

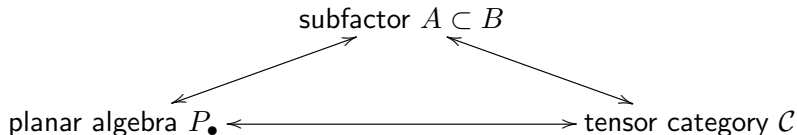
Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$
(with applications to subfactors at index $3 + \sqrt{5}$)
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Overview



- Planar algebras give a generators and relations approach to subfactors and tensor categories.
- From the above, we get an invariant called a fusion graph.

Question

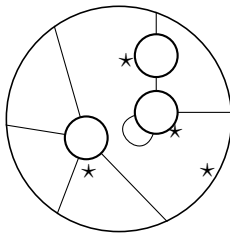
- (Unreasonable) Which graphs are fusion graphs?
- What is a reasonable way to classify fusion categories?

Planar algebras [Jon99]

Definition

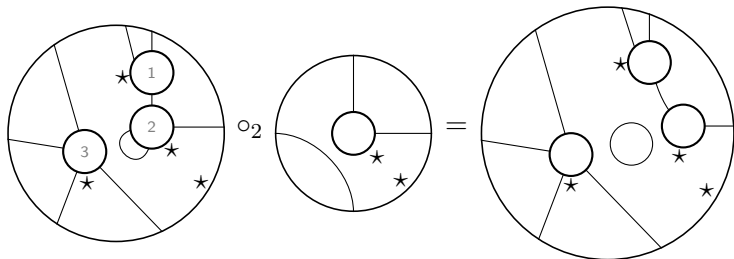
A planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval \star on each boundary disk



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

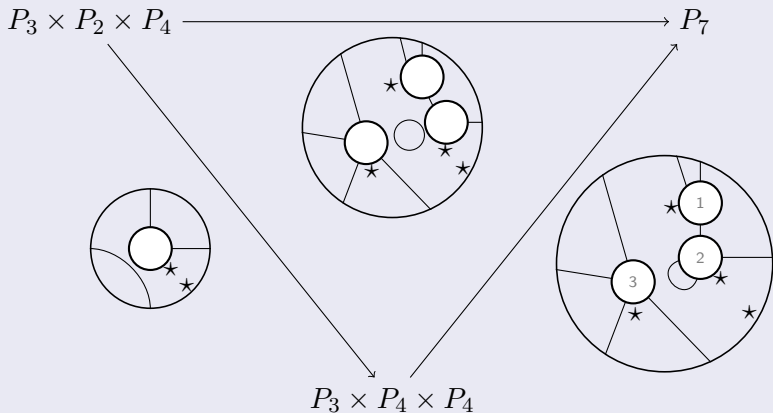


Definition

The *planar operad* consists of all planar tangles (up to isotopy) with the operation of composition.

Definition

A *planar algebra* is a family of vector spaces P_k , $k = 0, 1, 2, \dots$ and an action of the planar operad.



Example: Temperley-Lieb

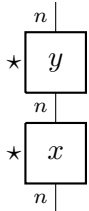
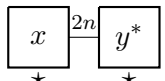
$TL_n(\delta)$ is the complex span of non-crossing pairings of n points arranged around a circle, with formal addition and scalar multiplication.

$$TL_6(\delta) = \text{Span}_{\mathbb{C}} \left\{ \begin{array}{c} \star \\ \text{Diagram 1} \\ \star \end{array}, \begin{array}{c} \star \\ \text{Diagram 2} \\ \star \end{array}, \begin{array}{c} \star \\ \text{Diagram 3} \\ \star \end{array}, \begin{array}{c} \star \\ \text{Diagram 4} \\ \star \end{array}, \begin{array}{c} \star \\ \text{Diagram 5} \\ \star \end{array} \right\}.$$

Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ .

$$\begin{array}{c} \star \\ \text{Large Circle with Vertical Line and Small Circle} \\ \star \end{array} \left(\begin{array}{c} \star \\ \text{Small Circle with Vertical Line} \\ \star \end{array} \right) = \begin{array}{c} \star \\ \text{Large Circle with Vertical Line and Two Small Circles} \\ \star \end{array} = \delta^2 \begin{array}{c} \star \\ \text{Large Circle with Vertical Line} \\ \star \end{array}$$

Some special tangles/properties

- multiplication: $x \cdot y =$

 $(TL_{2n}$ is an algebra)
- adjoint is reflection: $\left(\star \begin{array}{|c|} \hline \text{diagonal with top-left and bottom-right arcs} \\ \hline \end{array} \right)^* = \star \begin{array}{|c|} \hline \text{diagonal with top-right and bottom-left arcs} \\ \hline \end{array}$
- trace: $\text{Tr}_{2n}(x) = \star \begin{array}{|c|} \hline \text{box x with a loop on the right} \\ \hline \end{array} = n \begin{array}{|c|} \hline \text{box x with a loop on the left} \\ \hline \end{array}$ (spherical)
- sesquilinear form: $\langle x, y \rangle = \text{Tr}_{2n}(y^* x) =$


Jones' index rigidity theorem

Jones' index rigidity theorem [Jon83]

Suppose the sesquilinear form on TL_{2n} given by $\langle x, y \rangle := \text{Tr}_{2n}(y^*x)$ is positive semi-definite for every $n \geq 0$. Then

$$\delta \in \left\{ 2 \cos \left(\frac{\pi}{k} \right) \mid k \geq 3 \right\} \cup [2, \infty).$$

Jones' index rigidity theorem

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$$\delta \in \underbrace{\left\{ 2 \cos \left(\frac{\pi}{k} \right) \mid k \geq 3 \right\}}_{\text{semi-definite}} \cup \underbrace{[2, \infty)}_{\text{definite}}.$$

Factor (or fantastic) planar algebras

Definition

A planar algebra P_\bullet is a factor planar algebra if it is:

- Finite dimensional: $\dim(P_k) < \infty$ for all k
- Evaluable: $\dim(P_0) = 1$

- Sphericity: $\text{Tr}_2(X) = \left(\star \textcircled{X} \right) = \left(\textcircled{X} \star \right)$

- Positivity: each P_j has an adjoint $*$ such that the sesquilinear form on P_{2k} given by $\langle x, y \rangle_{2k} := \text{Tr}_{2k}(y^*x)$ is positive definite for all $k \geq 0$.

From these properties, it follows that closed circles count for a multiplicative constant δ .

Skein relations


If the sesquilinear form is semi-definite, we quotient out the length zero vectors.

Example: A_2

The fantastic planar algebra A_2 is the quotient of Temperley-Lieb when $\delta = 2 \cos(\pi/3) = 1$ by the following skein relations:

$$\begin{array}{l} \boxed{\bigcirc} = 1 \\ \boxed{\parallel\parallel} = \boxed{\cup \cup} \end{array}$$

Example: T_2 Example: T_2

Generated by a trivalent vertex: 

Skein relations:

$$\boxed{\bigcirc} = \boxed{\bigcirc \text{ with a horizontal line through it}} = \tau = \frac{1 + \sqrt{5}}{2}$$

$$\boxed{\bigcirc \text{ with a line from the bottom}} = 0$$

$$\boxed{\bigcirc \text{ with a line from the bottom}} = \boxed{\text{two vertical lines}}$$

$$\boxed{\text{two vertical lines}} = \frac{1}{\tau} \boxed{\text{two vertical lines with a cup and a cap}} + \boxed{\text{trivalent vertex with lines from top and bottom}}$$

$$\boxed{\text{trivalent vertex with lines from top and bottom}} = \boxed{\text{trivalent vertex with lines from top and bottom}}^* = \boxed{\text{trivalent vertex with lines from top and bottom}}$$

Example: Free product, tensor product

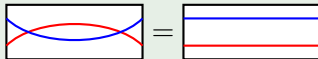
Example: Free product $A_2 * T_2$

All non-crossing string diagrams with red and blue strings satisfying the previous relations.




Example: Tensor product $A_2 \boxtimes T_2$

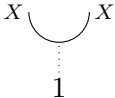
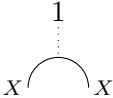
All crossing string diagrams with red and blue strings satisfying the previous relations, and a Reidemeister two relation

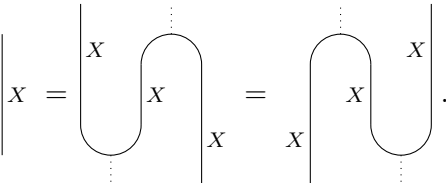


Tensor categories to planar algebras

Given a rigid C^* -tensor category, e.g., a unitary fusion category, and a 'nice' object X , we can construct a planar algebra.

- $\mathcal{PA}(\mathcal{C}, X)_n = \text{Hom}(1, X^{\otimes n})$: 

- $\text{ev}_X =$  and $\text{coev}_X =$ 

zig-zag relation: 

Tensor categories to planar algebras (cont.)

- unitary implies positive and spherical:

$$\langle f, g \rangle = \begin{array}{c} \boxed{f} \text{---}^n \boxed{g^*} \\ \star \qquad \qquad \star \end{array}$$

$$\text{tr}(f) = \begin{array}{c} \text{---}^n \\ \boxed{f} \\ \text{---}^n \end{array} = \begin{array}{c} \boxed{f} \\ \text{---}^n \\ \text{---}^n \end{array}$$

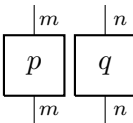
- spherical implies pivotal:

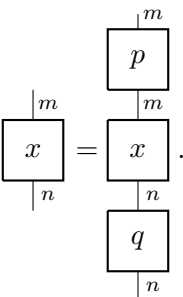
$$\begin{array}{c} \text{---}^n \\ \boxed{f} \\ \text{---}^n \end{array} = \begin{array}{c} \boxed{f} \\ \text{---}^n \\ \text{---}^n \end{array} .$$

Planar algebras to tensor categories

Given a factor planar algebra, can construct its rigid C^* -tensor category of projections.

- Objects are (formal direct sums of) projections

- Tensoring is horizontal concatenation $p \otimes q =$


- $\text{Hom}(p, q) = \{x \mid x = qxp\}$, i.e.,
 

- Composition of morphisms is vertical stacking.

Planar algebras to tensor categories (cont.)

- Duality is rotation by π

$$\boxed{\bar{p}} = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{p} \\ | \\ \text{---} \end{array} .$$

- The adjoint $*$: $\text{Hom}(p, q) \rightarrow \text{Hom}(q, p)$ is the adjoint in P_{\bullet} .

Theorem

- $P_{\bullet} \rightarrow \text{Pro}(P_{\bullet}) \rightarrow \mathcal{PA}(\text{Pro}(P_{\bullet}), |)$ is the identity.
- $(\mathcal{C}, X) \rightarrow \mathcal{PA}(\mathcal{C}, X) \rightarrow \text{Pro}(\mathcal{PA}(\mathcal{C}, X))$ is an equivalence.

Fusion graphs

Definition

Given a rigid C^* tensor category \mathcal{C} and a 'nice' object X , we define Γ_X , the fusion graph with respect to X , as follows:

- Vertices: equivalence classes of simple objects
- Edges: If P is simple, $P \otimes X = \bigoplus_{Q \text{ simple}} N_{P,X}^Q Q$.
There are $N_{P,X}^Q$ edges between simples $P, Q \in \mathcal{C}$.

Example: A_2

Two simples $1, \theta$, and $\theta \otimes \theta = 1$, so $\Gamma_\theta = 1 \text{ --- } \theta$.


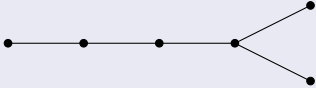


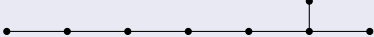
Example: T_2

Two simples $1, \tau$, and $\tau \otimes \tau = 1 \oplus \tau$, so $\Gamma_\tau = 1 \text{ --- } \tau \text{ --- } \tau$.

Planar algebras with $\delta < 2$

Theorem

The factor planar algebras with $\delta < 2$ are as follows:

name	principal graph	#	constructed
A_n		1	[Jon83]
D_{2n}		1	[Ocn88, Kaw95]
T_n		1	[KO02, EO04]
E_6		2	[Ocn88, BN91]
E_8		2	[Ocn88, Izu94]

Composing fusion categories

Interpolating between tensor products and free products of fusion categories.

Simplest examples of fusion categories have 2 objects.

- $A_2 - A_2$ (Warmup)
- $A_2 - T_2$ (Main motivation - Bisch-Haagerup 1994)
- $T_2 - T_2$ (Bonus!)

Two copies of A_2

Take two copies of A_2 :

$$\alpha = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right. \text{ and } \theta = \left| \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right.$$

where $\alpha \otimes \alpha \cong 1$ and $\theta \otimes \theta \cong 1$. We have the following skein relations:



$$\begin{array}{cc} \boxed{\text{green circle}} = 1 & \boxed{\text{blue circle}} = 1 \\ \boxed{\text{3 vertical green lines}} = \boxed{\text{2 green semi-circles}} & \boxed{\text{3 vertical blue lines}} = \boxed{\text{2 blue semi-circles}} \end{array}$$

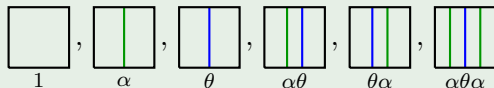
Simple objects

Proposition

Suppose \mathcal{C} is generated by α, θ . Then either \mathcal{C} is the free product $A_2 * A_2$, or there is an $n \in \mathbb{N}$ such that $(\alpha\theta)^n \cong 1$, but $(\alpha\theta)^{n-1} \not\cong 1$. Any word in α, θ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

Example

If $n = 3$, then (representatives for) the simple objects are



Even though $\theta\alpha\theta$ is simple, it is isomorphic to $\alpha\theta\alpha$.

Relations for U

Proposition

U satisfies the following skein relations:

- $UU^* = |||$ and $U^*U = |||$
- Rotation relation:

The diagram shows the rotation relation for the operator U . It consists of three terms separated by equals signs. The first term is a circle labeled U with a star symbol $*$ to its left. Three blue strands enter from the left and exit to the right. Two of these strands loop back around the circle: one loops clockwise and the other counter-clockwise. The second term is a circle labeled U^* with a star symbol $*$ to its left. Three green strands enter from the left and exit to the right. The third term is a circle labeled U with a star symbol $*$ to its left. Three green strands enter from the left and exit to the right. Two of these strands loop back around the circle: one loops clockwise and the other counter-clockwise. The entire equation is $*U = *U^* = \omega_U^{-1} *U$.

for some n -th root of unity ω_U .

- Jellyfish relations:

The diagram shows the jellyfish relations for the operator U . It consists of two equations. The first equation is $*U = *U^*$. On the left, a circle labeled U with a star symbol $*$ to its left has a blue arc (the jellyfish) connecting its top two strands. Below the circle is the label $2n|$. On the right, a circle labeled U^* with a star symbol $*$ to its left has a small blue arc below it. The second equation is $*U = \omega_U *U^*$. On the left, a circle labeled U with a star symbol $*$ to its left has a green arc (the jellyfish) connecting its top two strands. Below the circle is the label $2n|$. On the right, a circle labeled U^* with a star symbol $*$ to its left has a small green arc below it. The entire equation is $*U = \omega_U *U^*$.

Bigelow-Morrison-Peters-Snyder [BMPS12]

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

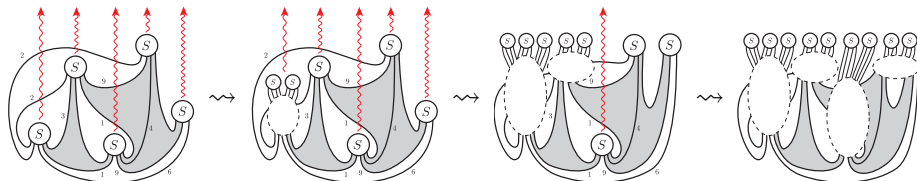
- $$\star \text{ [Diagram: } f^{(2n+2)} \text{ with cap } S \text{ and line } 2n-1 \text{]} = i \frac{\sqrt{[n][n+2]}}{[n+1]} \star \text{ [Diagram: } f^{(2n+2)} \text{ with two } S \text{ circles and lines } n+1 \text{]} ,$$
- $$\star \text{ [Diagram: } f^{(2n+4)} \text{ with cap } S \text{ and line } 2n \text{]} = \frac{[2][2n+4]}{[n+1][n+2]} \star \text{ [Diagram: } f^{(2n+4)} \text{ with three } S \text{ circles and lines } n+1, 2, n+1 \text{]} ,$$

- capping S gives zero, and
- (Absorption) $S^2 = f^{(n)}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

- 1 First, pull all generators to the outside using the jellyfish relations



- 2 Second, reduce the number of generators using the capping and absorption (multiplication) relations.

$\text{Vec}_{D_{2n}}^\omega$ and $A_{2n-1}^{(1)}$


Theorem


These relations are consistent and sufficient to evaluate all closed diagrams. Hence there are exactly n distinct categories satisfying $(\parallel)^n \cong 1$. These are $\text{Vec}_{D_{2n}}^\omega$.

Remark

If we draw a black string for $X = \alpha \oplus \theta$,

$$| = | + |,$$

then the fusion graph Γ_X is $A_{2n-1}^{(1)}$ 

Equivariantization $(| \leftrightarrow |)$ gives $D_{n+2}^{(1)}$ 

What about A_2 and T_2 ?

- Can we interpolate between tensor and free product for A_2 and T_2 ?
- This question was asked by Bisch and Haagerup in 1994.

Bisch-Haagerup Fish

- Possible subfactors $A_3 \boxtimes A_4 \leq \mathcal{BHF}_n \leq A_3 * A_4$.
- Possible fusion categories $A_2 \boxtimes T_2 \leq \frac{1}{2}\mathcal{BHF}_n \leq A_2 * T_2$.

$$\mathcal{BHF}_1 = A_3 \boxtimes A_4 = \left(\text{Diagram 1}, \text{Diagram 2} \right)$$

$$\mathcal{BHF}_2 = \left(\text{Diagram 3}, \text{Diagram 4} \right)$$

$$\mathcal{BHF}_3 = \left(\text{Diagram 5}, \text{Diagram 6} \right)$$

$$\vdots$$

$$\mathcal{BHF}_n = \left(\text{Diagram 7}, \text{Diagram 8} \right)$$

$$\vdots$$

$$\mathcal{BHF}_\infty = A_3 * A_4 = \left(\text{Diagram 9}, \text{Diagram 10} \right)$$

Skein relations

Suppose \mathcal{C} is generated by θ, ρ with $\theta \otimes \theta \cong 1$ and $\rho \otimes \rho \cong 1 \oplus \rho$.

$$\theta = \left| \right. \text{ and } \rho = \left| \right.$$

We have the following skein relations:

$$\begin{array}{l}
 \boxed{\text{blue circle}} = 1 \\
 \boxed{\text{blue vertical lines}} = \boxed{\text{blue cup and cap}} \\
 \boxed{\text{red circle}} = \boxed{\text{red circle with horizontal line}} = \tau = \frac{1 + \sqrt{5}}{2} \\
 \boxed{\text{red circle with bottom line}} = 0 \\
 \boxed{\text{red circle with top line}} = \boxed{\text{red vertical line}} \\
 \boxed{\text{red vertical lines}} = \frac{1}{\tau} \boxed{\text{red cup and cap}} + \boxed{\text{red Y-junction}} \\
 \boxed{\text{red U-junction}} = \boxed{\text{red Y-junction}}^* = \boxed{\text{red Y-junction}}
 \end{array}$$

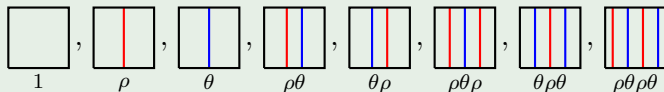
Simple objects

Proposition

Suppose \mathcal{C} is generated by ρ, θ . Then either \mathcal{C} is the free product $A_2 * T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho\theta)^n \cong (\theta\rho)^n$, but $(\rho\theta)^{n-1} \not\cong (\theta\rho)^{n-1}$. Any word in ρ, θ of length $\leq 2n$ is a simple object. Words of length $< 2n$ give distinct simples.

Example

If $n = 2$, then (representatives for) the simple objects are



Even though $\theta\rho\theta\rho$ is simple, it is isomorphic to $\rho\theta\rho\theta$.

Another generator

- Free product $A_2 * T_2$ has no extra relations.
- In the tensor product $A_2 \boxtimes T_2$, $|$ and $|$ commute:

$$\boxed{\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}} : \begin{array}{|l} | \\ | \end{array} \xrightarrow{\cong} \begin{array}{|l} | \\ | \end{array}$$

- For $1 < n \leq \infty$, we have $(\begin{array}{|l} | \\ | \end{array})^n \cong (\begin{array}{|l} | \\ | \end{array})^n$:

$$\star \left(\begin{array}{c} | \\ | \\ \bigcirc U \\ | \\ | \end{array} \right) : (\begin{array}{|l} | \\ | \end{array})^n \xrightarrow{\cong} (\begin{array}{|l} | \\ | \end{array})^n$$

where we draw $|$ for $(\begin{array}{|l} | \\ | \end{array})^{n-1}$.

Relations for U

- Reidemeister relations:

$$\begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{c} \star \\ \circlearrowleft \\ U^* \end{array} \begin{array}{c} \star \\ \circlearrowright \\ U \end{array} \quad \text{and} \quad \begin{array}{|c|} \hline \text{---} \\ \hline \text{---} \\ \hline \end{array} = \begin{array}{c} \star \\ \circlearrowright \\ U \end{array} \begin{array}{c} \star \\ \circlearrowleft \\ U^* \end{array} .$$

- Rotation relations:

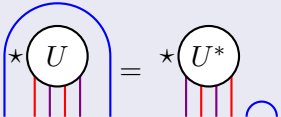
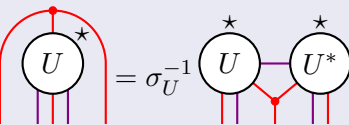
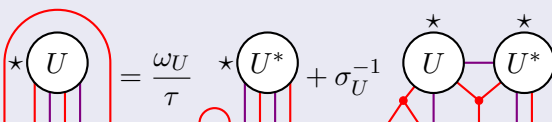
$$\begin{array}{c} \overbrace{(| \rangle \langle | \rangle)^{n-1}} \\ \vdots \\ \star \circlearrowleft U \\ \vdots \\ \underbrace{(| \rangle \langle | \rangle)^{n-1}} \end{array} = \begin{array}{c} \star \circlearrowleft U^* \\ \vdots \\ \vdots \end{array} = \omega_U^{-1} \begin{array}{c} \star \circlearrowright U \\ \vdots \\ \vdots \end{array}$$

where ω_U is a $2n$ -th root of unity.

Jellyfish relations

Theorem

U satisfies the following jellyfish relations:

- 1 
- 2 
- 3 

Here $\sigma_U^2 = \omega_U$. Switching U with $-U$ switches the sign of σ_U .

Existence and uniqueness for $n = 1, 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Liu 2013]

\mathcal{BHF}_n exists and is unique for $n = 1, 2, 3, \infty$.

\mathcal{BHF}_n does not exist for $4 \leq n < \infty$.

Theorem [Izumi-Morrison-Penneys 2013]

$\frac{1}{2}\mathcal{BHF}_n$ exists and is unique for $n = 1, 2, 3, \infty$.

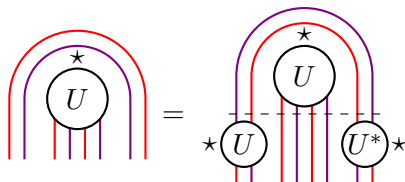
$\frac{1}{2}\mathcal{BHF}_n$ does not exist for $4 \leq n \leq 10$.

Both proofs discovered simultaneously and independently.

- IMP's method - construction for $n = 1, 2, 3$ ad hoc, only eliminates $4 \leq n \leq 10$. Conjecturally eliminates all $4 \leq n < \infty$.
- Liu's method - uniform construction, eliminates all $4 \leq n < \infty$.

Uniqueness and nonexistence

- For $n = \infty$, no more relations, so planar algebra is unique.
- When $n < \infty$, we consider the following diagram:

























- First, we pull the U upward using the jellyfish relations. Then we compare the results.
- For $n = 1, 2, 3$, we get $\omega_U = 1$, so planar algebra is unique.
- For $4 \leq n \leq 10$, the results are inconsistent. We conjecture the results are inconsistent for all $4 \leq n < \infty$.

What about T_2 and T_2 ?

Suppose \mathcal{C} is generated by ρ, μ with $\rho \otimes \rho \cong 1 \oplus \rho$ and $\mu \otimes \mu \cong 1 \oplus \mu$.

$$\rho = \begin{array}{|c} \hline \\ \hline \end{array} \quad \text{and} \quad \mu = \begin{array}{|c} \hline \\ \hline \end{array}$$

We have the following skein relations:

	=		=	$\tau = \frac{1 + \sqrt{5}}{2}$		=		=	$\tau = \frac{1 + \sqrt{5}}{2}$			
	=	0				=	0					
	=					=						
	=	$\frac{1}{\tau}$		+				=	$\frac{1}{\tau}$		+	
	=		*	=				=		*	=	

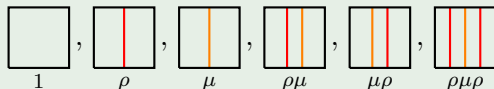
Simple objects

Proposition

Suppose \mathcal{C} is generated by ρ, μ . Then either \mathcal{C} is the free product $T_2 * T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho\mu)^n \cong 1$, but $(\rho\mu)^{n-1} \not\cong 1$. Any word in ρ, μ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

Example

If $n = 3$, then (representatives for) the simple objects are



Even though $\mu\rho\mu$ is simple, it is isomorphic to $\rho\mu\rho$.

Skein relations

Again, we have another generator U when $2 \leq n < \infty$.

Proposition

U satisfies the following skein relations:

- $UU^* = |||$ and $U^*U = |||$
- Rotation relation:

$$\star U = \star U^* = \omega_U^{-1} \star U$$

for some n -th root of unity ω_U .

Theorem

U satisfies the following jellyfish relations:

$$\begin{aligned}
 \textcircled{1} \quad & \text{Diagram 1} = \sigma_U^{-1} \text{Diagram 2} \\
 \textcircled{2} \quad & \text{Diagram 3} = \frac{\omega_U}{\tau} \text{Diagram 4} + \sigma_U^{-1} \text{Diagram 5} \\
 \textcircled{3} \quad & \text{Diagram 6} = \text{Diagram 7} \\
 \textcircled{4} \quad & \text{Diagram 8} = \frac{1}{\tau} \text{Diagram 9} + \text{Diagram 10}
 \end{aligned}$$

Detailed description of the diagrams:

- Diagram 1:** A circle labeled U with a star above it. A red arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 2:** Two circles, U and U^* , both with stars above them. They are connected by two horizontal strands. A red arc connects the top of the U circle to the top of the U^* circle. There are four vertical strands below each circle.
- Diagram 3:** A circle labeled U with a star above it. A red arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 4:** A circle labeled U^* with a star above it. A red arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 5:** A circle labeled U^* with a star above it. A red arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 6:** A circle labeled U^* with a star above it. An orange arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 7:** Two circles, U^* and U , both with stars above them. They are connected by two horizontal strands. A red arc connects the top of the U^* circle to the top of the U circle. There are four vertical strands below each circle.
- Diagram 8:** A circle labeled U^* with a star above it. An orange arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 9:** A circle labeled U with a star above it. An orange arc connects the top of the circle to the top of the first strand. There are four vertical strands below the circle.
- Diagram 10:** Two circles, U^* and U , both with stars above them. They are connected by two horizontal strands. An orange arc connects the top of the U^* circle to the top of the U circle. There are four vertical strands below each circle.

Existence and uniqueness for $n = 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Izumi-Morrison-Penneys 2013]

This $T_2 - T_2$ category exists and is unique for $n = 2, 3, \infty$.
Does not exist for $4 \leq n \leq 10$.

Theorem [Liu 2013]

A similar, but much better result for subfactors.
Existence and uniqueness for $n = 2, 3, \infty$.
Non-existence for $4 \leq n < \infty$.

- Again, IMP's method only eliminates $4 \leq n \leq 10$.
Conjecturally eliminates all $4 \leq n < \infty$.
- Liu's method is uniform, eliminates all $4 \leq n < \infty$.

What next?

What about




- Subfactors between $A_3 \boxtimes A_5$ and $A_3 * A_5$
- Fusion categories between $A_2 \boxtimes \frac{1}{2}A_5$ and $A_2 * \frac{1}{2}A_5$
($\frac{1}{2}A_5 = \text{Rep}(S_3)$)






The situation is much harder since $\frac{1}{2}A_5$ has three objects $1, \rho, \alpha$ with $\rho \otimes \rho \cong 1 \oplus \rho \oplus \alpha$.

I am exploring certain cases of these in joint work with Liu:

The end

Thank you for listening!
(Preprints coming soon)

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