Exactly 1-3T subfactors w/ index at most 6.2.

Goals:
1. SFTs + invariants
2. SF classification program - 1-3T case.
3. Examples - braided + “6-braided” categories.

Big picture:

\[ \begin{array}{ccc}
B & \rightarrow & P_0 \\
\downarrow & & \uparrow \\
A & \leftarrow & (C_+, P_-)
\end{array} \]

Subfactors:
ACB until inclusion of II_1-factors

- A factor \( ACB(H) \) is a vNa \( (A=A') \) w/ trivial center \( (A',A)\approx Z(A)=C_1 \).
- A factor is type II_1 if it is separable and has a trace.
- Can think of it as a complex \( * \)-algebra, simple.

Index: finite if \( \text{index} \) is a finite proj. module.

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\[ [B:A] = n + m([B] \in K_0(CA)^+) \]

\[ = n + m(p) \text{ where } A \cong \bigoplus A_i \oplus A_{p} \text{ (fractional point).} \]

Standard invariant: unitary 2-category “rep theory of SF”

Objects: \( A, B \)

1-maps: \( \otimes B \), split into simple \( A-A, A-B, B-A, B-B \)

“Semi-simplicity comes from analysis”

2-maps: intertwiners. \( \text{Hom}(P, Q) \) is divided
rigid structure: dual of bimodule is contragredient.
- have evaluation, coevaluation, pivotality
unitary structure: intertwiners are bimod maps
- there is an adjoint intertwiner
- also from analysis.

- a unitary 2-category + choice of 1-morphism
(for us it is \( \mathcal{B}_B \)) gives a planar algebra
via the usual diagrammatic calculus.
- gives a generators + relations approach to constructing subfactors.

Principal graphs:
"induction-restriction graphs"

\( P^- \) biparte graphs
\( P^+ \):
- even vertices: simple \( A \rightarrow A \) bimods / \( \alpha \)
- odd vertices: simple \( A \rightarrow B \) bimods / \( \alpha \)
- edges: \( \dim(\text{Hom}(P \otimes B, Q)) \) edges from simple \( P \rightarrow Q \)
- \( \beta \)

\( P^- \):
restriction graphs: \( B \rightarrow B \) to \( B \rightarrow A \) by \( \otimes R \).

\( \ast \) rigid structure gives duality
\( A \rightarrow A \rightarrow A \rightarrow A \)
\( B \rightarrow B \rightarrow B \rightarrow B \)
\( A \rightarrow B \rightarrow B \rightarrow B \)

SF classification program
i) enumerate graph pairs \((P^+, P^-)\), apply obstructions.
ii) construct examples when graphs survive.
iii) fit exotic/exceptional examples into families.
recent survey of Jones-Harrison-Snyder in Bull. AMS.
- complete classification of standard invariants to $5$
- some results above $5$, not many!

Supertransitivity: $P_E$ is $k$-supertransitive (ST) if

- $A_n$ is $k$-ST $\iff k > 0$
- $D_n$ is $(2n-3) - ST$, not $(2n-2) - ST$

- Morrison-Peters classified $1$-ST w/ index below $3+\sqrt{5}$
- Liu classified at $3+\sqrt{5}$, partial proof by Iani-Harrison-Penneys (project during last ANU visit)
- in joint work w/ Liu + Morrison, we classify up to index 6.2 [except at index 6]

Intermediates: exactly $1$-ST if:

- $+B_3$ irreducible
- $A_4 \cong A_4 \oplus [B_4 A_4]$ reducible

- if 3 intermediate SF: $A C D C B$, then $B_4 A_4$ is reducible. $B \cong A \oplus [B_4 A_4] \oplus [B_3 D_4]$
- index multiplicative. If 3 nontrivial intermediate, index $\leq 6.2$, $(C D: A_7, C B: D_7) = (2, 2), (2, 7^2), (2, 3)$
- $2 = 15/3$. 

Table:

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<th>Step</th>
<th>Index &lt; 4</th>
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- Composite standard invariants at $2^{\times} = 4$ and $2^{\times} = 3 + \sqrt{5}$ are completely classified.
- At index $0$, they are wild. "No $\mathbb{Z}_2 \times \mathbb{Z}_3$"
- All 1-\text{st} std. inv. at index $0$ are composite.

Thm (Liu-Monton-Pereyra): If a standard invariant is exactly 1-\text{st} w/o intermediate, index in $(3 + \sqrt{5}, 0.27)$, then it is one of 3 or index $3 + 2\sqrt{5}$.

Map of 1-\text{st} SF's:

- 3 examples at $3 + 2\sqrt{5}$ are actually unitary fusion cats.

1. Rep($SO(3)_E$) or $SU(2)_E$ on $(g, \nu, S) = (SU(2), V, \nu)$
   - generated by a "quadratic breaking"

- satisfying the following relations:
\( \binom{11}{2} \quad \frac{1}{n} = \frac{1}{n} \quad \text{and} \quad \frac{1}{n} = u \)

2. unit \( / R1 \)
\( n = a \quad \frac{1}{n} = a \quad \frac{1}{n} = a \quad \text{at C} \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \)

3. rotate \( \alpha = x \quad \alpha = x \quad \text{needs } g + \text{lower order terms} \)

4. twist \( / R2 \)
\( 6 \quad \alpha = x \quad 6 \quad 6 \)

5. quadratic \( \phi \quad \text{span } \{ 11, \frac{1}{n}, x \} \quad 3 \)

- Can evaluate closed diagrams via skein-template algorithm \( \text{"also a Hecke alg. argument"} \)

\[ \Rightarrow 3 \text{ at most 1 category with these relms.} \]

6. \( \text{"6-braided" / twisted variations. } 6 = \pm i \)

- Sprinkle in some 6's in above relations.

- Again, at most 1 category for \( 6 = \pm i \).

\( \text{\"do these also come from quantum groups?\"} \quad \text{\"yes, Lue's current work\"} \)

To show existence, find representation in a graph planar algebra (GPA)

To show uniqueness, show that the GPA repin of a planar algebra by graph \( \begin{array}{c} \text{\.de} \end{array} \) is one of above 3 examples.