Rigidity of corona algebras

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Let $H = \ell^2(\mathbb{N})$, S the unilateral shift, $Q = B(H)/K(H)$, and $\pi : B(H) \to Q$ the quotient map.

Question (BDF, 1970's)

Is there an automorphism of Q which sends $\pi(S)$ to $\pi(S)^*$? (Equivalently, an automorphism of Q which induces the automorphism $n \mapsto -n$ of $K_1(Q) = \mathbb{Z}$.)

Note that inner automorphisms of Q preserve Fredholm index, so such an automorphism would have to be outer.

Question (BDF)

Is there an outer automorphism of Q?

Theorem (Phillips-Weaver, 2007)

Assume the Continuum Hypothesis (CH). Then there is an outer automorphism of Q.

Theorem (Farah, 2011)

Assume the Open Coloring Axiom (OCA). Then every automorphism of Q is inner.

It's still unknown whether there can be an automorphism of Q which sends $\pi(S)$ to $\pi(S^*)$, in any model of set theory.

The *Continuum Hypothesis* states that $2^{\aleph_0} = \aleph_1$, or in other words, every $A \subseteq \mathbb{R}$ satisfies either $|A| \leq |\mathbb{N}|$ or $|A| = |\mathbb{R}|$.

Notation: given a set X we write $[X]^2$ for the set of unordered pairs $\{x, y\}$ of elements of X.

The *Open Coloring Axiom* states that for every $X \subseteq \mathbb{R}$ and every partition $[X]^2 = R \cup B$ into symmetric sets R and B, where R is relatively open in $[X]^2$ (identified with $X \times X$ minus the diagonal), either

- there is an uncountable $A \subseteq X$ such that $[A]^2 \subseteq R$, or
- there is a partition $X = \bigcup_{n=1}^{\infty} Y_n$ such that for all n, $[Y_n]^2 \subseteq B$.

OCA is just one combinatorial consequence of a more complicated axiom, the *Proper Forcing Axiom* (PFA).

Where CH often implies that uncountable structures are wild and pathological, PFA often implies a strong rigidity on these uncountable structures. For instance:

Theorem (Moore, 2006)

Assume PFA. Then there is a list of five canonical uncountable linear orders $L_1, \ldots, L_5$ such that every uncountable linear order contains an isomorphic copy of one of $L_1, \ldots, L_5$. (A 5-element basis.)

Theorem (Sierpinski, 1932)

Assume CH. Then the minimal size of a basis for the uncountable linear orders is $2^{2^{\aleph_0}}$.

The history of CH and PFA with C*-algebras was already there before Phillips-Weaver and Farah:

Theorem (W. Rudin, 1950's)

Assume CH. Then there is an automorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ which is not induced by a function $\mathbb{N} \to \mathbb{N}$.

Theorem (Shelah, 1980's)

Assume PFA. Then every automorphism of $\beta\mathbb{N} \setminus \mathbb{N}$ is induced by a function $\mathbb{N} \to \mathbb{N}$.

So assuming PFA,

- every automorphism of $B(H)/K(H)$ is inner, and
- every automorphism of $C(\beta\mathbb{N} \setminus \mathbb{N}) \simeq \ell^\infty/c_0$ is induced by a function $\mathbb{N} \to \mathbb{N}$.

Each of $B(H)/K(H)$ and ℓ^∞/c_0 is a *corona algebra*, $M(A)/A$.

Question

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For this we'll need a definition:

Definition

Given C*-algebras A and B and a function $\varphi : M(A)/A \to M(B)/B$, the *graph* of φ is

$$\Gamma_\varphi = \{(a, b) \in M(A) \times M(B) \mid \varphi([a]) = [b]\}$$

Recall that the *strict topology* on $M(A)$ is generated by the seminorms $x \mapsto \|xa\|$ and $x \mapsto \|ax\|$, where a ranges over A.

If A is separable then $M(A)_1$ is Polish in its strict topology.

Definition

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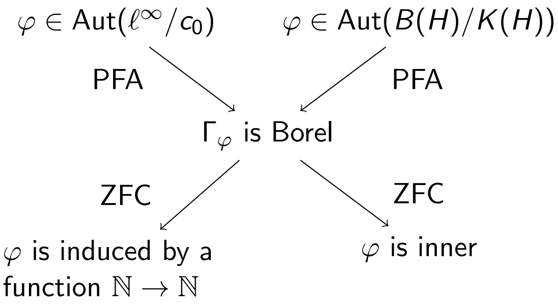

Conjecture (Coskey-Farah)

Let A be a separable C-algebra.*

1. *Assume CH. Then there is an automorphism of $M(A)/A$ whose graph is not Borel.*
2. *Assume PFA. Then every automorphism of $M(A)/A$ has Borel graph.*

Theorem (Coskey-Farah)

Assume CH. Let A be a separable C-algebra which is either simple or stable. Then $M(A)/A$ has $2^{2^{\aleph_0}}$-many automorphisms. (And only $2^{\aleph_0}$-many of them can have Borel graphs.)*

Let's sketch this in an easy case. Say $A = \bigoplus_n M_{2^\infty}$. Then

$$M(A)/A = \prod_n M_{2^\infty} / \bigoplus_n M_{2^\infty}$$

For every function $f : \mathbb{N} \to \mathbb{N}$, $M(A)/A$ contains a unital copy of

$$D_f = \prod M_{2^{f(n)}}(\mathbb{C}) / \bigoplus M_{2^{f(n)}}(\mathbb{C})$$

Moreover, $M(A)/A = \bigcup_f D_f$.

We have $D_f \subseteq D_g$ if and only if $\exists n_0\, \forall n \geq n_0\; f(n) \leq g(n)$ (written $f \leq^* g$).

Assuming CH, we can build a sequence f_α $(\alpha < \omega_1)$ such that $\alpha < \beta$ implies $f_\alpha \leq^* f_\beta$, and

$$M(A)/A = \bigcup_\alpha D_{f_\alpha}$$

Now build unitaries $u_\alpha \in D_{f_\alpha}$ such that $u_\alpha^* u_\beta \in D'_{f_\beta} \cap D_{f_\alpha}$ for all $\beta < \alpha$. This determines an automorphism $\varphi_{\vec{u}}$ of $M(A)/A$.

At every stage α, there are at least two choices for u_α. Hence there are at least $2^{\aleph_1}$-many distinct automorphisms $\varphi_{\vec{u}}$.

□

Theorem (M.-Vignati)

Assume PFA and let A and B be separable C-algebras which satisfy the Metric Approximation Property and have an increasing approximate identity of projections. Then every isomorphism $M(A)/A \to M(B)/B$ has a Borel graph.*

A Banach space X has the *(Metric) Approximation Property* if for every finite $F \subseteq X$ and $\epsilon > 0$, there is a continuous linear $T : X \to X$ with finite rank such that for all $x \in F$,

$$\|T(x) - x\| < \epsilon$$

(and $\|T\| \leq 1$.)

To give you some context:

- Nuclear C*-algebras have the MAP.
- $C_r^*(\mathbb{F}_n)$ has the MAP. (Haagerup)
- $B(H)$ does not have the AP. (Szankowski)
- There is a separable C*-algebra without the AP. (Szankowski)
- It is not known whether $C^*(\mathbb{F}_2)$ has the AP or MAP.

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This proves the second Coskey-Farah conjecture for a large class of C*-algebras.

Now what can we say about the structure of isomorphisms with a Borel graph?

Theorem (M.-Vignati)

Assume PFA. Suppose A_n, B_n are separable, unital C-algebras with the MAP, which have no nontrivial central projections. Then the following are equivalent.*

1. $\prod A_n / \bigoplus A_n \simeq \prod B_n / \bigoplus B_n$.
2. *Up to a permutation of the indices, A_n is ϵ-$*$-isomorphic to B_n, where $\epsilon \to 0$ as $n \to \infty$.*

Definition

Let A and B be C*-algebras. An ϵ-$*$-*isomorphism* is a map $\alpha : A \to B$ which satisfies all of the properties of a $*$-isomorphism, up to ϵ:

$$\begin{aligned}
\forall x, y \in A_1, |\lambda|, |\mu| \leq 1 \quad & \|\alpha(\lambda x + \mu y) - \lambda\alpha(x) - \mu\alpha(y)\| \leq \epsilon \\
\forall x, y \in A_1 \quad & \|\alpha(xy) - \alpha(x)\alpha(y)\| \leq \epsilon \\
\forall x \in A_1 \quad & \|\alpha(x^*) - \alpha(x)^*\| \leq \epsilon \\
\forall x \in A_1 \quad & \big|\|\alpha(x)\| - \|x\|\big| \leq \epsilon \\
\forall y \in B_1 \quad \exists x \in A_1 \quad & \|\alpha(x) - y\| \leq \epsilon
\end{aligned}$$

(α might not be linear or continuous or even measurable...)

Question

Suppose ϵ is small and A and B are ϵ-$*$-isomorphic. Does it follow that A and B must be isomorphic? What if A and B are simple, separable, nuclear, etc?

This kind of question is well-studied in certain nice cases, in particular when

- the ϵ-$*$-isomorphism is already linear, or
- A and B are subalgebras of $B(H)$ which are close in the Kadison-Kastler metric.

Theorem (M.-Vignati)

There is a universal constant K such that if A is finite-dimensional, B is any C-algebra, and $\phi : A \to B$ is an ϵ-$*$-homomorphism, then there is a $*$-homomorphism $\psi : A \to B$ with*

$$\|\phi - \psi\| < K\sqrt{\epsilon}$$

Theorem (M.-Vignati)

There is an $\epsilon > 0$ such that if A is unital, separable and AF, and B is any C-algebra ϵ-$*$-isomorphic to A, then A and B are $*$-isomorphic.*

Proposition (M.-Vignati)

There is an $\epsilon > 0$ such that if A and B are unital, purely infinite and simple, and A and B are ϵ-$*$-isomorphic, then $K_*(A) \simeq K_*(B)$.

Corollary

Assume PFA. Suppose that $\prod A_n / \bigoplus A_n$ *is isomorphic to* $\prod B_n / \bigoplus B_n$*, where none of the* A_n*'s or* B_n*'s has a nontrivial central projection, and either*

1. *each* A_n *and* B_n *is a UCT Kirchberg algebra, or*
2. *each* A_n *is separable, unital and AF, and* B_n *is separable and unital.*

Then up to a permutation of the indices, $A_n \simeq B_n$ *for large enough* n*.*

Thank you!