Generalized gauge actions, KMS states, and Hausdorff dimension for higher-rank graphs

Elizabeth Gillaspy
University of Montana

joint with C. Farsi, S. Kang, N. Larsen, J. Packer

OHIO Workshop, 28 April 2018

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry;

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e:r(e)=v} t_e t_e^*$.

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \qquad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e:r(e)=v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \qquad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e:r(e)=v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

$C^*(E)$ is <u>universal</u> for representations of $\{t_e, t_v\}_{v \in E^0, e \in E^1}$

Graph C^*-algebras

From a directed graph $E = (E^0, E^1, r, s)$, get a C^*-algebra $C^*(E)$:

$$C^*(E) = \langle \{t_e, t_v : v \in E^0, e \in E^1\} \rangle, \quad \text{where}$$

- t_e is a partial isometry; t_v is a projection;
- $t_e^* t_e = t_{s(e)}$;
- for any $v \in E^0$, $t_v = \sum_{e: r(e) = v} t_e t_e^*$.

It then follows that $t_e t_f = 0$ unless $s(e) = r(f)$.

$C^*(E)$ is universal for representations of $\{t_e, t_v\}_{v \in E^0, e \in E^1}$; any collection of partial isometries and projections $\{s_e, s_v\}_{v,e} \subseteq B(\mathcal{H})$ satisfying the above conditions generates a quotient of $C^*(E)$.

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$.

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra:

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra: writing $E^\infty = \{(e_n)_{n\in\mathbb{N}} \subseteq E^1 : s(e_i) = r(e_{i+1}) \ \forall \ i\}$ and $\sigma : E^\infty \to E^\infty$ given by $\sigma((e_n)_n) = (e_{n+1})_n$,

$$\mathcal{G}_E = \{(x, j-k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \sigma^j(x) = \sigma^k(y)\}.$$

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra: writing $E^\infty = \{(e_n)_{n\in\mathbb{N}} \subseteq E^1 : s(e_i) = r(e_{i+1}) \ \forall\ i\}$ and $\sigma : E^\infty \to E^\infty$ given by $\sigma((e_n)_n) = (e_{n+1})_n$,

$$\mathcal{G}_E = \{(x, j-k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \sigma^j(x) = \sigma^k(y)\}.$$

- E^∞ is a Cantor set;

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra: writing $E^\infty = \{(e_n)_{n \in \mathbb{N}} \subseteq E^1 : s(e_i) = r(e_{i+1}) \ \forall \ i\}$ and $\sigma : E^\infty \to E^\infty$ given by $\sigma((e_n)_n) = (e_{n+1})_n$,

$$\mathcal{G}_E = \{(x, j-k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \sigma^j(x) = \sigma^k(y)\}.$$

- E^∞ is a Cantor set; [IK13] connection between KMS states on $C^*(E)$ and Hausdorff structures on E^∞.

Graph C^*-algebras

- Many structural aspects of $C^*(E)$ (ideals, unit, K-theory) are visible in E.
- [KPR98] If $C^*(E)$ is simple and unital, it is either AF or purely infinite.
- [Dri00] For any AF algebra A, there exists a graph E with $A \otimes \mathcal{K} \cong C^*(E) \otimes \mathcal{K}$. However, [EKRT14] the question $A \cong C^*(E)$ is more complicated.
- [KPRR97] $C^*(E) \cong C^*(\mathcal{G}_E)$ is also a groupoid C^*-algebra: writing $E^\infty = \{(e_n)_{n\in\mathbb{N}} \subseteq E^1 : s(e_i) = r(e_{i+1}) \ \forall \ i\}$ and $\sigma : E^\infty \to E^\infty$ given by $\sigma((e_n)_n) = (e_{n+1})_n$,
$$\mathcal{G}_E = \{(x, j-k, y) \in E^\infty \times \mathbb{Z} \times E^\infty : \sigma^j(x) = \sigma^k(y)\}.$$
- E^∞ is a Cantor set; [IK13] connection between KMS states on $C^*(E)$ and Hausdorff structures on E^∞.
- [ERRS16] $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ iff a finite number of moves will convert E into F.

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow k$-dimensional rectangles in Λ;

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow k$-dimensional rectangles in Λ; degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path."

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow$ k-dimensional rectangles in Λ; degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path."

$C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$;

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow k$-dimensional rectangles in Λ; degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path."

$C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$; Unit space $\mathcal{G}_\Lambda^{(0)} \cong \Lambda^\infty$ is a Cantor set.

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow$ k-dimensional rectangles in Λ; degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path."

$C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$; Unit space $\mathcal{G}_\Lambda^{(0)} \cong \Lambda^\infty$ is a Cantor set.

$C^*(\Lambda)$ has more flexible structure than $C^*(E)$;

Higher-rank graphs

Higher-rank graphs (k-graphs) Λ are a k-dimensional generalization of directed graphs.

Introduced by [KP00] to give examples of combinatorial, computable C^*-algebras, more general than $C^*(E)$.

Paths in $E \rightsquigarrow k$-dimensional rectangles in Λ; degree functor $d : \Lambda \to \mathbb{N}^k$ tells us the size of the "path."

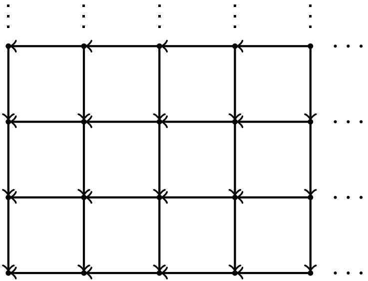
$C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$; Unit space $\mathcal{G}_\Lambda^{(0)} \cong \Lambda^\infty$ is a Cantor set.

$C^*(\Lambda)$ has more flexible structure than $C^*(E)$; more options than AF/purely infinite for simple algebras, more varied K-theory [Eva08], etc.

Higher-rank graphs

Definition

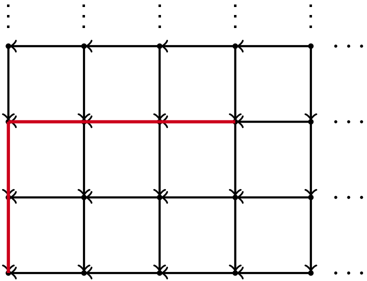
A k-graph is a countable category Λ with a degree map $d : \Lambda \to \mathbb{N}^k$ such that, if $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ with $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$.



Higher-rank graphs

Definition

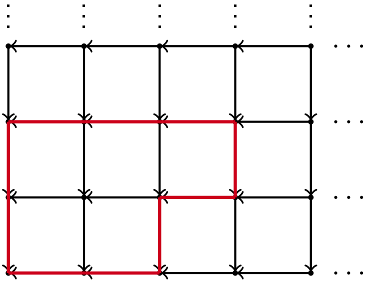
A k-graph is a countable category Λ with a degree map $d : \Lambda \to \mathbb{N}^k$ such that, if $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ with $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$.



Higher-rank graphs

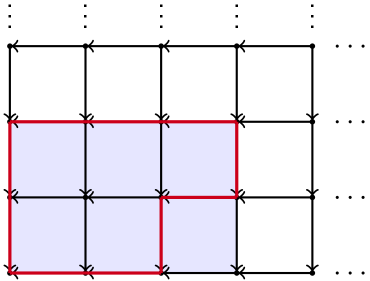
Definition

A k-graph is a countable category Λ with a degree map $d : \Lambda \to \mathbb{N}^k$ such that, if $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ with $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$.

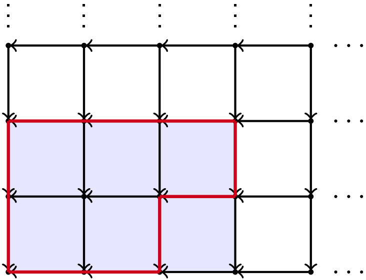


Definition

A k-graph is a countable category Λ with a degree map $d : \Lambda \to \mathbb{N}^k$ such that, if $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ with $d(\mu) = m, d(\nu) = n$, and $\lambda = \mu\nu$.



Higher-rank graphs



In our example,

$$d(\lambda) = (3,2) = (0,2) + (3,0) = (2,0) + (0,1) + (1,0) + (0,1),$$

so each of these possible factorizations must give us the same element λ.

Examples of k-graphs

- A directed graph E is a k-graph:

- A directed graph E is a k-graph: $k = 1$

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.
  $d(\lambda) = |\lambda|$ is the number of edges in λ.

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.
  $d(\lambda) = |\lambda|$ is the number of edges in λ.

- $\mathbb{N}^k$ is a k-graph:

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.
  $d(\lambda) = |\lambda|$ is the number of edges in λ.

- $\mathbb{N}^k$ is a k-graph: $d = id$.

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.
  $d(\lambda) = |\lambda|$ is the number of edges in λ.

- $\mathbb{N}^k$ is a k-graph: $d = id$. Think of $\mathbb{N}^k$ as a category with one object.

Examples of k-graphs

- A directed graph E is a k-graph: $k = 1$
  More precisely, the category of paths in E is a 1-graph.
  $d(\lambda) = |\lambda|$ is the number of edges in λ.

- $\mathbb{N}^k$ is a k-graph: $d = id$. Think of $\mathbb{N}^k$ as a category with one object.

- One can also think of a k-graph as a (quotient of a) directed graph, with k different colors of edges.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\text{edges of color } i \text{ from } w \text{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v \in \Lambda^0}$ of ℓ^1-norm 1.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\text{edges of color } i \text{ from } w \text{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v \in \Lambda^0}$ of ℓ^1-norm 1.

In fact, [HLRS15] a finite Λ is strongly connected iff there is a finite sum of finite products of $\{A_i\}_{1 \leq i \leq k}$ which is irreducible.

Strongly connected k-graphs

A (higher-rank) graph is strongly connected if $v\Lambda w \neq \emptyset$ for all $v, w \in \Lambda^0$.

Theorem ([HLRS15])

If Λ is finite and strongly connected, then the adjacency matrices $\{A_i : 1 \leq i \leq k\} \subseteq M_{\Lambda^0}(\mathbb{N})$,

$$A_i(v, w) = |v\Lambda^{e_i} w| = \#\{\textit{edges of color } i \textit{ from } w \textit{ to } v\}$$

share a unique positive eigenvector $(x_v^\Lambda)_{v\in\Lambda^0}$ of ℓ^1-norm 1.

In fact, [HLRS15] a finite Λ is strongly connected iff there is a finite sum of finite products of $\{A_i\}_{1\leq i\leq k}$ which is irreducible.

Note that $A_iA_j = A_jA_i \ \forall\ i, j$ by the factorization rule.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

If $|v\Lambda^{e_i}| > 2$ for at least one i, the space Λ^∞ of infinite paths in Λ is naturally a Cantor set – compact, perfect, totally disconnected.

Infinite paths and Cantor sets

An infinite path in a k-graph Λ is (the equivalence class of) an infinite string of composable edges (range but no source) which contains infinitely many edges of any color.

If $|v\Lambda^{e_i}| > 2$ for at least one i, the space Λ^∞ of infinite paths in Λ is naturally a Cantor set – compact, perfect, totally disconnected.

The collection of sets

$$Z(\lambda) = \{x \in \Lambda^\infty : x = \lambda y\},$$

where $\lambda \in \Lambda$ is a finite path (morphism) in Λ, is a compact open basis for the topology on Λ^∞ making Λ^∞ into a Cantor set.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Definition

For (X, d) a metric space and $s \in \mathbb{R}_{\geq 0}$, the Hausdorff measure of dimension s of a compact subset Z of X is

$$H^s(Z) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{U_i \in F} (\text{diam } U_i)^s : |F| < \infty, \bigcup_i U_i = Z, \text{ diam } U_i < \epsilon \ \forall i \right\}.$$

Hausdorff structure on Λ^∞

There are many ways to metrize the cylinder set topology on Λ^∞. For many choices of metric, we get the same (nontrivial) Hausdorff measure on Λ^∞.

Definition

For (X, d) a metric space and $s \in \mathbb{R}_{\geq 0}$, the Hausdorff measure of dimension s of a compact subset Z of X is

$$H^s(Z) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{U_i \in F} (\text{diam } U_i)^s : |F| < \infty, \bigcup_i U_i = Z, \text{ diam } U_i < \epsilon \ \forall i \right\}.$$

Moreover, $\exists!\, s \in \mathbb{R} : t < s \Rightarrow H^t(X) = \infty$ and $t > s \Rightarrow H^t(X) = 0$.

We call s the Hausdorff dimension of X.

$\mathbb{R}_+$-functors

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

$\mathbb{R}_+$-functors

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

Definition ([McN15])

Let Λ be a higher-rank graph. An $\mathbb{R}_+$-functor on Λ is a function $y : \Lambda \to \mathbb{R}_{\geq 0}$ such that

$$y(\lambda\nu) = y(\lambda) + y(\nu).$$

Note that this forces $y(v) = 0$ for all vertices v.

$\mathbb{R}_+$-functors

We study metrics on Λ^∞ which arise from putting "weights" on Λ.

Definition ([McN15])

Let Λ be a higher-rank graph. An $\mathbb{R}_+$-functor on Λ is a function $y : \Lambda \to \mathbb{R}_{\geq 0}$ such that

$$y(\lambda\nu) = y(\lambda) + y(\nu).$$

Note that this forces $y(v) = 0$ for all vertices v.

Proposition (Farsi-G-Kang-Larsen-Packer)

For any weight functor y on a strongly connected finite k-graph Λ, and any $\beta \geq 0$, the matrices $\{B_i(y,\beta)\}_{1\leq i\leq k} \in M_{\Lambda^0}$ given by

$$B_i(y,\beta)_{v,w} = \sum_{\lambda \in v\Lambda^{e_i} w} e^{-\beta y(\lambda)}$$

have a unique positive common eigenvector $\xi^{y,\beta}$ of ℓ^1-norm 1.

Examples and Notation for $\mathbb{R}_+$-functors

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor.

Examples and Notation for $\mathbb{R}_+$-functors

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

Examples and Notation for $\mathbb{R}_+$-functors

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \; \forall \; \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

Define

$$\rho(B(y, \beta)) := (\rho(B_1(y, \beta)), \rho(B_2(y, \beta)), \ldots, \rho(B_k(y, \beta))).$$

Examples and Notation for $\mathbb{R}_+$-functors

The function $y : \Lambda \to \mathbb{R}_{\geq 0}$ given by $y(\lambda) = 0 \ \forall \ \lambda$ is a $\mathbb{R}_+$-functor. In fact, in this case, $B_i(y, \beta) = A_i$ for all β.

Define

$$\rho(B(y,\beta)) := (\rho(B_1(y,\beta)), \rho(B_2(y,\beta)), \ldots, \rho(B_k(y,\beta))).$$

Then, for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$, define

$$\rho(B(y,\beta))^n := \rho(B_1(y,\beta))^{n_1} \cdot \rho(B_2(y,\beta))^{n_2} \cdot \cdots \cdot \rho(B_k(y,\beta))^{n_k}.$$

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Kang-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)} \right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i.

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Kang-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)} \right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i. Then,

$$d_{y,\beta}(x,z) := w_{y,\beta}(x \wedge z), \qquad \text{where } x \wedge z = \max\{\lambda : x, z \in Z(\lambda)\},$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology.

Hausdorff structure and $\mathbb{R}_+$-functors

Theorem (Farsi-G-Kang-Larsen-Packer)

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. For any $\lambda \in \Lambda$, define

$$w_{y,\beta}(\lambda) = e^{-y(\lambda)} \left(\rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}\right)^{1/\beta}.$$

Suppose moreover that $\rho(B_i(y,\beta)) > \max_{v,w}\{B_i(y,\beta)_{v,w}\}$ for at least one i. Then,

$$d_{y,\beta}(x,z) := w_{y,\beta}(x \wedge z), \qquad \text{where } x \wedge z = \max\{\lambda : x, z \in Z(\lambda)\},$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Also, $(\Lambda^\infty, d_{y,\beta})$ has Hausdorff dimension β and Hausdorff measure

$$\mu_{y,\beta}(Z(\lambda)) = H^\beta(Z(\lambda)) = w_{y,\beta}(\lambda)^\beta = e^{-\beta y(\lambda)} \rho(B(y,\beta))^{-d(\lambda)} \xi^{y,\beta}_{s(\lambda)}.$$

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.
Taking $y = 0$ in the previous Theorem gives:

Corollary (FGKLP)

For any finite strongly connected k-graph, and any $\beta \in (0, \infty)$, the function

$$d_\beta(x, z) := \left(\rho(\Lambda)^{-d(x \wedge z)} x^\Lambda_{s(\lambda)}\right)^{1/\beta}$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology.

Corollary

For strongly connected finite k-graphs, the authors of [HLRS15] described a measure M on Λ^∞:

$$M(Z(\lambda)) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)},$$

where $\rho(\Lambda) = (\rho(A_1), \rho(A_2), \ldots, \rho(A_k))$, and x^Λ is the common Perron–Frobenius eigenvector of $A_1, \ldots, A_k$.
Taking $y = 0$ in the previous Theorem gives:

Corollary (FGKLP)

For any finite strongly connected k-graph, and any $\beta \in (0, \infty)$, the function

$$d_\beta(x, z) := \left(\rho(\Lambda)^{-d(x \wedge z)} x^\Lambda_{s(\lambda)}\right)^{1/\beta}$$

is an ultrametric on Λ^∞ which metrizes the cylinder set topology. Moreover, the ultrametric spaces $(\Lambda^\infty, d_\beta)$ all have the same Hausdorff measure, namely M.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.

KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09]

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS$_\beta$ states are given by Galois groups for large β.

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS$_\beta$ states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS$_\beta$ states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"
- For graph algebras and related constructions, KMS states for the gauge action can often be computed by integration against the Hausdorff measure of Λ^∞; cf. [Nes13, Tho14, HLRS15]

KMS states for C^*-algebras

To an action α of $\mathbb{R}$ on a C^*-algebra A, which extends to an analytic action of $\mathbb{C}$ on $\mathcal{A} \subseteq A$, we associate KMS states.
KMS states arose in statistical physics, but have interesting implications more generally (eg, number theory and operator algs):

- Dynamical systems and KMS states for number fields $\rightsquigarrow$ Kroneckers Jugendtraum (Hilbert 12) [BC95, CMR05, LLN09] In particular, the symmetries of the KMS_β states are given by Galois groups for large β.
- A sudden change in the number of KMS_t states for (A, α) $\leftrightsquigarrow$ "symmetry breaking"
- For graph algebras and related constructions, KMS states for the gauge action can often be computed by integration against the Hausdorff measure of Λ^∞; cf. [Nes13, Tho14, HLRS15]

Given an $\mathbb{R}_+$-functor on Λ, we obtain an associated action on $C^*(\Lambda)$, and compute the associated KMS states.

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

$\mathbb{R}_+$-functors, generalized gauge actions, KMS states

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)}\right)^{it/\beta} s_\lambda.$$

$\mathbb{R}_+$-functors, generalized gauge actions, KMS states

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)} \right)^{it/\beta} s_\lambda.$$

Generalized gauge actions [Nes13, Tho14, McN15]

$\mathbb{R}_+$-functors, generalized gauge actions, KMS states

$C^*(\Lambda)$ is the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz-Krieger relations.

Natural action α (gauge action) of $\mathbb{T}^k$ on $C^*(\Lambda)$: $\alpha_z(s_\lambda) = z^{d(\lambda)} s_\lambda$.

From an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$, get an action of $\mathbb{R}$ on $C^*(\Lambda)$:

$$\alpha_t^{y,\beta}(s_\lambda) = e^{ity(\lambda)} \left(\rho(B(y,\beta))^{d(\lambda)}\right)^{it/\beta} s_\lambda.$$

Generalized gauge actions [Nes13, Tho14, McN15]

Definition

A positive linear map $\phi : C^*(\Lambda) \to \mathbb{C}$ is a KMS state at (inverse) temperature t for $\alpha^{y,\beta}$ if, for all $\lambda, \eta, \nu, \rho \in \Lambda$,

$$\phi(s_\lambda s_\eta^* s_\nu s_\rho^*) = \phi(\alpha_{it}^{y,\beta}(s_\nu s_\rho^*) s_\lambda s_\eta^*).$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

Theorem (Farsi-G-Kang-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y, \beta)) > 1$ for some $1 \le i \le k$. Then

$$\phi(a) = \int_{\Lambda^\infty} \Phi(a)\, d\mu_{y,\beta}$$

is a KMS state at inverse temperature β for $(C^*(\Lambda), \alpha^{y,\beta})$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

Write $\Phi : C^*(\Lambda) \to C_0(\Lambda^\infty)$ for the usual conditional expectation,

$$\Phi(s_\lambda s_\mu^*) = \begin{cases} \chi_{Z(\lambda)}, & \mu = \lambda \\ 0, & \text{else.} \end{cases}$$

(This suffices to define Φ since the Cuntz–Krieger relations imply that $C^*(\Lambda)$ is densely spanned by $\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda\}$.)

Theorem (Farsi-G-Kang-Larsen-Packer, [Tho14])

Let Λ be a strongly connected finite k-graph, with an $\mathbb{R}_+$-functor y and $\beta \in \mathbb{R}_{>0}$. Suppose $\rho(B_i(y, \beta)) > 1$ for some $1 \leq i \leq k$. Then

$$\phi(a) = \int_{\Lambda^\infty} \Phi(a)\, d\mu_{y,\beta}$$

is a KMS state at inverse temperature β for $(C^*(\Lambda), \alpha^{y,\beta})$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\mathrm{Per}\,\Lambda = \{m - n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ *admits KMS states at inverse temperature t iff* $\alpha^{y,\beta} = \alpha^{y,t}$.

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(\operatorname{Per}\Lambda)$:

KMS states for $(C^*(\Lambda), \alpha^{y,\beta})$

When Λ is strongly connected, the KMS states of $C^*(\Lambda)$ are closely linked to the periodicity group of Λ:

$$\operatorname{Per}\Lambda = \{m-n \in \mathbb{Z}^k : \exists\, \mu, \nu \in \Lambda \text{ s.t.} d(\mu) = m, d(\nu) = n, Z(\mu) = Z(\nu)\}.$$

Motivation: if λ is a loop without entry, $Z(\lambda) = Z(r(\lambda))$. The length of the loop is its periodicity.

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(\operatorname{Per}\Lambda)$: the KMS state associated to $z \in \widehat{C^*(\operatorname{Per}\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin \operatorname{Per}\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \text{else.} \end{cases}$$

Connections with the literature

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS$_\beta$ states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\,\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\,\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\,\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \textit{else.} \end{cases}$$

Connections with the literature

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & \textit{else.} \end{cases}$$

- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.

Connections with the literature

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS_β states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs;

Connections with the literature

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS$_\beta$ states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs; in this case, ϕ is the unique KMS$_\beta$ state.

Connections with the literature

Theorem (Farsi-G-Kang-Larsen-Packer)

$(C^*(\Lambda), \alpha^{y,\beta})$ admits KMS states at inverse temperature t iff $\alpha^{y,\beta} = \alpha^{y,t}$. Moreover, the simplex of KMS$_\beta$ states for $\alpha^{y,\beta}$ is affinely isomorphic to the state space of $C^*(Per\Lambda)$: the KMS state associated to $z \in \widehat{C^*(Per\Lambda)} \subseteq \mathbb{T}^k$ is

$$\phi_z(s_\lambda s_\nu^*) = \begin{cases} 0, & d(\lambda) - d(\nu) \notin Per\Lambda \\ z^{d(\lambda)-d(\nu)} \mu_{y,\beta}(Z(\lambda)), & else. \end{cases}$$

- For the classical gauge action on strongly connected k-graphs, [HLRS15] we have the same simplex of KMS states.
- McNamara [McN15] studied coordinate-wise irreducible k-graphs; in this case, ϕ is the unique KMS$_\beta$ state.

References I

- J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, Selecta Math. **1** (1995), no. 3.
- A. Connes, M. Marcolli, and N. Ramachandran, KMS states and complex multiplication, Selecta Math. (N.S.) **11** (2005), no. 3-4, 325–347.
- D. Drinen, Viewing AF-algebras as graph algebras, Proc. Amer. Math. Soc. **128** (2000), 1991–2000.
- S. Eilers, T. Katsura, E. Ruiz, and M. Tomforde, Identifying AF-algebras that are graph C^*-algebras, J. Funct. Anal. **266** (2014), no. 6, 3968–3996.
- S. Eilers, G. Restorff, E. Ruiz, and A. P. W. Sørensen, The complete classification of unital graph C^*-algebras: Geometric and strong, arXiv:1611.07120, 2016.

References II

D.G. Evans, On the K-theory of higher rank graph C^*-algebras, New York J. Math. **14** (2008), 1–31.

A. an Huef, M. Laca, I. Raeburn, and A. Sims, KMS states on the C^*-algebra of a higher-rank graph and periodicity in the path space, J. Funct. Anal. **268** (2015), 1840–1875.

M. Ionescu and A. Kumjian, Hausdorff measures and KMS states, Indiana Univ. Math. J. **62** (2013), no. 2, 443–463.

A. Kumjian and D. Pask, Higher rank graph C^*-algebras, New York J. Math. **6** (2000), 1–20.

A. Kumjian, D. Pask, and I. Raeburn, Cuntz-Krieger algebras of directed graphs, Pacific J. Math. **184** (1998), no. 1, 161–174.

References III

- A. Kumjian, D. Pask, I. Raeburn, and J. Renault, Graphs, groupoids and Cuntz-Krieger algebras, J. Funct. Anal. **144** (1997), 505–541.
- M. Laca, N.S. Larsen, and S. Neshveyev, On Bost-Connes type systems for number fields, J. Number Theory **129** (2009), no. 2, 325–338.
- R. McNamara, KMS states of graph algebras with a generalised gauge dynamics, Ph.D. thesis, University of Otago, 2015.
- S. Neshveyev, KMS states on the C^*-algebra of non-principal groupoids, J. Operator Theory **70** (2013), no. 2, 513–530.
- K. Thomsen, KMS weights on groupoid and graph C^*-algebras, J. Funct. Anal. **266** (2014), no. 5, 2959–2988.