Most of the following information comes from Loday’s *Cyclic Homology*.

I. The Cyclic, Simplicial, and Semi-Simplicial Categories

**The Cyclic Category.** The category $c\Delta$ has the following presentation:

Objects: $[n] = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$.

Morphisms: composites of

\[
\delta_i : [n] \rightarrow [n + 1] \text{ for } 0 \leq i \leq n + 1,
\]

\[
\sigma_j : [n - 1] \rightarrow [n] \text{ for } 0 \leq j \leq n,
\]

\[
\tau : [n] \rightarrow [n]
\]

such that the following relations hold:

(i) $\delta_j \delta_i = \delta_i \delta_j - 1$ for $i < j$,

(ii) $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$,

(iii) $\sigma_j \delta_i = \begin{cases} 
\delta_i \sigma_{j-1} & \text{for } i < j \\
\text{id}_{[n]} & \text{for } i = j, j + 1 \\
\delta_{i-1} \sigma_j & \text{for } i > j + 1,
\end{cases}$

(iv) $\tau^{n+1} = \text{id}_{[n]}$,

(v) $\tau \delta_i = \begin{cases} 
\delta_{i-1} \tau & \text{for } 1 \leq i \leq n \\
\delta_0 & \text{for } i = 0,
\end{cases}$

(vi) $\tau \sigma_i = \begin{cases} 
\sigma_{i-1} \tau & \text{for } 1 \leq i \leq n \\
\sigma_0 \tau_2 & \text{for } i = 0.
\end{cases}$

We can take $\delta_i$ to be the strictly increasing injection that skips $i$, $\sigma_j$ the increasing surjection such that $j, j + 1 \mapsto j$, and $\tau$ the cyclic permutation such that $\tau(k) = k + 1 \mod (n + 1)$ for $0 \leq k \leq n$.

**The Simplicial Category.** The category $s\Delta$ is the subcategory of $c\Delta$ generated by the $\delta_i$’s and the $\sigma_j$’s.

**The Semi-Simplicial Category.** The category $ss\Delta$ is the subcategory of $c\Delta$ (or $s\Delta$) generated by the $\delta_i$’s.

II. Cyclic, Simplicial, and Semi-Simplicial Objects

**Definition:** A cyclic (respectively simplicial, semi-simplicial) object in a category $A$ is a functor

$$X_\bullet : c\Delta^{op} \rightarrow A$$

(respectively $s\Delta^{op}$, $ss\Delta^{op}$). Denote the category of cyclic (respectively simplicial, semisimplicial) $A$-objects by $cA$ (respectively $sA$, $ssA$), and note that $cA = \text{Fun}(c\Delta^{op}, A)$, the category of functors from $c\Delta^{op}$ to $A$ (respectively $sA = \text{Fun}(s\Delta^{op}, A)$, $ssA = \text{Fun}(ss\Delta^{op}, A)$).
Usually we write $X_n = X_\bullet([n])$, $d_i = X_\bullet(\delta_i)$, $s_j = X_\bullet(\sigma_j)$, and $t = X_\bullet(\tau)$. Note that

\[
  d_i : X_n \longrightarrow X_{n-1} \quad \text{for} \quad 0 \leq i \leq n, \quad n > 1 \\
  s_j : X_n \longrightarrow X_{n+1} \quad \text{for} \quad 0 \leq j \leq n, \quad \text{and} \\
  t : X_n \longrightarrow X_n.
\]

We have the following relations among these maps:

(i) \( d_i d_j = d_{j-1} d_i \) for \( i < j \),
(ii) \( s_i s_j = s_{j+1} s_i \) for \( i \leq j \),
(iii) \( d_i s_j = \begin{cases} 
  s_{j-1} d_i & \text{if } i < j \\
  \text{id} & \text{if } i = j, j+1 \\
  s_j d_{i-1} & \text{if } i > j + 1 
\end{cases} \)
(iv) \( t_{n+1}^n = \text{id} \),
(v) \( d_i t = \begin{cases} 
  t d_{i-1} & \text{for } 1 \leq i \leq n \\
  d_n & \text{for } i = 0, 
\end{cases} \)
(vi) \( s_i t = \begin{cases} 
  t s_{i-1} & \text{for } 1 \leq i \leq n \\
  t^2 s_n & \text{for } i = 0. 
\end{cases} \)

From this point on, \( k \) will be a commutative ring. Note that if \( C_\bullet : c\Delta^{\text{op}} \to k\text{MOD} \), the category of left \( k \)-modules, we set \( t = (-1)^n C_\bullet (\tau) \) to satisfy Loday’s sign convention, which accounts for the sign of the cyclic permutation in \( S_{n+1} \). Hence (v) and (vi) above are replaced with:

(v) \( d_i t = \begin{cases} 
  -t d_{i-1} & \text{for } 1 \leq i \leq n \\
  (-1)^n d_n & \text{for } i = 0 
\end{cases} \) and
(vi) \( s_i t = \begin{cases} 
  -t s_{i-1} & \text{for } 1 \leq i \leq n \\
  (-1)^n t^2 s_n & \text{for } i = 0. 
\end{cases} \)

III. Various Categories and Functors

Given a category \( A \), we will write \( a \in A \) to mean \( a \in \text{Ob}(A) \) and \( f \in A(a, b) \) if \( f : a \to b \) for \( a, b \in A \).

We already know about the categories \( \text{ss\Delta}, \text{s\Delta}, \) and \( \text{c\Delta}; \text{ssA}, \text{sA}, \) and \( \text{cA} \) for a given category \( A \); and the category \( \text{Fun}(A, B) \) for given categories \( A \) and \( B \).

**Definition:** Categories will be denoted by the sans-serif font: \( \text{ABC} \), etc.

(1) \( \text{Grp} \) is the category of groups, and \( \text{AB} \) is the category of abelian groups, which will sometimes be referred to as \( \mathbb{Z} \)-modules since the term “cyclic abelian group” would be ambiguous.

(2) \( \text{Set} \) is category of sets.

(3) \( \text{Top} \) is the category of topological spaces with continuous maps.

(4) \( k\text{MOD} \) (respectively \( \text{MOD}_k \)) is the category of left (respectively right) \( k \)-modules.

(5) \( \text{CPLX} \) will denote the category of chain complexes of abelian groups, and \( k\text{CPLX} \) (respectively \( k\text{CPLX}_k \)) will denote the category of chain complexes in \( k\text{MOD} \) (respectively \( k\text{MOD}_k \)).

(6) \( k\text{ALG} \) is the category of \( k \)-algebras.

(7) \( \text{Cat} \) is the category of small categories.

Sometimes we add a subscript to the category to denote certain properties.
**Definition:** $U$ will denote the forgetful functor. We have obvious forgetful functors

$$\text{ss}A \xleftarrow{U} \text{s}A \xleftarrow{U} \text{c}A.$$  

In fact, there are left adjoints to these functors.

**Definition:** The functor $F = \text{FREE}: \text{Set} \to \text{AB}$ is given by $X \mapsto \mathbb{Z}(X)$, the free abelian group on the elements of $X$.

$F$ extends to a functor $F: \text{ssSet} \to \text{ssAB}$ by $F(X_n) = \mathbb{Z}(X_n)$ and extending the maps $\mathbb{Z}$-linearly. Similarly, we can look at the functor $F: \text{ssSet} \to \text{ss}_k\text{MOD}$. We can also replace ss with s or c in the above discussion.

$F$ is left adjoint to $U: \text{AB} \to \text{Set}$.

**Definition:** The functor $\text{ALT}: \text{ss}_k\text{MOD} \to \text{iCPLX}$ is given by $C_\bullet \mapsto (C_\bullet, b)$ where $C_n$ is the same as before, and $b$ is the alternating sum of the $d_i$’s:

$$b = \sum_{i=0}^{n} (-1)^{i}d_i.$$  

Once again, we can replace ss with s or c in the above discussion.

**IV. Examples**

(1) **Singular Homology.** Let $X$ be a topological space, and let $\Delta^n = \text{co}\{e_0, \ldots, e_n\} \subset \mathbb{R}^{n+1}$ be the standard $n$-simplex. For $0 \leq i \leq n + 1$, define

$$\delta_i: \Delta^n \to \Delta^{n+1}$$

by

$$\sum_{j=0}^{n} \alpha_j e_j \mapsto \sum_{j=0}^{n} \alpha_j e_{\delta_i(j)}$$

where $\alpha_0, \ldots, \alpha_n \in [0, 1]$ with $\alpha_0 + \cdots + \alpha_n = 1$. Define the semi-simplicial set

$$\Delta_\bullet(X): \text{ss}\Delta \to \text{SET}$$

by $S_n(X) = \{\text{continuous } f: \Delta^n \to X\}$ and $d_i = \Delta_\bullet(\delta_i) = \delta_i^*\bullet$, i.e. $d_i(f) = f \circ \delta_i$:

$$X \xleftarrow{f} \Delta^{n+1} \xleftarrow{\delta_i} \Delta^n.$$  

Applying the functor $F: \text{ssSET} \to \text{ssAB}$, we have a semi-simplicial $\mathbb{Z}$-module $S_\bullet(X) = F(\Delta_\bullet(X))$. Now applying the functor $\text{ALT}: \text{ssAB} \to \text{iCPLX}$ and the functor $H_n: \text{CPLX} \to \text{AB}$ gives the $n$th singular homology of $X$. Hence, singular homology is the composite of the following functors:

$$\text{AB} \xleftarrow{H_n} \text{CPLX} \xleftarrow{\text{ALT}} \text{ssAB} \xleftarrow{F} \text{ssSET} \xleftarrow{\Delta_\bullet} \text{Top}.$$
(2) **Group Homology.** Let $A$ be a small category.

**Definition:** The nerve of $A$ is the simplicial set $N_\bullet(A)$ is given by
\[
N_0(A) = \text{Ob}(A),
\]
\[
N_1(A) = \left\{ b \xrightarrow{f} a \mid a, b \in A \right\},
\]
\[
N_2(A) = \left\{ c \xrightarrow{g} b \xleftarrow{f} a \mid a, b, c \in A \right\},
\]
and so forth. The maps $d_i$ are given by delting the $i^{th}$ object, e.g.
\[
d_0 \left( c \xrightarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g} b,
\]
\[
d_1 \left( c \xrightarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g \circ f} a, \text{ and }
\]
\[
d_2 \left( c \xrightarrow{g} b \xleftarrow{f} a \right) = b \xleftarrow{f} a.
\]
The maps $s_j$ are given by adding the identity morphism for the $j^{th}$ object, e.g.
\[
s_0 \left( c \xleftarrow{g} b \xrightarrow{f} a \right) = c \xleftarrow{g} b \xleftarrow{\text{id}_a} a,
\]
\[
s_1 \left( c \xleftarrow{g} b \xrightarrow{f} a \right) = c \xleftarrow{g} b \xleftarrow{\text{id}_b} b \xrightarrow{f} a, \text{ and }
\]
\[
s_2 \left( c \xleftarrow{g} b \xrightarrow{f} a \right) = c \xleftarrow{\text{id}_c} c \xleftarrow{g} b \xrightarrow{f} a.
\]
If $A$ has only one object, $\text{Ob}(A) = \{\ast\}$, then $N_\bullet(A)$ is the cyclic set where the map $t$ is given by the cyclic permutation, e.g.
\[
t \left( \ast \xrightarrow{h} \ast \xrightarrow{g} \ast \xrightarrow{f} \ast \xrightarrow{h} \ast \right) = \ast \xrightarrow{g} \ast \xrightarrow{f} \ast \xrightarrow{h} \ast.
\]
Suppose now that $A$ is a group, i.e. $\text{Ob}(A) = \{\ast\}$ and all morphisms are invertible. Then applying the functors $U, F, ALT, \text{ and } H_n$, we get group homology. That is, if $G = N_1(A)$, then $H_n(G)$ is given by the following composite of functors:
\[
\begin{align*}
\text{AB} & \xleftarrow{H_n} \text{CPLX} \xleftarrow{ALT} \text{cAB} \xleftarrow{F} \text{cSET} \xleftarrow{N_\ast} \text{Cat}_\ast \xleftarrow{\text{Grp}},
\end{align*}
\]
where $\text{Cat}_\ast$ is the category of all small categories with one object.

(3) **Hochschild Homology.** Let $A$ be a unital $k$-algebra, and let $M$ be an $A - A$-bimodule. Define the simplicial module $C_\bullet(A, M)$ by $C_n(A, M) = M \otimes A^\otimes n$ where $\otimes$ means $\otimes_k$. Define the $d_i$’s by
\[
d_0 (a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_n,
\]
\[
d_i (a_0 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \text{ for } 0 < i < n, \text{ and }
\]
\[
d_n (a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n) = a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}
\]
for $a_0 \in M$ and $a_1, \ldots, a_n \in A$. Define the $s_i$’s by
\[
s_i (a_0 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.
\]
If \( M = A \), then \( C_\bullet(A, A) \) is a cyclic module where \( t \) is given by
\[
 t (a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.
\]
Taking \( ALT \) and \( H_n \) gives the \( n \)th Hochschild homology \( H_n(A, M) \), so Hochschild homology is a composite of the following functors:
\[
 AB \xleftarrow{H_n} k\text{CPLX} \xrightarrow{ALT} s_k\text{MOD} \xrightarrow{C\bullet(-,M)} k\text{ALG}_{\text{unital}}
\]
If \( M = A \), we write \( HH_n(A) = H_n(A, A) \), and the above sequence is altered by replacing \( s_k\text{MOD} \) with \( c_k\text{MOD} \).

If \( G: k\text{ALG}_{\text{unital}} \rightarrow A \) is a functor where \( A \) is an abelian category, we can extend \( G \) to a functor \( \tilde{G}: k\text{ALG} \rightarrow A \). For \( I \in k\text{ALG} \), we can form its unitalization \( \tilde{I} = k \oplus I \), and define \( \tilde{G}(I) = \text{coker}(G(i)) \), where \( i: k \rightarrow \tilde{I} \) in the natural inclusion:
\[
 0 \leftarrow I \xleftarrow{p} \tilde{I} \xleftarrow{i} k \leftarrow 0.
\]
Hence, Hochschild homology extends to a functor \( k\text{ALG} \rightarrow AB \).

V. Cyclic Homology

Given a cyclic module \( C_\bullet \), there is a classical complex arising from calculations in group homology:
\[
 \cdots \xleftarrow{1-t} C_n \xleftarrow{N} C_n \xleftarrow{1-t} C_n \xleftarrow{N} \cdots
\]
where \( N = 1 + t + t^2 + \cdots + t^n \) is the “norm map.” A group action on a space \( X \) is a continuous map
\[
 G \times X \rightarrow X.
\]
We can think of this classical complex “acting” on \( (C_\bullet,b) = ALT(C_\bullet) \) to get the cyclic bicomplex \( CC_{**} \):

\[
 \begin{array}{cccccccc}
 b & b & b & b & b & b & b & b \\
 C_2 & C_2 & C_2 & C_2 & C_2 & C_2 & C_2 & C_2 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 C_1 & C_1 & C_1 & C_1 & C_1 & C_1 & C_1 & C_1 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 C_0 & C_0 & C_0 & C_0 & C_0 & C_0 & C_0 & C_0 \\
 \end{array}
\]
where \( b' = d_0 - d_1 + \cdots + (-1)^{n-1} d_{n-1} \). The relations of a cyclic module imply \( b(1-t) = (1-t)b' \) and \( b'N = Nb \), so the bicomplex above is well defined.

**Definition:** Then \( n \)th cyclic homology of a cyclic module \( C_\bullet \) is the \( n \)th total homology of the associated cyclic bicomplex \( CC_{**} \):
\[
 HC_n(C_\bullet) = H_n(\text{Tot}_*(CC_{**})).
\]
Cyclic homology is thus the composite of the following functors:
\[
 AB \xleftarrow{H_n} k\text{BCPLX} \xrightarrow{c_k\text{MOD}}
\]
We have an additional method of taking homology of a cyclic module by the functors described before, which will be denoted by $HH_n$:  

$$HH_n(C_\bullet) = H_n(ALT(C_\bullet)) = H_n(C_\bullet, b).$$

From this bicomplex construction, we can reconstruct Connes' initial construction of cyclic homology and many of the tools he developed for calculation. First, note that the complex $(C_\bullet, -b')$ is contractible via contraction $-s = -(1)^n t_s n$. We will need the following lemma:

**Killing Contractible Complexes Lemma:** Let $(A_\bullet \oplus A'_\bullet, d) \in k\text{CPLX}$

$$
\cdots \longleftrightarrow A_{n-1} \oplus A'_{n-1} \overset{d=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}{\longrightarrow} A_n \oplus A'_n \longleftrightarrow \cdots
$$

such that $(A'_\bullet, \delta) \in k\text{CPLX}$ is contractible via contraction $h$. Then $(A_\bullet, \alpha - \beta h \gamma) \in k\text{CPLX}$, and the inclusion

$$(\text{id}, -h \gamma): (A_\bullet, \alpha - \beta h \gamma) \longrightarrow (A_\bullet \oplus A'_\bullet, d)$$

is a quasi-isomorphism.

(1) **Connes' Periodicity Exact Sequence.** Let $CC_\bullet$ be the cyclic bicomplex associated to the cyclic module $C_\bullet$. Let $CC^{(2)}_\bullet$ be the bicomplex obtained from $CC_\bullet$ by looking at only the first two columns, and let $CC^{[2]}_\bullet$ be the bicomplex obtained from $CC_\bullet$ by shifting the bicomplex two columns to the right. We then get an exact sequence of bicomplexes

$$0 \longrightarrow CC^{[2]}_\bullet \longleftrightarrow CC_\bullet \longleftrightarrow CC^{(2)}_\bullet \longleftrightarrow 0$$

which yields an exact sequence in total complexes, which in turn, yields a long exact sequence in homology via the snake lemma:

$$
\cdots \longrightarrow HH_{n-1}(C_\bullet) \longleftrightarrow HC_{n-2}(C_\bullet) \longleftrightarrow HC_n(C_\bullet) \longleftrightarrow HH_n(C_\bullet) \longleftrightarrow HC_{n-1}(C_\bullet) \longleftrightarrow \cdots.
$$

Note that

$$H_n(\text{Tot}_s(CC^{(2)}_\bullet)) \cong HH_n(C_\bullet) = H_n(C_\bullet, b)$$

by the lemma, and clearly

$$H_n(\text{Tot}_s(CC^{[2]}_\bullet)) = HC_{n-2}(C_\bullet).$$

(2) **Connes' Bicomplex.** By applying the lemma to the $-b'$ columns of the cyclic bicomplex which are contractible via $-s$, we get Connes' bicomplex $BC_\bullet$ where $B = (1-t)sN$:

```
\begin{array}{c}
\downarrow b \\
C_2 \leftarrow B \leftarrow C_1 \leftarrow C_0 \\
\downarrow b \\
C_1 \leftarrow B \leftarrow C_0 \\
\downarrow b \\
\cdots \leftarrow C_0
\end{array}
```
Connes' initial definition of cyclic homology was given as the homology of a complex instead of a total complex of a bicomplex. Since $b(1 - t) = (1 - t)b'$, we have that $b$ descends to a map

$$\cdots \xrightarrow{b} C_{n+1}^\lambda \xrightarrow{b} C_n^\lambda \xrightarrow{b} C_{n-1}^\lambda \xrightarrow{\cdots}$$

where $C_n^\lambda = C_n / \text{im}(1 - t)$. Define $H_n^\lambda(C_c) = H_n(C_c, \tilde{b})$.

**Theorem:** If $Q \subseteq k$, there is a canonical isomorphism $H_n^\lambda(C_c) \cong HC_n(C_c)$.

**Proof.** Let

$$h' = \frac{1}{n+1} \text{id}, \quad h = -\frac{1}{n+1} \sum_{i=1}^{n} it^i$$

be maps $C_n \to C_n$:

$$C_n \xleftarrow{\text{(1-t)}} C_n \xrightarrow{h'} N C_n \xleftarrow{\text{(1-t)}} C_n \xrightarrow{h} N C_n \xleftarrow{\text{\ldots}}$$

From the relations of a cyclic module, $h'N + (1-t)h = \text{id}$ and $Nh' + h(1-t) = \text{id}$. Hence, the rows of $CC^\times$ are acyclic augmented complexes with $H_0 = C_n^\lambda$, and there is a first quadrant spectral sequence $\II E_{p,q}^0 \Rightarrow HC_{p+q}(C_c)$ with

- $\II E_{p,q}^0 = CC_{p,q}$ and $d^0 = dh$,
- $\II E_{p,q}^1 = H(CC_{p,q}, d^h) = \delta_{p,0} C_q^\lambda$ and $d^1 = [d''], $ and
- $\II E_{p,q}^2 = \II E_{p,q}^\infty = H(\delta_{p,0} C_q^\lambda, [d'']) = \delta_{p,0} H_q(C_c^\lambda, \tilde{b})$ and $d^2 = 0$.

\[ \square \]

**VI. Elementary Computations**

(1) **Hochschild Homology.** Let $A \in k\text{ALG}_{\text{unital}}$ and $M \in A\text{MOD}_A$. Clearly

$$H_0(A, M) = M_A = M/(am - ma),$$

the module of coinvariants of $M$ by $A$. If $M = A$, then $HH_0(A) = A/[A, A]$. If $A$ is commutative, $HH_0(A) = A$. If $A = k$, we have the cyclic module $C_c(k)$ given by $C_n(k) = k^{\otimes (n+1)} \cong k$, and each $d_i$ is the identity. Hence, the Hochschild complex $C_c(k)$ is given by

$$k \xleftarrow{0} k \xrightarrow{1} k \xleftarrow{0} k \xrightarrow{1} \cdots$$

and we have $HH_0(k) = k$ and $HH_n(k) = 0$ for all $n > 0$.  

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We have $C_n(A) = A^\otimes n+1$ for $A \in k\text{ALG}_{\text{unital}}$, and Connes’ boundary map $B: A^\otimes n+1 \to A^\otimes n+2$ is given explicitly by

$$B(a_0 \otimes \cdots \otimes a_n) = (1 - t)(-1)^n ts_n \sum_{i=0}^{n} t^i(a_0 \otimes \cdots \otimes a_n)$$

$$= (1 - t)(-1)^n ts_n \sum_{i=0}^{n} (-1)^i(a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1})$$

$$= (1 - t)(-1)^n t \sum_{i=0}^{n} (-1)^i(a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes 1)$$

$$= (1 - t)(-1)^n(-1)^{n+1} \sum_{i=0}^{n} (-1)^i(1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1})$$

$$= \sum_{i=0}^{n} (-1)^{ni+1}(1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1})$$

$$+ (-1)^{n(i+1)+1}(a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}).$$

It is clear that $HC_0(A) = HH_0(A)$. If $A = k$, Connes’ bicomplex is given by

$$\begin{array}{c c c c}
1 & 0 & 1 & 0 \\
& 6 & 0 & 1 & -2 & k \\
& & 0 & k & -2 & k \\
& & & 1 & 0 \\
& & & 0 & k \\
& & & 0 & k \\
& & & & & k
\end{array}$$

Note that $B: C_{2n} \to C_{2n+1}$ is multiplication by $-2(2n + 1)$ and $B: C_{2n+1} \to C_{2n+2}$ is the zero map. Letting $d^{\text{odd}}$ denote $d: \text{Tot}_{2n+1}(BC_{**}) \to \text{Tot}_{2n}(BC_{**})$ and $d^{\text{even}}$ denote $d: \text{Tot}_{2n}(BC_{**}) \to \text{Tot}_{2n-1}(BC_{**})$, it is clear that $d^{\text{odd}} = 0$ and $d^{\text{even}}$ is onto with kernel isomorphic to $k$. Hence

$$HC_n(k) \cong \begin{cases} k & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

**Theorem:** Hochschild and cyclic homology are Morita invariant, i.e. if $A$ and $B$ are Morita equivalent $k$-algebras, then $HH_n(A) \cong HH_n(B)$ and $HC_n(A) \cong HC_n(B)$.

**Corollary:** $HH_0(M_n(k)) = k$ and $HH_n(k) = 0$ for all $n > 0$, and

$$HC_n(M_n(k)) = \begin{cases} k & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$