Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} * \mathcal{D}$

(with applications to subfactors at index $3 + \sqrt{5}$)

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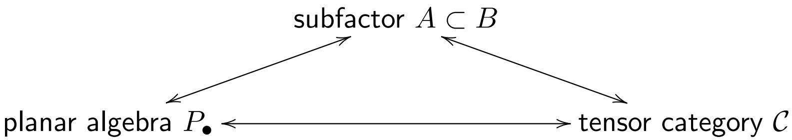
Overview



- Planar algebras give a generators and relations approach to subfactors and tensor categories.
- From the above, we get an invariant called a <u>fusion graph</u>.

Question

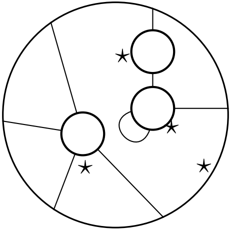
- (Unreasonable) Which graphs are fusion graphs?
- What is a reasonable way to classify fusion categories?

Planar algebras [Jon99]

Definition

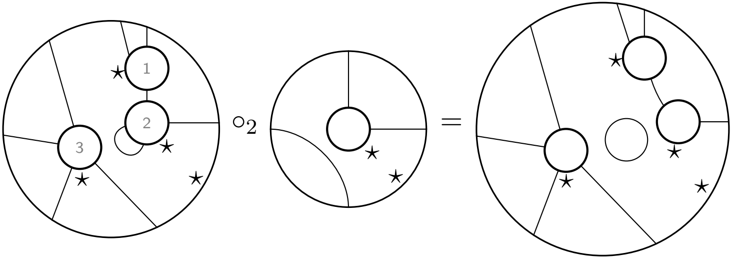
A planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk



Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

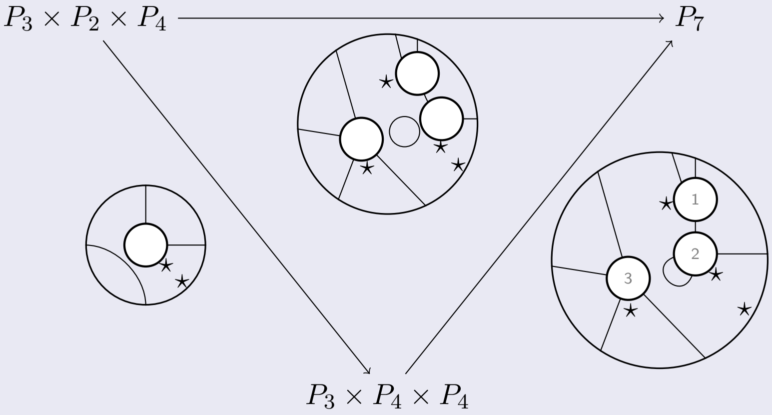


Definition

The *planar operad* consists of all planar tangles (up to isotopy) with the operation of composition.

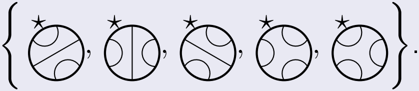
Definition

A *planar algebra* is a family of vector spaces P_k, $k = 0, 1, 2, \ldots$ and an action of the planar operad.

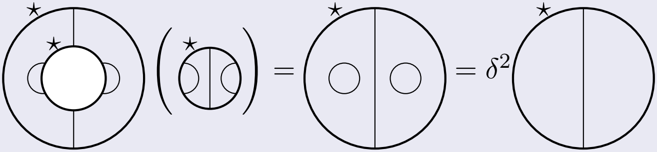


Example: Temperley-Lieb

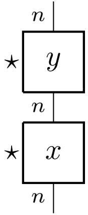
$TL_n(\delta)$ is the complex span of non-crossing pairings of n points arranged around a circle, with formal addition and scalar multiplication.

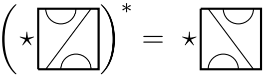
$TL_6(\delta) = \mathrm{Span}_{\mathbb{C}}$ 

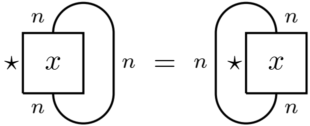
Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of δ.

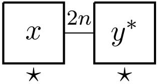


Some special tangles/properties

- multiplication: $x \cdot y =$ 

$(TL_{2n}$ is an algebra$)$
- adjoint is reflection: 

- trace: $\mathrm{Tr}_{2n}(x) =$ 

(spherical)
- sesquilinear form: $\langle x, y \rangle = \mathrm{Tr}_{2n}(y^* x) =$ 

Jones' index rigidity theorem

Jones' index rigidity theorem [Jon83]

Suppose the sesquilinear form on TL_{2n} given by $\langle x, y\rangle := \mathrm{Tr}_{2n}(y^* x)$ is positive semi-definite for every $n \geq 0$. Then

$$\delta \in \left\{ 2\cos\left(\frac{\pi}{k}\right) \middle| k \geq 3 \right\} \cup [2, \infty).$$

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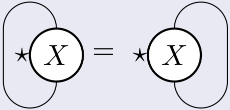
$$\delta \in \underbrace{\left\{2\cos\left(\frac{\pi}{k}\right)\middle| k \geq 3\right\}}_{\text{semi-definite}} \cup \underbrace{[2, \infty)}_{\text{definite}}.$$

Factor (or fantastic) planar algebras

Definition

A planar algebra $P_\bullet$ is a factor planar algebra if it is:

- Finite dimensional: $\dim(P_k) < \infty$ for all k
- Evaluable: $\dim(P_0) = 1$
- Sphericality: $\operatorname{Tr}_2(X) =$ 

- Positivity: each P_j has an adjoint $*$ such that the sesquilinear form on P_{2k} given by $\langle x, y\rangle_{2k} := \operatorname{Tr}_{2k}(y^* x)$ is positive definite for all $k \geq 0$.

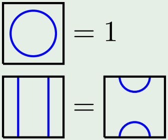
From these properties, it follows that closed circles count for a multiplicative constant δ.

Skein relations

If the sesquilinear form is semi-definite, we quotient out the length zero vectors.

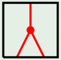
Example: A_2

The fantastic planar algebra A_2 is the quotient of Temperley-Lieb when $\delta = 2\cos(\pi/3) = 1$ by the following skein relations:

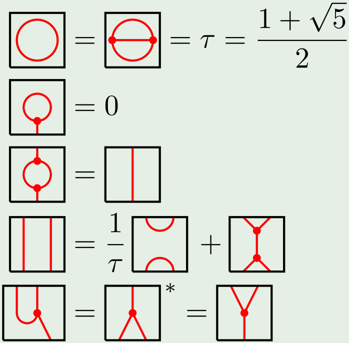


Example: T_2

Example: T_2

Generated by a trivalent vertex: 

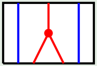
Skein relations:



Example: Free product, tensor product

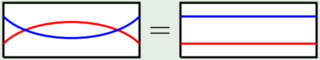
Example: Free product $A_2 * T_2$

All non-crossing string diagrams with red and blue strings satisfying the previous relations.



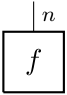
Example: Tensor product $A_2 \boxtimes T_2$

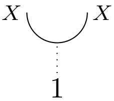
All crossing string diagrams with red and blue strings satisfying the previous relations, and a Reidemeister two relation

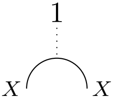


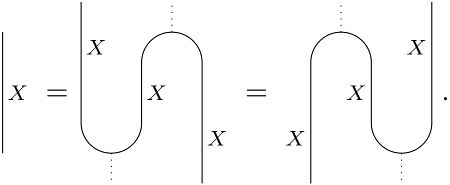
Tensor categories to planar algebras

Given a rigid C^*-tensor category, e.g., a unitary fusion category, and a 'nice' object X, we can construct a planar algebra.

- $\mathcal{PA}(\mathcal{C}, X)_n = \mathrm{Hom}(1, X^{\otimes n})$: 

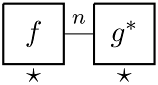
- $\mathrm{ev}_X =$ 

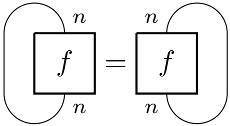
 and $\mathrm{coev}_X =$ 

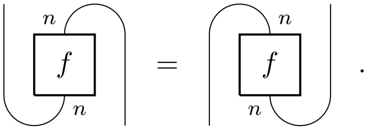
zig-zag relation: 

Tensor categories to planar algebras (cont.)

- unitary implies positive and spherical:

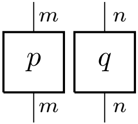
$\langle f, g \rangle =$ 

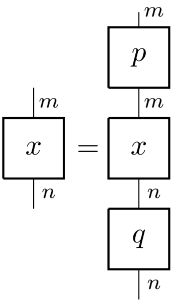
$\mathrm{tr}(f) =$ 

- spherical implies pivotal: 

Planar algebras to tensor categories

Given a factor planar algebra, can construct its rigid C^*-tensor category of projections.

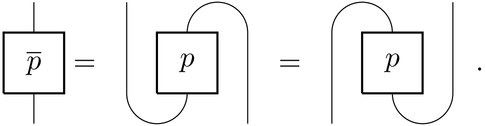
- Objects are (formal direct sums of) projections
- Tensoring is horizontal concatenation $p \otimes q =$ 

- $\mathrm{Hom}(p,q) = \{x | x = qxp\}$, i.e.,  .

- Composition of morphisms is vertical stacking.

Planar algebras to tensor categories (cont.)

- Duality is rotation by π



- The adjoint $* : \mathrm{Hom}(p,q) \to \mathrm{Hom}(q,p)$ is the adjoint in $P_\bullet$.

Theorem

- $P_\bullet \to \text{Pro}(P_\bullet) \to \mathcal{PA}(\text{Pro}(P_\bullet), |)$ is the identity.
- $(\mathcal{C}, X) \to \mathcal{PA}(\mathcal{C}, X) \to \text{Pro}(\mathcal{PA}(\mathcal{C}, X))$ is an equivalence.

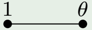
Fusion graphs

Definition

Given a rigid C^* tensor category $\mathcal{C}$ and a 'nice' object X, we define Γ_X, the fusion graph with respect to X, as follows:

- Vertices: equivalence classes of simple objects
- Edges: If P is simple, $P \otimes X = \bigoplus_{Q \text{ simple}} N_{P,X}^Q Q$.
  There are $N_{P,X}^Q$ edges between simples $P, Q \in \mathcal{C}$.

Example: A_2

Two simples $1, \theta$, and $\theta \otimes \theta = 1$, so $\Gamma_\theta =$ 

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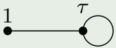
Example: T_2

Two simples $1, \tau$, and $\tau \otimes \tau = 1 \oplus \tau$, so $\Gamma_\tau =$ 

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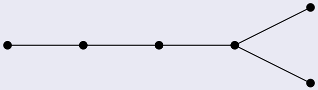
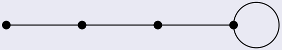
Planar algebras with $\delta < 2$

Theorem

The factor planar algebras with $\delta < 2$ are as follows:

name	principal graph	#	constructed
A_n		1	[Jon83]
D_{2n}		1	[Ocn88, Kaw95]
T_n		1	[KO02, EO04]
E_6		2	[Ocn88, BN91]
E_8		2	[Ocn88, Izu94]

Composing fusion categories

Interpolating between tensor products and free products of fusion categories.

Simplest examples of fusion categories have 2 objects.

- $A_2 - A_2$ (Warmup)
- $A_2 - T_2$ (Main motivation - Bisch-Haagerup 1994)
- $T_2 - T_2$ (Bonus!)

Two copies of A_2

Take two copies of A_2:

$\alpha =$ and $\theta =$

where $\alpha \otimes \alpha \cong 1$ and $\theta \otimes \theta \cong 1$. We have the following skein relations:

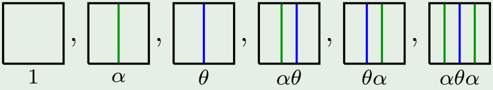
Simple objects

Proposition

Suppose $\mathcal{C}$ is generated by α, θ. Then either $\mathcal{C}$ is the free product $A_2 * A_2$, or there is an $n \in \mathbb{N}$ such that $(\alpha\theta)^n \cong 1$, but $(\alpha\theta)^{n-1} \ncong 1$. Any word in α, θ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

Example

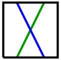
If $n = 3$, then (representatives for) the simple objects are

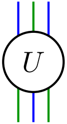


Even though $\theta\alpha\theta$ is simple, it is isomorphic to $\alpha\theta\alpha$.

Another generator

- Free product $A_2 * A_2$ has no extra relations.
- In the tensor product $A_2 \boxtimes A_2$, | and | commute:

 $: || \xrightarrow{\cong} ||$

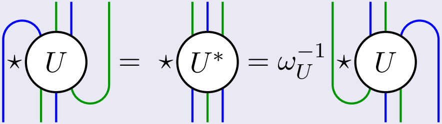
- If there is such an $n \in \mathbb{N}$, then we have an isomorphism $U : (||)^n \to 1$.
- For $n = 3$: $\star$  $: (|||) \xrightarrow{\cong} (|||)$.

Relations for U

Proposition

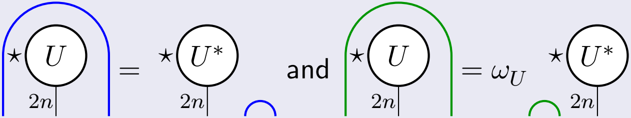
U satisfies the following skein relations:

- $UU^* =$ ||| and $U^*U =$ |||
- Rotation relation:



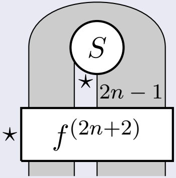
for some n-th root of unity ω_U.

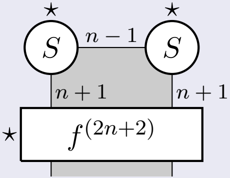
- Jellyfish relations:

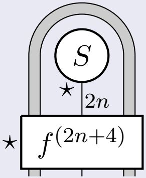


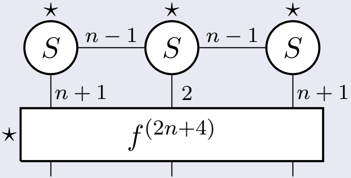
Bigelow-Morrison-Peters-Snyder [BMPS12]

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

- 

$= i\dfrac{\sqrt{[n][n+2]}}{[n+1]}$


,
- 

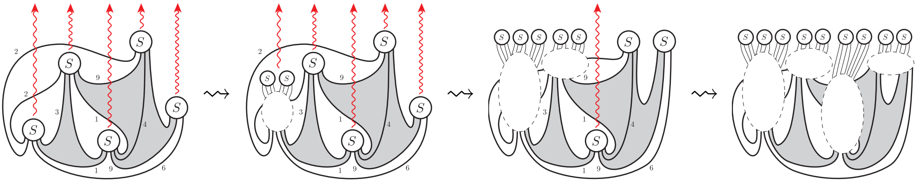
$= \dfrac{[2][2n+4]}{[n+1][n+2]}$


,
- capping S gives zero, and
- (Absorption) $S^2 = f^{(n)}$.

The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations



2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.

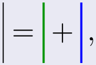
$\text{Vec}^{\omega}_{D_{2n}}$ and $A^{(1)}_{2n-1}$

Theorem

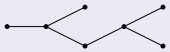
These relations are consistent and sufficient to evaluate all closed diagrams. Hence there are exactly n distinct categories satisfying $(||)^n \cong 1$. These are $\text{Vec}^{\omega}_{D_{2n}}$.

Remark

If we draw a black string for $X = \alpha \oplus \theta$,



then the fusion graph Γ_X is $A^{(1)}_{2n-1}$ 

Equivariantization $(| \leftrightarrow |)$ gives $D^{(1)}_{n+2}$ 

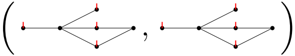
What about A_2 and T_2?

- Can we interpolate between tensor and free product for A_2 and T_2?
- This question was asked by Bisch and Haagerup in 1994.

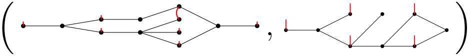
Bisch-Haagerup Fish

- Possible subfactors $A_3 \boxtimes A_4 \leq \mathcal{BHF}_n \leq A_3 * A_4$.
- Possible fusion categories $A_2 \boxtimes T_2 \leq \frac{1}{2}\mathcal{BHF}_n \leq A_2 * T_2$.

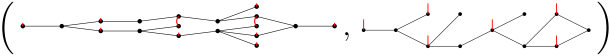
$\mathcal{BHF}_1 = A_3 \boxtimes A_4 =$



$\mathcal{BHF}_2 =$

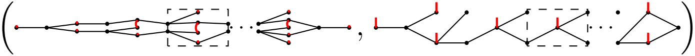


$\mathcal{BHF}_3 =$



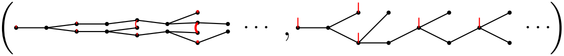
$\vdots$

$\mathcal{BHF}_n =$



$\vdots$

$\mathcal{BHF}_\infty = A_3 * A_4 =$

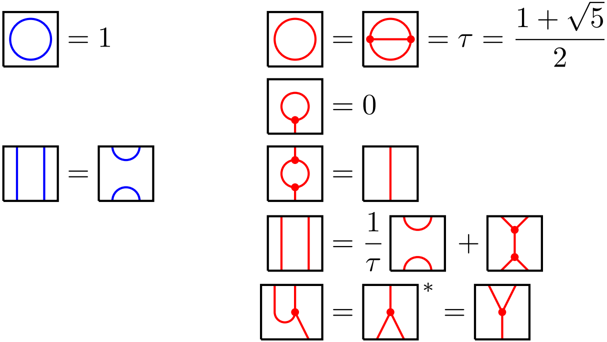


Skein relations

Suppose $\mathcal{C}$ is generated by θ, ρ with $\theta \otimes \theta \cong 1$ and $\rho \otimes \rho \cong 1 \oplus \rho$.

$\theta =$ and $\rho =$

We have the following skein relations:



Simple objects

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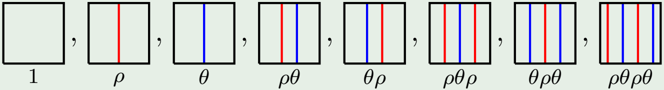
Suppose $\mathcal{C}$ is generated by ρ, θ. Then either $\mathcal{C}$ is the free product $A_2 * T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho\theta)^n \cong (\theta\rho)^n$, but $(\rho\theta)^{n-1} \not\cong (\theta\rho)^{n-1}$. Any word in ρ, θ of length $\leq 2n$ is a simple object. Words of length $< 2n$ give distinct simples.

Example

If $n = 2$, then (representatives for) the simple objects are



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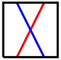
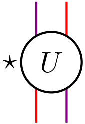
Another generator

- Free product $A_2 * T_2$ has no extra relations.
- In the tensor product $A_2 \boxtimes T_2$, $|$ and $|$ commute:

 $: \| \xrightarrow{\cong} \|$

- For $1 < n \leq \infty$, we have $(\|)^n \cong (\|)^n$:

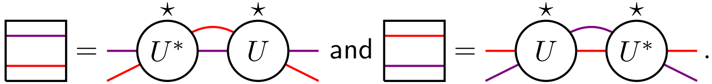


$: (\|)^n \xrightarrow{\cong} (\|)^n$

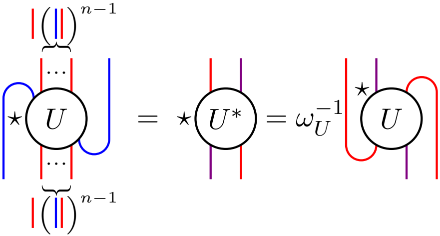
where we draw $|$ for $(\|)^{n-1}|$.

Relations for U

- Reidemeister relations:



- Rotation relations:

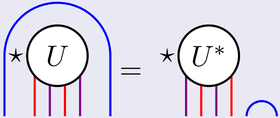


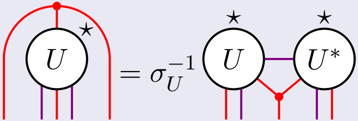
where ω_U is a $2n$-th root of unity.

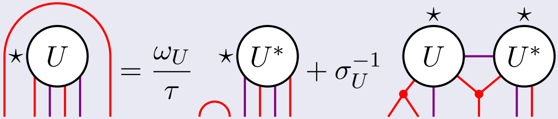
Jellyfish relations

Theorem

U satisfies the following jellyfish relations:

1. 

2. 

3. 

Here $\sigma_U^2 = \omega_U$. Switching U with $-U$ switches the sign of σ_U.

Existence and uniqueness for $n = 1, 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Liu 2013]

$\mathcal{BHF}_n$ exists and is unique for $n = 1, 2, 3, \infty$.
$\mathcal{BHF}_n$ does not exist for $4 \leq n < \infty$.

Theorem [Izumi-Morrison-Penneys 2013]

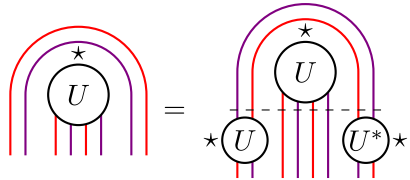
$\frac{1}{2}\mathcal{BHF}_n$ exists and is unique for $n = 1, 2, 3, \infty$.
$\frac{1}{2}\mathcal{BHF}_n$ does not exist for $4 \leq n \leq 10$.

Both proofs discovered simultaneously and independently.

- IMP's method - construction for $n = 1, 2, 3$ ad hoc, only eliminates $4 \leq n \leq 10$. Conjecturally eliminates all $4 \leq n < \infty$.
- Liu's method - uniform construction, eliminates all $4 \leq n < \infty$.

Uniqueness and nonexistence

- For $n = \infty$, no more relations, so planar algebra is unique.
- When $n < \infty$, we consider the following diagram:



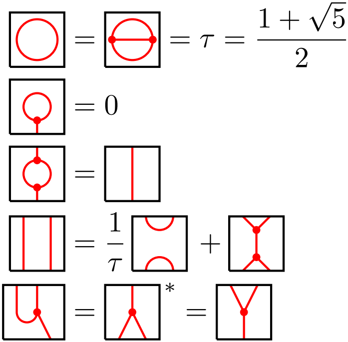
- First, we pull the U upward using the jellyfish relations. Then we compare the results.
- For $n = 1, 2, 3$, we get $\omega_U = 1$, so planar algebra is unique.
- For $4 \leq n \leq 10$, the results are inconsistent. We conjecture the results are inconsistent for all $4 \leq n < \infty$.

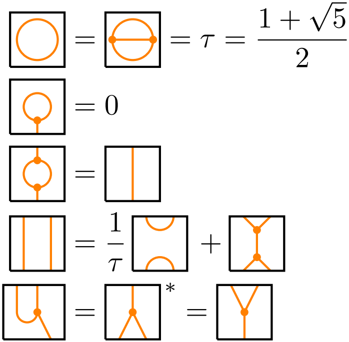
What about T_2 and T_2?

Suppose $\mathcal{C}$ is generated by ρ, μ with $\rho \otimes \rho \cong 1 \oplus \rho$ and $\mu \otimes \mu \cong 1 \oplus \mu$.

$\rho =$ and $\mu =$

We have the following skein relations:





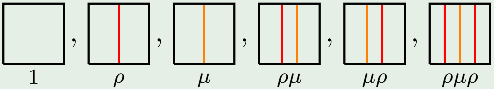
Simple objects

Proposition

Suppose $\mathcal{C}$ is generated by ρ, μ. Then either $\mathcal{C}$ is the free product $T_2 * T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho\mu)^n \cong 1$, but $(\rho\mu)^{n-1} \ncong 1$. Any word in ρ, μ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

Example

If $n = 3$, then (representatives for) the simple objects are



Even though $\mu\rho\mu$ is simple, it is isomorphic to $\rho\mu\rho$.

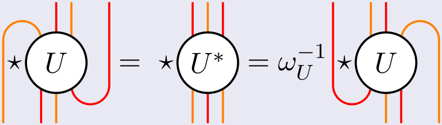
Skein relations

Again, we have another generator U when $2 \leq n < \infty$.

Proposition

U satisfies the following skein relations:

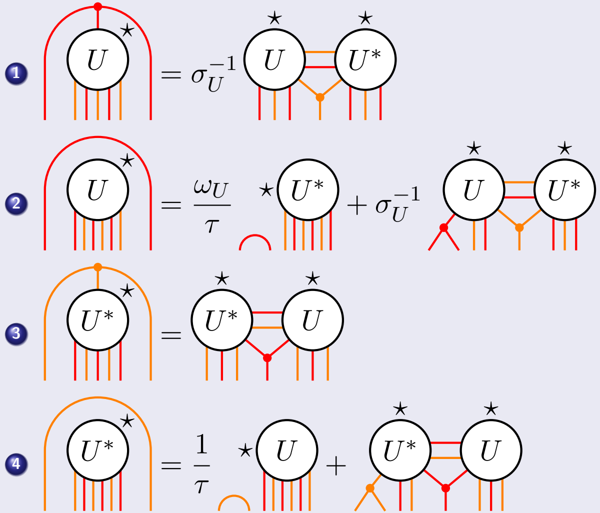
- $UU^* = |||$ and $U^*U = |||$
- Rotation relation:



for some n-th root of unity ω_U.

Theorem

U satisfies the following jellyfish relations:



Existence and uniqueness for $n = 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Izumi-Morrison-Penneys 2013]

This $T_2 - T_2$ category exists and is unique for $n = 2, 3, \infty$.
Does not exist for $4 \leq n \leq 10$.

Theorem [Liu 2013]

A similar, but much better result for subfactors.
Existence and uniqueness for $n = 2, 3, \infty$.
Non-existence for $4 \leq n < \infty$.

- Again, IMP's method only eliminates $4 \leq n \leq 10$. Conjecturally eliminates all $4 \leq n < \infty$.
- Liu's method is uniform, eliminates all $4 \leq n < \infty$.

What next?

What about

- Subfactors between $A_3 \boxtimes A_5$ and $A_3 * A_5$
- Fusion categories between $A_2 \boxtimes \frac{1}{2}A_5$ and $A_2 * \frac{1}{2}A_5$ ($\frac{1}{2}A_5 = \mathrm{Rep}(S_3)$)

The situation is much harder since $\frac{1}{2}A_5$ has three objects $1, \rho, \alpha$ with $\rho \otimes \rho \cong 1 \oplus \rho \oplus \alpha$.
I am exploring certain cases of these in joint work with Liu:

The end

Thank you for listening!

(Preprints coming soon)

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