Commutants of multifusion categories

David Penneys, OSU
joint with André Henriques

AMS Special Session on Fusion categories and applications

April 1, 2017
## Categorical analogies

Tensor categories categorify algebras.

<table>
<thead>
<tr>
<th>algebra $A$</th>
<th>tensor category $C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite dimensional algebra</td>
<td>fusion category</td>
</tr>
<tr>
<td>center $Z(A)$</td>
<td>Drinfel’d center $Z(C)$</td>
</tr>
<tr>
<td>commutant $Z_B(A)$ of $A$ in $B$</td>
<td>commutant $Z_D(C)$ of $C$ in $D$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$B(H)$</th>
<th>$\text{Bim}(R)$, all bimodules</th>
</tr>
</thead>
<tbody>
<tr>
<td>commutant $A' := Z_{B(H)}(A)$</td>
<td>commutant $C' := Z_{\text{Bim}(R)}(C)$</td>
</tr>
<tr>
<td>von Neumann algebra $A = A''$</td>
<td>bicommutant category $C \cong C''$</td>
</tr>
</tbody>
</table>

Bicommutant categories categorify von Neumann algebras.
Categorifying basic theorems

In previous work with Henriques, we proved the *categorified finite dimensional bicommutant theorem*.

**Theorem [HP15]**

Suppose $\mathcal{C}$ is a unitary fusion category embedded in $\text{Bim}(R)$, where $R$ is a non type I factor. Then

$$\mathcal{C}'' \cong \mathcal{C} \otimes_{\text{Vec Hilb}} \text{Hilb} \cong \text{Hilb}(\mathcal{C}).$$

Today, we will prove the categorification of:

*Two Morita equivalent finite dimensional von Neumann algebras embedded in $B(H)$ have isomorphic commutants.*
Unitary multifusion categories

Definition
A $k \times k$ unitary multifusion category is a rigid C*-tensor category $\mathcal{C}$ satisfying:

- $\mathcal{C}$ is idempotent complete,
- $\mathcal{C}$ has finitely many isomorphism classes of simple objects,
- $1_{\mathcal{C}} = \bigoplus_{i=1}^{k} 1_i$ where each $1_i$ is simple, and
- $\mathcal{C}$ is indecomposable.

Proposition
Every unitary $k \times k$ multifusion category has a fully faithful tensor embedding $\mathcal{C} \hookrightarrow \text{Bim}(R^\oplus k)$ which is dimension preserving.

- The proof uses a modification of Ocneanu compactness [JS97].
Graphical calculus

Fix a finite set $\text{Irr}(\mathcal{C})$ of representatives of irreducibles.

- Shaded regions denote irreducible summands of $1_{\mathcal{C}}$.
- Morphisms $f : x \otimes y \rightarrow z$ are represented by coupons.
- For all simple $x \in \mathcal{C}_{i,j}$, $y \in \mathcal{C}_{j,k}$, and $z \in \mathcal{C}_{k,i}$, $\text{Hom}(1, x \otimes y \otimes z)$ is a finite dimensional Hilbert space with inner product $\langle f, g \rangle = g^* \circ f$.

Choose dual bases:

$$e_i \in \text{Hom}(1, x \otimes y \otimes z) \text{ and } e^i \in \text{Hom}(1, z \otimes y \otimes x)$$

We represent the canonical element by colored nodes

$$x \ y \ z \ \otimes \ z \ x \ y := \sqrt{d_x d_y d_z} \cdot \sum_{\alpha} e_{\alpha} \otimes e^\alpha$$

The canonical element is independent of choice of basis.
Important relations

(Bigon 1)

(Bigon 2)

(Fusion)

(I=H)

We’ll use Snyder convention and ignore all scalars.
Commutant $\mathcal{C}'$ of $\mathcal{C}$ in $\text{Bim}(R^{\oplus k})$

The commutant $\mathcal{C}' \subset \text{Bim}(R^{\oplus k})$ of $\mathcal{C} \subset \text{Bim}(R^{\oplus k})$ has:

- Objects are pairs $(X, e_X)$ where $X \in \text{Bim}(R^{\oplus k})$, and $e_X$ is a unitary half braiding with $\mathcal{C}$

$$e_{X,c} = \begin{tikzpicture}[baseline=-0.65ex]
\node (X) at (0,0) {$X$};
\node (c) at (1,0) {$c$};
\draw[->] (X) to node[above]{$\mathcal{C}$} (c);
\draw[->] (X) to node[right]{$e_X$} (c);
\end{tikzpicture} : X \boxtimes c \to c \boxtimes X$$

These half braidings must satisfy compatibility conditions.

- Morphisms $f : (X, e_X) \to (Y, e_Y)$ are bimodule maps $f : X \to Y$ which commute with the half braidings:

$$\begin{tikzpicture}[baseline=-0.65ex]
\node (f) at (0,0) {$f$};
\node (X) at (0,1) {$X$};
\node (Y) at (0,2) {$Y$};
\draw[->] (f) to node[left]{$X$} (X);
\draw[->] (f) to node[right]{$Y$} (Y);
\draw[->,dashed] (X) to node[below]{$c$} (Y);
\end{tikzpicture} = \begin{tikzpicture}[baseline=-0.65ex]
\node (f) at (0,0) {$f$};
\node (X) at (0,1) {$X$};
\node (Y) at (0,2) {$Y$};
\draw[->] (f) to node[left]{$X$} (X);
\draw[->] (f) to node[right]{$Y$} (Y);
\draw[->,dashed] (X) to node[below]{$c$} (Y);
\end{tikzpicture}$$

$\mathcal{C}'$ is a tensor category, but it is usually not braided.
Describing $\mathcal{C}'$ for unitary multifusion

Suppose $(X, e_X) \in \mathcal{C}' \subset \text{Bim}(R^{\oplus k})$.
Write $X = (X_{i,j})$ where $X_{i,j}$ is an $R_i - R_j$ bimodule.

Easy facts about $(X, e_X) \in \mathcal{C}'$

1. $X \boxtimes 1_j \cong 1_j \boxtimes X \boxtimes 1_j$ implies $X_{i,j} = 0$ for $i \neq j$.

2. Writing $X = \bigoplus_{i=1}^k X_i$, we can write $e_X$ as a family of natural isomorphisms $(e^i_X)$ given on $c_{i,j} \in C_{i,j}$ by

\[
\begin{array}{c}
\begin{array}{c}
\text{c}_{i,j} \\
\text{X}_j \\
\text{X}_i \\
\text{c}_{i,j}
\end{array}
\end{array}
\begin{array}{c}
\text{X}_j \\
\text{c}_{i,j} \\
\text{X}_i \\
\text{c}_{i,j}
\end{array}
= e^i_X \in \text{Hom}_{R_i - R_j}(X_i \boxtimes_{R_i} c_{i,j} \to c_{i,j} \boxtimes_{R_j} X_j)
\]

3. We have a projection functor $P_j : \mathcal{C}' \to \mathcal{C}'_j$ by

$(X, e_X) \mapsto (X_j, e^j_X)$. 
**Induction functor** $\text{Bim}(R) \to C'$

We have a way to construct lots of objects in $C'$. We always use the shading $\square = R_1$.

$$
\Phi : \text{Bim}(R_1) \to C' \quad \Phi(\Lambda) = (\Phi(\Lambda), e_{\Phi(\Lambda)}).
$$

$$
\Phi(\Lambda) := \bigoplus_{j=1, \ldots, k} \sum_{c \in \text{Irr}(C_j, 1)} c \otimes \Lambda \otimes \overline{c} \in \text{Bim}(R^{\oplus k})
$$

$$
e_{\Phi(\Lambda), a} := \sum_{i, j \in \{1, \ldots, k\}} \sum_{b \in \text{Irr}(C_j, 1)} \sqrt{d_{a}^{-1} b} ; a \in C_{i,j}
$$

**Proposition**

The functor $\Phi : \text{Bim}(R_1) \to C'$ is dominant.
A canonical projector

For \((X_1, e_{X_1}) \in C'_1\), we have a canonical projector in \(\text{End}_{C'_1}(\Phi(X_1))\):

\[
p_{X_1} := \frac{1}{D} \sum_{j \in \{1, \ldots, k\}} \sum_{a \in \text{Irr}(C_1)} \sqrt{d_a} \sum_{x, y \in \text{Irr}(C_{j,1})} \sqrt{d_x d_y}
\]

\[
= \frac{1}{D} \sum_{j \in \{1, \ldots, k\}} \sum_{x, y \in \text{Irr}(C_{j,1})} \sqrt{d_x d_y}
\]
Equivalence

We have functors

\[
\text{Bim}(R^\oplus k) \ni \mathcal{C} \xrightarrow{P_1} \mathcal{C}_1' \subset \text{Bim}(R_1)
\]

\[
\text{Bim}(R_1) \ni \mathcal{C}_1' \xrightarrow{\Phi} \mathcal{C}' \in \text{Bim}(R^\oplus k)
\]

We get another functor \( p\Phi : \mathcal{C}_1' \to \mathcal{C}' \) by applying \( \Phi \) and then applying the canonical projector.

**Theorem**

The functors \( P_1 \) and \( p\Phi \) witness an equivalence of categories

\[
\text{Bim}(R_1) \supseteq \mathcal{C}_1' \cong \mathcal{C}' \subseteq \text{Bim}(R^\oplus k).
\]

**Sketch of one direction.**

We get a natural isomorphism \( u : p\Phi \circ P_1 \Rightarrow \text{id} \) where

\[
u_X : p_X\Phi(X_1) \to X \text{ is given by}
\]

\[
u_X = \frac{1}{\sqrt{D}} \sum_{j \in \{1, \ldots, k\}} \sum_{x \in \text{Irr}(C_j, 1)} \sqrt{d_x}.
\]
The main corollary

We can now prove our main result as a corollary.

**Corollary**

If $C_1 \subset \text{Bim}(R_1)$ and $C_2 \subset \text{Bim}(R_2)$ are two Morita equivalent unitary fusion categories, then $C'_1 \cap \text{Bim}(R_1) \cong C'_2 \cap \text{Bim}(R_2)$.

**Proof.**

Let $\mathcal{M} \subset \text{Bim}(R_1, R_2)$ be an equivalence unitary $C_1 - C_2$ bimodule category. We can form a $2 \times 2$ unitary multifusion category by

$$
C = \begin{pmatrix}
C_1 & \mathcal{M} \\
\mathcal{M}^* & C_2
\end{pmatrix} \subset \text{Bim}(R_1 \oplus R_2).
$$

Now we apply the previous theorem twice:

$$
C'_1 \cap \text{Bim}(R_1) \cong C' \cap \text{Bim}(R_1 \oplus R_2) \cong C'_2 \cap \text{Bim}(R_2).
$$
Thank you for listening!

Slides available at:
https://people.math.osu.edu/penneys.2/PenneysAMS2017.pdf

Previous article *Bicommutant categories from fusion categories* with André Henriques available at:
http://arxiv.org/abs/1511.05226

New article *Commutants of multifusion categories* with André Henriques coming soon!