THE COMMUTATOR STRUCTURE OF OPERATOR IDEALS OHIO CONFERENCE APRIL 28, 2018

GARY WEISS

Abstract. Abstract: The additive commutators of operators belonging to two-sided ideals of B(H) are characterized. For ideals I and J, the space, [I, J], of all finite sums of (I, J)-commutators is characterized and found to equal [IJ, B(H)]. An historical survey of this subject will be presented along with open problems and some recent progress. Time permitting I will discuss recent work on the subideal structure of B(H), that is, ideals inside the compacts K(H), and on B(H)-semigroup ideals on a problem of Radjavi concerning semigroups with automatic selfadjoint ideals.

1. Ancient history

Commutators: operators of the form $AB - BA : A, B \in B(H)$,

$C(\text{a class}) := \{AB - BA \mid A, B \in \text{that class}\}$

E.g., a mathematical formulation of Heisenberg's Uncertainty Principle;
The product rule for $(xf)' = xf' + f$ reframed: $I = \frac{d}{dx}M_x - M_x\frac{d}{dx}$
where the operators act on the class of differentiable functions.

C(B(H))
$I \notin C(B(H))$ Wintner/Wielandt 47/49 different proofs.
Pf. $I = AB - BA \Rightarrow I = A(B + nI) - (B + nI)A\ \forall n.\ \therefore S = B + nI$ invertible for some n.
Then $I = AS - SA \Rightarrow AS = I + SA \Rightarrow \sigma(AS) = 1 + \sigma(SA)$ (Spectral Mapping Th).

But similarity $SA = S(AS)S^{-1} \Rightarrow AS, SA$ have same spectrum.
Recall now: All $B(H)$ operators have nonempty compact spectrum.
Then $x \in \sigma(AS) \Rightarrow 1 + x \in \sigma(AS) \Rightarrow 2 + x \in \sigma(AS) \Rightarrow \cdots$,
so spectrum $\sigma(AS)$ is unbounded, a contradiction to compactness.

Characterization of C(B(H)) (Arlen Brown-Carl Pearcy 69, major contribution):
All $B(H)$ except the thin operators are NOT, i.e., $\lambda + K(H)$ with $\lambda \neq 0$.
($K(H)$ = the compacts.)
$I \neq AB - BA$ applied to Calkin algebra $B(H)/K(H) \Rightarrow$ thin ops are not commutators.
Brown-Pearcy proved all others are.

Commutators, ideals and traces

$[B(H), B(H)] = B(H)$ (Halmos 52/54?)

Commutator ideal $[I, I]$ denotes the linear span of $C(I)$ for any two-sided ideal.
$[I, J]$ denotes the linear span of $\{AB - BA \mid A \in I, B \in J\}$ for any two-sided ideal pair.

Motivation for studying $[I, J]$: Traces can act only on ideals. Why?
A linear functional on an ideal I is a trace (i.e., invariant under unitary equivalence) if and only if it vanishes on $[I, B(H)]$.
$\Leftarrow$: $T - U^*TU = [U^*, UT - TU] \in [I, B(H)]$. $\Rightarrow$: $[A, B] = \Sigma_4 u_i[A, U_i]$ and $AU_i \cong U_iA$.

Important traces today: the standard trace on the trace class, various Dixmier traces, positive traces, continuous traces on Banach ideals.
Some current and recent researchers: Sukochev, Zanin, Dykema, Kalton

And each trace is cannonically a linear complex map on the algebraic quotient $I/[I, B(H)]$. So ideal I has no traces if and only if $I = [I, B(H)]$. When true? So back to the study of $[I, J]$.

Pearcy-Topping 71
For compacts, $[K(H), K(H)] = K(H)$; & Schatten p-classes, $[C_{2p}, C_{2p}] = C_p \ \forall p > 1$.

4 Seminal Questions:
1. Is $C(K(H)) = K(H)$?
Test Question:
Their key idea for $[K(H), K(H)] = K(H)$ was to prove

the rank one projection $P \in [K(H), K(H)]$. $\quad (P = \begin{pmatrix} 1 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix})$

So they asked: is $P \in C(K(H))$? (Turned out very hard. More to come on this.)

2. Is $C(C_{2p}) = C_p \ \forall p > 1$.

Trace obstruction: Recall products of Hilbert-Schmidt operators (C_2 operators) are trace class (C_1 operators) with $trAB = trBA$. So $C(C_2) \subset C_1^o$ (trace zero trace class operators) and consequently so also $[C_2, C_2] \subset C_1^o$.

3. Is $\mathbf{C(C_2) = C_1^o}$?
4. If not, what about $\mathbf{[C_2, C_2] = C_1^o}$?

Progress to date

1. $C(K(H)) = K(H)$? Still open but

a) $P \in C(K(H))$ with consequence $C(K(H), B(H)) = K(H)$ (J. Anderson 77)

b) 70's open problem: $C(K(H))$ contains some strictly positive compact operators.
(2006 Davidson-Marcoux-Radjavi-unpublished, and independently Patnaik-W 2012)

c) Nilpotent compact ops $\in C(K(H))$ (2017 Dykema-Amudhan Krishnaswamy-Usha)

2. $C(C_{2p}) = C_p, p > 1$ still open.

3 & 4. NO and the beginning of a long investigation involving commutators and traces, the main object of this talk. 1973-2004

2. My beginning

Evolving commutator matrix constructions & their solution operators norm bounds suggested "extremal" test question: $\operatorname{diag}(-\sum_1^\infty d_n, d_1, d_2, \dots) = AB - BA$, minimizing equal A,B Hilbert-Schmidt norm. Natural to focus on finite case, and extremal among these (i.e., maximizing known Hilbert-Schmidt norm bounds) are $\operatorname{diag}(-1, 1/N, \dots, 1/N)$.

Necessary bound: $\|A\|^2_{C_2} \geq 1$:

$$2\|A\|^2_{C_2} = 2\|A\|_{C_2}\|B\|_{C_2} \geq \|AB\|_{C_1} + \|BA\|_{C_1} \geq \|AB - BA\|_{C_1} = 2$$

Among $\operatorname{diag}(-1, 1/N, \dots, 1/N)$: $\operatorname{diag}(-1, 1/2, 1/2)$ nontrivally had minimum precisely 1, and so **the minimum question for $N = 3$ sat from 1973–1976.**
Computer simulations indicated otherwise.
Whether or not $\min \|A\|_{C_2} \to \infty$ as $N \to \infty$ in 73 turned out an essential test question. For me, the solution showcases the birth of staircase forms and block tridiagonal forms.

Compute the Hilbert-Schmidt norm minimum over $A \in M_4(\mathbb{C})$

$$\min\{\|A\|_{C_2} \mid AB - BA = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}\}$$

subject to scalar normalizing to insure $\|A\|_{C_2} = \|B\|_{C_2}$.

Theorem 2.1 (W 1980)**.**

$$\min\{\|A\|_{C_2} \mid AB - BA = \operatorname{diag}(-1, 1/3, 1/3, 1/3)\} = \sqrt{\frac{4}{3}}.$$

The minimum is attained:

$$A = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Proof that $\sqrt{\frac{4}{3}}$ is a lower bound was the breakthrough and will be given shortly.

This led to NO for 3 & 4 by determining which among this somewhat general class of diagonal trace class operators $(\operatorname{diag}(-\sum_1^\infty d_n, d_1, d_2, \dots))$ are commutators of Hilbert-Schmidt operators or finite linear combinations.

Theorem 2.2 (W 73, 80, 86)**.** *The following are equivalent.*

(i) $\operatorname{diag}(-\sum_1^\infty d_n, d_1, d_2, \dots) \in [C_2, C_2]$

(ii) $\operatorname{diag}(-\sum_1^\infty d_n, d_1, d_2, \dots) \in [C_1, B(H)]$

(iii) $\sum_1^\infty d_n \log n < \infty$.

In particular, if $\langle d_n \rangle = \langle \frac{1}{n \log^2 n} \rangle$, *then*

$$\operatorname{diag}(-\sum_1^\infty d_n, d_1, d_2, \dots) \in C_1^o \setminus [C_2, C_2].$$

Culminated years later into a totally general characterization of $[\mathbf{I}, \mathbf{J}]$:

Theorem 2.3 (Dykema, Figiel, Wodzicki, W, Advances 2004, announced PNAS 02)**.** *If I, J are two arbitrary $B(H)$-ideals, at least one proper, and $\mathbf{T} = \mathbf{T}^* \in IJ$, then*

$$T \in [I, J] \text{ if and only if } \operatorname{diag} \lambda(T)_a \in IJ.$$

($\lambda(T)_a$ denotes the arithmetic mean sequence formed from the eigenvalue sequence of T, arranged in order of decreasing moduli, counting multiplicities and when finite rank, ending in infinitely many zeros.)

Consequently, $[I, J] = [IJ, B(H)]$.

This characterizes all $[I, J]$ because it clearly is selfadjoint since ideals are selfadjoint by the polar decomposition, and so characterizing the real and imaginary parts of its commutators suffices for a characterization of $[I, J]$.

Proof of the "$\frac{4}{3}$" Theorem introducing also staircase forms.

Proof. Assume

$$AB - BA = \mathrm{diag}(-1, 1/3, 1/3, 1/3) \tag{2.1}$$

Solvable since finite matrix with trace 0.
WLOG normalizing by scalars: $\|A\|_{C_2} = \|B\|_{C_2}$.
The sequence $e_1, Ae_1, A^*e_1, e_2, e_3, e_4$ spans $\mathbb{C}^4$,
and the Gram-Schmidt process yields another basis for $\mathbb{C}^4$ with associated unitary U that fixes e_1 and for which Ad_U leaves invariant $\mathrm{diag}(-1, 1/3, 1/3, 1/3)$.
that is, $U^*\mathrm{diag}(-1, 1/3, 1/3, 1/3)U = \mathrm{diag}(-1, 1/3, 1/3, 1/3)$ (equivalently, this diagonal remains the same under this basis change).

This new basis puts A into "staircase" form: $U^*AU = \begin{pmatrix} * & * & * & 0 \\ * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$

Computing the diagonal entries of the commutator $AB - BA$ in terms of $A = (a_{ij})$ and $B = (b_{ij})$ one obtains the 4 equations:

$$\begin{aligned}
-1 &= a_{12}b_{21} - b_{12}a_{21} + a_{13}b_{31} - b_{13}a_{31} \\
\frac{1}{3} &= a_{21}b_{12} - b_{21}a_{12} + a_{23}b_{32} - b_{23}a_{32} + a_{24}b_{42} - b_{24}a_{42} \\
\frac{1}{3} &= a_{31}b_{13} - b_{31}a_{13} + a_{32}b_{23} - b_{32}a_{23} + a_{34}b_{43} - b_{34}a_{43} \\
\frac{1}{3} &= a_{42}b_{24} - b_{42}a_{24} + a_{43}b_{34} - b_{43}a_{34}
\end{aligned}$$

Summing the first 3 equations and taking the first equation yields the 2 equations:

$$\begin{aligned}
-1 &= a_{12}b_{21} - b_{12}a_{21} + a_{13}b_{31} - b_{13}a_{31} \\
\frac{1}{3} &= a_{42}b_{24} - b_{42}a_{24} + a_{43}b_{34} - b_{43}a_{34}
\end{aligned}$$

Subtracting:

$$-\frac{4}{3} = a_{12}b_{21} - b_{12}a_{21} + a_{13}b_{31} - b_{13}a_{31} - (a_{42}b_{24} - b_{42}a_{24} + a_{43}b_{34} - b_{43}a_{34})$$

Apply triangle then Hölder inequalities:

$$\begin{aligned}
\frac{4}{3} &\leq |a_{12}||b_{21}| + |b_{12}||a_{21}| + |a_{13}b_{31}| + |b_{13}||a_{31}| \\
&\qquad + |a_{42}||b_{24}| + |b_{42}||a_{24}| + |a_{43}||b_{34}| + |b_{43}||a_{34}| \\
&\leq \sqrt{|a_{12}|^2 + |a_{21}|^2 + |a_{13}|^2 + |a_{31}|^2 + |a_{42}|^2 + |a_{24}|^2 + |a_{43}|^2 + |a_{34}|^2} \\
&\quad \times \sqrt{|b_{21}|^2 + |b_{12}|^2 + |b_{31}|^2 + |b_{13}|^2 + |b_{24}|^2 + |b_{42}|^2 + |b_{34}|^2 + |b_{43}|^2} \\
&\leq \|A\|_{C_2}\|B\|_{C_2} = \|A\|_{C_2}^2.
\end{aligned}$$

The last inequality arises from observing that each a_{ij}, b_{ij} appears no more than once each in the first inequality, and some appear not at all. The last equality follows from the assumed scalar normalization to make $\|A\|_{C_2} = \|B\|_{C_2}$ in the equation (2.1). Without this normalization one has in general that $\|A\|_{C_2}\|B\|_{C_2} \geq \frac{4}{3}$. □

This motivated the general staircase form result needed for Theorem 2.2 above and reframed can be stated as a block-tridiagonal form:

Corollary 2.4. *If $A_1, \dots, A_N$ denotes any finite collection of operators in $B(H)$, then there exists a unitary operator U fixing e_1 so that $A_1, \dots, A_N$ transform simultaneously matrices with their n^{th} row and column nonzero in at most the first $n(2N+1)$ entries. If they are selfadjoint, then they are thinner-as above but nonzero for at most $n(N+1)$ entries.*

For a single selfadjoint matrix, this form with inducing change of basis unitary is:

$$U^*AU = \begin{pmatrix} * & * & * & 3 & 0 & 0 & 0 & 0 & \cdots \\ * & * & * & * & * & * & 6 & 0 & \cdots \\ * & * & * & * & * & * & * & * & \cdots \\ 3 & * & * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & * & * & \cdots \\ 0 & 6 & * & * & * & * & * & * & \cdots \\ 0 & 0 & * & * & * & * & * & * & \cdots \\ & & \vdots & & & & & & \end{pmatrix}$$

3. P = a single commutator of compact operators (J. Anderson 79)

Seminal unparalled contribution to the field:

$$[C, Z] = P = \begin{pmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$$

in terms of block tri-diagonal matrices

$$C = \begin{pmatrix} 0 & A_1 & & \\ B_1 & 0 & A_2 & \\ & B_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} 0 & X_1 & & \\ Y_1 & 0 & X_2 & \\ & Y_2 & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

where A_n and X_n are the $n \times (n+1)$ matrices of norm $\frac{1}{\sqrt{n}}$

$$A_n = \frac{1}{n} \begin{pmatrix} \sqrt{n} & 0 & & & \\ & \sqrt{n-1} & 0 & & \\ & & \ddots & \ddots & \\ & & & \sqrt{1} & 0 \end{pmatrix} \quad \text{and} \quad X_n = \frac{1}{n} \begin{pmatrix} 0 & \sqrt{1} & & & \\ & 0 & \sqrt{2} & & \\ & & \ddots & \ddots & \\ & & & 0 & \sqrt{n} \end{pmatrix}$$

while B_n and Y_n are the $(n+1) \times n$ matrices of norm $\frac{\sqrt{n}}{n+1}$

$$B_n = -\frac{1}{n+1} \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & \ddots & \\ & & \ddots & 0 \\ & & & \sqrt{n} \end{pmatrix} \quad \text{and} \quad Y_n = \frac{1}{n+1} \begin{pmatrix} \sqrt{n} & & & \\ 0 & \sqrt{n-1} & & \\ & 0 & \ddots & \\ & & \ddots & \sqrt{1} \\ & & & 0 \end{pmatrix}.$$

4. Impact and an introduction to my work of the last decade

1. DFWW gave birth to arithmetic mean ideals I_a, ${}_aI$ and combinations like ${}_a(I_a)$ and I_{a^2}

and to diagonal invariance:
Which ideals have all their operators' diagonals (in all bases) back in the ideal?

Yes: Trace class, Hilbert-Schmidts, compacts.
No: Finite rank operators-consider any nonzero entry rank one infinite matrix $(a_i b_j)$.

Characterization: the arithmetic mean closed ideals, i.e., ${}_a(I_a) = I$. Kaftal-W 2011, IUMJ

This got Kaftal and me interested in the general question of diagonals of operators, in particular the classical works of Schur-Horn and recent works of Arveson, Kadison and others on diagonals of operators.

Back to this shortly.

2. $B(H)$-Subideals (characterize ideals inside ideals I, starting with $K(H)$)

Observe all $B(H)$-ideals inside I are automatically subideals of I.
Hence the question: Which subideals of I are not $B(H)$-ideals?

Fong-Radjavi 83: principal ideals in K(H) exist that are not B(H) ideals.
The $K(H)$-principal ideal generated by diag $< \frac{1}{n} >$ is not a $B(H)$-ideal.

Patnaik-W 2012-13 IEOT, JOT: Characterizations for principal, finitely generated, and certain infinitely generated subideals depending on the continuum hypothesis.

Notable: When is a subideal J of a $B(H)$-ideal I itself a $B(H)$-ideal?

Answer: When J is I-soft, i.e., when $J \subset I$ and $IJ = J$.
Subject to constraint: I is generated by a set of cardinality $< c$.
(Without this constraint, question is open.)

(Softness was introduced by Mityagin-Pietsch but was unbeknownst to us.)

3. Diagonality and Schur-Horn theorems
(with Loreaux, Jasper, Patnaik, Kaftal - JFA, IUMJ, JOT, ...)

Diagonality:
Characterize the diagonals of operator A in all bases, i.e., diagonals of unitary orbit of A. Or a class of operators A.

The classical Schur-Horn theorem, early in the last millenium:
Sequence $y = < y_j >$ $(1 \leq j \leq n)$ is the diagonal of a normal $n \times n$ operator A with eigenvalues $x = < x_j >$ arranged in order of decreasing moduli if and only if

$$\sum_1^k y_j \leq \sum_1^k x_j, 1 \leq k < n \quad \text{with } \sum_1^n y_j = \sum_1^n x_j$$

(a step function area comparison commonly known as Hardy-Littlewood majorization).

Recent infinite dimensional investigations focused on positive compact operators. Some contributors last 20 years: Arveson, Kadison (his Pythagorean papers for projections), Gohberg, Marcus, Neumann. All proved approximate Schur-Horn theorems.
And recently for von Neumann algebra analogs, Ravishandran and Skoufranis et al.

Exact Schur-Horn theorem, Kaftal-W 2011 JFA:
if $A > 0$, then for A trace class, Schur-Horn holds true for $n = \infty$, and for A compact but not trace class, the same but without equality at end. E.g., for non-trace class case:

Theorem 4.1 (Kaftal-W)**.** *Let $A \in K(H)^+$ with $R_A = I$. Then*

$$E(\mathcal{U}(A)) = \{B \in \mathcal{D} \mid s(B) \prec s(A), \ \textit{with } R_B = I\}.$$

Loreaux-W 2015 JFA, for dim $ker A = \infty$, the characterization of eigenvalues of B involves an infinite ladder of majorization analogs, e.g., $\sum_1^{k+p} y_j \leq \sum_1^k x_j$.

Case: $1 \leq$ dim ker $A < \infty$ remains open. Unexpectedly harder than the i∞ case.

Loreaux-Jasper-W 2017 IUMJ A Thompson type Schur-Horn theorem + a characterization of the diagonals of the full class of unitaries.
I.e., Schur-Horn majorization theorem with added singular value constraints;
and for diagonals of the class of unitaries:

$$x \text{ is bounded} \quad \& \quad 2(1 - \inf|x_j|) \leq \sum(1 - |x_j|))$$

(reminiscent of Kadison's diagonals of projection condition and an infinite dimensional analog of Thompson).

Loreaux-W 2016 JOT Diagonals of idempotents. Motivated by Kadison's proj work.

4. Automatic selfadjoint semigroup ideals (ASI) on a B(H) problem of Radjavi, with S. Patnaik.

B(H) semigroups are simply classes closed under products.
And their ideals are subsets closed under products from inside and outside, analogous to two-sided ring ideals.

Radjavi's question: characterize those semigroups possessing only selfadjoint ideals.

B(H) semigroups with ASI must themselves be selfadjoint,
and semigroups are built from their singly generated semigroups.

So main focus: Which $S(T, T^*)$ have ASI?
(the singly generated selfadjoint semigroups generated by T).

For T rank 1,
NASC for $S(T, T^*)$ having all its ideals s.a: the trace-norm condition
$(trT)^n\overline{(trT)}^m||T||^{2p} = 1,$ for some $n, m, p \geq 1$.
In most cases $S(T, T^*)$ is simple, i.e., no proper ideals.

For normal ops N: $S(N, N^*)$ is ASI if and only if $N \cong$ unitary $\oplus 0$.

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