to $\otimes$

Examples:
- Free-particle moving in $M = \mathbb{R}$. $U \rightarrow A(U) = E(U) = \bigoplus_{\text{classical fields}} C(U)$,
  - Space of classical fields - classical phase space.
  - "Dense" phase space.
  - Really, secretly the exterior power.

Factorization alg. of classical observables.

$$H^*(\text{Obs classical}(U)) = \text{Sym}^*(H^*(\mathcal{E}(U))) = \mathbb{C}[q, \Lambda]$$
need a pairing \( E^*(\mathcal{O}) \otimes E^*(\mathcal{O}) \to \mathbb{C} \) degree = 1

symp. form \( \sum \phi_{ij} \prod \phi_{ij} \in \mathbb{C} \).

Poisson bracket \( \{ \cdot, \cdot \} \).

Can look at \( (\mathcal{O}, \cdot) \) to get map \( E^*(\mathcal{O}) \xrightarrow{\phi} \mathbb{C} \).

Poisson bracket on \( \text{Sym}^*(E^\bullet(\mathcal{O})) \).

BV-Laplacian \( \Delta_{\text{BV}}: \text{Sym}^* \to \text{Sym}^* \) by defined by

\[ \Delta_{\text{BV}}(f g) = \partial f \otimes g + f \otimes \partial g. \]

\( \text{H}^* (\text{Obs}^* (\mathcal{O})) \sim \text{Alg over } E^0(\mathcal{O}) \text{ of differential forms}. \)

Note: alg structure of \( \text{Sym}^* \) does not induce multi. above on \( \text{H}^* \).

Since \( \Delta_{\text{BV}} \) not a derivation.

If \( f, g \) have disjoint support, \( \otimes \partial g = 0 \), we can multiply.

Only multiply cycles w/ disjoint support.

Can represent 2 cycles on same integral by representatives on smaller manifold. Need to be careful. Pre-things careftuly.

\( \text{Ex: } \mathcal{O} = \mathbb{C}, \ E^0(\mathcal{O}) = \mathbb{R}^{2n}, \text{ natural pairing.} \)

\[ \text{H}^* (\text{Obs}^* (\mathcal{O})) \preceq \text{Alg over } \mathbb{C}, b_n \text{ (no \( \otimes \))}. \]

\[ \text{H}^0 (E^*(\mathcal{O})) = [E^0(\mathcal{O}) \oplus E^0(\mathcal{O})]^*, \text{ in Taylor sense}. \]

\[ \text{H}^0 (E^*(\mathcal{O})) = \mathbb{C} \] (in Taylor sense).

\( \Omega^* \to \hat{\Omega}^* \) ——> same field correspondence.
Relation to Functional Field Theories:

Twisted FT, $T$ twist factor, $E_T$ twisted FT.

Look at bordism category $T$ $E_T$

<table>
<thead>
<tr>
<th>Obj: $O$</th>
<th>$T(Y)$ alg.</th>
<th>$E(Y)_{T(Y)}$ mod.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morphism</td>
<td>$T(\Sigma)$</td>
<td>$E(\Sigma)_{T(\Sigma)}$</td>
</tr>
</tbody>
</table>

Thm (Dwyer-Kan-Rezk):

If $A$ is a finiteness alg., defined on d-manifolds equipped with a $T$ structure, then $A$ determines a twist functor $T_A$ and a $T_A$-twisted field theory $E_A$. (on d-dimensional manifolds of geometry $G$.)

Remark: Here, "alg." and "bimods" are interpreted in a derived sense.
T \to B \text{ map of } \omega - \text{categories.}

-the commutant of T inside B is $T'$ given by

- $\text{Obj}^a \overset{a}{\to} \text{Obj}^b$, $\beta_a : b \circ a = a \circ b \circ T$ s.t.

- $ba \rightarrow ba' \quad \text{e} \rightarrow \text{e}' \quad \text{m}
\downarrow \quad \downarrow \\
\beta_{a} \quad \beta_{a'}
\quad ab \rightarrow a'b

- $b a' \rightarrow a a' b$
\quad \beta_{a'} \quad \alpha_a \quad \beta_b
\quad aba' \rightarrow \alpha_{a'}

\text{Morphism} \overset{s}{\rightarrow} b \rightarrow b' \quad \text{s.t.} \quad \text{ba} \rightarrow b'a \quad \text{e} \rightarrow \text{e}' \quad \text{m}
\downarrow \quad \downarrow \\
\beta_{a} \quad \beta_{a'}
\quad ab \rightarrow a'b

\otimes : (b, \beta) \otimes (b', \beta') = (b \circ b', \quad bb' \rightarrow bb' \rightarrow b' \circ b')

\text{Map to } \overset{s}{ightarrow} (b, \beta) \rightarrow b'. \quad \text{Coboe identifies image of } T \circ B)

\text{Fact:} \text{ Canonical map } T \to T'. \text{ by } a \rightarrow (a), \text{ for each } (b, \beta), \text{ need } 30 \rightarrow a b \rightarrow b a. \text{ Just use } \beta)

\text{Lemma: } T \rightarrow T'^{\sim} \text{ is an equivalence of categories.}

\text{Specialize: } B = \text{Bim}(R) \quad R \text{ is hyperfinite III.}

\text{Def. } T \to \text{Bim}(R) \text{ is a truck if the natural map } T \to T'^{\sim} \text{ is an equivalence.}

\text{and } T \text{ is closed under contragredient?}
$Bim(R)$ is equipped with $2$ modules:

\[ f : H_1 \rightarrow H_2 \quad \Rightarrow \quad f^* : H_2 \rightarrow D(H_1) \]

\[ R \quad \otimes \quad R \quad \rightarrow \quad H \quad \text{where} \quad a \otimes b = b^g a \]

really want $T$ has unitary/rigidity structures,
and $T \rightarrow Bim(R)$ respects these structures.

Analogies between $B(H) + Bim(R)$:

<table>
<thead>
<tr>
<th>Trucks</th>
<th>WNs \ H</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>B(H)</td>
</tr>
<tr>
<td>$Bim(R)$</td>
<td>\text{topologies ...}</td>
</tr>
<tr>
<td>$\neq$</td>
<td>\text{L}^1(H) trace class</td>
</tr>
<tr>
<td>split bimods,</td>
<td>\text{L}^2(H) HS</td>
</tr>
<tr>
<td>$R \otimes R^{op}$ bimods,</td>
<td>$\text{abc-L}^2(H) \rightarrow \text{tr(cab)} \in C$</td>
</tr>
</tbody>
</table>

Thus: To every conformal net $A$ $[\iota(A) < \infty]$ there is
an associated truck $T(A)$. Moreover, $Z(T(A)) = \text{Rep}(A)$

Construction: 

Then every $A$-rep is an $R-R-R$ bimodule.
Among $L-R$ bimods, reps of $A$ are those satisfying:
1. The action of $A_2 \otimes A_3$ extends to $A_{23}$
2. The action of $A_4 \otimes A_4$ extends to $A_{44}$

$T(A):= \{ R-R \text{ bimods} \mid \text{Claim 1 holds}\} \quad \text{Full subset of } \text{Bimod}(R)$

Claim 2: $T(A)' = \{ R-R \text{ bimods} \mid \text{Claim 2 holds}\}$

$\Rightarrow T(T(A)) = T(T(A))$ (Claim 4)

Sketch of why these commute:

$H \in T(A), K \in T(A)'$, need $\exists \theta : H K \simeq K H$.

Claim: $I$ ideal in $T(A)$ analogous to $HS$.

$I_2 = \{ \text{elts of } T(A) \mid \text{the action of } A_4 \otimes A_4 \text{ extends}\}$

This is opposite to 2.

- Trunks are "analogous" of type I factors.
  - Fully classifiable.
Cauchy's Thm for MTC

- Let $G$ a finite gp. $p | | G \implies \exists x \in G \mid \text{ord}(x) = p$

  $\exp(G) = \text{least } N \text{ s.t. } x^N = e \forall x \in G$

  So if $p | | \exp(G)$, i.e., $\text{ord}(x)$ have some prime factors.

- Now $|G| = \sum_{V \in \text{Rep}(G)} \chi(V) \chi(V)^*$ where $\chi$ is char of $V$

  $V_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi(V)(g^n)$ where $V_n$ is char of $V$

  Observe: $V_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi(V)(g^n) = \text{dim}(V)$.

  $\chi(V)(g^n) = \text{least } n \text{ s.t. } V_n(V) = \text{dim}(V) + V \in \text{Rep}(G)$

- $E$ a strict spherical fusion category $(-) \otimes I \otimes E$ monoidal.

  Graphical calculus: $f \mapsto \frac{f}{f}$, etc.

  $\text{dim}(E) = \sum_{V \in \text{Rep}(E)} \chi(V)^2$

  $E^n = \text{Hom}_G(1, V^n) \otimes \text{by } f \mapsto \frac{f}{f}$

  $V_n(V) = Tr(E^n)$ trace of operators.

  FS $\exp(E) = \text{least } N \text{ s.t. } V_n(V) = \text{dim}(V) + V \in E$.

  Thm (N-Schauenburg) this $\#$ exists if $G$ finite. $G_{\text{con}}$

  Facts $\text{dim}(V) \in \mathbb{Z}[\mathbb{S}_n]$ cyclotomic integers.

  Con.: $\text{dim}(E) \in \mathbb{Z}[\mathbb{S}_n]$ Dedekind domain.

  Cauchy's thm (BNPw): $N = F \exp(E)$. Then $E(N)$ and $\langle \text{dim}(E) \rangle$

  in $\mathbb{Z}[\mathbb{S}_n]$ have same prime factors.
Fact (Bryant) \( \frac{\text{dim}(C)}{N} \) is algebraic integer.

Application to MTC.

Thm. (Vafa) \( T = (\theta_v) \) diagonal matrix, i.e. \( \text{Id}_N = \theta_v \) has finite order. (By non-deg. of br)\n
Thm. (N-Sch.) \( F \exp(C) = \text{ord}(T) \).

\[
N \prod \frac{2}{\theta_v} = 1 + v \in \mathbb{C}. \quad \theta_v = \text{id}.
\]

Thm (Bryant - Gelaki) \( C \) modular. \( \frac{\text{dim}(C)}{\text{dim}(\theta_v)^2} \) alg. integer.

\[
\text{dim}(C) = \prod \frac{2}{\theta_v} \geq \left( 1 + \frac{1}{\theta_v} \right)^2 - \text{dim}(C) = 0.
\]

\[
x = (\text{dim}(\theta_v)^2, \ldots, \text{dim}(\theta_v)^2, -\text{dim}(C)) \in \mathbb{Z}[S_N].
\]

If \( P | x_{i} \Rightarrow P | N \).

[Exercise 7.53] Let \( \mathcal{Q} \in \mathcal{K} \) be a finite set of prime ideals in a field \( K \).

Then, \( \mathcal{Q} \) only finite many tuples \((x_1, \ldots, x_n) \in K^N \).

\( P | x_{i} \Rightarrow P \in \mathcal{Q} \).

\( x = (1 + x_1 + \ldots + x_n) \) and no sub-sum is zero.

Consequently, \( \mathcal{Q} \) modular \& rank \( n \), \( F \exp(C) = N \), \( \mathcal{Q} \) only finitely many dim vectors in \( \mathbb{Z}[S_N] \).

If, any dim's not from \( \mathcal{Q} \), not many categories.

If, any br's, etc., \( \text{Lim}(\text{dim}(v \in \mathcal{Q})) \in \mathcal{C}(\mathbb{Q}[S_N], \mathbb{C}) \).

\( v \cdot 1 \neq N \) for \( \mathcal{Q} \).
YesCFT, j= \text{M. Bichlaff + R. Longo.}

Positive soln to Kangel-Runkel Conj, we'll see op. alg. CFT corollary.

**Chiral CFT**
\[ \theta \rightarrow A(\theta) \]

**Full CFT**
\[ \theta \rightarrow A(0) \]

**Boundary CFT**
\[ \theta \rightarrow A(0) \]

\[ A = \text{ACFT} \] completely reducible local conformal net, \( \rightarrow \text{Rep}(A) \) MTC.

\( (\text{K-Kong-Kreutzer}) \)

Extension \( A \text{CFT} \rightarrow B(\text{CFT}) \) using \( \text{Rep}(A) \), non-locality for \( B \).

\[ [A(\theta), B(\theta)] = 0 \]

"non-local extension" \( \rightarrow \) \( (\text{non-commutative}) \) Frob. alg. in \( \text{Rep}(A) \).

\[ \text{Vev: } V \hookrightarrow W, \quad \mathcal{W}[V, W] = \rho \in V, \quad \rho \in W. \]

\[ \text{Y(1, 2)} \text{ action on } W, \quad [Y(1, 2), Y(2, 3)] = \rho \in V, \quad \rho \in W. \]

\[ \text{non-local ext. of } A \rightarrow \text{B full CFT} \]

based on \( A \)

\[ \text{Klang-Runkel} \]

\[ \text{End - } \text{Sect 2.4, 2.5} \]

\[ \text{B(\theta)} = B(\theta) \text{ of } B(\text{CFT}). \]

**X-induction:**
\[ \text{NGM, w.o. bounds, braided fusion category assumption.} \]

\[ X_K \text{ given } \mu_\mu \otimes X_\mu = X \otimes M_\mu \text{, 3.0. from braiding.} \]

\[ \text{R-action of } M \text{ on } M, \text{ these commute.} \]

\[ \text{gives an } M-M \text{ module, functionally.} \]

\[ \text{choice of braiding } \gamma \rightarrow \mu \times \mu \text{ 2 functions.} \]

\[ \text{Remark: } B(1 \times 1) = \text{the } \text{vev.} \]

\[ \text{B-KZ } \mathbb{Z} \text{ is a modular invariant, \text{C1 covariance of} } \text{Sc(2, 2) w.r.t.} \text{ braiding.} \]
Lemaire gave a description of this situation via a commutative diagram. Exercise 2. (Using Pontryagin's construction method, restrict to one variable, and another using alg. obj is full form.)

Note: If \( Z \neq Z_n \), then this is LR-construction / Asymptotic Reduction.

Generalization of LR-construction.

Theorem (Biskoi-K-Longo). This is true.

Suppose \( \mathcal{P} \subseteq \mathcal{Q} \). Look at \( \mathcal{P} \). Pick a \( \mathcal{Q} \)-slice, same \( \Sigma \).

- Not sufficient for subfactors to be the same.

Product structure of alg. in MTC.

- Some small can be distinct (prod. structures).

Key step: [Some coefficients \( \mathcal{H} \), \( \mathcal{M} \) (Ham(\( \mathcal{A} \), \( \mathcal{M} \))).]

- Gives natural embedding \( \mathcal{H} \) (\( \mathcal{A} \), \( \mathcal{M} \)) \( \rightarrow \) Ham(\( \mathcal{A} \), \( \mathcal{M} \)).

- Need a characterization of the image.

Consequence:

Theorem 2: A bijective correspondence between monia-equivalence classes of non-local extensions and \( \mathcal{E} \)-classes of full CFT based on \( \mathcal{A} \).

Remark: Each (eq. class of) non-local extension corresponds to a \( \mathcal{E} \)-condition of boundary CFT.

Examples:

(1) Measure net in \( c = 1 - \infty \) discrete series. (all possible values \( < 1 \).)

\([\text{by (I)}] \) take \( \mathcal{A} \) in the closure.

- Some classification in (C) separately. non-local ext. \( \rightarrow \) full CFT

\( \mathcal{A} \) gives simplification at uniformism. \( \mathcal{K} \)-vanishing. \( \mathcal{K} \)-\( \mathcal{L} \).
Tensor, Segal CFT at the

Example: The free fermion.
- Construction (complex analysis)
- Calculation (QFT, etc.)
- Analytic properties (=> conformal nets)

Definition: Chiral Segal CFT with one label/sector.

Proposition: Hilbert space $F^2$
- Positive energy rep of Diff+(S^1) on $F^2$.
- $E(\Sigma)$ (disk subspace of $F^2$), $\Sigma$ a partitioned parameterized
  surface
- States: $\Sigma$, labels on $\Sigma$
- Cuts into $\Sigma_1, \Sigma_2$, then $E(\Sigma_2)E(\Sigma_1) = E(\Sigma)$
- Composition of ops.
- Reparametrization: $E(\Sigma)$ (all $Y_{out}$) $\rightarrow$ $E(\Sigma)$ (all $Y_{in}$)
- Holomorphic dependence $\rightarrow$ hole (line bundle $\omega$)
- Unitarity $E(\Sigma) = E(\Sigma)^*$.

Specific example:

General construction:
- $\Lambda H = \oplus \Lambda^N H$, $\Lambda^{0} H = C\mathbb{R}$.
- Creation/annihilation ops $a(\tau)w = f^{\tau} w$, $a(\tau)^{\dagger}$ annihilation
  - Pay a price for fermions, e.g., bosons (spin), but get that there are odd ops! $c(\tau)f^{\tau}c^{\dagger}(\tau)f^{\tau}$.
- $CA^\infty(H) = C^\infty(a(\tau), a(\tau)^{\dagger})$ universal self-adj.
  - $a(\tau), a(\tau)^{\dagger}$ = 0 $\rightarrow$ $C^\infty(a(\tau), a(\tau)^{\dagger})$.

Prop. $H = H^{\infty}(\Sigma)$
- Spin for $\Sigma$ odd.

$T_i : a(\tau^i) = a(\tau^i) + a(c^{\dagger}(\tau^i))^{\dagger}$. $F$ & flip (QFT structure)
- Take $\Sigma, \delta \cap \tau^i = 0$ for $\tau^i$
  - $a(\tau)^{\dagger}$ $\rightarrow$ $a(\tau)$
\[ (\mathbb{S}^1 \times \mathbb{C}) \times (\mathbb{S}^1 \times \mathbb{C}) \times \mathbb{C}^2 \]

\[ H^2(\Sigma) \cong H_{2\text{dim}} \oplus H_{\partial \Sigma} \]

\[ E(\Sigma) = \mathbb{C} \mathcal{T} : \mathcal{O} \mathcal{F} \rightarrow \mathcal{O} \mathcal{F} \]

Thm. This is a rigid CFT.

will talk about gluing + existence.

Def: \[ \text{Twist } T_\Sigma : \mathbb{C} \mathcal{T} \rightarrow \mathbb{C} \mathcal{T} \]

\[ a(\gamma_\text{in}) T_\Sigma = T_\Sigma a(\gamma_\text{out}) \]

\[ \mathcal{H}^2(\Sigma) : \mathcal{H}^2(\Sigma) \]

\[ \mathcal{H}^2(\Sigma) : \mathcal{H}^2(\Sigma) \]

\[ \text{Some regularity on } \partial \Sigma. \]

Comment: \( H^2(\Sigma) \) is a moduli of hole, 2-forms.

Geometric so can use the same pt!

Spin someone \( \Rightarrow \) 1 3 sections of complementary spin structure

Def: \[ X = \bigcirc \]

- can still define a Hardy space \( \mathcal{F} \) of square-integrable fields on \( 

\text{Q: does } X \text{ a bit of satisfying commutation relations?}

- don't expect it to be trace class (non-segment).
Theorem: $\exists \chi: F_0 \otimes F \to F$ s.t. $\forall \epsilon \in F_0$, $\chi(\epsilon \otimes \cdot) \in \mathcal{B}(F)$. 

(Proof:)

It parameterizes $\exists r \in \mathcal{N}$ such that $\chi(\epsilon \otimes \cdot)$ agrees with $\chi(\epsilon \otimes \cdot)$. (Confirmed net ...)

Uses connecting $\psi \leftrightarrow$ vertex alg.

Surface lives on $\exists$ of a Segal CFT -> tdd ops + confirmed $\chi$.

Sketch of idea:

- Convergence on dense domain, need a little bit above argument.
- Commutation rels. $T_\epsilon a(c, \alpha) = a(c, \alpha) T_\epsilon \Rightarrow T_\epsilon(f \otimes \cdot) = a(f \otimes \cdot) T_\epsilon$. 

\[
\text{Diagram:}
\begin{align*}
\circ & \quad \circ \quad \sim \quad \circ \\
\text{from convergence} & \quad \text{le} \quad \text{well-defined} \\
\text{convergence.}
\end{align*}
\]