Cyclic Homology Handout — Dave Penneys
Quantum Geometry Seminar — 4/28/08

Most of the following information comes from Loday's *Cyclic Homology*.

I. The Cyclic, Simplicial, and Semi-Simplicial Categories

The Cyclic Category. The category $\mathsf{c}\Delta$ has the following presentation:

Objects: $[n] = \{0 < 1 < \cdots < n\}$ for $n \in \mathbb{N}$.
Morphisms: composites of

$$\begin{aligned} \delta_i &\colon [n] \longrightarrow [n+1] \text{ for } 0 \leq i \leq n+1, \\ \sigma_j &\colon [n-1] \longrightarrow [n] \text{ for } 0 \leq j \leq n, \text{ and} \\ \tau &\colon [n] \longrightarrow [n] \end{aligned}$$

such that the following relations hold:

(i) $\delta_j \delta_i = \delta_i \delta_{j-1}$ for $i < j$,

(ii) $\sigma_j \sigma_i = \sigma_i \sigma_{j+1}$ for $i \leq j$,

(iii) $\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{for } i < j \\ \mathrm{id}_{[n]} & \text{for } i = j, j+1 \\ \delta_{i-1}\sigma_j & \text{for } i > j+1, \end{cases}$

(iv) $\tau^{n+1} = \mathrm{id}_{[n]}$,

(v) $\tau \delta_i = \begin{cases} \delta_{i-1}\tau & \text{for } 1 \leq i \leq n \\ \delta_n & \text{for } i = 0, \end{cases}$

(vi) $\tau \sigma_i = \begin{cases} \sigma_{i-1}\tau & \text{for } 1 \leq i \leq n \\ \sigma_n \tau^2 & \text{for } i = 0. \end{cases}$

We can take δ_i to be the strictly increasing injection that skips i, σ_j the increasing surjection such that $j, j+1 \mapsto j$, and τ the cyclic permutation such that $\tau(k) = k+1 \mod (n+1)$ for $0 \leq k \leq n$.

The Simplicial Category. The category $\mathsf{s}\Delta$ is the subcategory of $\mathsf{c}\Delta$ generated by the δ_i's and the σ_j's.

The Semi-Simplicial Category. The category $\mathsf{ss}\Delta$ is the subcategory of $\mathsf{c}\Delta$ (or $\mathsf{s}\Delta$) generated by the δ_i's.

II. Cyclic, Simplicial, and Semi-Simplicial Objects

Definition: A cyclic (respectively simplicial, semi-simplicial) object in a category A is a functor

$$X_\bullet \colon \mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{A}$$

(respectively $\mathsf{s}\Delta^{\mathsf{op}}$, $\mathsf{ss}\Delta^{\mathsf{op}}$). Denote the category of cyclic (respectively simplicial, semisimplicial) A-objects by cA (respectively sA, ssA), and note that $\mathsf{cA} = \mathsf{Fun}(\mathsf{c}\Delta^{\mathsf{op}}, \mathsf{A})$, the category of functors from $\mathsf{c}\Delta^{\mathsf{op}}$ to A (respectively $\mathsf{sA} = \mathsf{Fun}(\mathsf{s}\Delta^{\mathsf{op}}, \mathsf{A})$, $\mathsf{ssA} = \mathsf{Fun}(\mathsf{ss}\Delta^{\mathsf{op}}, \mathsf{A})$).

Usually we write $X_n = X_\bullet([n])$, $d_i = X_\bullet(\delta_i)$, $s_j = X_\bullet(\sigma_j)$, and $t = X_\bullet(\tau)$. Note that

$$\begin{aligned} d_i &\colon X_n \longrightarrow X_{n-1} \text{ for } 0 \le i \le n,\ \ n > 1 \\ s_j &\colon X_n \longrightarrow X_{n+1} \text{ for } 0 \le j \le n, \text{ and} \\ t &\colon X_n \longrightarrow X_n. \end{aligned}$$

We have the following relations among these maps:

(i) $d_i d_j = d_{j-1} d_i$ for $i < j$,

(ii) $s_i s_j = s_{j+1} s_i$ for $i \le j$,

(iii) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id} & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$

(iv) $t_n^{n+1} = \mathrm{id}$,

(v) $d_i t = \begin{cases} t d_{i-1} & \text{for } 1 \le i \le n \\ d_n & \text{for } i = 0, \end{cases}$

(vi) $s_i t = \begin{cases} t s_{i-1} & \text{for } 1 \le i \le n \\ t^2 s_n & \text{for } i = 0. \end{cases}$

From this point on, k will be a commutative ring. Note that if $C_\bullet \colon \mathsf{c}\Delta^{\mathsf{op}} \to {}_k\mathsf{MOD}$, the category of left k-modules, we set $t = (-1)^n C_\bullet(\tau)$ to satisfy Loday's sign convention, which accounts for the sign of the cyclic permutation in S_{n+1}. Hence (v) and (vi) above are replaced with:

(v) $d_i t = \begin{cases} -t d_{i-1} & \text{for } 1 \le i \le n \\ (-1)^n d_n & \text{for } i = 0 \end{cases}$ and

(vi) $s_i t = \begin{cases} -t s_{i-1} & \text{for } 1 \le i \le n \\ (-1)^n t^2 s_n & \text{for } i = 0. \end{cases}$

III. Various Categories and Functors

Given a category A, we will write $a \in \mathsf{A}$ to mean $a \in \mathrm{Ob}(\mathsf{A})$ and $f \in \mathsf{A}(a, b)$ if $f \colon a \to b$ for $a, b \in \mathsf{A}$.

We already know about the categories $\mathsf{ss}\Delta$, $\mathsf{s}\Delta$, and $\mathsf{c}\Delta$; ssA, sA, and cA for a given category A; and the category $\mathsf{Fun}(\mathsf{A}, \mathsf{B})$ for given categories A and B.

Definition: Categories will be denoted by the sans-serif font: ABC, etc.

(1) Grp is the category of groups, and AB is the category of abelian groups, which will sometimes be referred to as $\mathbb{Z}$-modules since the term "cyclic abelian group" would be ambiguous.
(2) Set is category of sets.
(3) Top is the category of topological spaces with continuous maps.
(4) ${}_k\mathsf{MOD}$ (respectively MOD_k) is the category of left (respectively right) k-modules.
(5) CPLX will denote the category of chain complexes of abelian groups, and ${}_k\mathsf{CPLX}$ (respectively CPLX_k) will denote the category of chain complexes in ${}_k\mathsf{MOD}$ (respectively MOD_k).
(6) ${}_k\mathsf{ALG}$ is the category of k-algebras.
(7) Cat is the category of small categories.

Sometimes we add a subscript to the category to denote certain properties.

Definition: U will denote the forgetful functor. We have obvious forgetful functors

$$\mathsf{ssA} \xleftarrow{U} \mathsf{sA} \xleftarrow{U} \mathsf{cA}.$$

In fact, there are left adjoints to these functors.

Definition: The functor $F = FREE\colon \mathsf{Set} \to \mathsf{AB}$ is given by $X \mapsto \mathbb{Z}\langle X\rangle$, the free abelian group on the elements of X.

F extends to a functor $F\colon \mathsf{ssSet} \to \mathsf{ssAB}$ by $F(X_n) = \mathbb{Z}\langle X_n\rangle$ and extending the maps $\mathbb{Z}$-linearly. Similarly, we can look at the functor $F\colon \mathsf{ssSet} \to \mathsf{ss}_k\mathsf{MOD}$. We can also replace ss with s or c in the above discussion.

F is left adjoint to $U\colon \mathsf{AB} \to \mathsf{Set}$.

Definition: The functor $ALT\colon \mathsf{ss}_k\mathsf{MOD} \to {}_k\mathsf{CPLX}$ is given by $C_\bullet \mapsto (C_*, b)$ where C_n is the same as before, and b is the alternating sum of the d_i's:

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

Once again, we can replace ss with s or c in the above discussion.

IV. Examples

(1) **Singular Homology.** Let X be a topological space, and let $\Delta^n = \mathrm{co}\{e_0, \ldots, e_n\} \subset \mathbb{R}^{n+1}$ be the standard n-simplex. For $0 \le i \le n+1$, define

$$\delta_i\colon \Delta^n \longrightarrow \Delta^{n+1} \text{ by}$$

$$\sum_{j=0}^{n} \alpha_j e_j \longmapsto \sum_{j=0}^{n} \alpha_j e_{\delta_i(j)}$$

where $\alpha_0, \ldots, \alpha_n \in [0,1]$ with $\alpha_0 + \cdots + \alpha_n = 1$. Define the semi-simplicial set

$$\Delta_\bullet(X)\colon \mathsf{ss}\Delta \to \mathsf{SET}$$

by $S_n(X) = \{\text{continuous } f\colon \Delta^n \to X\}$ and $d_i = \Delta_\bullet(\delta_i) = \delta_i^*$, i.e. $d_i(f) = f \circ \delta_i$:

$$X \xleftarrow{f} \Delta^{n+1} \xleftarrow{\delta_i} \Delta^n.$$

Applying the functor $F\colon \mathsf{ssSET} \to \mathsf{ssAB}$, we have a semi-simplicial $\mathbb{Z}$-module $S_\bullet(X) = F(\Delta_\bullet(X))$. Now applying the functor $ALT\colon \mathsf{ssAB} \to \mathsf{CPLX}$ and the functor $H_n\colon \mathsf{CPLX} \to \mathsf{AB}$ gives the n^{th} singular homology of X. Hence, singular homology is the composite of the following functors:

$$\mathsf{AB} \xleftarrow{H_n} \mathsf{CPLX} \xleftarrow{ALT} \mathsf{ssAB} \xleftarrow{F} \mathsf{ssSET} \xleftarrow{\Delta_\bullet} \mathsf{Top}.$$

(2) **Group Homology.** Let A be a small category.

Definition: The nerve of A is the simplicial set $N_\bullet(\mathsf{A})$ is given by

$$N_0(\mathsf{A}) = \mathrm{Ob}(\mathsf{A}),$$
$$N_1(\mathsf{A}) = \left\{ b \xleftarrow{f} a \,\middle|\, a, b \in \mathsf{A} \right\},$$
$$N_2(\mathsf{A}) = \left\{ c \xleftarrow{g} b \xleftarrow{f} a \,\middle|\, a, b, c \in \mathsf{A} \right\},$$

and so forth. The maps d_i are given by delting the i^{th} object, e.g.

$$d_0\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g} b\ ,$$
$$d_1\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g\circ f} a\ , \text{ and}$$
$$d_2\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = b \xleftarrow{f} a\ .$$

The maps s_j are given by adding the identity morphism for the j^{th} object, e.g.

$$s_0\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g} b \xleftarrow{f} a \xleftarrow{\mathrm{id}_a} a\ ,$$
$$s_1\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{g} b \xleftarrow{\mathrm{id}_b} b \xleftarrow{f} a\ , \text{ and}$$
$$s_2\left(c \xleftarrow{g} b \xleftarrow{f} a \right) = c \xleftarrow{\mathrm{id}_c} c \xleftarrow{g} b \xleftarrow{f} a\ .$$

If A has only one object, $\mathrm{Ob}(\mathsf{A}) = \{*\}$, then $N_\bullet(\mathsf{A})$ is the cyclic set where the map t is given by the cyclic permutation, e.g.

$$t\left(* \xleftarrow{h} * \xleftarrow{g} * \xleftarrow{f} * \right) = * \xleftarrow{g} * \xleftarrow{f} * \xleftarrow{h} *\ .$$

Suppose now that A is a group, i.e. $\mathrm{Ob}(\mathsf{A}) = \{*\}$ and all morphisms are invertible. Then applying the functors U, F, ALT, and H_n, we get group homology. That is, if $G = N_1(\mathsf{A})$, then $H_n(G)$ is given by the following composite of functors:

$$\mathsf{AB} \xleftarrow{H_n} \mathsf{CPLX} \xleftarrow{ALT} \mathsf{cAB} \xleftarrow{F} \mathsf{cSET} \xleftarrow{N_\bullet} \mathsf{Cat}_* \longleftarrow \mathsf{Grp},$$

where Cat_* is the category of all small categories with one object.

(3) **Hochschild Homology.** Let A be a unital k-algebra, and let M be an $A-A$-bimodule. Define the simplicial module $C_\bullet(A, M)$ by $C_n(A, M) = M \otimes A^{\otimes n}$ where $\otimes$ means $\otimes_k$. Define the d_i's by

$$d_0\left(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n\right) = a_0a_1 \otimes a_2 \otimes \cdots \otimes a_n,$$
$$d_i\left(a_0 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n\right) = a_0 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n \text{ for } 0 < i < n, \text{ and}$$
$$d_n\left(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n\right) = a_na_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

for $a_0 \in M$ and $a_1, \ldots, a_n \in A$. Define the s_i's by

$$s_i\left(a_0 \otimes \cdots \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n\right) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n$$

(where $1 \in A$). If $M = A$, then $C_\bullet(A, A)$ is a cyclic module where t is given by

$$t\,(a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

Taking ALT and H_n gives the n^{th} Hochschild homology $H_n(A, M)$, so Hochschild homology is a composite of the following functors:

$$\mathsf{AB} \xleftarrow{H_n} {}_k\mathsf{CPLX} \xleftarrow{ALT} \mathsf{s}_k\mathsf{MOD} \xleftarrow{C_\bullet(-,M)} {}_k\mathsf{ALG}_{\mathsf{unital}}$$

If $M = A$, we write $HH_n(A) = H_n(A, A)$, and the above sequence is altered by replacing $\mathsf{s_k MOD}$ with $\mathsf{c_k MOD}$.

If $G \colon {}_k\mathsf{ALG}_{\mathsf{unital}} \to \mathsf{A}$ is a functor where A is an abelian category, we can extend G to a functor $\widetilde{G} \colon {}_k\mathsf{ALG} \to \mathsf{A}$. For $I \in {}_k\mathsf{ALG}$, we can form its unitalization $\widetilde{I} = k \oplus I$, and define $\widetilde{G}(I) = \operatorname{coker}(G(i))$, where $i \colon k \to \widetilde{I}$ in the natural inclusion:

$$0 \longleftarrow I \xleftarrow{p} \widetilde{I} \xleftarrow{i} k \longleftarrow 0.$$

Hence, Hochschild homology extends to a functor ${}_k\mathsf{ALG} \to \mathsf{AB}$.

V. Cyclic Homology

Given a cyclic module $C_\bullet$, there is a classical complex arising from calculations in group homology:

$$\cdots \xleftarrow{1-t} C_n \xleftarrow{N} C_n \xleftarrow{1-t} C_n \xleftarrow{N} \cdots$$

where $N = 1 + t + t^2 + \cdots + t^n$ is the "norm map." A group action on a space X is a continuous map

$$G \times X \longrightarrow X.$$

We can think of this classical complex "acting" on $(C_*, b) = ALT(C_\bullet)$ to get the cyclic bicomplex CC_{**}:

$$\begin{array}{ccccccccc}
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_2 & \xleftarrow{1-t} & C_2 & \xleftarrow{N} & C_2 & \xleftarrow{1-t} & C_2 & \xleftarrow{N} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & C_1 & \xleftarrow{1-t} & C_1 & \xleftarrow{N} & \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} & C_0 & \xleftarrow{1-t} & C_0 & \xleftarrow{N} &
\end{array}$$

where $b' = d_0 - d_1 + \cdots + (-1)^{n-1} d_{n-1}$. The relations of a cyclic module imply $b(1-t) = (1-t)b'$ and $b'N = Nb$, so the bicomplex above is well defined.

Definition: Then n^{th} cyclic homology of a cyclic module $C_\bullet$ is the n^{th} total homology of the associated cyclic bicomplex CC_{**}:

$$HC_n(C_\bullet) = H_n(\operatorname{Tot}_*(CC_{**})).$$

Cyclic homology is thus the composite of the following functors:

$$\mathsf{AB} \xleftarrow{H_n} {}_k\mathsf{BCPLX} \longleftarrow \mathsf{c}_k\mathsf{MOD}$$

We have an additional method of taking homology of a cyclic module by the functors described before, which will be denoted by HH_n:

$$HH_n(C_\bullet) = H_n(ALT(C_\bullet)) = H_n(C_*, b).$$

From this bicomplex construction, we can reconstruct Connes' initial construction of cyclic homology and many of the tools he developed for calculation. First, note that the complex $(C_*, -b')$ is contractible via contraction $-s = -(-1)^n t s_n$. We will need the following lemma:

Killing Contractible Complexes Lemma: Let $(A_* \oplus A'_*, d) \in {}_k\mathsf{CPLX}$

$$\cdots \longleftarrow A_{n-1} \oplus A'_{n-1} \xleftarrow{\; d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \;} A_n \oplus A'_n \longleftarrow \cdots$$

such that $(A'_*, \delta) \in {}_k\mathsf{CPLX}$ is contractible via contraction h. Then $(A_*, \alpha - \beta h \gamma) \in {}_k\mathsf{CPLX}$, and the inclusion

$$(\mathrm{id}, -h\gamma) \colon (A_*, \alpha - \beta h\gamma) \longrightarrow (A_* \oplus A'_*, d)$$

is a quasi-isomorphism.

(1) **Connes' Periodicity Exact Sequence.** Let CC_{**} be the cyclic bicomplex associated to the cyclic module $C_\bullet$. Let $CC^{(2)}_{**}$ be the bicomplex obtained from CC_{**} by looking at only the first two columns, and let $CC[2]_{**}$ be the bicomplex obtained from CC_{**} by shifting the bicomplex two columns to the right. We then get an exact sequence of bicomplexes

$$0 \longleftarrow CC[2]_{**} \longleftarrow CC_{**} \longleftarrow CC^{(2)}_{**} \longleftarrow 0$$

which yields an exact sequence in total complexes, which in turn, yields a long exact sequence in homology via the snake lemma:

$$\cdots \longleftarrow HH_{n-1}(C_\bullet) \longleftarrow HC_{n-2}(C_\bullet) \longleftarrow HC_n(C_\bullet) \longleftarrow HH_n(C_\bullet) \longleftarrow HC_{n-1}(C_\bullet) \longleftarrow \cdots.$$

Note that

$$H_n(\mathrm{Tot}_*(CC^{(2)}_{**})) \cong HH_n(C_\bullet) = H_n(C_*, b)$$

by the lemma, and clearly

$$H_n(\mathrm{Tot}_*(CC[2]_{**})) = HC_{n-2}(C_\bullet).$$

(2) **Connes' Bicomplex.** By applying the lemma to the $-b'$ columns of the cyclic bicomplex which are contractible via $-s$, we get Connes' bicomplex BC_{**} where $B = (1-t)sN$:

$$\begin{array}{ccccc}
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} \\
C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\
C_1 & \xleftarrow{B} & C_0 & & \\
\downarrow{\scriptstyle b} & & & & \\
C_0 & & & &
\end{array}$$

(3) **Connes' Complex.** Connes' initial definition of cyclic homology was given as the homology of a complex instead of a total complex of a bicomplex. Since $b(1-t) = (1-t)b'$, we have that b descends to a map

$$\cdots \xleftarrow{\widetilde{b}} C^{\lambda}_{n-1} \xleftarrow{\widetilde{b}} C^{\lambda}_{n} \xleftarrow{\widetilde{b}} C^{\lambda}_{n+1} \xleftarrow{\widetilde{b}} \cdots$$

where $C^{\lambda}_n = C_n / \operatorname{im}(1-t)$. Define $H^{\lambda}_n(C_\bullet) = H_n(C^{\lambda}_*, \widetilde{b})$.

Theorem: If $\mathbb{Q} \subseteq k$, there is a canonical isomorphism $H^{\lambda}_n(C_\bullet) \cong HC_n(C_\bullet)$.

Proof. Let

$$h' = \frac{1}{n+1}\,\mathrm{id}, \quad h = \frac{-1}{n+1}\sum_{i=1}^{n} i t^i$$

be maps $C_n \to C_n$:

$$\begin{array}{ccccccccc}
C_n & \xleftarrow{(1-t)} & C_n & \xleftarrow{N} & C_n & \xleftarrow{(1-t)} & C_n & \xleftarrow{N} & \cdots \\
& \searrow^{h} & \downarrow \mathrm{id} & \searrow^{h'} & \downarrow \mathrm{id} & \searrow^{h} & \downarrow \mathrm{id} & \searrow^{h'} & \\
C_n & \xleftarrow{(1-t)} & C_n & \xleftarrow{N} & C_n & \xleftarrow{(1-t)} & C_n & \xleftarrow{N} & \cdots
\end{array}$$

From the relations of a cyclic module, $h'N + (1-t)h = \mathrm{id}$ and $Nh' + h(1-t) = \mathrm{id}$. Hence, the rows of CC_{**} are acyclic augmented complexes with $H_0 = C^{\lambda}_n$, and there is a first quadrant spectral sequence ${}^{II}E^0_{p,q} \Rightarrow HC_{p+q}(C_\bullet)$ with

$$\begin{aligned}
&{}^{II}E^0_{p,q} = CC_{p,q} \ \text{ and } \ d^0 = d_h, \\
&{}^{II}E^1_{p,q} = H(CC_{p,q}, d^h) = \delta_{p,0} C^{\lambda}_q \ \text{ and } \ d^1 = [d^v], \ \text{ and} \\
&{}^{II}E^2_{p,q} = {}^{II}E^{\infty}_{p,q} = H(\delta_{p,0} C^{\lambda}_q, [d^v]) = \delta_{p,0} H_q(C^{\lambda}_*, \widetilde{b}) \ \text{ and } \ d^2 = 0.
\end{aligned}$$

□

VI. Elementary Computations

(1) **Hochschild Homology.** Let $A \in {}_k\mathsf{ALG}_{\mathsf{unital}}$ and $M \in {}_A\mathsf{MOD}_A$. Clearly

$$H_0(A, M) = M_A = M / \langle am - ma \rangle,$$

the module of coinvariants of M by A. If $M = A$, then $HH_0(A) = A/[A,A]$. If A is commutative, $HH_0(A) = A$. If $A = k$, we have the cylic module $C_\bullet(k)$ given by $C_n(k) = k^{\otimes(n+1)} \cong k$, and each d_i is the identity. Hence, the Hochschild complex $C_*(k)$ is given by

$$k \xleftarrow{0} k \xleftarrow{1} k \xleftarrow{0} k \xleftarrow{1} k \longleftarrow \cdots$$

and we have $HH_0(k) = k$ and $HH_n(k) = 0$ for all $n > 0$.

(2) **Cyclic Homology.** We have $C_n(A) = A^{\otimes n+1}$ for $A \in {}_k\mathsf{ALG}_{\mathsf{unital}}$, and Connes' boundary map $B\colon A^{\otimes n+1} \longrightarrow A^{\otimes n+2}$ is given explicitly by

$$\begin{aligned}
B(a_0 \otimes \cdots \otimes a_n) &= (1-t)(-1)^n t s_n \sum_{i=0}^{n} t^i (a_0 \otimes \cdots \otimes a_n) \\
&= (1-t)(-1)^n t s_n \sum_{i=0}^{n} (-1)^{ni} (a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}) \\
&= (1-t)(-1)^n t \sum_{i=0}^{n} (-1)^{ni} (a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1} \otimes 1) \\
&= (1-t)(-1)^n (-1)^{n+1} \sum_{i=0}^{n} (-1)^{ni} (1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}) \\
&= \sum_{i=0}^{n} (-1)^{ni+1} (1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}) \\
&\qquad + (-1)^{n(i+1)+1} (a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-2}).
\end{aligned}$$

It is clear that $HC_0(A) = HH_0(A)$. If $A = k$, Connes' bicomplex is given by

$$\begin{array}{ccccccc}
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} \\
k & \xleftarrow{-6} & k & \xleftarrow{0} & k & \xleftarrow{-2} & k \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & \\
k & \xleftarrow{0} & k & \xleftarrow{-2} & k & & \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} & & & & \\
k & \xleftarrow{-2} & k & & & & \\
\downarrow{\scriptstyle 0} & & & & & & \\
k & & & & & &
\end{array}$$

Note that $B\colon C_{2n} \to C_{2n+1}$ is multiplication by $-2(2n+1)$ and $B\colon C_{2n+1} \to C_{2n+2}$ is the zero map. Letting d^{odd} denote $d\colon \text{Tot}_{2n+1}(BC_{**}) \longrightarrow \text{Tot}_{2n}(BC_{**})$ and d^{even} denote $d\colon \text{Tot}_{2n}(BC_{**}) \longrightarrow \text{Tot}_{2n-1}(BC_{**})$, it is clear that $d^{\text{odd}} = 0$ and d^{even} is onto with kernel isomorphic to k. Hence

$$HC_n(k) \cong \begin{cases} k & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Theorem: Hochschild and cyclic homology are Morita invariant, i.e. if A and B are Morita equivalent k-algberas, then $HH_n(A) \cong HH_n(B)$ and $HC_n(A) \cong HC_n(B)$.

Corollary: $HH_0(M_n(k)) = k$ and $HH_n(k) = 0$ for all $n > 0$, and

$$HC_n(M_n(k)) = \begin{cases} k & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$