Ocneanu compactness

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Abstract

We prove a more general version of the Ocneanu compactness theorem than that which appears in the literature. We adapt the proof that appears in Jones-Sunder to apply to a more general class of commuting squares.

1 Indices for inclusions of tracial von Neumann algebras

Suppose we have a unital inclusion of von Neumann algebras $M_0 \subset M_1$, and tr_1 is a faithful normal tracial state on M_1. Define $\operatorname{tr}_0 = \operatorname{tr}_1|_{M_0}$. Let $M_2 = \langle M_1, e_1 \rangle = JM_0J'$ be the basic construction algebra, and let Tr_2 on M_2 be the canonical commutant trace obtained from the right M_0-action on $L^2(M_1, \operatorname{tr}_1)$, which is the extension of $xe_1y \mapsto \operatorname{tr}_1(xy)$ [Pop94].

Definition 1.1. Recall from [Pop94, 1.1.4.c] or [JP11, Def. 2.8] that the inclusion $M_0 \subset (M_1, \operatorname{tr}_1)$ is called *Markov* if Tr_2 is finite, and the normalized trace $\operatorname{tr}_2 = \operatorname{Tr}_2(1)^{-1} \operatorname{Tr}_2$ on M_2 restricts to tr_1 on M_1. In this case, we define the *Markov index* of $M_0 \subseteq M_1$ to be $\operatorname{Tr}_1(1) \in [1, \infty)$.

Definition 1.2. Recall from [Wat90] that the inclusion $M_0 \subset M_1$ is said to have *index finite type* if there is a *(left) Pimsner-Popa basis* for M_1 over M_0, which is a finite set $\{b\} \subset M_1$ such that $\sum_b bE_{M_0}(b^*x) = x$ for all $x \in M_1$ [PP86]. In this case, the *Watatani index* is equal to $\sum_b bb^*$ and is independent of the choice of Pimsner-Popa basis.

Similarly, a *right Pimsner-Popa basis* for M_1 over M_0 is a finite set $\{\beta\} \subset M_1$ such that $\sum_\beta E_{M_0}(x\beta^*)\beta = x$ for all $x \in M_1$. Observe that when such a basis exists, the Watatani index is equal to $\sum_\beta \beta^*\beta$, since $\{\beta\}$ is a right Pimsner-Popa basis if and only if $\{\beta^*\}$ is a (left) Pimsner-Popa basis.

Remark 1.3. An inclusion $M_0 \subset (M_1, \operatorname{tr}_1)$ is called *strongly Markov* [JP11, Def. 2.8] (see also [BDH88]) if the inclusion both is Markov and has index finite type. In this case, the Markov index is equal to the Watatani index [Pop94, 1.1.4.c].

Example 1.4. By [Jon83, §3.2], an inclusion of finite dimensional tracial von Neumann algebras $M_0 \subset (M_1, \operatorname{tr}_1)$ is Markov if and only if the trace vectors $\vec{\lambda}_1$ for M_1 and $\vec{\lambda}_0$ for M_0 (whose j-th entry is the trace of a minimal projection in the j-th summand) satisfy $\Lambda^T\Lambda\vec{\lambda}_1 = d^2\vec{\lambda}_1$ and $\Lambda\Lambda^T\vec{\lambda}_0 = d^2\vec{\lambda}_0$. Here, Λ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion $M_0 \subset M_1$, whose (i, j)-th entry $\Lambda_{i,j}$ is the number of edges between the i-th even vertex/simple summand of M_0 and j-th odd vertex/simple summand of M_1. If all entries of $\vec{\lambda}_0, \vec{\lambda}_1$ are non-zero, then the scalar d satisfies $d^2 = \|\Lambda\Lambda^T\| = \|\Lambda^T\Lambda\|$.

2 Horizontally Markov commuting squares

Consider a quadrilateral of unital inclusions of finite dimensional von Neumann algebras

$$\begin{array}{ccc} M_0 & \subset & M_1 \\ \cup & & \cup \\ N_0 & \subset & N_1 \end{array} \tag{1}$$

together with a faithful tracial state tr_1 on M_1. Let $E_{M_0} : M_1 \to M_0$, $E_{N_1} : M_1 \to N_1$, and $E_{N_0} : M_0 \to N_0$ be the canonical trace preserving conditional expectations.

Definition 2.1. The quadrilateral (1) is called a *commuting square* if

$$E_{M_0} E_{N_1} = E_{N_0} \Longleftrightarrow E_{N_1} E_{M_0} = E_{N_0},$$

where all algebras are considered as subalgebras of M_1. A commuting square is called *horizontally Markov* if the inclusion $M_0 \subseteq (M_1, \mathrm{tr}_1)$ is Markov with index $d^2 = \mathrm{Tr}_2(1)$, and the following equivalent conditions hold (see Lemma 2.2 below):

(1) The inclusion $(N_0 \subseteq N_1, \mathrm{tr}_1|_{N_1})$ is also Markov with index d^2.

(2) The canonical $*$-algebra map $N_2 = N_1 f_1 N_1 \to M_2 = M_1 e_1 M_1$ given by $a f_1 b \mapsto a e_1 b$ for $a, b \in N_1$ is unital.

(3) There is a (left) Pimsner-Popa basis for N_1 over N_0 which is also a Pimsner-Popa basis for M_1 over M_0.

Lemma 2.2. *When $M_0 \subseteq (M_1, \mathrm{tr}_1)$ is Markov with index $d^2 = \mathrm{Tr}_2(1)$, the conditions (1) – (3) in the definition above are equivalent.*

Proof.

(1) ⇔ (2) : This can be easily deduced from [Pop90, Lem. 6.1]. First, we note that the inclusion $N_0 \subseteq (N_1, \mathrm{tr}_1|_{N_1})$ is always Markov. The canonical map $N_2 = N_1 f_1 N_1 \to M_2 = M_1 e_1 M_1$ by $a f_1 b \mapsto a e_1 b$ preserves the canonical commutant trace on N_2 given by $a f_1 b \mapsto \mathrm{tr}_1(ab)$. However, it may be the case that the Markov index of $N_0 \subseteq (N_1, \mathrm{tr}_1|N_1)$ may be strictly less than d^2. Second, the image of $1 \in N_2$ under the canonical map is an orthogonal projection in M_2 by the commuting square condition. Thus we see that this projection is equal to $1 \in M_2$ if and only if the Markov index of $N_0 \subseteq (N_1, \mathrm{tr}_1|N_1)$ is equal to d^2

(2) ⇔ (3) : If $N_1 f_1 N_1 = N_2$, then the trick of [Con80, Prop. 3(b)] allows us to write $1_{N_2} = \sum_b b f_1 b^*$, and $\{b\}$ is a Pimsner-Popa basis for N_1 over N_0. Now the canonical inclusion is unital if and only if $\sum_b b e_1 b^* = 1_{M_2}$ if and only if $\{b\}$ is a Pimsner-Popa basis for M_1 over M_0. □

Lemma 2.3. *Suppose we have a horizontally Markov commuting square as in (1). Then M_1 is linearly spanned by $N_1 M_0$. Hence there is a* right *Pimsner-Popa basis $\{\beta\}$ for M_0 over N_0 which is also a right Pimsner-Popa basis for M_1 over N_1.*

Proof. This follows from the proof of [JS97, Cor. 5.3.4 and Lem. 5.7.3]. Let $\{b\}$ be a Pimsner-Popa basis $\{b\}$ for N_1 over N_0 which is also a left Pimsner-Popa basis for M_1 over M_0. This immediately implies that

$$M_1 = \sum_b b M_0 \subseteq \operatorname{span} N_1 M_0 \subseteq M_1.$$

Now suppose $\{\beta\}$ is a right Pimsner-Popa basis for M_0 over N_0, so that $M_0 = \sum_\beta N_0\beta$. We then have that

$$M_1 = \operatorname{span} N_1 M_0 \subseteq \sum_\beta N_1 N_0 \beta = \sum_\beta N_1\beta \subseteq M_1.$$

We conclude that $\{\beta\}$ is also a right Pimsner-Popa basis for M_1 over N_1. Indeed, if $x \in M_1$, writing $x = \sum_\beta x_\beta \beta$ where each $x_\beta \in N_1$, since every $\beta \in M_0$, we have

$$\begin{aligned}\sum_\beta E_{N_1}(x\beta^*)\beta &= \sum_{\beta,\beta'} x_{\beta'} E_{N_1}(\beta'\beta^*)\beta = \sum_{\beta,\beta'} x_{\beta'} E_{N_1} E_{M_0}(\beta'\beta^*)\beta \\ &= \sum_{\beta,\beta'} x_{\beta'} E_{N_0}(\beta'\beta^*)\beta = \sum_{\beta'} x_{\beta'}\beta' = x.\end{aligned}$$

□

The following proposition is essentially a rewording of [Pop90, Lem. 6.1], and the proof is omitted.

Proposition 2.4. *Suppose we have a horizontally Markov commuting square as in (1). Consider the basic construction commuting square*

$$\begin{array}{ccc} M_1 & \subset & M_2 \\ \cup & & \cup \\ N_1 & \subset & N_2 \end{array} \qquad (2)$$

with the canonical Markov trace on M_2 where the inclusion $N_2 \subset M_2$ is given by the canonical map $af_1b \to ae_1b$. (Notice that viewing all algebras as subalgebras of M_2 identifies *$f_1 = e_1$!) Then the commuting square (2) is also horizontally Markov, and the Bratteli diagram for $N_2 \subset M_2$ is the same as the Bratteli diagram for the inclusion $N_0 \subset M_0$.*

3 Inductive limits of horizontally Markov commuting squares

Starting with a horizontally Markov commuting square as in (1), we iterate the basic construction to get an increasing sequence of horizontally Markov commuting squares. Let $M_\infty = \varinjlim M_n$, $N_\infty = \varinjlim N_n$, and $\operatorname{tr}_\infty = \varinjlim \operatorname{tr}_n$ on M_∞. Notice that tr_∞ is faithful on M_∞ since tr_n is faithful on M_n for all n by the horizontally Markov condition.

It is straightforward to verify that M_∞ acts on $H := L^2(M_\infty, \operatorname{tr}_\infty)$ by bounded operators, where $\|x^*x\|_{B(H)} \leq \|x^*x\|_{M_n}$ for $x \in M_n$. Let $M = M_\infty'' \subset B(H)$ and $N = N_\infty'' \subset B(H)$. Define tr on M by $\operatorname{tr}(x) = \langle x\Omega, \Omega\rangle$ where $\Omega \in H$ is the image of $1 \in M_\infty$.

Lemma 3.1. *The normal state* tr *on M is a faithful trace.*

Proof. Traciality follows by normality of tr, SOT density of M_∞ in M, and the Kaplansky Density Theorem. To show tr is faithful, we use the proof in [Jon10, §6.2], which we include for completeness and convenience.

Suppose $\operatorname{tr}(x^*x) = 0$. Then for all $m \in M_\infty$,

$$\|xm\Omega\|_2^2 = \|xR_m\Omega\|_2^2 = \|R_m x\Omega\|_2^2 \leq \|R_m\|^2 \|x\Omega\|_2^2 = \|R_m\|^2 \operatorname{tr}(x^*x) = 0$$

where for $a, m \in M_\infty$, $R_m a\Omega = am\Omega$ is the bounded right action as tr_∞ is a trace on M_∞. We conclude $x = 0$. □

We thus have an increasing sequence of horizontally Markov commuting squares with an inductive limit inclusion of finite von Neumann algebras with the canonical inductive limit faithful normal tracial state tr on M. Notice that at each iteration, identifying N_{n+1} with a subalgebra of M_{n+1} identifies the Jones projection f_n for $N_{n-1} \subset N_n$ with the Jones projection e_n for $M_n \subset M_{n-1}$.

$$\begin{array}{ccccccccccc} M_0 & \subset & M_1 & \overset{e_1}{\subset} & M_2 & \overset{e_2}{\subset} & M_3 & \overset{e_3}{\subset} & \cdots & \subset & M \\ \cup & & \cup & & \cup & & \cup & & & & \cup \\ N_0 & \subset & N_1 & \overset{e_1}{\subset} & N_2 & \overset{e_2}{\subset} & N_3 & \overset{e_3}{\subset} & \cdots & \subset & N \end{array} \tag{3}$$

Lemma 3.2. *The finite von Neumann algebras M and N are finite direct sums of II_1 factors.*

Proof. We prove the result for M and the result for N is similar. The Bratteli diagram for the inclusion $M_0 \subseteq M_1$ is a disjoint union of $\dim(Z(M_0) \cap Z(M_1))$ connected graphs. Denote the minimal projections of $Z(M_0) \cap Z(M_1)$ by $\{p\}$. Then the Bratteli diagram of each inclusion $pM_0 \subseteq pM_1$ is connected with Markov trace $x \mapsto \mathrm{tr}_1(px)/\mathrm{tr}_1(p)$. Iterating the basic construction, we have $M_n = \bigoplus pM_n$ for all n, and the von Neumann algebra generated by $\varinjlim pM_n$ in the GNS representation with respect to the unique Markov trace is clearly isomorphic to pM, as multiplication by p is SOT continuous. We know that each pM is a II_1 factor as the unique trace is faithful by an argument similar to Lemma 3.1. Thus $M = \bigoplus_p pM$ is a finite direct sum of II_1 factors. □

Lemma 3.3. *Each composite commuting square*

$$\begin{array}{ccc} M_0 & \subset & M_n \\ \cup & & \cup \\ N_0 & \subset & N_n \end{array}$$

in (3) above is horizontally Markov.

Proof. First, the composite square commutes by [JS97, Def. 5.1.7]. By the multistep basic construction [JP11, Lem. 2.19 and Prop. 2.20], the inclusion $M_0 \subseteq M_n \subseteq (M_{2n}, \mathrm{tr}_{2n}, p)$ is standard (see [JP11, Def. 2.14]) with Jones projection

$$p = d^{n(n-1)}(e_n e_{n-1} \cdots e_1)(e_{n+1} e_n \cdots e_2) \cdots (e_{2n-1} e_{2n-2} \cdots e_n).$$

Similarly, $N_0 \subseteq M_n \subseteq (N_{2n}, \mathrm{tr}_{2n}, p)$ is standard, as the Jones projections for the tower (N_i) are identified with the Jones projections e_i for the tower (M_i). Starting with a Pimsner-Popa basis B for N_1 over N_0 which is also a Pimsner-Popa basis for M_1 over M_0, by [JP11, Prop. 2.17],

$$\left\{ d^{-\frac{(n-1)n}{2}} (b_1 e_1 \cdots e_{n-1})(b_2 e_1 \cdots e_{n-2}) \cdots (b_{n-1} e_1)(b_n) \,\middle|\, b_1, \ldots, b_n \in B \right\}$$

is a Pimsner-Popa basis for N_n over N_0 which is also a Pimsner-Popa basis for M_n over M_0. We conclude that the basic construction commuting square

$$\begin{array}{ccccc} M_0 & \subset & M_n & \subset & M_{2n} \\ \cup & & \cup & & \cup \\ N_0 & \subset & N_n & \subset & N_{2n} \end{array}$$

is horizontally Markov. □

Proposition 3.4. *The inductive limit hyperfinite type* II_1 *inclusion* $N \subset M$ *from* (3) *is of index finite type. The Watatani index of* $N \subset M$ *is equal to the Watatani index of the inclusion* $N_0 \subset M_0$*, which is given by* $\sum_\beta \beta^*\beta$ *where* $\{\beta\}$ *is any right Pimsner-Popa basis for* M_0 *over* N_0*.*

Proof. The proof is identical to [JS97, Cor. 5.7.4]. By Lemmas 2.3 and 3.3, there is a right Pimsner-Popa basis $\{\beta\}$ for M_0 over N_0 which is also a Pimsner-Popa basis for M_n over N_n for every n. This means for every $x \in \bigcup_{n\geq 0} M_n$, we have $x = \sum_\lambda E_N(x\beta^*)\beta$. This equation clearly varies ultraweakly continuously in x, and thus $\{\beta\}$ is a right Pimsner-Popa basis for M over N. We conclude that the Watatani index of $N \subseteq M$ is equal to $\sum_\beta \beta^*\beta$. □

Remark 3.5. We call a commuting square as in (1) *horizontally connected* if one of the following two equivalent conditions holds:

- The Bratteli diagrams for the inclusions $M_0 \subseteq M_1$ and $N_0 \subseteq N_1$ are connected.
- $Z(M_0) \cap Z(M_1) = \mathbb{C}$ and $Z(N_0) \cap Z(N_1) = \mathbb{C}$.

In the event that our horizontally Markov commuting square is also horizontally connected, then every commuting square in the tower (3) is horizontally connected, and thus $N \subseteq M$ is a II_1 subfactor whose Jones index $[M : N]$ is equal to the Watatani index $\sum_\beta \beta^*\beta$, which is necessarily equal to $\|\Gamma\Gamma^T\|$ where Γ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion $N_0 \subset M_0$ [JS97, Lem. 5.3.3] or [Pop90, Cor. 6.2].

4 Proof of the Ocneanu compactness theorem

We now prove a more general version of Ocneanu Compactness [JS97, Thm. 5.7.1] by modifying the proof presented therein. While the outline of our proof is basically the same, the proofs of some technical intermediate steps are new; in particular, we highlight where we use *compactness*.

Theorem 4.1 (Ocneanu Compactness)**.** *Suppose we have a commuting square as in* (1) *which is horizontally Markov. Let* $N \subset M$ *be the inductive limit hyperfinite type* II_1 *inclusion. Then the relative commutant* $N' \cap M$ *is equal to* $N_1' \cap M_0$ *considered inside* M_1*.*

Proof. The outline of the proof follows [JS97, Thm. 5.7.1] closely. We postpone our modified proofs of the technical lemmas, which appear below.

(A) The same argument from [JS97, Thm. 5.7.1] shows that for all $1 \leq p < q$, $N_q' \cap M_p = N_1' \cap M_0$. This immediately implies that $N_1' \cap M_0 \subseteq N' \cap M$.

(B) Let $E_n : M \to M_n$ be the canonical trace preserving conditional expectation, and let Ω be the image of 1_M in $L^2(M)$. Let $x \in M$ and set $x_n = E_n(x)$. Then $\|x\Omega - x_n\Omega\|_2 \to 0$ as $n \to \infty$. We provide a proof in Lemma 4.2 below for the convenience of the reader.

(C) Starting with an $x \in N' \cap M$, we want to show that the sequence $(x_n)_{n\geq 0}$ from (B) is getting arbitrarily close to the finite dimensional subspace $N_1' \cap M_0$ in $\|\cdot\|_2$. We could then conclude by (B) that $x \in N_1' \cap M_0$. We break this step up as follows.

    (i) For $n \geq 0$, we have a map $\Phi_n : N_0' \cap M_n \to N_2' \cap M_{n+2}$ which sends $N_k' \cap M_n$ to $N_{k+2}' \cap M_{n+2}$ and $N_k' \cap N_n \to N_{k+2}' \cap N_{n+2}$ for all $0 \leq k \leq n$. We define this map explicitly in Definition 4.3 below, and we prove many properties about it in Proposition 4.4. Of particular importance are:

- each Φ_n is a $*$-algebra map, and
- for all $y \in N_0' \cap M_n$, $\Phi_n(E_{N_n}(y)) = E_{N_n}(\Phi_n(y))$.

(ii) In general, the maps Φ_n do *not* preserve the Markov trace, and thus the Φ_n are not isometries on $(N_0' \cap M_n)\Omega$. However, Φ_n gets closer to being an isometry as $n \to \infty$. Indeed, for $n \in \mathbb{N}$, we consider the composites $\Psi_n := \Phi_{n-1} \circ \cdots \circ \Phi_1 \circ \Phi_0$ which map $N_k' \cap M_n \to N_{2n+k}' \cap M_{2n+k}$ for all $0 \le k \le n$. By (Ci), Ψ_n maps the subalgebra $N_k' \cap N_n \to N_{2n+k}' \cap N_{2n+k}$.
Now setting $k = n = 0$, Ψ_n maps $Z(N_0) = N_0' \cap N_0$ to $Z(N_{2n}) = N_{2n}' \cap N_{2n}$, which is an isomorphic algebra as N_0 is Morita equivalent to N_{2n}. Thus starting with the trace $\tau_0 = \operatorname{tr}_0$ on $Z(N_0)$, we obtain a sequence of traces on $Z(N_0)$ by setting $\tau_n = \operatorname{tr}_{2n} \circ \Psi_n$. We show that each τ_n on $Z(N_0)$ is faithful, and that $(\tau_n)_{n \ge 0}$ converges to a faithful trace τ_∞ on $Z(N_0)$. We prove this result in Proposition 4.8 below in the language of densities with respect to the trace tr_0 on $Z(N_0)$.

(iii) Now since all faithful traces on a finite dimensional algebra are comparable, for every $n \in \mathbb{N}$, there is a $C_n > 0$ such that $C_n^{-1} \operatorname{tr}_0 \le \tau_n \le C_n \operatorname{tr}_0$ on $Z(N_0) = N_0' \cap N_0$. Since $\{\tau_n | n \in \mathbb{N}\} \cup \{\tau_\infty\}$ is *compact* by (Cii), there is a $C > 0$ *independent* of n such that $C^{-1} \operatorname{tr}_0 \le \tau_n \le C \operatorname{tr}_0$ for all $n \in \mathbb{N}$.

(iv) For all $y \in N_0' \cap M_0$,

$$\begin{aligned}\|\Psi_n(y\Omega)\|_2 &= \operatorname{tr}_{2n}(\Psi_n(y)^*\Psi_n(y)) \underset{\text{(Ci)}}{=} \operatorname{tr}_{2n}(\Psi_n(y^*y)) \\ &= \operatorname{tr}_{2n}(E_{N_{2n}}(\Psi_n(y^*y))) \underset{\text{(Ci)}}{=} \operatorname{tr}_{2n}(\Psi_n(E_{N_0}(y^*y))) = \tau_n(E_{N_0}(y^*y)).\end{aligned}$$

Thus for all $n \in \mathbb{N}$ and all $y \in N_0' \cap M_0$, by (Ciii) we have

$$C^{-1}\|y\Omega\|_2 \le \|\Psi_n(y\Omega)\|_2 \le C\|y\Omega\|_2.$$

(v) It is a simple algebraic calculation that $x_n \in N_n' \cap M_n \subset N_0' \cap M_n$ for all $n \ge 0$. We use the notation $H_n = (N_n' \cap M_n)\Omega$ for this finite dimensional Hilbert space, and we see from (A) that for all $n \in \mathbb{N}$,

$$H_n \cap H_{n+1} = (N_n' \cap M_n) \cap (N_{n+1}' \cap M_{n+1}) = N_{n+1}' \cap M_n = N_1' \cap M_0.$$

(vi) Since any two norms on a finite dimensional Hilbert space are equivalent, there is a constant $K > 0$ independent of n such that for all $y \in N_0' \cap M_0$,

$$\operatorname{dist}(y\Omega, H_0 \cap H_1) \le K(\operatorname{dist}(y\Omega, H_0) + \operatorname{dist}(y\Omega, H_1)).$$

(vii) Finally, we calculate for each $x_{2n} = E_{2n}(x) \in N_{2n}' \cap M_{2n}$, since $\Psi_n(H_k) = H_{2n+k}$,

$$\begin{aligned}
&\operatorname{dist}(x_{2n}\Omega, (N_1' \cap M_0)\Omega) \\
&\qquad = \operatorname{dist}(x_{2n}\Omega, H_{2n} \cap H_{2n+1}) && \text{(by (Cv))} \\
&\qquad \le C \operatorname{dist}(\Psi_n^{-1}(x_{2n})\Omega, H_0 \cap H_1) && \text{(by (Civ))} \\
&\qquad \le CK(\operatorname{dist}(\Psi_n^{-1}(x_{2n})\Omega, H_0) + \operatorname{dist}(\Psi_n^{-1}(x_{2n})\Omega, H_1)) && \text{(by (Cvi))} \\
&\qquad \le C^2K(\operatorname{dist}(x_{2n}\Omega, H_{2n}) + \operatorname{dist}(x_{2n}\Omega, H_{2n+1})) && \text{(by (Civ))} \\
&\qquad = C^2K \operatorname{dist}(x_{2n}\Omega, H_{2n+1}) && \text{(by (Cv))} \\
&\qquad \le C^2K \operatorname{dist}(x_{2n}\Omega, x_{2n+1}\Omega) && \text{(by (Cv))} \\
&\qquad \to 0 \text{ as } n \to \infty. && \text{(by (B))}
\end{aligned}$$

This completes the outline of the proof. □

We now provide the technical details of the above proof. The following lemma is well known. We include a proof for the reader's convenience.

Lemma 4.2. *Suppose R is a von Neumann algebra with a normal faithful tracial state* tr*, and $(R_n)_{n\geq 0}$ is an increasing sequence of unital $*$-subalgebras whose union is strongly dense in R. Let $E_n : R \to R_n$ be the unique trace preserving conditional expectation. Then for all $r \in R$, $\|r\Omega - E_n(r)\Omega\|_2 \to 0$ as $n \to \infty$.*

Proof. Fix $r \in R$, and consider the unital $*$-subalgebra $R^\circ = \bigcup_{n\geq 0} R_n \subset R$. Let X be the $\|\cdot\|_\infty$-closed ball of R of radius $\|r\|_\infty$. Recall from [Jon10, Prop. 9.1.1] that X is a complete metric space in $\|\cdot\|_2$, and the $\|\cdot\|_2$ topology on X agrees with the strong operator topology. Fix $\varepsilon > 0$, and let $B_\varepsilon(r)$ denote the open ball of radius ε in $\|\cdot\|_2$ about r. Pick an open neighborhood $U \subseteq R$ for the strong operator topology such that $U \cap X = B_\varepsilon(r) \cap X$.

By the Kaplansky density theorem, there is an $s \in R^\circ \cap X$ with $x \in B_\varepsilon(r)$. Let $N \in \mathbb{N}$ such that $s \in R_N$. Then since $E_n(r) \in R_n$ is the unique element in R_n closest to r in $\|\cdot\|_2$, for all $n \geq N$, $\|r\Omega - E_n(r)\Omega\|_2 \leq \|r\Omega - s\Omega\|_2 < \varepsilon$. □

4.1 The maps Φ_n

We now define the maps Φ_n which were the main tool for the difficult part of Theorem 4.1.

Definition 4.3. Let $\{b\}$ be a Pimsner-Popa basis for N_1 over N_0. Since the commuting square (1) is horizontally Markov, $\{b\}$ is also a basis for M_1 over M_0. We define Φ_n on $N_0' \cap M_n$ by $\Phi_n(x) = d^{2n}\sum_b be_1e_2\cdots e_{n+1}xe_n\cdots e_2e_1b^*$ (compare with the formula in [Bis97, Thm. 2.13]). Note that Φ_n on $N_0' \cap M_n$ is independent of the choice of basis as in [JP11, Rem. 2.30]. Whenever $z \in N_0' \cap M_n$,

$$\sum_b b \otimes b^* \mapsto \sum_b bzb^*$$

is well-defined, and the left hand side is independent of the choice of $\{b\}$. Since for every $u \in U(N_0)$ $\{ub\}$ is another Pimsner-Popa basis, we see that $\Phi_n(x) \in N_0' \cap M_{n+2}$.

Proposition 4.4. *The maps Φ_n on $N_0' \cap M_n$ enjoy the following properties:*

(1) For all $x \in N_0' \cap N_n$, $\Phi_n(x) \in N_0' \cap N_{n+2}$.

(2) $\Phi_n|_{M_k} = \Phi_k$ for all $0 \leq k \leq n$.

(3) Φ_n is a $*$-algebra map.

(4) For all $0 \leq k \leq n$, if $x \in N_k' \cap M_n$, then $\Phi_n(x) \in N_{k+2}' \cap M_{n+2}$.

(5) For all $x \in N_0' \cap M_n$, $E_{N_{n+2}}(\Phi_n(x)) = \Phi_n(E_{N_n}(x))$.

Proof.

(1) Since $\{b\} \subset N_1$, if $x \in N_0' \cap N_n$, then $\Phi_n(x) \in N_0' \cap N_{n+2}$.

(2) If $x \in M_k$, then $[x, e_j] = 0$ for all $k+1 \leq j \leq n+1$. Thus

$$\begin{aligned}\Phi_n(x) &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_k e_{k+1} \cdots e_n e_{n+1} e_n \cdots e_{k+1} x e_k \cdots e_2 e_1 b^* \\ &= d^{2k} \sum_b b e_1 e_2 \cdots e_{k+1} x e_k \cdots e_2 e_1 b^* \\ &= \Phi_k(x).\end{aligned}$$

(3) For all $x, y \in N_0' \cap M_n$, we have

$$\begin{aligned}\Phi_n(x)\Phi_n(y) &= d^{4n} \sum_{a,b} a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 a^* b e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{4n} \sum_{a,b} a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 E_{N_0}(a^* b) e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{4n} \sum_{a,b} a E_{N_0}(a^* b) e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 e_2 \cdots e_{n+1} y e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x y e_n \cdots e_2 e_1 b^* \\ &= \Phi_n(xy).\end{aligned}$$

(4) Suppose $x \in N_k' \cap M_n$ for $0 \leq k \leq n$, and suppose $y \in N_{k+2}$. We calculate

$$\begin{aligned}\Psi_n(x)y &= \Psi_n(x) y 1_{N_{k+2}} \\ &= \left(d^{2n} \sum_a a e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 a^* \right) y \left(d^{2k} \sum_b b e_1 e_2 \cdots e_k e_{k+1} e_k \cdots e_2 e_1 b^* \right) \\ &= d^{2(n+k)} \sum_{a,b} (a e_1 e_2 \cdots e_n x e_{n+1} e_n \cdots e_{k+2}) \underbrace{(e_{k+1} \cdots e_2 e_1 a^* y b e_1 e_2 \cdots e_{k+1})}_{z_{a,b} e_{k+1}} (e_k \cdots e_2 e_1 b^*)\end{aligned}$$

Since $e_{k+1} N_{k+2} e_{k+1} = N_k e_{k+1}$, for all $a \in \{a\}$ and $b \in \{b\}$, there is a $z_{a,b} \in N_k$ such that $(e_{k+1} \cdots e_2 e_1 a^* y b e_1 e_2 \cdots e_{k+1}) = z_{a,b} e_{k+1}$, as indicated in the underbrace above. Continuing the above calculation, we obtain

$$\begin{aligned}\Psi_n(x)y &= d^{2(n+k)} \sum_{a,b} (a e_1 e_2 \cdots e_n x e_{n+1} e_n \cdots e_{k+2})(z_{a,b} e_{k+1})(e_k \cdots e_2 e_1 b^*) \\ &= d^{2(n+k)} \sum_{a,b} a e_1 e_2 \cdots e_n x z_{a,b} e_{n+1} e_n \cdots e_2 e_1 b^* \qquad (4)\end{aligned}$$

Starting with $y\Phi_n(x)$, a similar calculation shows

$$y\Psi_n(x) = 1_{N_{k+2}} y \Psi_n(x) = d^{2(n+k)} \sum_{a,b} a e_1 e_2 \cdots e_n z_{a,b} x e_{n+1} e_n \cdots e_2 e_1 b^*. \qquad (5)$$

Since each $z_{a,b} \in N_k$ and $x \in N_k' \cap M_n$, (4) is equal to (5), and we are finished.

(5) Suppose $x \in N_0' \cap M_n$. Since

$$\begin{array}{ccc} M_n & \subset & M_{n+2} \\ \cup & & \cup \\ N_n & \subset & N_{n+2} \end{array}$$

is a commuting square, $E_{N_{n+2}}(x) = E_{N_n}(x)$. Since E_{n+2} is $N_{n+2} - N_{n+2}$ bilinear, we have

$$\begin{aligned} E_{N_{n+2}}(\Phi_n(x)) &= E_{N_{n+2}}\left(d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} x e_n \cdots e_2 e_1 b^* \right) \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} E_{N_{n+2}}(x) e_n \cdots e_2 e_1 b^* \\ &= d^{2n} \sum_b b e_1 e_2 \cdots e_{n+1} E_{N_n}(x) e_n \cdots e_2 e_1 b^* \\ &= \Phi_n(E_{N_n}(x)). \end{aligned}$$

□

4.2 Behavior of the traces τ_n

For $n \in \mathbb{N}$, we define $\Psi_n = \Phi_{n-1} \circ \cdots \circ \Phi_1 \circ \Phi_0$. We now observe the behavior of the sequence of traces $\tau_n := \mathrm{tr}_{2n} \circ \Psi_n$ on $Z(N_0) = N_0'' \cap N_0$, with $\tau_0 = \mathrm{tr}_0$ by convention. The following lemma is a straightforward calculation.

Lemma 4.5. *For all $x \in Z(N_0)$ and $n \in \mathbb{N}$,*

$$\tau_n(x) = d^{-2n} \sum_{b_1,\ldots,b_n \in B} \mathrm{tr}_0(x \cdot E_{N_0}(b_1^* E_{N_0}(b_2^* \cdots E_{N_0}(b_n^* b_n) \cdots b_2) b_1))$$

where B is any Pimsner-Popa basis for N_1 over N_0.

There is a unique $k \in \mathbb{N}$ such that $Z(N_0) \cong \mathbb{C}^k$. For $x \in Z(N_0)$, we denote by $\vec{x}$ the vector in $\mathbb{C}^k$ corresponding to x. We define:

- Λ is the bipartite adjacency matrix of the Bratteli diagram for the inclusion $N_0 \subset N_1$, i.e., $\Lambda_{i,j}$ is the number of times the i-th simple summand of N_0 is contained in the j-th simple summand of N_1.

- λ_i is the Markov trace (column) vector for N_i, whose j-th entry $\lambda_i(j)$ is the trace of a minimal projection in the j-th simple summand of N_i. This means $\Lambda\Lambda^T\lambda_0 = d^2\lambda_0$ and $\Lambda^T\Lambda\lambda_1 = d^{-2}\lambda_1$.

- m_i is the dimension (row) vector for N_i, i.e., the j-th simple summand of N_i is a full matrix algebra of size $m_i(j)$. Notice that $m_i\lambda_i = 1$ for $i = 0, 1$.

- $\Delta = \mathrm{diag}(m_0(i))_{i=1}^k$ is the diagonal $k \times k$ matrix whose (i,i)-th entry is $m_0(i)$.

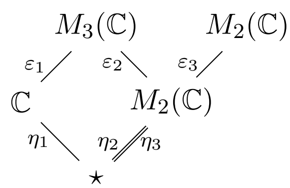
Example 4.6. For the A_4 inclusion $N_0 = \mathbb{C} \oplus M_2(\mathbb{C}) \subseteq M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) = N_1$ we have: $\Lambda = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\lambda_0 = \dfrac{1}{1+2\phi}\begin{bmatrix} 1 \\ \phi \end{bmatrix}$, $\lambda_0 = \dfrac{1}{2+3\phi}\begin{bmatrix} \phi \\ 1 \end{bmatrix}$, $m_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $m_1 = \begin{bmatrix} 3 & 2 \end{bmatrix}$, and $\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Proposition 4.7. *There is a Pimsner-Popa basis B for N_1 over N_0 such that for every $x \in Z(N_0)$, $\sum_{b \in B} E_{N_0}(b^* x b) \in Z(N_0)$. Moreover, under the isomorphism $Z(N_0) \cong \mathbb{C}^k$, we have $\sum_{b \in B} [E_{N_0}(b^* x b)]^{\rightarrow} = \Delta^{-1}\Lambda\Lambda^T\Delta\vec{x}$.*

Proof. We use the loop basis for $N_0 \subseteq N_1$ afforded by [JP11, §3.1-3.2]. We label the edges of Λ by ε, with source $s(\varepsilon)$ an even vertex corresponding to a simple summand of N_0, and target $t(\varepsilon)$ an odd vertex corresponding to a simple summand of N_1. We introduce a new vertex $\star$ with edges η, with each source $s(\eta) = \star$, and target $t(\eta)$ an even vertex corresponding to a simple summand of N_0. The number of edges η from $\star$ to the i-th even vertex is equal to $m_0(i)$. We denote by ε^* and η^* the edge with the reverse orientation.

We give an explicit basis for N_0 by loops of length 2 starting at $\star$, where adjoint is given by the conjugate linear extension of $[\eta_i\eta_j^*]^* = [\eta_j\eta_i^*]$, and multiplication is given by $[\eta_i\eta_j^*] \cdot [\eta_k\eta_\ell^*] = \delta_{j=k}[\eta_i\eta_\ell^*]$. We give an explicit basis for N_1 by loops of length 4 starting at $\star$, where adjoint is given by the conjugate linear extension of $[\eta_i\varepsilon_j\varepsilon_k^*\eta_\ell^*]^* = [\eta_\ell\varepsilon_k\varepsilon_j^*\eta_i^*]$, and multiplication is given by $[\eta_i\varepsilon_j\varepsilon_k^*\eta_\ell^*] \cdot [\eta_m\varepsilon_n\varepsilon_p^*\eta_q^*] = \delta_{\ell=m}\delta_{k=n}[\eta_i\varepsilon_j\varepsilon_p^*\eta_q^*]$. The trace tr_1 on N_1 is given by $\mathrm{tr}_1([\eta_i\varepsilon_j\varepsilon_k^*\eta_\ell^*]) = \delta_{\eta_i=\eta_\ell}\delta_{\varepsilon_j=\varepsilon_k}\lambda_1(t(\varepsilon_j))$, and the trace tr_0 on N_0 is given by $\mathrm{tr}_0([\eta_i\eta_j^*]) = \delta_{\eta_i=\eta_j}\lambda_0(t(\eta_i))$. The unital inclusion $N_0 \subseteq N_1$ is given by $[\eta_i\eta_j^*] \mapsto \sum_{s(\varepsilon)=t(\eta_i)}[\eta_i\varepsilon\varepsilon^*\eta_j^*]$, and the unique trace-preserving conditional expectation is given by $E_{N_0}([\eta_i\varepsilon_j\varepsilon_k^*\eta_\ell^*]) = \delta_{\varepsilon_j=\varepsilon_k}\left(\frac{\lambda_1(t(\varepsilon_j))}{\lambda_0(s(\varepsilon_j))}\right)[\eta_i\eta_\ell^*]$.

For example, the inclusion $N_0 = \mathbb{C} \oplus M_2(\mathbb{C}) \subseteq M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) = N_1$ from Example 4.6 could be represented in the loop basis as follows:



Now by [JP11, Prop. 3.22] and [JP11, Rem. 3.23], a Pimsner-Popa basis for N_1 over N_0 is given by $B = B_1 \amalg B_2$ where

$$B_1 = \left\{ \left(\frac{\lambda_0(s(\varepsilon_2))}{\lambda_1(t(\varepsilon_2))}\right)^{1/2} \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\eta^*] \middle| s(\varepsilon_1) = s(\varepsilon_2) \text{ and } t(\varepsilon_1) = t(\varepsilon_2) \right\}$$
$$B_2 = \left\{ \left(\frac{\lambda_0(s(\varepsilon_2))}{m_0(s(\varepsilon_2))\lambda_1(t(\varepsilon_2))}\right)^{1/2} [\eta_1\varepsilon_1\varepsilon_2^*\eta_2^*] \middle| s(\varepsilon_1) \neq s(\varepsilon_2) \right\}.$$

Now the minimal central projection in N_0 corresponding to the i-th simple summand is equal to $p_i = \sum_{t(\eta)=i}[\eta\eta^*]$. One calculates that $\sum_{b\in B_1} E_{N_0}(b^*p_ib) = \sum_j \Lambda_{i,j}^2 p_i$, while $\sum_{b\in B_2} E_{N_0}(b^*p_ib) = \sum_{i'\neq i}\sum_j \frac{m_0(i)}{m_0(i')}\Lambda_{i,j}\Lambda_{i',j}p_{i'}$. Hence we have that $\sum_{b\in B} E_{N_0}(b^*p_ib)$ is in $Z(N_0)$ with corresponding vector in $\mathbb{C}^k$ equal to $\Delta^{-1}\Lambda\Lambda^T\Delta\vec{e}_i$. Now since every element of $Z(N_0)$ is a linear combination of the p_i, the result follows. □

Equipped with this explicit Pimsner-Popa basis, we are prepared to analyze the traces τ_n. Note that an arbitrary tracial state τ on $Z(N_0)$ is always of the form $\tau(y) = \mathrm{tr}_0(y \cdot h)$ for some positive operator $h \in Z(N_0)$ with $\mathrm{tr}_0(h) = 1$ called the *density* of τ. Let $\mathcal{T}$ be the topological space of traces on $Z(N_0)$, and note that we may identify the *pointed* topological space $(\mathcal{T}, \mathrm{tr}_0)$ with $(\{h \in Z(N_0) | h \geq 0 \text{ and } \mathrm{tr}(h) = 1\}, 1_{Z(N_0)})$.

We see from Lemma 4.5, using the Pimsner-Popa basis B from Proposition 4.7, that the densities $h_n \in Z(N_0)$ of the τ_n are given inductively by $h_n = d^{-2}\sum_{b\in B} E_{N_0}(b^*h_{n-1}b)$ for all $n \in \mathbb{N}$. Letting $\vec{h}_n \in \mathbb{C}^k$ be the vector corresponding to $h_n \in Z(N_0)$, Proposition 4.7 tells us that

$\vec{h}_n = d^{-2}\Delta^{-1}\Lambda\Lambda^T\Delta\vec{h}_{n-1}$ for all $n \in \mathbb{N}$. Since the density h_0 of $\tau_0 = \mathrm{tr}_0$ is $1_{Z(N_0)}$, for all $n \in \mathbb{N}$,

$$\vec{h}_n = d^{-2n}\Delta^{-1}(\Lambda\Lambda^T)^n\Delta\vec{1}, \tag{6}$$

where $\vec{1} \in \mathbb{C}^k$ is the vector whose entries are all 1.

Proposition 4.8. *The traces τ_n are faithful and converge to a faithful trace τ_∞ on $Z(N_0)$.*

Proof. The density vector $\vec{h}_n = d^{-2n}\Delta^{-1}(\Lambda\Lambda^T)^n\Delta\vec{1}$ from (6) has strictly positive entries, and thus τ_n is faithful for all n. Second, the limit of $d^{-2n}\Delta^{-1}(\Lambda\Lambda^T)^n\Delta\vec{1}$ as $n \to \infty$ is well known to be $\Delta^{-1}\vec{\lambda}$, where $\vec{\lambda}$ is a suitably normalized Frobenius-Perron eigenvector for $\Lambda\Lambda^T$. Since $\vec{\lambda}_1$ had all strictly positive, $\vec{\lambda}$ has all entries strictly positive. Hence the densities $\vec{h}_n$ converge to $\vec{h}_\infty = \Delta^{-1}\vec{\lambda}$, which gives a faithful trace τ_∞ on $Z(N_0)$. (Note that τ_∞ is not tr_0 even if $\Delta = I_k$, since its density with respect to tr_0 is $\vec{\lambda}$, which is in general not $\vec{1}$.) □

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