Mason I: notes online at NCGOA website.

1. Axioms for VOA
2. Examples + existence thms.
3. Representations ⇝ modular data / invariants / forms

Axioms: k commutative ring, V a k-mod, w/ ∞ bilinear products $V \otimes V \longrightarrow V$ $\quad u(n)v \quad u,v \in V,\ n \in \mathbb{Z}$.

1. $\forall u,v \quad u(n)v = 0$ for $n \gg 0$.
2. $\exists\, 1 \in V$ vacuum vector $\quad u(-1)1 = u$, $\quad u(n)1 = 0 \quad (n \geq 0)$
3. Jacobi (Borcherds) identity: $\forall r,s,t \in \mathbb{Z}$

   [no (n)'s are assoc., commutative, kind of random...]

$\forall u,v,w \in V \qquad \sum_{i \geq 0} \binom{r}{i} \left[(u(t+i)v)(r+s-n)w \right]$

$$= \sum_{j \geq 0} (-1)^n \binom{t}{j} \Big\{ u(r+t-j)\underbrace{v(s+j)w}_{\text{do 1st}} - (-1)^t v(s+t-j)u(r+j)w \Big\}$$

Category of vertex k-alg's. Morphism $V \xrightarrow{\alpha} W$ must preserve (n) and vacuum $\quad [\alpha(u)](n)[\alpha(v)] = \alpha[u(n)v]$

$\alpha(1_V) = 1_W$

Ex: k-alg., k commutative assoc. alg w/ 1. $u(-1)v = uv$, $1_V = 1$.

Modes: $u \in V \quad u(n) : V \longrightarrow V$ k-linear endomorphism

n^{th} mode of u.

$$Y(u,z) = \sum_{n \in \mathbb{Z}} u(n) z^{-n-1} \quad \text{vertex operator}$$

$\in \mathrm{End}(V)[[z, z^{-1}]]$ formal generating fct. for these endo's.

$$Y(u,z)v = \sum_{n \in \mathbb{Z}} u(n)v\, z^{-n-1}$$ only finitely many negative z^i's

$$F(V) = \left\{ \sum_{n\in\mathbb{Z}} a(n) z^{-n-1} \quad a(n) \in \mathrm{End}(V) \;\middle|\; \begin{array}{l}\text{finitely many} \\ \text{neg powers of } z \text{ in} \\ \sum_n a(n) v z^{-n-1}\end{array} \right\}$$

- $\forall u \in V$, $Y(u,z) \in F(V)$.
- $Y : V \longrightarrow F(V)$ by $u \mapsto Y(u,z)$. k-lin map
  "state-field correspondence in QFT"
- $Y(u,z)\,\mathbb{1} = \sum_n u(n)\mathbb{1} z^{-n-1} = u + O(z)$
  "creation axiom"
- "translation covariance"
  Introduce $D : V \longrightarrow V$ by $u \mapsto u(-2)\mathbb{1}$.

$[D,u(n)] = \underbrace{D u(n) - u(n) D}_{\text{operators on } V} = -n u(n-1) \quad (\text{J.I.})$

in terms of vertex operators: $[D, Y(u,z)] = \sum_n [D,u(n)] z^{-n-1}$

$= \frac{\partial}{\partial z} Y(u,z)$ formal partial derivative.

- "locality" (perhaps most important!) for $t \gg 0$:

1 side of J.I. $\sum_{i\geq 0} (-1)^i \binom{t}{i} \left\{ \underbrace{u(n+t-i)\, \underbrace{v(s+i)w}_{\text{do 1st}}} - \underbrace{(-1)^t v(s+t-i) u(n+i) w} \right\} = 0.$

$(z_1 - z_2)^t Y(u, z_1) Y(v, z_2) - (z_2 - z_1)^t Y(v, z_2) Y(u, z_1) = 0.$

$[Y(u,z_1), Y(v,z_2)] = 0 \iff u(m)v(n) = v(n)u(m) \ \forall m,n.$

- not true in general, need to $\underline{\text{mult. by } (z_1 - z_2)^t}$ (large t).
- it's smallest t s.t. $u(t)v = 0$.
- write $Y(u,z_1) \underset{t}{\sim} Y(v,z_2)$. ; $\underbrace{[u(n+1), v(s)] - [u(n), v(s+1)] = 0}_{\text{if } t=1.}$

if $t=2$: $[u(n+2), v(s)] - 2[u(n+1), v(s+1)] + [u(n), v(s+2)] = 0.$

Thm: $(V, Y, 1, D)$ $1 \in V$, $D : V \to V$, $Y : V \to F(V)$ $u \mapsto Y(u,z)$

$Y(1,z) = Id$ (follows from axioms)

$Y(u,z)1 = u + O(z)$ $\forall u$.

$D1 = 0$

$[D, Y(u,z)] = \frac{\partial}{\partial z} Y(u,z)$

$Y(u,z) \sim Y(v,z)$

Then: $(V, \cancel{\ }, 1, \cancel{\ })$ is a vertex alg.

- std. ex's Virasoro, affine Kac-Moody
- look at subset of V which generates V

Thm: Assume given $(V, Y, 1, D)$ s.t. above hold $\forall u \in \mathcal{U} \subseteq V$.

D defined on V.

If $V = \operatorname{span} \langle u_{i_1}(n_1) u_{i_2}(n_2) \cdots u_{i_k}(n_k) 1 \rangle$

Then get a ! extension of Y, D so $(V, Y, 1, D)$ is a vertex algebra as before.

- usually start w/ $\mathcal{U}$ a finite set.
- get non-trivial ex's which are singly generated, eg. Heisenberg + Virasoro alg's.

Mason II:

Heisenberg Vertex Alg. CCR position x momentum $\frac{\partial}{\partial x}$ $\leadsto [\frac{\partial}{\partial x}, x] = id$.

- start w/ $x_1, \dots, x_n, \dots$ ∞ variables. $\leadsto [\frac{\partial}{\partial x_i}, i x_i] = i Id$. & renormalize.

$V = \mathbb{C}[x_i] \supseteq \mathcal{U} = \{x_1\}$.

$Y(x_1, z) = \sum_{n \geq 1} n x_n z^{n-1} + \partial x_n z^{-n-1}$ is a quantum field, $Y(x_1,z)1 = x_1 + O(z)$.

locality - $Y(x_1,z) \sim Y(x_1,z)$: call $a(n) = \partial x_n$ $n \geq 1$, $a(-n) = n x_n$.

$\leadsto Y(x_1, z) = \sum_n a(n) z^{-n-1}$. $[a(m), a(n)] = m \delta_{m+n,0} Id_V$.

$[a(r+2), a(s)] - 2[a(r+1), a(s+1)] + [a(r), a(s+2)] = 0$. $\Rightarrow \sim_2$.

$D = \sum (n+1) x_{n+1} \partial x_n$ $[D, Y(x_1, z)] = d_z Y(x_1, z)$.

- Heisenberg not rational VA
- note $\{a(n), \mathrm{Id}\}$ gives an affine Lie alg.
- WZW models in rational case are a direct generalization.

Virasoro: $\bigoplus_{n\in\mathbb{Z}} \mathbb{C}\,a(n) \oplus \mathbb{C}K \leftarrow$ Heisenberg.

$$Vir = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}K \qquad [L_n, K]=0 \qquad [L_m,L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}K$$

- central ext. of Witt alg. $0 \to \mathbb{C} \to Vir \to Witt \to 0$.
- L_n's are infinitesimal rotations of S^1.
- usually are proj. rep's, so to get reps to central extension.

Exercise: Vir module M. K acts as a scalar c called central charge.

Consider $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ on M. If $L(z) \in \mathcal{F}(M)$

is a quantum field, then $L(z) \sim_4 L(z)$. (long calculation)

- also $[L_{-1}, L(z)] = \partial_z L(z)$. (no VA structure though...)

Def of VOA: a VA $(V, Y, 1, D)$ as before, but
(of central charge c) $\exists\, \omega \in V$, and $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

where $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\, c\, \mathrm{Id}_V$.

- modes of field $Y(\omega,z)$ close on Vir w/ central charge c.
- now $D = L_{-1}$
- hard to not take next: $\langle L_{-1}, L_0, L_1\rangle \cong sl_2$. $\langle L_0\rangle = \underset{\text{Cartan}}{\mathfrak{h}}$

L_0 is a semi-simple operator on V. $Sp(L) \subseteq \mathbb{Z}$, bdd below.

- eigenspaces are f. dim'l.

Conformal grading on $V = \bigoplus_{n\geq n_0} V_n$ where $V_n = \{v\in V \mid L_0 v = nv\}$.
- get rid of D, put in ω.
- think of L_0 as a Hamiltonian, ~~so~~ n is Energy
  "26 bosonic CFT"
- unitarity $\Rightarrow n_0 = 0$ is positive energy.
- nicest theories have $V_0 = \mathbb{C}\mathbb{1}$ spanned by vacuum.

5

- important part is decomposition of $V = \bigoplus_{n \geq n_0} V_n$, b/c
  $\exists$ lots of quantum fields which give Vir at various central charges

<u>Heisenberg VOA</u>: $(V, \mathbb{1}, Y, \omega) = \langle x_1 \rangle$ $\omega = \frac{1}{2} x_1^2$.
only 1 ↗ ↖ lots of choices.

$Y(\frac{1}{2}x_1^2, z) =$ "normal ordered product" $\frac{1}{2} \overset{\circ}{\circ} Y(x_1, z)^2 \overset{\circ}{\circ} = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$.

central charge $c = 1$. V is an induced mod $\mathrm{Ind}_{\tilde{H}^+}^{H} \mathbb{C}\mathbb{1}$

<u>Vir VOA</u>: $Vir = \bigoplus \mathbb{C}L_n \oplus \mathbb{C}K$ $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = Y(\omega, z)$

$V = \langle L_{n_1} L_{n_2} \cdots L_{n_k} \mathbb{1} \mid n_i \leq -1 \rangle$

- take a 1 dim'l rep. $\mathrm{Ind}_{Vir_{\geq 0}} \mathbb{C}\mathbb{1}$ K acts by c.
- get VOA generated by ω

$V(c) = \langle L_{n_1} L_{n_2} \cdots L_{n_k} \mathbb{1} \mid n_k \leq -2 \rangle = \langle \omega \rangle$ $\omega = L_{-2}\mathbb{1}$.

[$L_{-1}\mathbb{1} = 0$. difference from this and Verma module]

- will show V_n's are f.dim'l + $V = \bigoplus V_n$ tomorrow.

<u>Facts</u>: In a VOA, $\mathbb{1} \in V_0$; $\omega = L_{-2}\mathbb{1} \in V_2$ spin 2 particle, like a graviton.

- $v \in V_k$ (v has weight k), $v(n)$ n^{th} mode $v(n): V_p \to V_{p+k-n-1}$
  do $L_0(v_n \omega) = (p+k-n-1) v_n \omega$ $L_0 v = kv$, $L_0 \omega = p\omega$, use J.I.
- zero mode of $v \in V_k$ $o(v) := v_{k-1} : V_p \to V_p$.

$q^{-\frac{c}{24}} \sum_p \mathrm{Tr}_{V_p}\, o(v) q^p$ formal q-expansion $q^{L_0 - \frac{c}{24}} \ldots$

- have associated $V \to \{q\text{-expansions}\}$ $Z(v, q)$
- correlation fcts. genus 1
- for a good (rational) VOA, these are modular forms. correlation fcts.
- need to understand rep theory of V.

Mason III:

VOA $\longrightarrow$ {Modular invariants, Modular data}   Q: Which ones?   ©

Think about formal series $q^{-c/24}\sum_{n\geq 0} \mathrm{Tr}_{V_n}\, o(v)\, q^n = Z_V(v,q)$

Heisenberg VOA: try $v=1$. $o(1)=\mathrm{Id}_V$. $Z_V(1,q)=q^{-c/24}\sum_{n\geq 0}\dim V_n\, q^n$

$V=$ Fock space $\mathbb{C}[x_1,x_2,\dots]$. $\deg(x_i)=i$. $V=\bigoplus_{n\geq 0} V_n$ $V_n=\mathrm{span}\langle x_1^{e_1}\cdots x_n^{e_n} \mid \sum_{i=1}^{n} i e_i = n\rangle$

$L_0=\sum_{n\geq 1} n x_n \partial x_n$ $L_0 V_n = n\,\mathrm{Id}\, V_n$. $\dim(V_n)=p(n)$ partition of n.

$\Rightarrow Z_V(1,q)=q^{-1/24}\sum_{n\geq 0} p(n)\, q^n$ "Partition fct"

(Euler) $= q^{-1/24}\prod_{n\geq 1}(1-q^n)^{-1} = \dfrac{1}{\eta(q)}$ modular form of weight $\frac{1}{2}$.

$q=e^{2\pi i \tau}$ τ in upper $\frac{1}{2}$ plane, $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$, $\eta\left(\dfrac{a\tau+b}{c\tau+d}\right)=\varepsilon(a,b,c,d)\sqrt{\tau}^{\,c\,\frac{1}{2}}\,\eta(\tau)$

24^{th} root of 1.

— Heisenberg is not rational CFT.

$Z_V(v,q)=?$

Ring of "quasimodular" forms $\mathbb{C}[P,Q,R]$ wt 2,4,6 Eisenstein series.

$$P=1-24\sum_{n\geq 1}\sigma_1(n)q^n \quad Q=1+240\sum_{n\geq 1}\sigma_3(n)q^n \quad R=1-504\sum_{n\geq 1}\sigma_5(n)q^n$$

where $\sigma_k(n)=\sum_{d\mid n} d^k$.

— algebra invariant under $\frac{d}{d\tau}$.

Thm: if $v\in$ Heisenberg VOA, $Z_V(v,q)=\dfrac{f}{\eta}$ quasi-modular form, where $f\in\mathbb{C}[P,Q,R]$.

Conversely, each such f appears in this way.

Virasoro VOA: $V(c)=\langle\omega\rangle$. $V(c)$ has basis $\{L_{-n_1}L_{-n_2}\cdots L_{-n_k}\mathbb{1} \mid n_1\geq\cdots\geq n_k\geq 2\}$

PBW basis from induced mod. construction.

$Z_{V(c)}(1,q)=q^{-c/24}\prod_{n=2}^{\infty}(1-q^n)^{-1}$ NOT quasimodular form, is mod. in $\mathbb{H}_+$.

$\mathbb{C}$-Vertex Alg's

$\uparrow$

VOA_c $\quad V \xrightarrow{\phi} V'$, $\omega \mapsto \omega'$ Same central charge. Morphisms not always injective.

$\cup$

Vir_c is initial object.

$\ker\phi \subseteq V$ ideal.

so $\ker\phi = \bigoplus_{n \geq 0} \ker\phi \cap V_n$.

VOA of CFT type: $V = \mathbb{C}1 \oplus V_1 \oplus \cdots$ positive energy.

- proper ideal $\subseteq \bigoplus_{n>0} V_n$. $\exists!$ maximal proper ideal! J.

Fact: For Heisenberg, $J = (0)$.

For $Vir(c)$, have nonzero max proper ideal, when c in discrete series.

Thm: $Vir(c)$. $\exists!$ max'l proper ideal J. $J \neq (0) \iff c$ in discrete series.

- if $J \neq (0)$, let $L(c) = Vir(c)/J$ simple VOA.
- its partition fct is a modular form.

$$Z_{L(c)}(1, q) = \text{modular fct. of weight } 0, \text{ on a congruence subgp of } SL(2,\mathbb{Z}).$$

$$\text{i.e., } Z(\tau) = Z\left(\frac{a\tau+b}{c\tau+d}\right) \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{matrix} m \equiv Id \\ \text{mod } N \end{matrix} \right\}$$

$V(c)$ not rational, $V(c)/J$ is rational for discrete series.

Def: A VOA V is rational if V-mod is semi-simple.

- if semi-simple, $\exists$ only finitely many iso-classes of simples.

Def: V-Module: M a space, mimic property of VOA where it makes sense.

$Y_M : V \longrightarrow F(M) \quad v \mapsto Y_M(v, z) = \sum v_M(n) z^{-n-1}, \ Y_M(1, z) = Id_M.$

+ Jacobi id. for $V \curvearrowright M$.

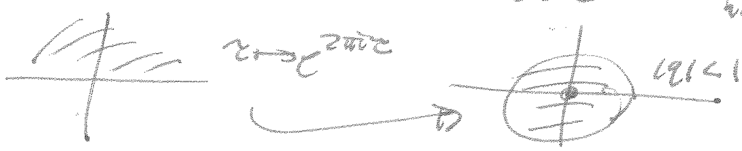
Simple module M over V, $M = \bigoplus_{n \geq 0} M_{n+\lambda}$ eigenspaces of L_0.

$M \neq (0)$, $M = M_\lambda \oplus M_{1+\lambda} \oplus \cdots$. M_λ called conformal weight of M.

Fact: If V rational, then $\lambda \in \mathbb{Q}$.

- in general, can get any λ, e.g. for Heisenberg.

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, $Z\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{n \geq n_0} b_n q^{n/N}$



Singularity at 0 at most a pole.

- invariant under congruence subgp + above!
- only a finite # of q expansions that can be obtained this way.
- put in "cusps" to compactify modular curve, only finitely many.
- space of q-expansions in $\mathbb{H}_+$ is f.dim., affords a rep'n of $SL(2,\mathbb{Z})$. (can have poles at cusps)
- called a $\vec{v}$-valued modular fct.

$$SL(2,\mathbb{Z}) \xrightarrow{\text{acts on}} \left\langle Z\left(\tfrac{a\tau+b}{c\tau+d}\right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \right\rangle$$

#

acts

$SL(2,\mathbb{Z})/\Gamma(N)$ congruence subgp

action factors through.

- if V a rational VOA (and C_2-cofinite) $\Rightarrow$ finitely many iso classes $M_1, \ldots, M_n$.
- look at $\langle Z_{M_1}(\tau), \ldots, Z_{M_n}(\tau) \rangle \circlearrowleft SL(2,\mathbb{Z})$

is a module over

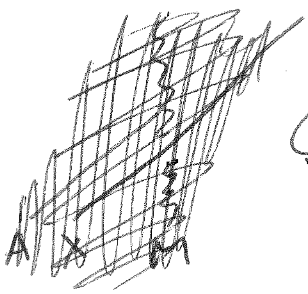
Conj: factors through a finite quotient...

- if V of CFT type, $V \cong V'$ (~~self~~ dual in cat. of mod's) module category is rigid + ?? $\rightsquigarrow$ action factors

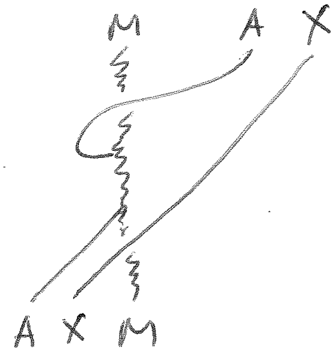
Conj: don't need C_2-cofinite...

Suppose A is a commutative algebra in a braided category.
Why does A-mod have an overbraiding?

We have a functor $\mathcal{C} \to A\text{-mod}$
$$X \mapsto A \otimes X$$
it's *these* objects that have an overbraiding. (Exercise — for D_{2n}, this means $\mathcal{P}$ has an overbraiding...)



<u>Claim</u>

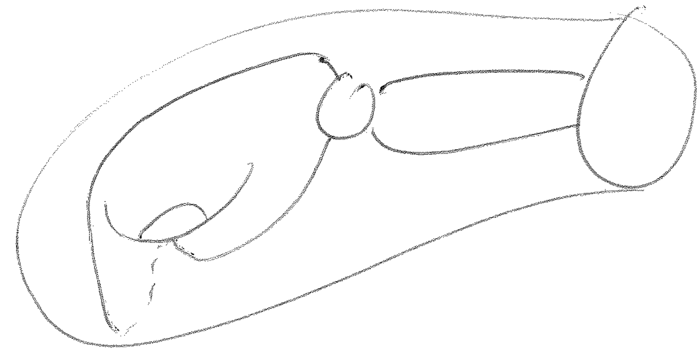


does the trick.

Two things to check —
- this is a map of A-modules.
- the obvious inverse is an inverse.

A better way to say this is that there's a functor
$$\mathcal{C} \to Z(A\text{-mod}).$$

Q What if the ambient $\mathcal{C}$ isn't braided, and instead A lifts to the centre?
Presumably the above only works for $X \in \mathcal{C}$ which lift.

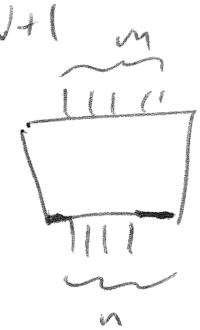


$P \longrightarrow \mathcal{C}$

Obj: ▨ □

$\mathrm{Hom}(\square \to \square) = 2\mathbb{N}$ if same shading

$2\mathbb{N}+1$

$\mathrm{Hom}(n \to m) =$ 

A-mod-A

A-mod-B

B-mod-A

B-mod-A

Gannon I: bridges VOA $\xleftrightarrow{CFT}$ SF

1. informal CFT
2. use CFT to motivate VOA's and conformal nets.
3. mathematical approaches to CFT

Informal QFT: in physics:

Step 1: Experimentalist measures something — $\frac{1}{2}$ lives — scattering cross-sections.

- compare experimental results w/ theory. Compute amplitudes.

  ↳ prob. distribution $\langle out | in \rangle$

Step 2: too damn hard. look at $|in\rangle$, $|out\rangle$ in limit $t \to \pm\infty$.

- entries of scattering matrix (S-matrix)

Step 3: LSZ reduction formulae which describe asymptotic amplitudes as $\int \langle \phi_1(x_1,t_1) \cdots \phi_n(x_n,t_n) \rangle dx_1^4 dx_2^4 \cdots$

correlation fcts / Green's fcts. — ~~# of particles~~ — quantum fields.

Step 4: That's impossible. So we Taylor expand in coupling constant.

→ Feynman integrals ⇝ effective descriptions of terms in Taylor expansion

Simplest class is 2d CFT   Witten — QFT ⇝ Math

States in classical mechanics: particle $x(t)$, $\dot{x}(t)$ state

observables are fcts. of state, e.g. energy $\frac{1}{2}m[\dot{x}(t)]^2 + V(x)$.

Classical field theory — fct on spacetime
— section of bundle on spacetime. } e.g.

- fields are as important as particles.

e.g. $\phi(x,t)$, $\dot{\phi}(x,t)$

Quantum field theory: algebraic spans of states $= S$ Fock space $\subseteq H \subseteq S^*$

Schwartz space — distributions

Observables are s.a. operators on $\mathcal{S}$

energy $\leftrightarrow -i\hbar\frac{\partial}{\partial t}$ momentum $\leftrightarrow -i\hbar\frac{\partial}{\partial x}$

position $\leftrightarrow$ multiply by coordinates...

QFT: states are generated by fields acting on vacuum $|0\rangle \in \mathcal{S}$

- operator-valued distribution on spacetime.

Correlation fcts are $\langle 0 | T[\phi_{i_1}(x_1,t_1)\cdots\phi_{i_n}(x_n,t_n)] | 0\rangle$

$\uparrow$ time ordering (convergence)

- quantum fields are $Y(u,z)$ from VOA's.
- modes $u(n)$ are fields smoothed out in an appropriate way.

$\hookrightarrow S = V$, spacetime is a tube - semi-infinite cylinder.

Feynman's integral: QM, particle in potential. $V(x)$ - Schrödinger's eq...

$K(x'';t_f;x';t_i)$ = amplitude of particle starting at $(x';t_i)$ finishing at $(x'';t_f)$

- formally solves Schrödinger's eq.

$$\psi(x'';t_f) = \int \underbrace{K(x'',t_f;x,t_i)}_{\text{kernel which solves the problem.}} \psi(x';t_i)\, d^3x$$

$$\rightsquigarrow K(\cdots) = \int_{\substack{\text{all paths}\\ x';t_i \to x'';t_f}} \exp\left(\tfrac{i}{\hbar}S(x(t))\right) \mathcal{D}x(t)$$

$\uparrow$ action

- matrix to 2×10^6 power, look at entries, they are #paths...

- now $S = S_{free} + \lambda S_{interaction}$ $\quad S_{int} = V$ potential.

$\uparrow$ coupling constant

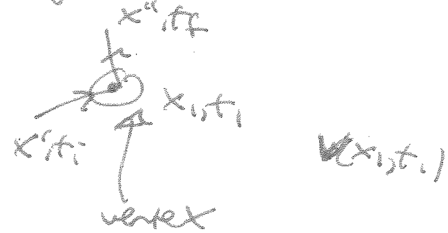
- expand as Taylor in λ.

$$\rightsquigarrow K(\cdots) = \int e^{S_{free}} \sum_n \left(\tfrac{i\lambda}{\hbar}\right)^n (SV)^n \mathcal{D}x$$

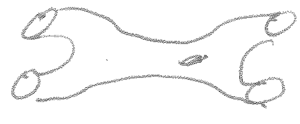
- reduce $\int_{paths}$ to stuff happening at discrete pts.

8

- look at Feynman diagrams - only thing that matters is "kinks" in diagram.
- only role of potentials is localized
- action is a vertex operator



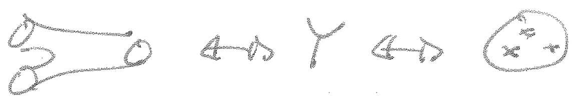
- particles arise from Poisson bracket.
  - build up space by using commutators.

String theory:  feynman $\int \rightsquigarrow \sum_{\text{genus}} \int_{\text{moduli}} \mathcal{D}$path

- need a conformal structure to get a # out.

$\rightsquigarrow$  as $t \to \infty$. pts tell us where we are in the moduli space.

- only interesting part is what happens at the vertices.

 sphere w/ 3 punctures.

- each term is a correlation fct. in CFT.
- CFT and string theory highly related...

2d CFT: next time.

Gannon II: axiomatization of CFT $\rightsquigarrow$ VOA. motivate.

Borcherd's VA $\to$ Moonshine module, $\to$ CFT $\}$ interesting examples.

2d CFT axioms: Wightman Axioms (1950's) Kac: VA's for beginners.
- Euclidean version (rather than Minkowski) Osterwalder-Schrader (1970's)
- apply to $\mathbb{P}^1$ Riemann Sphere.

Quantum field (QF) is an op-valued distribution on spacetime.

$f \in S(\mathbb{P}^1)$ smooth, dense subspace of L^2. $\phi(f): S \to S$ (Fock space)

I countable index set. $G_{i_1\cdots i_n}(z_1,\ldots,z_n) = \langle 0 | T(\phi_{i_1}(z_1)\cdots\phi_{i_n}(z_n)) | 0\rangle$.

think of $z = t+ix$. parametrized $\mathbb{P}^1 = \hat{\mathbb{C}}$. These $G_{\vec{i}}$'s are corr. fcts.

on C_n configuration space $= \{(z_1,\ldots,z_n) \in \hat{\mathbb{C}}^n \mid z_i \neq z_j \ i\neq j\}$.

Axioms: ① locality — $G_{\underset{\vec{i}}{i_1\cdots i_n}}(z_1,\ldots,z_n) = G_{\pi(\vec{i})}(z_{\pi(\vec{i})}) \quad \forall \pi \in S_n$.

2 pts: $\langle 0 | \phi_1(z_1)\phi_2(z_2) | 0\rangle$ is almost $\langle 0 | \phi_2(z_2)\phi_1(z_1)|0\rangle$

— up to singularities when $z_1 = z_2$ ⟶ dirac deltas + derivatives.

— in fact $(z_1 - z_2)^N(\phi_1(z_1)\phi_2(z_2) - \phi_2(z_2)\phi_1(z_1)) = 0$

② ~~poincare~~ Euclidean covariance — $E_2 = SO(2,0)\ltimes\mathbb{C}$. 3dim $\mathbb{R}$ gp.

$\exists$ conformal weights $h_i, \bar{h}_i$ $\forall i \in I$ real #s.

$G_{\vec{i}}(\vec{z}) = \prod_j \left(\frac{d\gamma}{dz}\right)^{h_j}\left(\frac{d\bar\gamma}{d\bar z}\right)^{\bar h_j} G_{\vec{i}}(\vec{w})$ Green's fcts respect E_2-action.

$\gamma \in E_2$, $\vec{w} = \gamma(\vec{z})$ diagonal action.

— also Möbius transformations.

$G_{ij}(z_1,z_2) = C_{ij}(z_1 - z_2)^{-h_1-h_2}(\bar z_1 - \bar z_2)^{-\bar h_1 - \bar h_2}$

③ unitarity — involution $*: I \to I$

$G_{\vec{i}}(\vec{z}) = G_{\vec{i}^*}(-\bar z_1, \ldots, -\bar z_n)$

$t+ix \mapsto -t+ix$

— inner prod. on wave packets. $f \in S(\hat{\mathbb{C}}^n)$ $\int_{C_n} f(\vec z)\phi_{i_1}(z_1)\cdots\phi_{i_n}(z_n)$

$$\langle \phi_{f,\vec{i}_{i_1\cdots i_n}}, \phi_{g,\vec{j}_{j_1\cdots j_m}}\rangle = \int_{C_{n+m}} G_{i_1^*\cdots i_n^*, j_1\cdots j_m}(z_1,\ldots,z_{n+m})\, \overline{f(z_1,\ldots,z_n)}\, g(z_1,\ldots,z_m)$$

"reflection positivity"

— condition is this is positive ~~semi~~ def inner prod.

(Re)construction Thm: Can build from this data.

Fock space S = span of wave packets / ker $\langle\cdot,\cdot\rangle$ H = completion.

$\forall i \in I$, $\phi_i(f): S \to S$. $\phi_i(f)(\phi_{g,i_1\cdots i_n}) =$ ~~$G_{\ldots}$~~

$1 = |0\rangle$ is the vacuum vector, zero pt. fct. $[\phi_{1,0}] \in S$.

- get unitary rep. of E_2.
  $\Rightarrow$ get an energy operator - Hamiltonian ≥ 0 positive spectrum.

analytic continuation $G_{i_1 \cdots i_n}(z_1, \ldots, z_n) = \langle 0 | T(\phi_{i_1}(z_1) \cdots \phi_{i_n}(z_n)) | 0 \rangle$.

- it analytically extends to $H_{i_1 \ldots i_n}(z_1, z_1', \ldots, z_n, z_n')$
- separately holomorphic on $z_1, \ldots, z_n$ and $z_1', \ldots, z_n'$.
- restrict to $\bar{z}_i = z_i'$, get G back.

- more axioms like state-field correspondence.

conformal symmetry on $\hat{\mathbb{C}}$, conf. symmetries are Möbius transformations.

- conserved current, conserved charge. $T(z), \bar{T}(\bar{z})$ currents

  L_n charges

  Classical symmetry broken, but gently. virasoro ...

  $\leadsto Vir \oplus \overline{Vir}$ 2 copies, one adjoint of other in unitary case.

- in a rational CFT, can extend to $A \otimes \bar{A}$ VOA's.
  fcts of z fcts of z'. Chiral algebras.

$\leadsto H = \bigoplus H_{h,\bar{h}} = \bigoplus_{h,\bar{h}} H_h \otimes H_{\bar{h}}$ not nec. irred., reps of Vir $\overline{Vir}$.

- want this to be a finite sum $\bigoplus_{\lambda,\mu} H_\lambda \otimes \bar{H}_\mu$

$\Rightarrow H_{i_1 \ldots i_n}(z_1, \ldots, z_n, z_1', \ldots, z_n') = \sum \tilde{H}(z_1, \ldots, z_n) \tilde{\tilde{H}}(z_1', \ldots, z_n')$

correlation fct. holomorphic anti-holomorphic.

mathematical artifacts, chiral blocks.

- carry f. dim'l reps of mapping class gps.
- chiral blocks for ⊚ carry action of $SL(2,\mathbb{Z})$, MCG of $\mathbb{T}^2$.

Gannon III: full CFT - correlation fcts are more important than quantum fields

- chiral halves (VOA)

Haag-Kastler axioms (1960s) Haag "Local quantum physics"; Kawahigashi (2007):

to each region $\mathcal{O}$ in spacetime $\mathbb{R}^{3,1}$ minkowski $\leadsto A(\mathcal{O})$ observables.

$A(\mathcal{O}) \subset B(H)$ same H $\forall \mathcal{O}$, $A(\mathcal{O})$ is hyperfinite III_1.

- if $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated, $[A(\mathcal{O}_1), A(\mathcal{O}_2)] = 0$.

⇝ get 2 Diff(S^1)'s "light cones" ⇝ project to chiral halves.

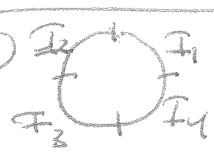
↳ local conformal net on S^1, projective rep'n of Diff(S^1).

– Lie alg of Diff(S^1) is v. fields on S^1, complexify ⇝ Witt alg.,
take central ~~extension~~ (since it's a proj. rep) ⇝ Vir(c).
↳ L_0 positive op!

⇝ understand Rep(A).

– can talk about a character of $\varphi \in Rep(A)$, which is $\sum_n \dim(K_n) q^n = ch_\varphi$ (eigenspace for L_0)

Complete rationality: ∘ ch_H converges, H vacuum rep, $|q|<1$. $q = e^{2\pi i \tau}$

②  $A(I_1 \cup I_3) \subseteq A(I_2 \cup I_4)'$ finite index

Thm (Kawahigashi–Longo–Müger) If net rational, category of reps, Rep(A) is MTC.

Feng Xu: (2000s) completely rational conformal net A, finite gp G, then A^G equivariantization is completely rational.

Conjecture:

sufficiently nice conformal nets ⟷ sufficiently nice VOA's "unitary"

completely rational conformal nets ⟷ rational? VOA's

subfactor (finite index, finite depth) ⟷ fusion category

– when FC has nondeg. braiding ⇝ MTC ⟷ braided SFs.

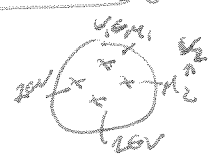
↑ $Z(\mathcal{C})$ ↑ LR/Symm Env.

fusion $\mathcal{C}$ ⟷ SF

Lessons for SF from VOA:

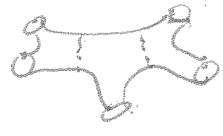
Zhu's algebra

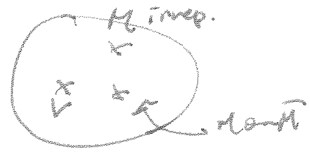
C_2-algebra

 etc. ⟷ chiral block (correlation fct.

$Y \in Hom(M_1 \otimes M_2, M_3)$ } $Y \in I_{xx}^{x}$ 1 dim'l.

$Y(u_1, z)u_2 \in M_3$

– decompose  ⇝ ⟨all intertwiners⟩



$\longleftrightarrow \; Y^{M}_{V\otimes M}$ 1 dim'l intertwiner space
— action of V on M.

— assigning $\vec{v}$ to each puncture / each $\vec{v}$ space $\rightsquigarrow \langle u_3 | Y^{M}_{V\otimes M} u_2 \rangle$
— gives map to $\mathrm{Hom}(V, M\otimes\bar{M})$, large kernel.
— is a fct. on $A = V/\text{subspace}$ (independent of M)
— fin dim C^*-alg ~~for~~ a rational VOA, $\cong \bigoplus_M \mathrm{Hom}(M_0, M_0)$ lowest weight.
— non-unitary (Evans+Gannon)
— Logarithmic ...

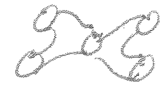
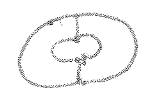
Lessons for VOA from SF:

— Haagerup, not braided. take double, find character $\vec{v}$'s, $c=8$.
↳ new construction of VOA's. — splitting of $c=1$ + doubles (D_{odd})
— Constructing SF B $\sqrt{\ }$ of MTC, take $2x/\mathbb{Z}$ to get MTC's.

— Subfactor $\longrightarrow$ full CFT   no analog for VOA's.
↳ α-induction $\quad A(I) \underset{\text{res.}}{\overset{\text{ind?}}{\subset}} B(I)$

Wasserman I: Basic objects of CFT: vertex alg's, rep's + intertwiners. ②

- alg geometry → $\mathbb{C}$ alg curves w/ singularities at worst 2x pts or nodes
- holomorphic approach — using Riemann surfaces.
  - to Riemann surface w/ ∂ circles $\Longleftrightarrow$ trace class op's.
- op. alg. — vNa's, LG, Diff S^1 + Connes fusion on bimod's.

"closed string theory"  "open string theory"  open problem.

3 sets of lecture notes: dpmms website.
- 2006
- 2 yrs / 3 yrs back
- one most recent.

One on arXiv.

plan: ① Singular $\int$ op's.
② Quantization
③ CFT on Riemann Surfaces

Hilbert transform on S^1: $L^2(S^1)$ $e_n = e^{in\theta} = z^n$

$H^2(S^1) \ni f = \sum a_n e^{in\theta} \Longleftrightarrow a_n = 0$ $n < 0$. L^2 ∂-values of holo. fcts in $\mathbb{D}$.

Proj $L^2(S^1) \to H^2(S^1)$. Szegö proj.

$$f = \left(\begin{array}{cc|cc} a_0 & a_1 & a_2 & \\ a_{-1} & \ddots & a_1 & a_2 \\ \hline a_{-2} & a_{-1} & a_1 & \\ & a_{-2} & a_0 & \\ & & a_{-1} & \end{array}\right)$$ sum of squares of off diag. blocks $\sum |n| |a_n|^2$.

$T(f) = P m(f) P$ Toeplitz op. is Fredholm if $f \neq 0$ on S^1.

(here $f: S^1 \to \mathbb{C}$, continuous)

Commutator $[m(f), P] = (1-P)\, m(f)\, P - P\, m(f)\, (1-P)$

$H = i(2P-1)$ Hilbert transform $[H, m(f)]$ op. w/ smooth kernel $\int_0^{2\pi} K(\theta,\varphi)\, \xi(\varphi)\, d\varphi$

trace class $|T| = (T^*T)^{1/2}$ $\|T\|_1 = \mathrm{Tr}(|T|) < \infty$. $\|T\|_2 = \mathrm{Tr}(T^*T)$. $\to L^p(H)$.

If $f \in H^2$, Cauchy $\Rightarrow$ $F(z) = F(re^{i\theta}) = \frac{1}{2\pi i} \int \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\varphi})\, d\varphi}{1 - re^{i\varphi}}$

- take limit as $r \to 1$. Want to set $r = 1$, get singularity.

<u>Define truncated Hilb. trans</u>: $H_\varepsilon f(\theta) = \frac{1}{2\pi} \int_{\varepsilon \le |\varphi| \le \pi} \frac{f(\theta-\varphi)}{1-e^{i\varphi}} d\varphi$

<u>Check</u>: on $e^{in\theta}$ trig polys, $H_\varepsilon f \to Hf$ unif.

$\overline{H_\varepsilon f} = -u^{-1} H_\varepsilon (u\bar{f}) \quad u = m(z).$

If $\|H_\varepsilon\| \le M$, get $H_\varepsilon \xrightarrow{SOT} H$.

Check $(1-e^{i\theta})^{-1} \sim i\theta^{-1}$. Compute Fourier Trans. $i\theta^{-1}\chi\{\varepsilon \le |\theta| \le \pi\}$.

– show Fourier coefficients are uniformly bdd. Are $\left|\int_a^b \frac{\sin t}{t} dt\right|$ Int. By Parts.

<u>Compute</u>: $[m(g), H]$ $\quad \frac{g(\theta)-g(\varphi)}{e^{i\theta}-e^{i\varphi}}$ singularity disappears. C^∞ if g is C^∞.

If $h \in \mathrm{Diff}(S^1)$ def $Vf(\theta) = f(h(\theta))$. Then

$V^{-1}HV - H \quad \leftrightarrow \quad \frac{h'(\varphi)e^{ih(\varphi)}}{e^{ih(\varphi)}-e^{ih(\theta)}} - \frac{e^{i\varphi}}{e^{i\varphi}-e^{i\theta}}$

<u>L^p</u>: Continuity of H on L^p.

Classically, this is the M. Riesz thm., use Poisson integral.

$f(\theta) = \sum a_n e^{in\theta} \quad \mathbb{P}_r f = \sum a_n r^{|n|} e^{in\theta}$ convolution w/

$K_r(\theta) = \sum |r|^n e^{in\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$

– note $\|\mathbb{P}_r\|_p$ unif. bdd. $\|\mathbb{P}_r f - f\|_p \to 0$ as $r \nearrow 1$.

But $H_{1-r} = H\mathbb{P}_r$ truncated H's in terms of Poisson transforms.

– just need to show H bdd on L^p.

<u>Cotlar</u>: $(Hf)^2 = f^2 + 2H(fHf)$ $\quad$ Cont $L^2 \to$ Cont $L^4 \to$ Cont L^8 etc. rest, interpolate. (Hadamard 3 line thm)

<u>Application</u>: Conformal welding φ diffeo $\equiv$

– find $f: \bar{D} \to \bar{\Omega}$ conf. trans.
$\quad g: \bar{D}^c \to \bar{\Omega}^c$ $\Big\}$ s.t. $\varphi = g^{-1}f$. $\quad f(e^{i\theta}) = g(\varphi(e^{i\theta}))$.

$f(z) = a_0 + a_1 z + \cdots$

$g(z) = z + b_1 z^{-1} + b_2 z^{-2} + \cdots$

<u>Sol'n</u>: $F = g|_{S^1}$ $\quad P(F) = e^{i\theta}, \; P(F\circ\varphi) = F\circ\varphi$ $\Big] (1-K_\varphi)F = e^{i\theta}$. inv.

$K_\varphi = V(\varphi)^{-1} P V(\varphi) - P$

- This tells you how to glue Riemann surfaces given a diffeo! ③

<u>Hilb. transform on a ~~closed~~ curve:</u> (Ω) $H^{\varepsilon}_{\partial\Omega} = \frac{1}{\pi i}\int_{|s-t|\geq\varepsilon} \frac{f(t)}{z(t)-z(s)}\dot{z}(t)\,dt$

- Scale length so it's 2π. Then $H^{\varepsilon}_{\partial\Omega} - H^{\varepsilon}_{S^1}$ has smooth kernel.

Then $H = i(2E-1)$ $E^2 = E$, but $E \neq E^*$. $E^* - E$ cpt op.

<u>Kerzman-Stein:</u> $P = E(1+E-E^*)^{-1}$. $\perp$ proj onto $E(H) = H^2(\partial\Omega)$.

$EP = P$, $PE = E$ so $PE^* = P$ $P(1+E-E^*) = E$.

<u>Cayley trans:</u> $\mathbb{R} \to S^1$ $C(x) = \frac{x-i}{x+i}$ on S^1 $\xrightarrow[I]{\text{upm } H}$

$H_{S^1} \to H_{\mathbb{R}}$, but do direct computation for $\mathbb{R}$.

- poisson is $e^{-s|t|}$ in Fourier, all is similar as before.

<u>Hilb transform on $\mathbb{C}$:</u> (aka Riesz transforms, Beurling transform)

Riesz transform R^k is mult. by $e^{-ik\theta}$ in Fourier on $L^2(\mathbb{C})$.

$\widehat{Rf}(\zeta) = \frac{\bar{\zeta}}{\zeta}\hat{f}(\zeta)$ "manifestly contin"

$M_k(z) = \frac{k}{2\pi i^k}\frac{z^k}{|z|^{k+2}}$ $R^k f(w) = \lim_{\varepsilon\to 0}\int_{|z-w|\geq\varepsilon} M_k(w-z)f(z)\,dx\,dy$.

- Poisson mult. $e^{-s|\zeta|}$.

L^p: Calderon-Zygmund $\mathbb{C} = \mathbb{R}\times\mathbb{R}$, $L^2(\mathbb{C}) = L^2(\mathbb{R})\otimes L^2(\mathbb{R})$

$H\otimes 1 = H^{(1)}$.

$u_\theta f(z) = f(e^{i\theta}z)$; $R = \frac{1}{2\pi}\int e^{-i\theta}u_\theta H^{(1)}u_\theta^* \,d\theta$.

$H^{(1)}$ cont. on $L^p \Rightarrow R$ cont. on L^p.

<u>Wasserman II:</u> Beurling transform T mult. by $\frac{\bar{\zeta}}{\zeta}$ in Fourier on $L^2(\mathbb{C})$.

Since $\frac{\bar{\zeta}}{\zeta} = \left(\frac{\bar{\zeta}}{|\zeta|}\right)^2$, $T = R^2$. $T_\varepsilon f(w) = -\frac{1}{\pi}\int\frac{f(w)}{(w-z)^2}\,dx\,dy$ $|z-w|\geq\varepsilon$.

$T\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial}{\partial z}Tf$ $\frac{\partial g}{\partial \bar{z}} = k$.

- like Hilb trans, T has compatibility w/ conformal change of coord. φ.

$\varphi: D \to \Omega$ $T(\Omega) - T(D)$ has smooth kernel.

(2)

$$K(w,z) = \frac{1}{\pi}\left[\frac{\varphi'(w)\varphi'(z)}{(\varphi(z)-\varphi(w))^2} - \frac{1}{(z-w)^2}\right]$$ smooth.

Define $A^2(\Omega)$ Bergman space of L^2 holo. fcts. on Ω.

$A^2(\Omega^c)$ T restricts to a unitary on $\overline{A^2(\Omega)\oplus A^2(\Omega^c)}$

conj. cplx structure. $A^2(\Omega)\oplus A^2(\Omega^c)$

– write T as $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ called Grunsky operator.

K^2 s.a. pos. op., trace class. λ_n^2 Fredholm e.vals of Ω.

If Ω and Ω^c arise from $\varphi = g^{-1}\circ f$. $\det(1-K^2) = \frac{1}{12\pi}(2\|\log \partial_z \log \frac{f}{z}\|_D^2 - \|\partial_z \log f\|_D^2 +$ same for g, D^c)

turns out to be $\propto$ to 4^{th} power of $\langle \pi(\varphi)\Omega, \Omega\rangle|^2$

(rep'n of $\mathrm{Diff}(S^1)$ for $\mathbb{C}$-fermions)

<u>Beltrami eq.</u>: metric dx^2+dy^2 on Ω,

pull back by $D \xrightarrow{h} \Omega$ (diffeo, not conformal), $ds^2 = |h_z dz + h_{\bar z} d\bar z|$

$= |h_z|^2 \, |dz + \frac{h_{\bar z}}{h_z} d\bar z|^2$

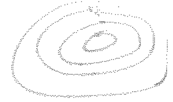
– Jacobian is $|h_z|^2 - |h_{\bar z}|^2 > 0$.

$\mu(z) = \frac{h_{\bar z}}{h_z}$ Beltrami coeff, $\|\mu\|_\infty < 1$. $\mu(z)$

$f: D \to D$, $f: \mathbb{C} \to \mathbb{C}$, $D \to D$, $S^1 \to S^1$, $D^c \to D^c$.

– solve Beltrami eq. $\frac{f_{\bar z}}{f_z} = \mu$.

– arrange for metric on D to extend to a smooth metric on $\mathbb{C}\cup\{\infty\}$.

– for ellipses  and 0 ∞ $\subset$ etc need const. in L^p $p>2$.

$\frac{\partial f}{\partial \bar z} = \mu \frac{\partial f}{\partial z}$ $g = f - z$, $h = \frac{\partial g}{\partial z} = \frac{\partial f}{\partial z} - 1$ $\leadsto h = T(\mu h) + T\mu$.

$\Rightarrow h = (1 - T\mu)^{-1} T\mu$.

<u>Pt 2</u>: ~~Quantization~~.

• Vertex alg of a single real fermion. (wick formula + Taylor's thm)

Bosons
• Möbius semigp
• Trace-class semigp
• Oscillator rep'n
– ∞-dim'l extension theory $\leadsto$ $\mathrm{Diff}(S^1)$.

<u>Mehler</u> kernel on $L^2(\mathbb{R})$
$K(x,y) = e^{-(Ax^2+Bxy+Dy^2)}$
$\mathrm{Im}\begin{pmatrix} A & B \\ B & D\end{pmatrix} > 0$.

• Fermionic const. of rep's of loop gp's + $\mathrm{Diff}(S^1)$

• Fermionic semigp.

<u>vertex alg of single real fermion</u>: ψ_r $(r \in \mathbb{Z}+\frac12)$ $\{\psi_r, \psi_s\} = \psi_r\psi_s + \psi_s\psi_r = \delta_{r+s,0}$

$\psi_r^* = \psi_{-r}$.

– energy op. $[L_0, \psi_r] = -r\psi_r$

$f = \sum a_n e^{in\theta}$ $\psi(f) = \sum a_r \psi_r$.

$\psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}}$. (by school of Sato.) ⑤

– $\exists!$ pos. energy rep^n irred., gen by vacuum Ω w/ $L_0\Omega = 0$.

$\psi_{i_1} \cdots \psi_{i_r} \Omega$ "exterior alg" $i_r > \cdots > i_1 \geq \frac{1}{2}$. Fock space $\mathbb{F}$

Define $L_k = \frac{1}{2} \sum r \psi_{-r} \psi_{r+k}$ $k \neq 0$, $L_0 = \sum_{r>0} r \psi_{-r} \psi_r$, get Virasoro alg.

$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{24}(m^3 - m)\delta_{m+n,0}$ $(c = \frac{1}{2})$,

$[L_n, \psi_r] = -r \psi_{r+n}$; $L_n^* = L_{-n}$.

– Virasoro acts irred. on $\mathbb{F}_{ev}$, $\mathbb{F}_{odd}$, $\mathbb{F}_{\pm}$ (give explicitly using ψ for sl_2 gen by $L_0, L_{\pm 1}$)

– $\partial^{(n)} = \frac{1}{n!}(\partial_z)^n$ field $A(z) = \sum a(n) z^{-n-1}$ A^+ = non-neg powers of z. raising / creation

A^- = neg powers of z lowering / annihilation ops

Wick normal ordering. $:A(w)B(w): = A^+(z)B(w) \pm B(w)A^-(z)$.

$:A_1 A_2 A_3: = :A_1(:A_2 A_3:):$ and so on.

$:A_1 \cdots A_n: = \sum \pm A_{i_1}^+ \cdots A_{i_k}^+ A_{j_1}^- \cdots A_{j_{n-k}}^-$.

$z_1 \cdots z_n$ no singularities for $z_i = z_j$.

So state $\psi_{i_n} \cdots \psi_{i_1} \Omega \longleftrightarrow \partial^{(i_n - \frac{1}{2})}\psi \cdots \partial^{(i_1 - \frac{1}{2})}\psi$.

$a \longleftrightarrow V(a,z)$ $(Y(a,z)$ notation$)$

$V(a,z)\Omega|_{z=0} = a$.

– immediate that $[L_0, V(a,z)] = z\frac{dV}{dz} + \alpha V$ if $L_0 a = \alpha a$.

$[L_{-1}, V(a,z)] = \frac{dV}{dz} = V(L_{-1}a, z)$.

– check locality + associativity. Both immediate from Wick's formula, use Taylor's formula.

$$\underset{|z|>|w|}{V(a,z)V(b,w)} = \underset{|w|>|z|}{\pm V(b,w)V(a,z)} = \underset{|w|>|z-w|}{V(V(a,z-w)b, w)}$$

Taylor: $f(z) = f(z-w+w) = \sum (z-w)^k \partial_w^{(k)} f(w)$

Wick: $:A_1 \cdots A_M: :B_1 \cdots B_N: = \sum \pm (A_{i_1} B_{j_1}) \cdots (A_{i_k} B_{j_k}) \underbrace{:A_1 \cdots A_M}_{A_i\text{'s removed}} \underbrace{B_1 \cdots B_N:}_{B_j\text{'s removed}}$

$(A(z)B(w)\Omega, \Omega)$ $(\psi(z)\psi(w)\Omega, \Omega) = \frac{1}{z-w}$ $|z|>|w|$

Izumi I: Known fusion Categories/Subfactors ①

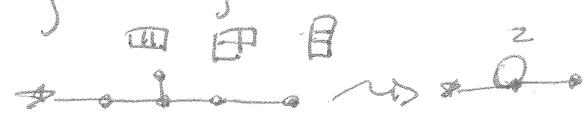
Defn: $\mathcal{C}$ near gp category - has fusion gp G, ρ simple,
$gh = g\otimes h$, $g\otimes\rho = \rho\otimes g = \rho$ $\forall g\in G$. $\bar{\rho}=\rho$, $\bar{g}=g^{-1}$.

$$\rho^2 = \bigoplus_g g \oplus m\rho.$$

Ex: Fibonacci, even pt. of $SU(2)_3$ $\rho^2 = 1\oplus\rho$.

Ex: Ising model $\rho^2 = 1\oplus\alpha$

Ex: Rep(S_3) $\rho^2 = 1\oplus\rho\oplus\alpha$

Ex: even pt. E_6  $\rho^2 = 1\oplus 2\rho\oplus\alpha$.

Tambara-Yamagami '98. $m=0$. Then G abelian, classified by $(G,\langle\cdot,\cdot\rangle,\varepsilon)$ where $\varepsilon=\pm1$, $\langle\cdot,\cdot\rangle$ nondeg. symmetric bicharacter. $G\times G\to S^1$.

Ostrik '03 If $G=\{e\}$ $\Rightarrow$ $m=1$, only 1 unitary (one more non-unitary)

Etingof-Gelaki-Ostrik '04 $G=\mathbb{Z}/n$, $m=n-1$. Then $n=p^a-1$, and $\widehat{\mathbb{Z}/n}\cong\mathbb{F}_{p^a}^\times$ canonically. Classified by a subset of $H^3(\mathbb{F}_{p^a}\rtimes\mathbb{F}_{p^a}^\times, S^1)$
$\omega\in$ s.t. $\omega|_{\mathbb{F}_{p^a}^\times}=1$.

G	$\mathbb{Z}/2$	$\mathbb{Z}/3$, $\mathbb{Z}/7$	others
$\#\mathcal{C}$	3	2	1

<u>Thm:</u> Assume $n \neq 0$, $n = \#G$, $d = \dim(\rho) = \frac{m + \sqrt{m^2 + 4n}}{2}$.

① when $d \in \mathbb{Q}$, then either ① $m = n-1$ and $G = \mathbb{Z}/_{n}$ (EGO), or

② $n = 2^{2a+1}$, $m = 2^a$, G extra special 2-gp.

$[G, G] = Z(G) = \mathbb{Z}/_2$ ⤳ $\underline{\#\mathcal{C} = 3.}$

$\forall G, \exists 3\, \mathcal{C}$.

ex: D_8, Q_8.

② when $d \notin \mathbb{Q}$, G abelian, $n | m$ $m = ln$ $l = 1, 2, \ldots$ — Evans-Gannon

when $l = 1$, $\exists$ explicit polynomial eq's (I. 2001) — Preprint of Ostrik.

$l = 2$, $G = \{e\}$, no such category

$G = \mathbb{Z}/_2$, no such category

$G = \mathbb{Z}/_3$, $\exists$ a category.

— will focus on case $l = 1$. use Cuntz alg's to write eq's, construct $\mathcal{C}$.

<u>Def:</u> Cuntz alg $\mathcal{O}_n = C^*(S_1, \ldots, S_n)$ s.t. $S_i^* S_j = \delta_{ij}$ $\sum S_i S_i^* = 1$.

<u>Suppose:</u> $(\mathcal{C}, \otimes) \subset \mathrm{End}(M)$, M type III factor.

$\otimes$ $\quad \rho \circ \sigma$ composition.

Hom $\quad (\rho, \sigma) = \{T \in M \mid T\rho(x) = \sigma(x) T \ \forall x \in M\}$

$\oplus$ $\quad [\rho \oplus \sigma](x) = S_1 \rho(x) S_1^* + S_2 \sigma(x) S_2^*$ $\langle S_1, S_2 \rangle \cong \mathcal{O}_2 \subset M$.

↖ well defined up to equivalence

FPdim $\quad d(\rho) = [M : \rho(M)]^{1/2}$.

$\sim$ $\quad \rho \sim \sigma \iff \exists u \in M$ s.t. $\mathrm{Ad}\, u \circ \sigma = \rho$

inv. obj $\quad \alpha$ automorphism (simple currents)

$\mathrm{Hom}(\rho, \sigma) \longrightarrow \mathrm{Hom}(\rho \circ \alpha, \sigma \circ \alpha)$ $\qquad (\rho, \sigma) \subset (\rho\alpha, \sigma\alpha)$

$T \longmapsto T \otimes 1_\alpha$ $\qquad T \longmapsto T$

$\mathrm{Hom}(\rho, \sigma) \longrightarrow \mathrm{Hom}(\alpha \circ \rho, \alpha \circ \sigma)$ $\qquad (\rho, \sigma) \not\subset (\alpha\rho, \alpha\sigma)$

$T \longmapsto 1_\alpha \otimes T$ $\qquad \overset{\psi}{T} \longmapsto \alpha(T)$

$\boxed{T}\,|_\alpha = T$, but $\alpha|\,\boxed{T} = \alpha(T)$

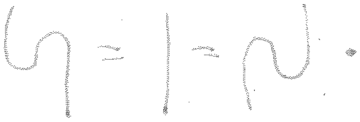
rigidity $\quad \dim(\rho) < \infty$. Then $\exists \bar\rho \in \mathrm{End}(M)$, Sometimes

$r_\rho \in (\mathrm{id}, \bar\rho\rho)$, $\bar r_\rho \in (\mathrm{id}, \rho\bar\rho)$

s.t. $\bar r_\rho^* \rho(r_\rho) = \frac{1}{d(\rho)}$, $r_\rho^* \bar\rho(\bar r_\rho) = \frac{1}{d(\rho)}$

Use diagrams $\bar{\rho} \cap \rho = \sqrt{d(\rho)}\, r_\rho \quad \rho \cap \bar{\rho} = \sqrt{d(\rho)}\, \bar{r}_\rho$ ($r_\rho, \bar{r}_\rho$ isometries)

$\bar{\rho} \cup \rho = \sqrt{d(\rho)}\, r_\rho^* \quad \rho \cup \bar{\rho} = \sqrt{d(\rho)}\, \bar{r}_\rho^*$

Then these eq's are .

When $\rho = \bar{\rho}$, $\bar{r}_\rho = \begin{cases} r_\rho & \rho \text{ \underline{real}} \\ -r_\rho & \rho \text{ \underline{pseudoreal}} \end{cases}$ [2dim'l $SU(2)$]

<u>Warm up:</u>

Assume $\mathcal{C} \subset End(M)$ Tambara-Yamagami, i.e. $Irr(\mathcal{C}) = G \cup \{\rho\}$.
$\{\alpha_g\}$

$\alpha_g \circ \alpha_h = \alpha_{gh}$, $\alpha_g \circ \rho \cong \rho \quad \rho^2 \cong \bigoplus_{g \in G} \alpha_g$.

- Fix ρ. One lift from eq. class. May assume $\alpha_g \circ \rho = \rho$ exactly.

(if $\rho = Ad u_g \circ \alpha_g \circ \rho$, just rename $\alpha_g = Ad u_g \circ \alpha_g$)

$\alpha_g \circ \alpha_h \circ \rho = \alpha_g \circ \rho = \rho$ or $\alpha_g \alpha_h \circ \rho = Ad(u_{g,h})\alpha_{gh} \circ \rho = Ad(u_{g,h}) \circ \rho$

Since $(\rho, \rho) = \mathbb{C} \Rightarrow u_{g,h} \in \mathbb{C} \Rightarrow \alpha_g \circ \alpha_h = \alpha_{gh}$.

- associator is always trivial.

- Fix $S_0 = r_\rho \in (id, \rho^2)$. $\underbrace{\alpha_g(S_0)}_{= S_g} \in \alpha_g(id, \rho^2) = (\alpha_g, \alpha_g \circ \rho^2) = (\alpha_g, \rho^2)$

$\Rightarrow \{S_g\}$ satisfy Cuntz alg. rels. $\quad S_g^* S_h = \delta_{g,h}$, $\sum S_g S_g^* = 1$

- all info of $\mathcal{C}$ contained in α_g, ρ restricted to $\mathcal{O}_n \subset M$ $n = \#G$.

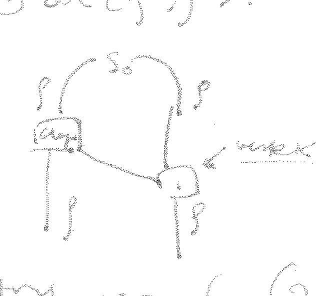
$\alpha_h(S_g) = S_{hg}$. Want to show $\rho(S_g) = $ poly. of $\{S_h\}$.

<u>Izumi Thm:</u> $(\rho^2, \rho^2) = \bigoplus_g \mathbb{C} S_g S_g^*$. $\quad \rho\alpha_g \sim \rho \Rightarrow \exists u_g \in (\rho, \rho\alpha_g)$ unitary.

$(\rho^2, \rho\alpha_g \rho) = (\rho^2, \rho^2)$

gives proj. rep. of G on (ρ^2, ρ^2).

- Consider $u_g S_0$



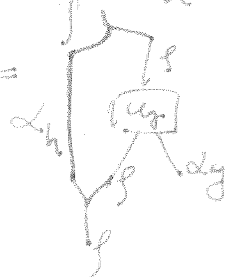
$\in (id, \rho^2)$. May assume $u_g S_0 = S_0$.

$\{u_g\}_{g \in G}$ unitary rep of G in (ρ^2, ρ^2) $\quad u_g = \sum_{h \in G} \chi_h(g) S_h S_h^*$

$\chi_0(g) = 1$.

Claim: $\alpha_h(u_g) = \chi_{h^{-1}}(g)\, u_g$,
and $h \longmapsto \chi_h \in \widehat{G}$ is a gp. homom.

Pf: $\alpha_h(u_g) =$



$\in (\rho, \rho\alpha_g) \Rightarrow \; = c\, u_g$.

Now $c = S_e^* \alpha_h(u_g) S_e$
$= \alpha_h(S_{e}^* u_g S_{e}) = \chi_{h^{-1}}(g)$.

Claim: $\rho(u_g) S_h = S_{hg^{-1}}$

$= c\, S_{hg^{-1}}$

$c = S_{hg^{-1}}^* \rho(u_g) S_h$.

$c = \alpha_h(S_{g^{-1}}^* \rho(u_g) S_0)$

Look at:

$\rho(S_{g^{-1}}^* \rho(u_g) S_0) = \rho(S_{g^{-1}}^*)\rho^2(u_g)\rho(S_0)$

$= u_g^{-1} \frac{\varepsilon}{d} \sum_h \chi_h(g) S_h^* \rho^2(u_g) \cdot \frac{\varepsilon}{d} \sum_k S_k$

$= \frac{1}{d^2} u_g^{-1} \sum \chi_h(g) \alpha_h(u_g)$

$= \frac{1}{d^2} u_g^{-1} \sum_h \chi_h(g) \overline{\chi_h(g)}\, u_g = 1$

we know $S_0^* \rho(S_0) = \frac{\varepsilon}{d}$ $\quad \varepsilon = \pm 1$, $d = \dim(\rho)$.

$\rho(S_0) = \sum S_g S_g^* \rho(S_0)$

$= \sum S_g\, \alpha_g(S_0^* \rho(S_0))$

$= \frac{\varepsilon}{d} \sum S_g$.

$\rho(S_g) = \rho(\alpha_g(S_0)) = u_g\, \rho(S_0)\, u_g^*$

$= \frac{\varepsilon}{d} \sum_h \chi_h(g) S_h\, u_g^*$

$\boxed{\text{Ex.}}$

Show: $\chi_h(g) = \overline{\chi_g(h)}$:

Pf: $u_h\, \rho(u_g)\, S_0 = u_h S_{g^{-1}} = \chi_{g^{-1}}(h) S_{g^{-1}} = \overline{\chi_g(h)}\, S_{g^{-1}}$.

but $\quad = \rho(\alpha_h(u_g))\, u_h S_0 = \chi_{h^{-1}}(g)\rho(u_g) S_0 = \overline{\chi_h(g)}\, S_{g^{-1}}$.

Non-degeneracy: $\rho(S_0)^* \rho(S_g)\, u_g^* = \rho(S_0^* S_g)\, u_g = \delta_{g,0}\, u_g = \delta_{g,0} = \frac{1}{d^2}\mathrm{Tr}(\lambda_g)$ $\quad$ regular rep. $\quad d^2 = n$

$= \frac{\varepsilon}{d} \sum_h S_h^* \frac{\varepsilon}{d} \sum \chi_k(g) S_k = \frac{1}{d^2} \sum_h \chi_h(g)$

$\Rightarrow \mathrm{Tr}(\lambda_g) = \sum_h \chi_h(g)$ $\quad$ exhausts $\widehat{G}$!

$\Rightarrow G$ abelian, $\{\chi_h\} = \widehat{G}$.

$\langle h, g\rangle = \chi_h(g)$ symmetric, nondeg. bicharacter, $\varepsilon = \pm 1$.

$\alpha_g(S_h) = S_{gh}$; $\rho(S_g) = u_g\left(\frac{\varepsilon}{d}\sum_h \langle h,g\rangle S_h\right) u_g^*$; $u_g = \sum_h \langle h,g\rangle S_h S_h^*$

- Reconstruction comes for free! Use formulas to define $\alpha_g \in \mathrm{End}(\mathcal{O}_n)$, get $\rho \in \mathrm{End}(\mathcal{O}_n)$. Check satisfy correct rel'ns.

$\Rightarrow \langle \alpha_g, \rho\rangle \subset \mathrm{End}(\mathcal{O}_n)$.

- to get $\subset \mathrm{End}(M)$, look at graded gauge action, $\exists!$ KMS, do GNS construction.

General case: $\mathcal{C} \subset \mathrm{End}(M)$ where $\mathcal{C}$ ~~is~~ near-gp, $m \neq 0$.

$\Rightarrow$ may assume by previous argument that $\alpha_g \circ \rho = \rho$, $\alpha_g(S_h) = S_{gh}$

$u_g \in (\rho, \rho\circ\alpha_g)$ $\quad u_g S_h = \chi_h(g) S_h$ characters, $\chi_g(h) = \chi_h(g)$.

- in general don't have nondegeneracy! Sometimes is degenerate.

Look at $K = (\rho, \rho^2)$, Hilb space w/ $\langle v, w\rangle = w^* v$ inner prod.

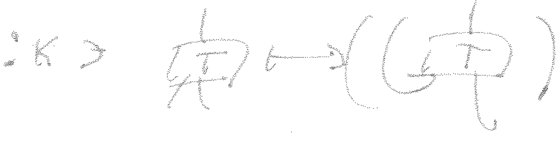
Choose $\{T_i\}_{i=1}^m$ ONB. $\rho^2 = \bigoplus \alpha_g \oplus m\rho$, so $(\rho^2, \rho^2) \cong \bigoplus_g \mathbb{C} \oplus M_m(\mathbb{C})$

$\Rightarrow (\rho^2, \rho^2) = \bigoplus \mathbb{C} S_g S_g^* \oplus \mathrm{span}_{i,j}\{T_i T_j^*\}$. $\quad u_g \in (\rho^2, \rho^2)$

$u_g = \sum \chi_h(g) S_h S_h^* + V_g$, where $\{V_g\}$ unitary rep'n $B(K)$

Frobenius reciprocity: two natural maps.

$J_1 : K \ni T \longmapsto (\ldots)^* \in K$ antiunitary, $J_1^2 = \varepsilon$.



$J_2 : K \ni T \longmapsto (\ldots)^* \in K$, $J_2^2 = \varepsilon$.

J_1 ~~...~~ $= \sqrt{d}\, T^* \rho(S_0)$. $\quad$ have $\mathcal{O}_{n+m} = C^*\{S_g, T_i\}$

$\rho(S_0) = \left(\sum S_g S_g^* + \sum T_i T_i^*\right)\rho(S_0) = \sum S_g \alpha_g(S_0^* \rho(S_0)) + \sum T_i (T_i^* \rho(S_0))$

$= \frac{\varepsilon}{d}\sum S_g + \frac{1}{\sqrt{d}}\sum T_i J_1(T_i)$ $\quad$ ONB.

$\rho(S_g) = \left[\frac{\varepsilon}{d}\sum \chi_h(g) S_h + \frac{1}{\sqrt{d}}\sum V_g T_i J_1(T_i)\right] u_g^*$

$$\rho(S_0^*)\rho(S_g)u_g = n^{-1}\mathrm{Tr}(\lambda_g)$$

$$= \left(\frac{\varepsilon}{d}\sum S_h^* + \frac{1}{\sqrt{d}}\sum J_1(T_i)^* T_i^*\right)\left(\frac{\varepsilon}{d}\sum \chi_h(g) S_h + \frac{1}{\sqrt{d}}\sum v_g T_k J_1(T_k)\right)$$

$$= \frac{1}{d^2}\sum \chi_h(g) + \frac{1}{d}\underbrace{\sum J_1(T_i)^* \underbrace{(T_i^* v_g T_k)}_{\langle v_g T_k, T_i\rangle} J_2(T_k)}_{}$$

$$= \frac{1}{d}\sum \langle v_g T_k, T_i\rangle \langle J_2(T_k), J_2(T_i)\rangle$$

$$= \frac{1}{d^2}\sum \chi_h(g) + \frac{1}{d}\mathrm{Tr}(v_g). \qquad d^2 = n + md.$$

$$\Rightarrow \boxed{(n+md)\,\mathrm{Tr}(\lambda_g) = n\sum_h \chi_h(g) + nd\,\mathrm{Tr}(v_g)}$$

$d \notin \mathbb{Q} \Rightarrow \mathrm{Tr}(\lambda_g) = \sum \chi_h(g)$, $\frac{m}{n}\mathrm{Tr}(\lambda_g) = \mathrm{Tr}(v_g)$

$\Rightarrow G$ abelian, and $m = bn$.

for $d \in \mathbb{Q}$, $m = n-1$, characters are all 1, completely degenerate.

$m = 2^a$, $n = 2^{a+1}$, factor something though... get something non degenerate...

<u>What goes wrong for $\rho(M) \subseteq M$?</u>

$\lambda_g \circ \rho = \rho \Rightarrow M \supset M^G \supset \rho(M)$. But $\rho \circ \lambda_g = \mathrm{Ad}\, u_g \circ \rho$.

$M \supset \rho(M) \rtimes G \supset \rho(M)$. 2 intermediates.

<u>In case I</u>, when $d \in \mathbb{Q}$, ① and $m = n-1$, $M \supset M^G = \rho(M) \rtimes G \supset \rho(M)$

all info encoded here.

② when $n = 2^{2a+1}$, $m = 2^a$. $M \supset M^{[G,G]} \supset M^G \supset \rho(M)$.

$\rho(M) \rtimes G \quad \rho(M) \rtimes [G,G]$

still can do computation.

<u>In case II. $d \notin \mathbb{Q}$</u>

$$\begin{array}{ccc} M & \supset & M^G \\ \cup & & \cup \\ \rho(M)\rtimes G & \supset & \rho(M) \end{array}$$

commuting square.

<u>Much more complicated!</u>

Teichner I: Functorial field theories + Ring spectra (1)

w/ Stephan Stolz, many contributors.

Goal: construct spaces of field theories. |d-GFT| of "d-dim'l geometric field theories" + compute for $d=0,1$.

↳ topological, Euclidean, conformal, quantum

- need higher categories + classifying spaces.

Thm: δ stands for "discrete" (only know very left of table (small ...))

$|0\text{-TFT}_\delta| \simeq H\mathbb{C}_\delta$ ordinary $\mathbb{C}$-cohomology

$|0\text{-TFT}| \simeq H\mathbb{C} \simeq *$

$|0|n\text{-TFT}| \simeq H\mathbb{C}_\delta$, $n \geq 1$ super symmetries.

etc. ⋮

Conj: $|2|1\text{-EFT}| \simeq$ spectrum of topological modular forms.

Chiral CFT: mathematically described by < VOA, chiral conformal nets, Segal CFT's

- don't have many examples.

Hope that above are equivalent!

Segal CFT: 2-dim conformal functorial field theory.

$\mathrm{Fun}^{\otimes}({}^2\text{-CBord}_d, \mathrm{Hilb})$ over $\mathbb{C}$. ~~Vect~~

obj(^{2}-CBord$_d$) are closed oriented conformal 1-mflds. $\sqcup S^1$

Mor$(\circ\circ\, ; \circ\circ\circ)$ are conformal surfaces w/ ∂ split into IN+OUT

conformal welding - can glue along a diffeomorphism

- think of  small collars on S^1, need for smooth, Riemann, etc.

- looking at collars, can see IN+OUT.

Convention: cobordisms have ∂ equipped w/ collars so IN - "outside", OUT - "inside"

- when mflds not oriented, this tells us how to glue...

- works up to conformal isometry. cplx structure w/o orientation.

- very hard to construct. Find HS ops, must compose...

Generalize: A flavor of a functorial field theory is specified by a dimension d, geometry $G \curvearrowright M$ dim d. (space → states, spacetime → evolution of state space ...)

Ex's for G: Conformal structure, T smooth structure, T^{or} smooth structure + orientation, E Euclidean, Q Lorentzian. ②

also pick the codim of subMflds to study. Call it C. (locality)

$\rightsquigarrow$ get a $(d-C)$ category $\begin{bmatrix} d \\ d-1 \\ \vdots \\ C \end{bmatrix}$ -GBord, $\amalg$ $\in$ $(d-C)$-Cat$^{\otimes}$
symmetric monoidal — symm. mon.

$\begin{pmatrix} d \\ \vdots \\ C \end{pmatrix}$-G.FT $=$ Fun$^{\otimes}\left(\begin{smallmatrix} d \\ \vdots \\ dc \end{smallmatrix} \text{ G Bord}, \underbrace{(d-C)\text{ Vect}, \otimes}_{\text{another choice}} \right)$

Picture for TFT's: $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$-T^{or}FT $\overset{\text{ex(S')}}{\simeq}$ commutative Frobenius Alg's (f.dim'l) (folklore)
↑restrict ↑center (1-vect, ⊗ alg.)

Chris Schommer-Pries: $\begin{smallmatrix} 2 \\ 1 \\ 0 \end{smallmatrix}$-T^{or}FT $\underset{\text{ev(pt)}}{\simeq}$ semi-simple Frob. Alg's (f.dim'l)

2-vect = (Alg, Bimods, Intertwiners), ⊗alg.

| T ⟶ f. dim'l |
C.S.P. C.D., Bruce Bartlett, Jamie Vicary
does not use cobord. hypothesis

C.S.P., C. Douglas, N. Snyder

$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$-T^{or}FT $\overset{\text{ev(S')}}{\simeq}$ MTC
↑res. ↑center

$\begin{smallmatrix} 3 \\ 2 \\ 1 \\ 0 \end{smallmatrix}$-T^{or}FT $\underset{\text{ev(pt)}}{\simeq}$ fusion Cat's.

(3-vect, ⊗) = Tensor-cat's.

use's Lurie's Cobordism hypothesis

What's assoc. MTC for Segal CFT?

"Twisted" anomalous Segal CFT Z

$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$ C Bord (3 conformal structures) $\xrightarrow{\text{forget geom.}}$ $\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$-TBord $\xrightarrow{\text{choose an MTC}}$ Linear Cat's

$\Downarrow \boxed{?Z} \Uparrow$ 1

do these Z's exist?

note: central charge does not factor.

(examples?)

"WZW model twisted by Chern Simons..."
— equivalent to a weak CFT "modular functor"
— use ... $\rightsquigarrow$ get untwisted $\to$ ... = Vect

Lecture II: d-GFT = Fun$^{\otimes}$(d-GBord, d-Vect, ⊗), a d-category.

Goal: Define a space, ∞-loop space ("spectrum"), ring spectrum (not the spectrum of a ring) "Brave new ring", derived alg. geom.

C category $\rightsquigarrow$ space BC classifying space. "planar alg on a line"

BC $= \coprod_{n \geq 0}$ BC$_n$ as a set, BC$_n$ spaces

$B\mathcal{C}_n = \{ \underset{c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_n}{\overset{f_1 \quad f_2 \quad \cdots \quad f_n}{\text{———————}}} \ \mathbb{R} \}$ $c_i \in \mathrm{Obj}(\mathcal{C})$, $f_i \in \mathrm{mor}(c_{i-1}, c_i)$. $f_i \neq \mathrm{id}$.

↑ actual $\mathbb{R}$-line!

(e.g. $\mathcal{C} = */\!/G = \mathcal{C}(*, e)$ no don't label gaps, put g's → bar cplx)

attach $B\mathcal{C}_n$ to lower n strata if

- two pts collide
- one point disappears in $\pm\infty$.

- Morrison-Walker ~~D*~~-like 1-category. (Eilenberg-MacLane)

Examples: $\mathcal{C} = */\!/\mathbb{Z}$ or $\mathbb{N}_0$ Then $B\mathcal{C} \underset{h.eq.}{\simeq} S^1$, true. $B\mathcal{C} \longleftarrow S^1$ $\longmapsto \prod z_i^{m_i}$

$m_i \in \mathbb{N}_0$ live at pts $z_i \in S^1$. If $z \in S^1$ then $z \mapsto$ (z)

Note that $B(\mathbb{N}_0, +) \xrightarrow{\simeq} B(\mathbb{Z}, +)$.

Def: An ∞-loop space (=: naive Ω-spectrum) is a seq. of pointed spaces $E_0 \xrightarrow[\sigma_0]{\simeq} \Omega_* E_1 \xrightarrow{\simeq} \Omega_*^2 E_2 \to \cdots \to \Omega_*^k E_k$ $k \geq 0$

(not a property, more structure) $E_1 \xrightarrow[\sigma_1]{\simeq} \Omega_* E_2$ $E_k \xrightarrow[\sigma_k]{\simeq} \Omega_* E_{k+1}$

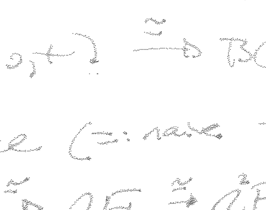
$\boxed{\text{E is connective if } E_1 \text{ connected, } E_2 \text{ is 1-connected, etc.}}$

Example: Claim that S^1 is an ∞-loop space (so is $\mathbb{Z}$)

Prop: Any topological ab. gp A is an ∞-loop space. $\overset{a_1 \ a_2 \ a_3}{\underset{b_1 \quad b_2}{\text{———}}} \mapsto \underset{a_1 b_1 \quad b_2}{\overset{a_2 \quad a_3}{\text{———}}}$ abelian ⇓ continuous

Fact: BA is also a top. ab. gp.

- can form $B(BA) = B^2A$, another top. ab. gp.

$B^2A = \{$ 

$\} = B(\ \cdot\ $ 2-category.)

B^kA is configurations of n pts in $\mathbb{R}^k$ labeled by A.

$= \mathrm{Conf}(\mathbb{R}^k; A)$, is k-th space is a connective spectrum.

Pf: $G \xrightarrow{\simeq} \Omega_* BG$ by $g \mapsto (z \mapsto (z)\,1)$ "one particle loop"

- not cyclic bar cplx!
- map backward is "holonomy map" (1) mark pt., record which g's pass the pt, multiply them...
- possibly some conditions "well-posed"

Exercise: $B^2(\mathbb{N}_0) = \mathrm{Conf}(\mathbb{C}; \mathbb{N}_0) \cong \mathbb{C}P^\infty$ homeomorphic via the

Fundamental thm of algebra. ↑ $\simeq K(\mathbb{Z},2)$ non-zero $\mathbb{C}$ poly's up to scaling $/\mathbb{C}^\times$

- map is take the roots. $\mathbb{C}P^\infty \simeq$ to lines in H.

$\mathbb{C}P^\infty \simeq K(\mathbb{Z},2) = B^2(\mathbb{Z})$; $S^1 \simeq B(\mathbb{Z})$.

"de-looping shifts homotopy grps by 1"

Defn: $HA = (A, BA, B^2A, \dots)$ Eilenberg MacLane spectrum for A; connective.

because $H^n(X;A) \simeq [X, B^nA]$. ← $\pi_k(HA) = \pi_k A$.
A discrete, then $K(A,n)$.

Defn: $\pi_k(E) := \pi_k(E_0, *)$ [illegible] of E.
∞-loop space

Define $E^k(X) = [X, E_k]$ the "E-cohomology"

- if you plug in a pt., might not get zero.
- cohomology theory w/o this axiom.
- this is represented by a ~~space, called a~~ spectrum.
- in a cohomology theory, this is suspension iso.

little disk operads act on these spaces as well! are E_n-spaces (like E_n-alg's).

Comparison: $0\text{-}G.FT = \mathrm{Fun}^{\otimes}(0\text{-}G\,\mathrm{Bord}, 0\text{-vect}, \otimes)$

↑ pts w/ disj. union. cpt 0-mflds.
- is free symm. monoidal cat on $*$.

$\simeq$ 0-vect by ev(pt.)

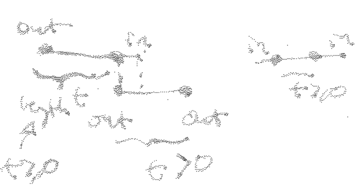
$1\text{-vect}_{\otimes} = (\mathbb{C}\text{-v spaces}, \otimes)$ so $0\text{-vect} = \Omega_{\mathbb{C}}(1\text{-vect}) = \mathbb{C}$.

Claim $|0\text{-}G.FT_H| = H(\mathbb{C}, +)$

- in general, we require $(d\text{-Vect}, \otimes, \oplus)$. Then $(d\text{-}G.FT, \oplus)$ is a symm. monoidal d-category. Next the underlying spectrum; more interesting.
  ↳ construct classifying spectrum → K_{alg} → K_{top}.
- now $H(\mathbb{C},+) = \begin{cases} \text{ordinary [illegible] in discrete case} \\ * \text{ in top. case.} \end{cases}$
- finishes 0-dim'l case.

Technical III: $|1\text{-}T^{op}TF| \simeq K^{alg}(\mathbb{C}_\delta)$ alg. K-theory
$|1\text{-}T^{op}FT| \simeq ku$, top. conn. K-theory.

1-EFT
↖ Euclidean



1-EBord $\xrightarrow{F}$ (1-Vect, ⊗)
‖
$(\text{Top. vect}_{\mathbb{R}}, \otimes_{proj})$
↳ Cont'l

Apply $F(\underset{pt}{\bullet}) = V$, top. v.sp. $F(\underset{t=0}{\bullet\!\!-\!\!\bullet}) = V \underset{proj}{\otimes} V \xrightarrow[\langle\cdot,\cdot\rangle]{} \mathbb{R}$ inner prod.

– note that proj $\otimes$ gives trace-class ops. For Hilb, $\langle\cdot,\cdot\rangle$ does not extend to ~~co~~ continuously.
$\otimes \to$ HS ops.

$F(\underset{t}{\bullet\!\!-\!\!\bullet}) = e^{-t\Delta}$ continuous semigp, generator Δ "Laplace"

– if $\langle\cdot,\cdot\rangle$ pos.def inner prod., ~~then~~ then $e^{-t\Delta}$ is trace class, b/c.

$F(\underset{t}{\bullet\!\!-\!\!\bullet}) \in V \underset{proj}{\otimes} V \cong$ trace class ops.

Note: $\lim_{t\to 0} F(\underset{t}{\bullet\!\!-\!\!\bullet}) = \lim_{t\to 0} e^{-t\Delta} = id.$

1 not trace class. $\}$ use ∞-ord!

But $\lim_{t\to\infty} F(\underset{t}{\overset{out\ \ out}{\bullet\!\!-\!\!\bullet}})$ is trace class, b/c homotopy many $sp(\Delta) \to \infty$.

– contractible space $|1-|$

– for 1/1-EFT, $F(\underset{pt}{\bullet}) = \mathbb{Z}/2$-graded $\mathcal{V}$. Super vector/Hilb spaces.

$F(\bullet\!\!-\!\!\bullet) = e^{-tD^2}$ D odd. spin mfld is target – Dirac op.

– D has a symm. spectrum. concentrate on 0, push outward.

– here is 1-D TFT, why we get same argum for 1/1-EFT.

"de-looping"

$A \longmapsto (A, BA, \ldots, B^nA, \ldots)$ $\quad \mathrm{Conf}(\mathbb{R}^n, A)$ $\quad \Omega^n(\cdot) \simeq A.$

$\cap$

Ab $\longrightarrow$ Spectra

Gr ? $\cap$ $\qquad$ Gr ? $\cap$

AbSemi$_0$ $\xrightarrow{H=K_0}$ Spectra$'$

add iso morphisms $\downarrow$ $\qquad K_1$ extends K_0

SymCat. (setting mult., add iso)

$\leftarrow$ same, but $A \to BA$ not nec. a h.eq.! $\quad$ $\nearrow$ gp. completion $\quad$ afterward, get equivalences!

eg: $A = (\mathbb{N}, +)$ $\quad B\mathbb{N}_0 \simeq S^1$

$\mathbb{N} \to \Omega B\mathbb{N}_0$ not h.eq.

$\Omega S^1 \simeq \mathbb{Z}$. "gp completion"

Have map ~~Spectra~~ Spectra$' \to$ Spectra by $(E_0, E_1, \ldots) \mapsto (\Omega E_1, E_1, \ldots)$

AbSemi$_0 \to$ Ab by completion.

– Can get a spectrum from a sym. monoidal category.

Why does this work philosophically?

want to extend further SymCat $\to$ Sym 2-cat, get $K_2 \to$ Spectra$'$
add 2-morph's.

– can extend to Sym d-Cat, K_d to Spectra!

$\qquad (d\text{-GFT}, \oplus)$

Def: $Ld\text{-GFT} := \underline{Gr} K_d(d\text{-GFT})$

Can add Top to Ab, AbSemi$_0$.
Top SymCat, Top Sym d-Cat
topology only on domain
– or (∞, d)-category, model, etc. ...

6

Rem: K_1 constructed by Graeme Segal using Γ-spaces as a model for connective spectra (think wk top ab) Γspcs $\cong$ Spectra$_{conn.}$

Modern Philosophy: Why does K_1 exist? (to make precise, use Γ-spaces)

$(A, BA, B^2A, \dots)$ $\quad BA = B\begin{pmatrix} A \\ \Downarrow \\ * \end{pmatrix}$ A as a Category.

$B^2A = B\begin{pmatrix} A \\ \Downarrow \\ * \\ \Downarrow \\ * \end{pmatrix}$ a 2-cat from A.
- both compositions are $+$.

What when Sym.$\oplus$-cat? (unitary $\oplus$) $\quad$ etc.

$$K_1(\mathcal{C}) = (B\mathcal{C}, B(\Sigma(\mathcal{C},\oplus)), \dots, B(\Sigma^n(\mathcal{C},\oplus)), \dots)$$

$$\left(B\begin{pmatrix}\mathcal{C},\oplus \\ \Downarrow \\ \mathcal{C}_0\end{pmatrix}, B\begin{pmatrix}(\mathcal{C}, 0, \oplus) \\ \Downarrow \\ (\mathcal{C}, 0, \oplus) \\ \Downarrow \\ *\end{pmatrix}\right),$$

for braided can go to 3, higher you must be symmetric!

Requirement: ∞-Cat $\xrightarrow{B}$ Top

(Tom Leinster - survey on 2-cats.)

Clark Barwick + Chris S.P.: unicity of higher cat's, some candidates...
[two notions of framed dualizable cat's, some duality built in....]

Example: [Quillen] R a ring. Take Proj_R f.g. proj. Mods., only invertible hom's of morphisms. $K_1(\mathrm{Proj}_R, \oplus) = K^{alg}(R)$.

$K_n^{alg}(R) = \pi_n(K^{alg}(R))$ $\quad$ π_n of spectrum.

Rmk: R comm $\Rightarrow$ Ring spectrum! $\otimes_R$.

Reproduce from field theories:

Show $1\text{-TFT} = \mathrm{Fun}^{\otimes}(1\text{-Bord}^{or}, (1\text{-vect}, \otimes))$

in discrete case, take $\mathcal{A}$ to be $(R\text{-Mod}, \otimes_R)$ R commutative.

Lemma: this category is equivalent to Proj_R. Map is $ev(pt_+) = ev(\bullet_+)$

In fact, $\mathrm{Fun}^{\otimes}(1\text{-Bord}, \underbrace{(\mathcal{C}, \otimes)}_{\text{Sym. mon.}}) \underset{ev(pt_+)}{\simeq}$ dualizable obj + inv. morphisms.

Lurie's Cobordism Hyp + Chris S.P. its generalization to d-cat's, fully dualizable obj's, inv's..

works by Morse theory:

Lemma: 1-Bord^{or} has the presentation as a Sym Cat.

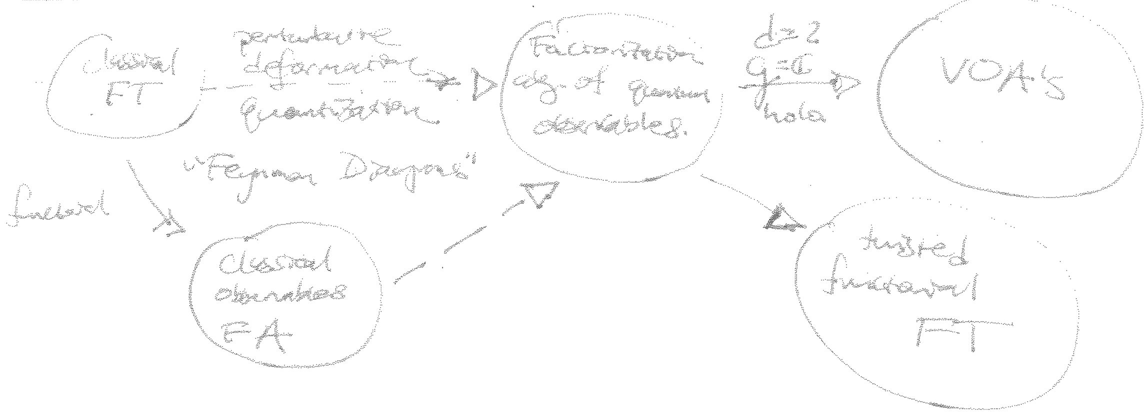
gen's
rel's
$\left(\rightleftarrows, \frown, \smile \;\middle|\; \text{zigzag} = id = \text{zigzag}\right)$ $\quad$ Pf. has Morse theory for 1-mflds.

Morse theory gives gen's $\quad$ Cerf theory gives rel's.

Cor: $\mathrm{Fun}(1\text{-Bord}^{or}, (\mathcal{C}, \otimes, 1)) \simeq ((V_+), V_-, ev: V_+ \otimes V_- \to 1, coev: 1 \to V_- \otimes V_+)$ s.t. N-th
$\simeq \mathcal{C}$ dualizable, inv. morphisms.

C.S.P. gen's + rel's for 2-cats. in thesis.

Stolz – Factorization Alg's + Functional FT's. (Costello + Gwilliam)



Def: A pre factorization alg. on a mfld M (spacetime) is an assignment

• $M \supset U$ open $\longmapsto A(U)$ dg-v.sp. [chain complex]

$A(U)$ a cochain cplx $H^*(A(U))$ physical observables.

• $\underbrace{U_1, \dots, U_k}_{\text{disjoint}} \subset V \longmapsto \left(A(U_1) \otimes \cdots \otimes A(U_k) \xrightarrow{m} A(V)\right)$

Properties: compatible w/ composition.

– Extra properties to be a genuine factorization alg:

• $\underbrace{U_1, \dots, U_k}_{\text{disjoint}} \subset V \quad A(U_1) \otimes \cdots \otimes A(U_k) \xrightarrow{\sim} A(U_1 \sqcup \cdots \sqcup U_k)$ iso in H^*

• "Cosheaf property" adjusted to $\otimes$.

Examples:

① Free particle moving in $\mathbb{R}$. $M = \mathbb{R}$. $U \longmapsto A(U) =$

$$\mathcal{E}(U) = \left\{ \overset{0}{C^\infty(U)} \xrightarrow{\Delta} \overset{1}{C^\infty(U)} \text{ all else } 0. \right\} \quad H^i(\mathcal{E}(a,b)) = \begin{cases} 0 & i \neq 0 \\ f: \mathbb{R} \to \mathbb{R} \text{ Geodesics} & i = 0. \end{cases}$$

– space of classical fields – classical phase space.

– "derived" phase space.

$$\mathbb{R} \underset{\text{open}}{\supset} U \longmapsto \mathrm{Obs}^{\text{classical}}(U) = \{\text{poly. fcts. on } \mathcal{E}(U)\} = \mathrm{Sym}^*(\mathcal{E}(U)^*)$$

dual

Graded Sym. power

– really secretly the exterior powers…

Factorization alg of classical observables.

$$H^*(\mathrm{Obs}^{\text{classical}}(U)) = \mathrm{Sym}^*(\underbrace{H^*(\mathcal{E}(U)^*)}_{\mathbb{C}\{p,q\}}) = \mathbb{C}[p,q]$$

position + momentum.

② Factorization alg. of quantum observables:

need a pairing $\mathcal{E}_c(U) \otimes \mathcal{E}_c(U) \xrightarrow[\langle\cdot,\cdot\rangle]{} \mathbb{C}$ degree -1

$\mathcal{E}_c$ cpt. support

skew-symmetric $(\varphi_1,\psi_1) \otimes (\varphi_2,\psi_2) \longmapsto \int_U \varphi_1\psi_2 - \varphi_2\psi_1 \in \mathbb{C}$.

"Symplectic form" on $\mathcal{E}(U)$. ⇝ poisson bracket ⇝ Lie alg.

— can look at $\langle x,\cdot\rangle$ to get map $\mathcal{E}_c(U)[1] \xrightarrow{\sim} \mathcal{E}(U)^*$

shift deg. ⇝ Iso in cohomology

⇝ Poisson bracket on $\mathrm{Sym}^*(\mathcal{E}_c(U))$ $\cong \mathrm{Sym}^*(\mathcal{E}_c(U)^*)$.

⇝ BV-Laplacian $\Delta_{BV}: \mathrm{Sym}^k \longrightarrow \mathrm{Sym}^{k-2}$, !ly determined by

- $\Delta_{BV}(fg) = (\Delta f)g \pm f\Delta g + \{f,g\}$.

$M \supset U \longmapsto \mathrm{Obs}^q(U) := \left(\mathrm{Sym}^*(\mathcal{E}_c(U)[1])[[\hbar]],\ Q + \underbrace{\hbar\Delta_{BV}}_{\text{deformation of differential.}}\right)$

$H^*(\mathrm{Obs}^q((a,b))) \cong$ Alg. over $\mathbb{C}[[\hbar]]$ gen by p,q, $\underbrace{\text{s.t. } [p,q] = \hbar}_{}$

Weyl alg., usual quantum business.

<u>Note</u>: alg. structure of Sym^* does <u>not</u> induce multi. above on H^*;

since Δ_{BV} <u>not</u> a derivation.

— if fcts have disjoint support, $\{f,g\} = 0$ ⇝ can multiply.

— only multiply cocycles w/ disjoint support.

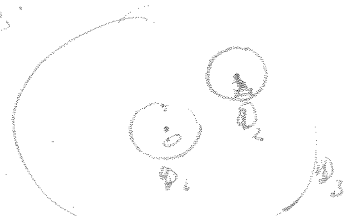
— can represent 2 cocycles on same interval by representatives on smaller intervals. Need to be careful, pick things carefully.

<u>Ex</u>: $M = \mathbb{C}$, $\mathcal{E}(U) = \begin{matrix}\Omega^{0,0}(U) \\ \oplus \\ \Omega^{1,0}(U)\end{matrix} \xrightarrow{\bar\partial} \begin{matrix}\Omega^{0,1}(U) \\ \oplus \\ \Omega^{1,1}(U)\end{matrix}$ $\exists$ natural pairing.

$H^*(\mathrm{Obs}^q_S(\mathbb{D})) \cong \mathbb{C}[a_n, b_n \mid n \in \mathbb{Z}] =: V$

$H^0(\mathcal{E}(U)^*) = [\mathcal{O}(U) \oplus \Omega_{hol}(U)]^*$ a_n taylor/laurent at 0. $\mathcal{O}(U)$ b_n — — — — . $\Omega_{hol}(U)$

<u>To VAs</u>:



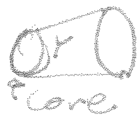
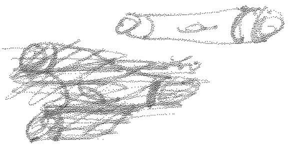
$\mathbb{D}_1, \mathbb{D}_2 \longrightarrow \mathbb{D}_3$ induces $\mathrm{Obs}^q(\mathbb{D}_1) \otimes \mathrm{Obs}^q(\mathbb{D}_2) \to \mathrm{Obs}^q(\mathbb{D}_3)$

— take H^* $\quad \begin{matrix}\cong \\ V\end{matrix} \otimes \begin{matrix}\cong \\ V\end{matrix} \longrightarrow \hat{V}[[z, z^{-1}]]$

state-field correspondence

Relation to Functional Field Theories:

twisted FT. T twist functor, E a T-twisted FT.

Look at bordism category		T	E
obj:	 Y core	$T(Y)$ alg.	$E(Y)_{T(Y)}$ module.
morphisms		${}_{T(Y_1)}T(\Sigma)_{T(Y_2)}$ bimod.	$E(Y_1)_{T(Y_1)} \xleftarrow{E(\Sigma)} E(Y_0)_{T(Y_0)}$; take $\otimes$: $\otimes\, {}_{T(Y_1)}T(\Sigma)_{T(Y_0)}$, $T(Y_0)$-linear

Thm (Dwyer-S-Teichner)

If A is a factorization alg., defined on d-mflds equipped with a G-structure. Then A determines a twist functor T_A and a T_A-twisted field theory E_A. (on d-dim'l mflds w/ geometry G.)

Warning: Here "alg's" + "bimod's" are interpreted in a derived sense.

Henriques: A categorification of the notion of vNa (0)

$T \to B$ map of $\otimes$-categories.

- the commutant of T inside B is T' given by

Obj: pairs $b \in B$, $\beta_a : b \otimes a \simeq a \otimes b$ $\forall a \in T$ s.t.

- $$\begin{tikzcd} ba \arrow[r] \arrow[d, "\beta_a"'] & ba' \arrow[d, "\beta_{a'}"] \\ ab \arrow[r] & a'b \end{tikzcd}$$ $\forall a \to a'$ in T.

- $$\begin{tikzcd} baa' \arrow[r, "\beta_{aa'}"] \arrow[d, "\beta_a"'] & aa'b \\ aba' \arrow[ur, "\beta_{a'}"'] & \end{tikzcd}$$ $\forall a, a' \in T$

Morphisms: $b \to b'$ s.t.
$$\begin{tikzcd} ba \arrow[r] \arrow[d, "\beta_a"'] & b'a \arrow[d, "\beta'_a"] \\ ab \arrow[r] & ab' \end{tikzcd}$$
$\forall a \in T$

$\otimes$: $(b,\beta) \otimes (b',\beta') = (b \otimes b',\ bb'a \xrightarrow{\beta'} bab' \xrightarrow{\beta} abb')$

Map to B: $(b,\beta) \to b$. (above identifies image of T in B)

Fact: $\exists$ canonical map $T \to T''$. by $a \mapsto (a$, for each (b,β), need iso $ab \to ba$. Just use β^{-1})

Lemma: $T'' \to T''''$ is an equivalence of categories.

$$\begin{tikzcd} & T'''' \arrow[dr] & & & \\ T'' \arrow[ur, bend left] \arrow[r] & & B & T''' \arrow[l] \arrow[d, bend left] & \\ T \arrow[u] \arrow[urr] & & & T' \arrow[ull] \arrow[u, bend left] & \end{tikzcd}$$

One step later than usual.

Specialize: $B = \mathrm{Bim}(R)$ R is hyperfinite III_1.

Def: $T \to \mathrm{Bim}(R)$ is a truck if the natural map $T \to T''$ is an equivalence.

[and T is closed under contragredient]

Bim(R) is equipped w/ 2 involutions:

$f: H_1 \to H_2 \rightsquigarrow f^*: H_2 \to H_1$

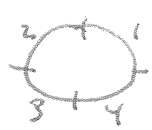
${}_R H_R \longrightarrow \overline{H}$; $a\bar{\xi}b = \overline{b^*\xi a^*}$

- really want T has unitary/rigid structure, and $T \longrightarrow$ Bimod(R) respects these structures.

Analogies btwn $B(H)$ + $\mathrm{Bim}(R)$:

Trucks	vNa's
R	H
$\mathrm{Bim}(R)$	$B(H)$
??	topologies ...
$\nexists$	$\mathcal{L}^1(H)$ trace class
split bimod's. $R\bar{\otimes}R^{op}$-mod's	$\mathcal{L}^2(H)$ HS
$H_1 \underset{R\bar{\otimes}R^{op}}{\boxtimes} H_2$ ↖Hilb	$a,b \in \mathcal{L}^2(H) \rightsquigarrow tr(ab) \in \mathbb{C}$

Thm: To every conformal net A $[\mu(A) < \infty]$ there is an associated truck $T(A)$. Moreover, $Z(T(A)) = \mathrm{Rep}(A)$

Construction: 

$\underbrace{A(I_1 \cup I_2) = A_{12}}_{=R}$ abbreviate identify w/ A_{34}^{op} via

Then every A-rep is an R-R bimodule.

Among R-R bimod's, rep's of A are those satisfying:

① The action of $A_2 \underset{alg}{\otimes} A_3$ extends to A_{23}

② The action of $A_1 \underset{alg}{\otimes} A_4$ extends to A_{14}

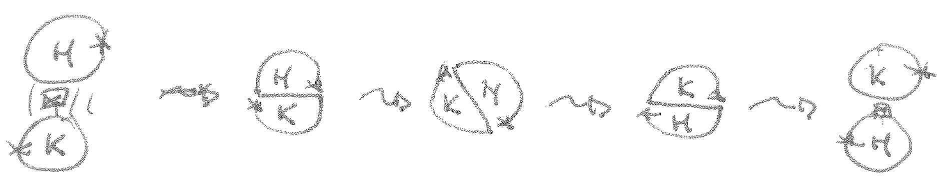
$T(\mathcal{A}) := \{ R\text{-}R \text{ bimod's} \mid ① \text{ holds} \}$ Full subcat of $\mathrm{Bimod}(R)$.

<u>Claim:</u> $T(\mathcal{A})' = \{ R\text{-}R \text{ bimods} \mid ② \text{ holds} \}$

$\rightsquigarrow Z(T(\mathcal{A})) = \mathrm{Rep}(\mathcal{A})$. $\quad (T(\mathcal{A})' \cap T(\mathcal{A}) = Z(T(\mathcal{A})) \ldots)$

<u>Sketch of why these commute:</u>

$H \in T(\mathcal{A})$, $K \in T(\mathcal{A})'$, need iso $HK \xrightarrow{\sim} KH$.



<u>Claim:</u> $\exists$ ideal in $T(\mathcal{A})$ analogous to HS.

$I_2^{T(\mathcal{A})} = \{ \text{elts of } T(\mathcal{A}) \mid \text{the action of } A_1 \underset{alg}{\otimes} A_4 \text{ extends to } A_1 \bar{\otimes} A_4 \}$

[this is opposite to ②]

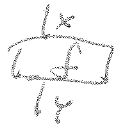
- trucks are "analogs" of type I factors.
  - fully dualizable.

Ng: Cauchy's Thm for MTC

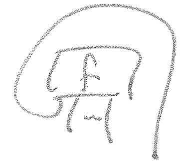
- G a finite gp, $p \mid |G|$. $\exists x \in G \mid \mathrm{ord}(x) = p$.
  $\exp(G)$ = least N s.t. $x^N = e$ $\forall x \in G$.
  So if $p \mid |G| \Leftrightarrow p \mid \exp(G)$; i.e., $|G|, \exp(G)$ have same prime factors.
- Now $|G| = \sum_{V \in \mathrm{Irr}(G)} (\dim V)^2$. $\mathrm{Exp}(G)$ can be realized by FS ind.
  $\nu_n(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g^n)$ where χ_V is char of V.
  Obvious: $\nu_N(V) = \frac{1}{|G|} \sum_{g \in G} \chi_V(e) = \dim(V)$.

Ex: $\exp(G)$ = least integ n s.t. $\nu_n(V) = \dim V$ $\forall V \in \mathrm{Rep}(G)$.

- $\mathcal{C}$ a strict spherical fusion category $(-)^{**} \simeq \mathrm{Id}_{\mathcal{C}}$ monoidally.

Graphical Calculus. $f: X \to Y$  etc.

$$\dim(\mathcal{C}) = \sum_{V \in \mathrm{Irr}(\mathcal{C})} \dim(V)^2.$$

$E_V^n : \mathrm{Hom}_{\mathcal{C}}(1, V^n) \circlearrowleft$ by $f \longmapsto$ 

$\nu_n(V) = \mathrm{Tr}(E_V^n)$ trace of operators.

$\mathrm{FSexp}(\mathcal{C})$ = least pos. N s.t. $\nu_n(V) = \dim(V)$ $\forall V \in \mathcal{C}$.

Thm (N-Schauenburg) this # exists + is finite. $N < \infty$.

Fact: $\dim(V) \in \mathbb{Z}[\zeta_N]$ cyclotomic integers.

Cor: $\dim(\mathcal{C}) \in \mathbb{Z}[\zeta_N]$, Dedekind domain.

Cauchy's thm: (BNRW) $N = \mathrm{FSexp}(\mathcal{C})$. Then $\langle N \rangle$ and $\langle \dim(\mathcal{C}) \rangle$ in $\mathbb{Z}[\zeta_N]$ have same prime ideal factors.

Fact (Etingof) $\frac{\dim(\mathcal{C})^5}{N}$ is algebraic integer. gives one direction.

Application to MTC.

Thm (Vafa) $T = (\theta_i)$ diagonal matrix $\theta_i \mathrm{id}_{V_i} =$ has finite order. (By non-deg. of braiding)

Thm (N-Sch.) $\mathrm{FSexp}(\mathcal{C}) = \mathrm{ord}(T)$.

N [$=$ | $\forall V \in \mathcal{C}$. $\theta_V^N = \mathrm{id}$.

Thm (Etingof - Gelaki) $\mathcal{C}$ modular. $\frac{\dim \mathcal{C}}{(\dim V_i)^2}$ alg. integ. $\dim(V_i) \in \mathbb{R}$.

$\dim(\mathcal{C}) = \sum_0^{n-1} (\dim V_i)^2 \Rightarrow \left(1 + \sum_1^{n-1} (\dim V_i)^2 - \dim \mathcal{C}\right) = 0.$

$x = ((\dim V_1)^2, \dots, (\dim V_{n-1})^2, -\dim \mathcal{C})$ in $\mathbb{Z}[\zeta_N]$.

If $\mathfrak{P} \mid (x_i) \Rightarrow \mathfrak{P} \mid (N)$.

[Evertse '70s] Let Q be a finite set of prime ideals in a #field K. $\forall n \in \mathbb{N}$, $\exists$ only finitely many tuples $(x_1, \dots, x_n)$ in K s.t.

① $\mathfrak{P} \mid (x_i) \Rightarrow \mathfrak{P} \in Q$.

② $0 = 1 + x_1 + \dots + x_n$, and no sub-sum is zero.

Consequently: $\forall$ modular $\mathcal{C}$ rank n, $\mathrm{FSexp}(\mathcal{C}) = N$, $\exists$ only finitely many dim vectors in $\mathbb{Z}[\zeta_N]$.

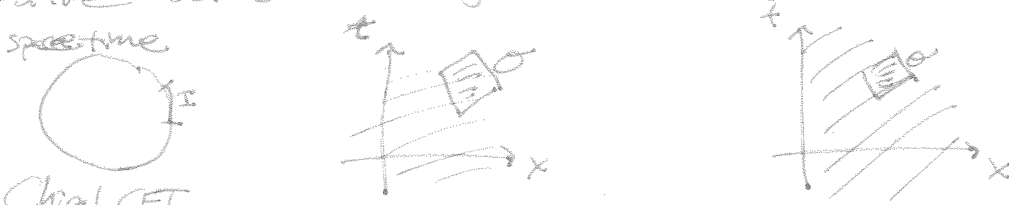
$\leadsto$ f. many dim's $\leadsto$ f. many fusion rules $\leadsto$ f. many categories $\leadsto$ f. many braidings, etc... - must bd. N for fixed rk.

Lem: (Bong-Lin-N.) $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ maps to S_n $n = \mathrm{rank}(\mathcal{C})$. $\boxed{\text{kernel kernel } 2\text{-gp}}$

Yasu: jt. w/ M. Bischoff + R. Longo.

Positive sol'n to Kong-Runkel conj., we'll see op. alg. CFT corollary.

spacetime



Chiral CFT $I \mapsto A(I)$ | Full CFT $O \mapsto A(O)$ | Boundary CFT $O \mapsto A(O)$

$A = \{A(I)\}$ completely rational local conformal net. $\Rightarrow$ Rep(A) MTC. (K-Longo-Müger)

- extension $A(I) \subset B(I)$ arising from a rep of A, non-locality for B.

$[A(I_1), B(I_2)] = 0$  , no commutator for $B(I_1), B(I_2)$

"non-local extension" $\longleftrightarrow$ (non-commutative) Frob. alg in Rep(A).

VOA: V a VOA, $W = \bigoplus^{\text{finite}} n_i M_i$ M_i's modules / V, $M_0 = V$, $n_0 = 1$.

$\leadsto Y(v,z)$ acting on W, $(z_1 - z_2)^N [Y(v,z_1), Y(u,z_2)] = 0$ $v \in V$, $u \in W$.

non-local ext. of A $\xleftrightarrow{LR}$ boundary CFT based on A $\xrightarrow{LR}$ full CFT based on A ; $\xleftarrow{\text{Carpi-K-L}}$ (from full CFT to boundary CFT)

$\updownarrow$ Longo-Rehren

non-comm alg. in Rep(A) $\xrightarrow{\text{Fröhlich-Fuchs-Runkel-Schweigert}}$ full center $\leftarrow$ SF ; $\xleftarrow{\text{Kong-Runkel}}$

full CFT $\rightarrow$ SF $\leftarrow$ have natural subfactor

conj. $\rightarrow$ KR

alg. in $\otimes$-cat.

$A(I) \otimes A(J)$
$\cap$
$\tilde{B}(I \times J)$ in full CFT.

finite index. Rehren

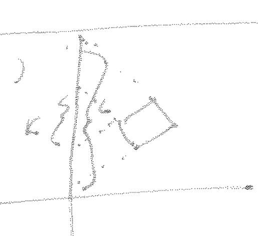
sized decomp. gives $\bigoplus_{\lambda, \mu \in \text{Rep}(A)} Z_{\lambda\mu}\, \lambda \otimes \bar{\mu}$ — modular inv. Z from ...

$\leadsto$ Full CFT by moving ∂ to left.

- in other direction, can create ∂.

Start w/ $B(I) \supset A(I)$

$B_+(O) := B(K)' \cap B(L)$.



$\leadsto$ family of vNa's on half space. Boundary 2CFT.

α-induction: $N \subset M$ N-N bimod's braided fusion category assumption.

${}_N X_N$: given ${}_N M_N \otimes X_N \simeq {}_N X \otimes_N M_N$ BO. from braiding.

L-action of M R-action of M, these commute.

- gives an M-M bimodule, functorially.
- choice of braiding $\pm$ $\leadsto \alpha^{\pm}$ 2 functors. $\dim(\alpha^+_\lambda, \alpha^-_\mu) =: Z_{\lambda,\mu}$

Böckenhauer-E-K: Z is a modular invariant. (i.e. commutant of $SL(2,\mathbb{Z})$ rep from braiding)

Rehren gave a descripn. of this subfactor via a commutative Frob.-alg. (2)
Kong-Runkel conj that these SF are the same (Rehren's from modular invts w/ $\alpha^{\pm}$ induction, and another using alg. obj in full center)

<u>Note:</u> If $Z_{\lambda\mu} = \delta_{\lambda\mu}$, then this is LR-construction/asymptotic inclusion.
- generalization of LR-construction.

<u>Thm</u> (Bischoff-K-Longo) This is true.

Suppose $P \subset Q$, look at $p \otimes p$. $P \subset Q$ and ~~[illegible]~~ $P \subset Q_1$ same $\Rightarrow$ $\rho_Q = \rho_{Q_1}$

- not sufficient for subfactors to be the same.

product structure of alg in MTC.
- same object can have distinct prod. structures.

<u>Key step:</u> $Z_{\lambda\mu}$ coefficients $= \dim(\mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-))$.
- Gives natural embedding $\mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-) \hookrightarrow \mathrm{Hom}(\lambda, \mu)$.
- need a characterization of the image.

Consequence:
<u>Thm:</u> $\exists$ a bijective correspondence btwn Morita-equivalence
(*) classes of non-local extensions$^{\text{of } A}$ and Eq. classes of full CFT based on A.

<u>Remark:</u> Each (eq. class of) non-local extension corresponds to a ∂-condition of boundary CFT.

<u>Examples:</u>
① Virasoro net w/ $c = 1 - \frac{6}{m(m+1)}$ discrete series. (all possible values < 1)
$\{ \mathrm{Vir}_c(I) \}$ take this A in the above.
- ~~has~~ same classification in (*) separately.   non-local ext. $\xrightarrow{\text{easy}}$ Full CFT $\swarrow$ paper ADO
- gives simplification of work in ← ··· inverse ⟶   K-L-Pennig R. ← ··· K.L.

Tener: Segal CFT at the ∂. ⓘ

Ex: The free fermion.
- construction (complex analysis)
- calculation ($\Rightarrow$ VOA's)
- analytic properties ($\Rightarrow$ conformal nets)*

Def: Chiral (spin) Segal CFT with ~~one~~ two label/sector.

Data:
- Hilbert space F
- positive energy rep'n of $\mathrm{Diff}_+^{(2)}(S^1)$ on F. $\mathcal{U}$
-  ∂ partitioned + parametrized $\rightsquigarrow E(\Sigma)$ 1-dim'l subspace of trace class ops $\bigotimes_{\partial_{in}} F \to \bigotimes_{\partial_{out}} F$

(+ spin structure)
(+ 2 labels on $\partial\Sigma$)

Axioms:
- cutting into Σ_1, Σ_2, then $E(\Sigma_2)E(\Sigma_1) = E(\Sigma)$ composition of ops.
- reparametrization $\rightsquigarrow \mathcal{U}(\gamma_{out})^* E(\Sigma)\, \mathcal{U}(\gamma_{in})^*$
- holomorphic dependence  w holo. line bundle $/w$.
- unitarity $E(\overline{\Sigma}) = E(\Sigma)^*$.

specific example. $H = L^2(S^1)$

General construction:
- $\Lambda H = \bigoplus \Lambda^i H$, $\Lambda^0 H = \mathbb{C}\Omega$.
- creation/annihilation ops $a(f)\omega = f \wedge \omega$, $a(f)^*$ annihilation

- pay a price for fermions over bosons (spin), but get that these are bdd ops! $\|a(f)\| = \|f\|_2$.

- $CAR(H) = C^*(a(f), a(f)^*)$ universal subalg.

$$\{a(f), a(g)\} = 0 \qquad \{a(f)^*, a(g)\} = \langle g, f \rangle 1.$$

$P_+H = H^2(\mathbb{D})$
$\overline{\text{span}\{z^n \mid n \geq 0\}}$

- given p a proj on H, get a rep $(\pi_p, \underbrace{\Lambda(pH \oplus \overline{(1-p)H})}_{F})$ * flip cplx structure

$$\pi_p(a(f)) = a(pf) + a(\overline{(1-p)f})^*.$$

- take $\Omega \in \Lambda^0$, $\begin{cases} a(f)^*\Omega = 0 & f \in pH \\ a(g)\Omega = 0 & g \in (1-p)H \end{cases}$

• $u(\gamma)\, a(f)\, u(\gamma)^* = a(\gamma \cdot f)$ characterizes $\mathrm{Diff}(S^1)$ action. ②

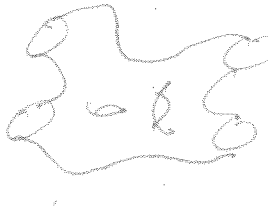
$(\gamma^{-1} f)(z) = |\gamma'(z)|^{1/2} f(\gamma(z))$.

$\xi = a(z^{n_1}) \cdots a(z^{n_k}) \cdot a(z^{m_1})^* \cdots a(z^{m_l})^* \Omega$

$n_i < 0,\ m_i \geq 0$.

• $L_0 \xi = \left(\sum |n_i - \tfrac{1}{2}| + \sum |m_i - \tfrac{1}{2}| \right) \xi$. (f.d. eigenspaces)

• $\forall \Sigma$, an operator:



$H^2(\Sigma) \subseteq \bigoplus_{\partial_i \subset \partial} L^2(S^1)$

$\| \{ F \circ \text{reparametrization} \mid F \in \mathcal{O}(\Sigma) \}$ (F holo)

$H^2(\Sigma) \subseteq H_{\partial in} \oplus H_{\partial out}$ — Some regularity on ∂.

$H_{\partial in} = \bigoplus_{\partial_i \subset \partial in} L^2(S^1)$, $H_{\partial out} = \bigoplus_{\partial_i \subset \partial out} L^2(S^1)$

$E(\Sigma) = \Big\{ T : \bigotimes_{in} F \to \bigotimes_{out} F \;\Big|\; a(f_{out})^* T = T a(f_{in})^*,\ (f_{in}, f_{out}) \in H^2(\Sigma);\ a(g_{out}) T = T a(g_{in}),\ (g_{in}, -g_{out}) \in H^2(\Sigma)^\perp \Big\}$.

<u>Thm</u>: This is a ~~legal~~ CFT.

— will talk about gluing + existence.

<u>Pf ideas</u>:

$\Sigma_1 \uparrow \Sigma_2$

∂out

$T_i \in E(\Sigma_i)$ WTS $T_2 T_1 \in E(\Sigma)$

— take $(f_{in}, f_{out}) \in H^2(\Sigma)$.

$a(f_{in})^* T_2 T_1 = T_2\, a(f_{out})^* T_1 = T_2 T_1\, a(f_{in})^*$

now take $(g_{in}, g_{out}) \in H^2(\Sigma)^\perp$

$a(g_{out}) T_2 T_1 = T_2 \underbrace{a(g_{out})}_{\text{doesn't exist a priori}} T_1 = T_2 T_1\, a(g_{in})$

<u>Lemma</u>: $H^2(\Sigma)^\perp$ is ∂ values of holo. 1-forms.

— geometric, so can use the same pf!

— spin structure $\Rightarrow \perp$ is sections of complementary spin structure

<u>Def</u>: ~~Thm~~

$X =$  think 

— can still define a Hardy space $\{ F \circ \text{parametrization} \mid F \text{ holo in nbhd } X \}$

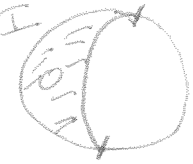
<u>Q</u>: does $\exists$ a bdd op satisfying commutation relations?

— don't expect it to be trace class (no sewing)

(8)

<u>Thm</u>[*]: $\exists\, T_x : \mathcal{F}_0 \otimes \mathcal{F} \longrightarrow \mathcal{F}$ s.t. $\forall \xi \in \mathcal{F}_0$, $T_x(\xi \otimes -) \in B(\mathcal{F})$.

$\mathcal{F}_0 = \Theta^{alg}\Lambda(\cdot)$

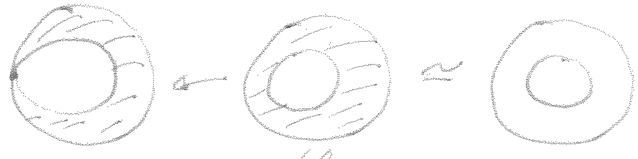


parametrizations agree $\Big\}$ $vN(T_x(\xi\otimes -)) = \{a(f), a(f)^* \mid f \in L^2(I)\}''$

(Conformal net ....)

- uses connections w/ vertex alg.
- Surface lives on ∂ of a Segal CFT $\rightsquigarrow$ bdd ops + conformal nets.

<u>Sketch of idea:</u>



$= \underbrace{u(\gamma)}\, r^{L_0}\, \underbrace{u(\gamma)}^*$

contraction.

- convergence on dense domain, need a $\|\cdot\|$ bdd above argument.
- commutation rel'ns. $T_\varepsilon\, a(f_{\varepsilon,in}) = a(f_{\varepsilon,out})\, T_\varepsilon \rightsquigarrow T a(f_{in}) = a(f_{out})\, T$.