Commutants of multifusion categories

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Categorical analogies

Tensor categories categorify algebras.

algebra A	tensor category $\mathcal{C}$
finite dimensional algebra	fusion category
center $Z(A)$	Drinfel'd center $\mathcal{Z}(\mathcal{C})$
commutant $Z_B(A)$ of A in B	commutant $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{D}$
$B(H)$	$\mathrm{Bim}(R)$, all bimodules
commutant $A' := Z_{B(H)}(A)$	commutant $\mathcal{C}' := \mathcal{Z}_{\mathrm{Bim}(R)}(\mathcal{C})$
von Neumann algebra $A = A''$	bicommutant category $\mathcal{C} \cong \mathcal{C}''$

Bicommutant categories categorify von Neumann algebras.

Categorifying basic theorems

In previous work with Henriques, we proved the *categorified finite dimensional bicommutant theorem*.

Theorem [HP15]

Suppose $\mathcal{C}$ is a unitary fusion category embedded in $\mathrm{Bim}(R)$, where R is a non type I factor. Then

$$\mathcal{C}'' \cong \mathcal{C} \otimes_{\mathsf{Vec}} \mathsf{Hilb} \cong \mathsf{Hilb}(\mathcal{C}).$$

Today, we will prove the categorification of:
Two Morita equivalent finite dimensional von Neumann algebras embedded in $B(H)$ have isomorphic commutants.

Unitary multifusion categories

Definition

A $k \times k$ unitary multifusion category is a rigid C*-tensor category $\mathcal{C}$ satisfying:

- $\mathcal{C}$ is idempotent complete,
- $\mathcal{C}$ has finitely many isomorphism classes of simple objects,
- $1_{\mathcal{C}} = \bigoplus_{i=1}^{k} 1_i$ where each 1_i is simple, and
- $\mathcal{C}$ is indecomposable.

Proposition

Every unitary $k \times k$ multifusion category has a fully faithful tensor embedding $\mathcal{C} \hookrightarrow \mathrm{Bim}(R^{\oplus k})$ which is dimension preserving.

- The proof uses a modification of Ocneanu compactness [JS97].

Graphical calculus

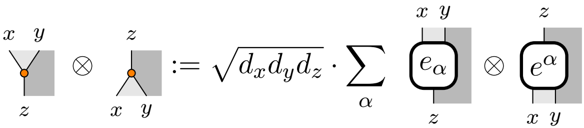
Fix a finite set $\mathrm{Irr}(\mathcal{C})$ of representatives of irreducibles.

- Shaded regions denote irreducible summands of $1_{\mathcal{C}}$.
- Morphisms $f : x \otimes y \to z$ are represented by coupons.
- For all simple $x \in \mathcal{C}_{i,j}$, $y \in \mathcal{C}_{j,k}$, and ,$z \in \mathcal{C}_{k,i}$, $\mathrm{Hom}(1, x \otimes y \otimes z)$ is a finite dimensional Hilbert space with inner product $\langle f, g \rangle = g^* \circ f$.

Choose dual bases:

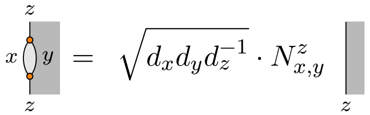
$$e_i \in \mathrm{Hom}(1, x \otimes y \otimes z) \text{ and } e^i \in \mathrm{Hom}(1, \overline{z} \otimes \overline{y} \otimes \overline{x})$$

We represent the canonical element by colored nodes

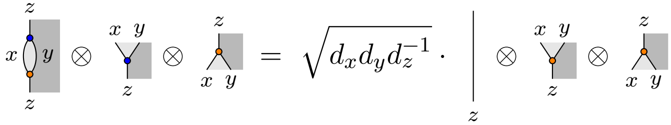


The canonical element is independent of choice of basis.

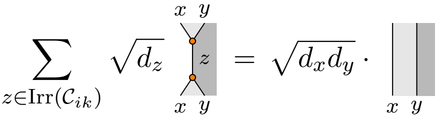
Important relations



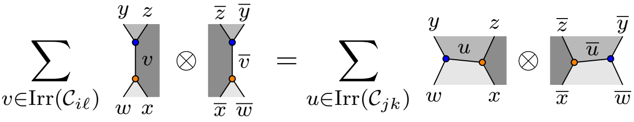
(Bigon 1)



(Bigon 2)



(Fusion)



(I=H)

We'll use Snyder convention and ignore all scalars.

Commutant $\mathcal{C}'$ of $\mathcal{C}$ in $\mathrm{Bim}(R^{\oplus k})$

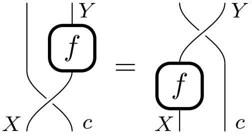
The commutant $\mathcal{C}' \subset \mathrm{Bim}(R^{\oplus k})$ of $\mathcal{C} \subset \mathrm{Bim}(R^{\oplus k})$ has:

- Objects are pairs (X, e_X) where $X \in \mathrm{Bim}(R^{\oplus k})$, and e_X is a unitary half braiding with $\mathcal{C}$

  $e_{X,c} =$ 

  $: X \boxtimes c \to c \boxtimes X$

  These half braidings must satisfy compatibility conditions.
- Morphisms $f : (X, e_X) \to (Y, e_Y)$ are bimodule maps $f : X \to Y$ which commute with the half braidings:

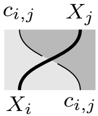
$\mathcal{C}'$ is a tensor category, but it is usually not braided.

Describing $\mathcal{C}'$ for unitary multifusion

Suppose $(X, e_X) \in \mathcal{C}' \subset \mathrm{Bim}(R^{\oplus k})$.
Write $X = (X_{i,j})$ where $X_{i,j}$ is an $R_i - R_j$ bimodule.

Easy facts about $(X, e_X) \in \mathcal{C}'$

1. $X \boxtimes 1_j \cong 1_j \boxtimes X \boxtimes 1_j$ implies $X_{i,j} = 0$ for $i \neq j$.
2. Writing $X = \bigoplus_{i=1}^{k} X_i$, we can write e_X as a family of natural isomorphisms (e_X^i) given on $c_{i,j} \in \mathcal{C}_{i,j}$ by



$$= e_X^i \in \mathrm{Hom}_{R_i - R_j}(X_i \boxtimes_{R_i} c_{i,j} \to c_{i,j} \boxtimes_{R_j} X_j)$$

3. We have a projection functor $P_j : \mathcal{C}' \to \mathcal{C}'_j$ by $(X, e_X) \mapsto (X_j, e_X^j)$.

Induction functor $\mathrm{Bim}(R) \to \mathcal{C}'$

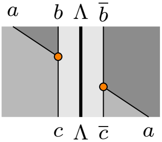
We have a way to construct lots of objects in $\mathcal{C}'$.

We always use the shading ▢ $= R_1$.

$$\underline{\Phi} : \mathrm{Bim}(R_1) \to \mathcal{C}' \qquad \underline{\Phi}(\Lambda) = (\Phi(\Lambda), e_{\Phi(\Lambda)}).$$

$$\Phi(\Lambda) := \bigoplus_{\substack{j=1,\dots,k \\ c \in \mathrm{Irr}(\mathcal{C}_{j,1})}} c \boxtimes \Lambda \boxtimes \bar{c} \in \mathrm{Bim}(R^{\oplus k})$$

$$e_{\Phi(\Lambda),a} := \sum_{\substack{i,j \in \{1,\dots,k\} \\ \text{▨} = R_i \\ \blacksquare = R_j}} \; \sum_{\substack{b \in \mathrm{Irr}(\mathcal{C}_{j,1}) \\ c \in \mathrm{Irr}(\mathcal{C}_{i,1})}} \sqrt{d_a^{-1}}$$



; $a \in \mathcal{C}_{i,j}$

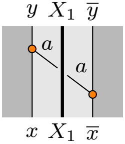
Proposition

The functor $\Phi : \mathrm{Bim}(R_1) \to \mathcal{C}'$ is dominant.

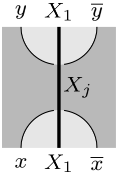
A canonical projector

For $(X_1, e_{X_1}) \in \mathcal{C}_1'$, we have a canonical projector in $\mathrm{End}_{\mathcal{C}'}(\Phi(X_1))$:

$$p_{X_1} := \frac{1}{D} \sum_{\substack{j\in\{1,\dots,k\} \\ \square = R_j}} \sum_{\substack{a\in\mathrm{Irr}(\mathcal{C}_1) \\ x,y\in\mathrm{Irr}(\mathcal{C}_{j,1})}} \sqrt{d_a}$$



$$= \frac{1}{D} \sum_{\substack{j\in\{1,\dots,k\} \\ \square = R_j}} \sum_{x,y\in\mathrm{Irr}(\mathcal{C}_{j,1})} \sqrt{d_x d_y}$$



Equivalence

We have functors

$$\mathrm{Bim}(R^{\oplus k}) \ni \mathcal{C}' \xrightarrow{P_1} \mathcal{C}_1' \subset \mathrm{Bim}(R_1)$$

$$\mathrm{Bim}(R_1) \ni \mathcal{C}_1' \xrightarrow{\Phi} \mathcal{C}' \in \mathrm{Bim}(R^{\oplus k})$$

We get another functor $p\Phi : \mathcal{C}_1' \to \mathcal{C}'$ by applying Φ and then applying the canonical projector.

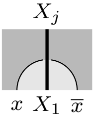
Theorem

The functors P_1 and $p\Phi$ witness an equivalence of categories $\mathrm{Bim}(R_1) \supseteq \mathcal{C}_1' \cong \mathcal{C}' \subseteq \mathrm{Bim}(R^{\oplus k})$.

Sketch of one direction.

We get a natural isomorphism $u : p\Phi \circ P_1 \Rightarrow \mathrm{id}$ where $u_X : p_X\Phi(X_1) \to X$ is given by

$$u_X = \frac{1}{\sqrt{D}} \sum_{\substack{j\in\{1,\ldots,k\} \\ \text{▨}=R_j}} \sum_{x\in\mathrm{Irr}(\mathcal{C}_{j,1})} \sqrt{d_x}$$



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The main corollary

We can now prove our main result as a corollary.

Corollary

If $\mathcal{C}_1 \subset \mathrm{Bim}(R_1)$ and $\mathcal{C}_2 \subset \mathrm{Bim}(R_2)$ are two Morita equivalent unitary fusion categories, then $\mathcal{C}_1' \cap \mathrm{Bim}(R_1) \cong \mathcal{C}_2' \cap \mathrm{Bim}(R_2)$.

Proof.

Let $\mathcal{M} \subset \mathrm{Bim}(R_1, R_2)$ be an equivalence unitary $\mathcal{C}_1 - \mathcal{C}_2$ bimodule category. We can form a 2×2 unitary multifusion category by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{M} \\ \mathcal{M}^* & \mathcal{C}_2 \end{pmatrix} \subset \mathrm{Bim}(R_1 \oplus R_2).$$

Now we apply the previous theorem twice:

$$\mathcal{C}_1' \cap \mathrm{Bim}(R_1) \cong \mathcal{C}' \cap \mathrm{Bim}(R_1 \oplus R_2) \cong \mathcal{C}_2' \cap \mathrm{Bim}(R_2). \qquad \square$$

Thank you for listening!

Slides available at:
`https://people.math.osu.edu/penneys.2/PenneysAMS2017.pdf`

Previous article *Bicommutant categories from fusion categories* with André Henriques available at:
`http://arxiv.org/abs/1511.05226`

New article *Commutants of multifusion categories* with André Henriques coming soon!