

# Commutants of multifusion categories

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# Categorical analogies

Tensor categories categorify algebras.

algebra $A$ finite dimensional algebra center $Z(A)$ commutant $Z_B(A)$ of $A$ in $B$	tensor category $\mathcal{C}$ fusion category Drinfel'd center $\mathcal{Z}(\mathcal{C})$ commutant $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{D}$
$B(H)$ commutant $A' := Z_{B(H)}(A)$ von Neumann algebra $A = A''$	$\text{Bim}(R)$ , all bimodules commutant $\mathcal{C}' := \mathcal{Z}_{\text{Bim}(R)}(\mathcal{C})$ bicommutant category $\mathcal{C} \cong \mathcal{C}''$

Bicommutant categories categorify von Neumann algebras.

# Categorifying basic theorems

In previous work with Henriques, we proved the *categorified finite dimensional bicommutant theorem*.

## Theorem [HP15]

Suppose  $\mathcal{C}$  is a unitary fusion category embedded in  $\text{Bim}(R)$ , where  $R$  is a non type I factor. Then

$$\mathcal{C}'' \cong \mathcal{C} \otimes_{\text{Vec}} \text{Hilb} \cong \text{Hilb}(\mathcal{C}).$$

Today, we will prove the categorification of:

*Two Morita equivalent finite dimensional von Neumann algebras embedded in  $B(H)$  have isomorphic commutants.*

# Unitary multifusion categories

## Definition

A  $k \times k$  unitary multifusion category is a rigid  $C^*$ -tensor category  $\mathcal{C}$  satisfying:

- ▶  $\mathcal{C}$  is idempotent complete,
- ▶  $\mathcal{C}$  has finitely many isomorphism classes of simple objects,
- ▶  $1_{\mathcal{C}} = \bigoplus_{i=1}^k 1_i$  where each  $1_i$  is simple, and
- ▶  $\mathcal{C}$  is indecomposable.

## Proposition

Every unitary  $k \times k$  multifusion category has a fully faithful tensor embedding  $\mathcal{C} \hookrightarrow \text{Bim}(R^{\oplus k})$  which is dimension preserving.

- ▶ The proof uses a modification of Ocneanu compactness [JS97].

# Graphical calculus

Fix a finite set  $\text{Irr}(\mathcal{C})$  of representatives of irreducibles.

- ▶ Shaded regions denote irreducible summands of  $1_{\mathcal{C}}$ .
- ▶ Morphisms  $f : x \otimes y \rightarrow z$  are represented by coupons.
- ▶ For all simple  $x \in \mathcal{C}_{i,j}$ ,  $y \in \mathcal{C}_{j,k}$ , and  $z \in \mathcal{C}_{k,i}$ ,  $\text{Hom}(1, x \otimes y \otimes z)$  is a finite dimensional Hilbert space with inner product  $\langle f, g \rangle = g^* \circ f$ .

Choose dual bases:

$$e_i \in \text{Hom}(1, x \otimes y \otimes z) \text{ and } e^i \in \text{Hom}(1, \bar{z} \otimes \bar{y} \otimes \bar{x})$$

We represent the canonical element by colored nodes

$$\begin{array}{c} x \quad y \\ \diagdown \quad / \\ \bullet \\ / \quad \backslash \\ z \end{array} \otimes \begin{array}{c} z \\ / \quad \backslash \\ \bullet \\ \backslash \quad / \\ x \quad y \end{array} := \sqrt{d_x d_y d_z} \cdot \sum_{\alpha} \begin{array}{c} x \quad y \\ \diagdown \quad / \\ \bullet \\ / \quad \backslash \\ z \end{array} e_{\alpha} \otimes \begin{array}{c} z \\ / \quad \backslash \\ \bullet \\ \backslash \quad / \\ x \quad y \end{array} e^{\alpha}$$

The canonical element is independent of choice of basis.

# Important relations

$$\begin{array}{c} z \\ \circ \\ | \\ \circ \\ z \end{array} \begin{array}{c} x \\ | \\ y \end{array} = \sqrt{d_x d_y d_z^{-1}} \cdot N_{x,y}^z \begin{array}{c} | \\ z \end{array} \quad (\text{Bigon 1})$$

$$\begin{array}{c} z \\ \circ \\ | \\ \circ \\ z \end{array} \begin{array}{c} x \\ | \\ y \end{array} \otimes \begin{array}{c} x \\ \circ \\ | \\ y \\ z \end{array} \otimes \begin{array}{c} z \\ \circ \\ | \\ y \\ x \end{array} = \sqrt{d_x d_y d_z^{-1}} \cdot \begin{array}{c} | \\ z \end{array} \otimes \begin{array}{c} x \\ \circ \\ | \\ y \\ z \end{array} \otimes \begin{array}{c} z \\ \circ \\ | \\ y \\ x \end{array} \quad (\text{Bigon 2})$$

$$\sum_{z \in \text{Irr}(\mathcal{C}_{ik})} \sqrt{d_z} \begin{array}{c} x \\ \circ \\ | \\ \circ \\ y \end{array} \begin{array}{c} x \\ | \\ y \end{array} = \sqrt{d_x d_y} \cdot \begin{array}{c} | \\ x \end{array} \begin{array}{c} | \\ y \end{array} \quad (\text{Fusion})$$

$$\sum_{v \in \text{Irr}(\mathcal{C}_{i\ell})} \begin{array}{c} y \\ \circ \\ | \\ \circ \\ w \end{array} \begin{array}{c} z \\ | \\ v \\ | \\ x \end{array} \otimes \begin{array}{c} \bar{z} \\ \circ \\ | \\ \circ \\ \bar{x} \end{array} \begin{array}{c} \bar{y} \\ | \\ \bar{v} \\ | \\ \bar{w} \end{array} = \sum_{u \in \text{Irr}(\mathcal{C}_{jk})} \begin{array}{c} y \\ \circ \\ | \\ \circ \\ w \end{array} \begin{array}{c} z \\ | \\ u \\ | \\ x \end{array} \otimes \begin{array}{c} \bar{z} \\ \circ \\ | \\ \circ \\ \bar{x} \end{array} \begin{array}{c} \bar{y} \\ | \\ \bar{u} \\ | \\ \bar{w} \end{array} \quad (I=H)$$

We'll use Snyder convention and ignore all scalars.

## Commutant $\mathcal{C}'$ of $\mathcal{C}$ in $\text{Bim}(R^{\oplus k})$

The commutant  $\mathcal{C}' \subset \text{Bim}(R^{\oplus k})$  of  $\mathcal{C} \subset \text{Bim}(R^{\oplus k})$  has:

- ▶ Objects are pairs  $(X, e_X)$  where  $X \in \text{Bim}(R^{\oplus k})$ , and  $e_X$  is a unitary half braiding with  $\mathcal{C}$

$$e_{X,c} = \begin{array}{c} \diagup \\ X \quad \diagdown \\ \quad \quad \quad c \end{array} : X \boxtimes c \rightarrow c \boxtimes X$$

These half braidings must satisfy compatibility conditions.

- ▶ Morphisms  $f : (X, e_X) \rightarrow (Y, e_Y)$  are bimodule maps  $f : X \rightarrow Y$  which commute with the half braidings:

The diagram shows an equality between two expressions. On the left, a vertical line labeled  $Y$  at the top passes through a box labeled  $f$ . Below the box, the line splits into two strands labeled  $X$  and  $c$ . These strands cross each other in a half-braiding configuration. On the right, the strands  $X$  and  $c$  cross first, then the line labeled  $Y$  passes through the box labeled  $f$ . The two diagrams are separated by an equals sign.

$\mathcal{C}'$  is a tensor category, but it is usually not braided.

## Describing $\mathcal{C}'$ for unitary multifusion

Suppose  $(X, e_X) \in \mathcal{C}' \subset \text{Bim}(R^{\oplus k})$ .

Write  $X = (X_{i,j})$  where  $X_{i,j}$  is an  $R_i - R_j$  bimodule.

Easy facts about  $(X, e_X) \in \mathcal{C}'$

1.  $X \boxtimes 1_j \cong 1_j \boxtimes X \boxtimes 1_j$  implies  $X_{i,j} = 0$  for  $i \neq j$ .
2. Writing  $X = \bigoplus_{i=1}^k X_i$ , we can write  $e_X$  as a family of natural isomorphisms  $(e_X^i)$  given on  $c_{i,j} \in \mathcal{C}_{i,j}$  by


$$\begin{array}{c} c_{i,j} \quad X_j \\ \text{---} \\ \text{---} \\ \text{---} \\ X_i \quad c_{i,j} \end{array} = e_X^i \in \text{Hom}_{R_i - R_j}(X_i \boxtimes_{R_i} c_{i,j} \rightarrow c_{i,j} \boxtimes_{R_j} X_j)$$

3. We have a projection functor  $P_j : \mathcal{C}' \rightarrow \mathcal{C}'_j$  by  $(X, e_X) \mapsto (X_j, e_X^j)$ .



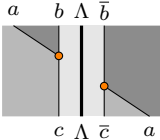
# Induction functor $\text{Bim}(R) \rightarrow \mathcal{C}'$

We have a way to construct lots of objects in  $\mathcal{C}'$ .

We always use the shading  =  $R_1$ .

$$\underline{\Phi} : \text{Bim}(R_1) \rightarrow \mathcal{C}' \quad \underline{\Phi}(\Lambda) = (\Phi(\Lambda), e_{\Phi(\Lambda)}).$$

$$\Phi(\Lambda) := \bigoplus_{\substack{j=1, \dots, k \\ c \in \text{Irr}(\mathcal{C}_{j,1})}} c \boxtimes \Lambda \boxtimes \bar{c} \in \text{Bim}(R^{\oplus k})$$

$$e_{\Phi(\Lambda), a} := \sum_{\substack{i, j \in \{1, \dots, k\} \\ \text{light gray} = R_i \\ \text{dark gray} = R_j}} \sum_{\substack{b \in \text{Irr}(\mathcal{C}_{j,1}) \\ c \in \text{Irr}(\mathcal{C}_{i,1})}} \sqrt{d_a^{-1}} \quad ; \quad a \in \mathcal{C}_{i,j}$$


## Proposition

The functor  $\Phi : \text{Bim}(R_1) \rightarrow \mathcal{C}'$  is dominant.

## A canonical projector

For  $(X_1, e_{X_1}) \in \mathcal{C}'_1$ , we have a canonical projector in  $\text{End}_{\mathcal{C}'}(\Phi(X_1))$ :

$$\begin{aligned}
 p_{X_1} &:= \frac{1}{D} \sum_{\substack{j \in \{1, \dots, k\} \\ \text{●} = R_j}} \sum_{\substack{a \in \text{Irr}(\mathcal{C}_1) \\ x, y \in \text{Irr}(\mathcal{C}_{j,1})}} \sqrt{d_a} \begin{array}{c} y \quad X_1 \quad \bar{y} \\ \text{---} | \text{---} | \text{---} \\ \text{●} \quad \quad \quad \text{●} \\ \text{---} | \text{---} | \text{---} \\ x \quad X_1 \quad \bar{x} \end{array} \\
 &= \frac{1}{D} \sum_{\substack{j \in \{1, \dots, k\} \\ \text{●} = R_j}} \sum_{x, y \in \text{Irr}(\mathcal{C}_{j,1})} \sqrt{d_x d_y} \begin{array}{c} y \quad X_1 \quad \bar{y} \\ \text{---} | \text{---} \\ \text{---} | \text{---} \\ x \quad X_1 \quad \bar{x} \end{array}
 \end{aligned}$$

# Equivalence

We have functors

$$\text{Bim}(R^{\oplus k}) \ni \mathcal{C}' \xrightarrow{P_1} \mathcal{C}'_1 \subset \text{Bim}(R_1)$$

$$\text{Bim}(R_1) \ni \mathcal{C}'_1 \xrightarrow{\Phi} \mathcal{C}' \in \text{Bim}(R^{\oplus k})$$

We get another functor  $p\Phi : \mathcal{C}'_1 \rightarrow \mathcal{C}'$  by applying  $\Phi$  and then applying the canonical projector.

## Theorem

The functors  $P_1$  and  $p\Phi$  witness an equivalence of categories  $\text{Bim}(R_1) \supseteq \mathcal{C}'_1 \cong \mathcal{C}' \subseteq \text{Bim}(R^{\oplus k})$ .

## Sketch of one direction.

We get a natural isomorphism  $u : p\Phi \circ P_1 \Rightarrow \text{id}$  where  $u_X : p_X\Phi(X_1) \rightarrow X$  is given by

$$u_X = \frac{1}{\sqrt{D}} \sum_{\substack{j \in \{1, \dots, k\} \\ \bullet = R_j}} \sum_{x \in \text{Irr}(\mathcal{C}_{j,1})} \sqrt{d_x} \begin{array}{c} X_j \\ \hline x \quad X_1 \quad \bar{x} \end{array} .$$

# The main corollary

We can now prove our main result as a corollary.

## Corollary

If  $\mathcal{C}_1 \subset \text{Bim}(R_1)$  and  $\mathcal{C}_2 \subset \text{Bim}(R_2)$  are two Morita equivalent unitary fusion categories, then  $\mathcal{C}'_1 \cap \text{Bim}(R_1) \cong \mathcal{C}'_2 \cap \text{Bim}(R_2)$ .

## Proof.

Let  $\mathcal{M} \subset \text{Bim}(R_1, R_2)$  be an equivalence unitary  $\mathcal{C}_1 - \mathcal{C}_2$  bimodule category. We can form a  $2 \times 2$  unitary multifusion category by

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1 & \mathcal{M} \\ \mathcal{M}^* & \mathcal{C}_2 \end{pmatrix} \subset \text{Bim}(R_1 \oplus R_2).$$

Now we apply the previous theorem twice:

$$\mathcal{C}'_1 \cap \text{Bim}(R_1) \cong \mathcal{C}' \cap \text{Bim}(R_1 \oplus R_2) \cong \mathcal{C}'_2 \cap \text{Bim}(R_2). \quad \square$$

# Thank you for listening!

Slides available at:

[https:](https://people.math.osu.edu/penneys.2/PenneysAMS2017.pdf)

[//people.math.osu.edu/penneys.2/PenneysAMS2017.pdf](https://people.math.osu.edu/penneys.2/PenneysAMS2017.pdf)

Previous article *Bicommutant categories from fusion categories*  
with André Henriques available at:

<http://arxiv.org/abs/1511.05226>

New article *Commutants of multifusion categories* with André  
Henriques coming soon!