Fusion categories between $\mathcal{C} \boxtimes \mathcal{D}$ and $\mathcal{C} \ast \mathcal{D}$

(with applications to subfactors at index $3 + \sqrt{5}$)

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Planar algebras give a generators and relations approach to subfactors and tensor categories.

From the above, we get an invariant called a fusion graph.

**Question**

- (Unreasonable) Which graphs are fusion graphs?
- What is a reasonable way to classify fusion categories?
Planar algebras [Jon99]

**Definition**

A planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk
Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

\[
\begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2}
\end{array}
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
\text{3}
\end{array} =
\begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\end{array}
\begin{array}{c}
\text{2}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array}
\end{array}
\begin{array}{c}
\circ
\end{array}
\begin{array}{c}
\text{2}
\end{array}
\]

Definition

The *planar operad* consists of all planar tangles (up to isotopy) with the operation of composition.
Definition

A planar algebra is a family of vector spaces $P_k$, $k = 0, 1, 2, \ldots$ and an action of the planar operad.
**Example: Temperley-Lieb**

$TL_n(\delta)$ is the complex span of non-crossing pairings of $n$ points arranged around a circle, with formal addition and scalar multiplication.

\[
TL_6(\delta) = \text{Span}_\mathbb{C}\left\{ \begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\end{array}\right\}.
\]

Planar tangles act on $TL$ by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of $\delta$. 

\[
\begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\end{array}\end{array} = \begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\end{array} = \delta^2 \begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\begin{array}{c}
\ast
\end{array}
\end{array}
\end{array}
\]
Some special tangles/properties

- **Multiplication:** $x \cdot y = \begin{array}{c} n \\ m \end{array} \begin{array}{c} n \\ m \end{array} \begin{array}{c} n \\ m \end{array} \begin{array}{c} n \\ m \end{array} (\text{TL}_{2n} \text{ is an algebra})$

- **Adjoint is reflection:** $\left( \begin{array}{c} * \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} * \\ n \end{array} \right)^* = \begin{array}{c} * \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} * \\ n \end{array}$

- **Trace:** $\text{Tr}_{2n}(x) = \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} = n \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} (\text{spherical})$

- **Sesquilinear form:** $\langle x, y \rangle = \text{Tr}_{2n}(y^*x) = \begin{array}{c} n \\ 2n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} n \\ n \end{array} \begin{array}{c} * \\ * \end{array}$
Jones’ index rigidity theorem [Jon83]

Suppose the sesquilinear form on $TL_{2n}$ given by $\langle x, y \rangle := Tr_{2n}(y^*x)$ is positive semi-definite for every $n \geq 0$. Then

$$\delta \in \left\{ 2 \cos \left( \frac{\pi}{k} \right) \mid k \geq 3 \right\} \cup [2, \infty).$$
Jones’ index rigidity theorem [Jon83]

Suppose the sesquilinear form on $TL_{2n}$ given by

$$\langle x, y \rangle := \text{Tr}_{2n}(y^*x)$$

is positive semi-definite for every $n \geq 0$. Then

$$\delta \in \left\{ 2 \cos \left( \frac{\pi}{k} \right) \mid k \geq 3 \right\} \cup [2, \infty).$$

- semi-definite
- definite
Definition

A planar algebra $P$ is a factor planar algebra if it is:

- Finite dimensional: $\dim(P_k) < \infty$ for all $k$
- Evaluable: $\dim(P_0) = 1$
- Sphericality: $\text{Tr}_2(X) = \star X = \star X$
- Positivity: each $P_j$ has an adjoint $\ast$ such that the sesquilinear form on $P_{2k}$ given by $\langle x, y \rangle_{2k} := \text{Tr}_{2k}(y^* x)$ is positive definite for all $k \geq 0$.

From these properties, it follows that closed circles count for a multiplicative constant $\delta$. 
If the sesquilinear form is semi-definite, we quotient out the length zero vectors.

**Example: $A_2$**

The fantastic planar algebra $A_2$ is the quotient of Temperley-Lieb when $\delta = 2 \cos(\pi/3) = 1$ by the following skein relations:

\[
\begin{align*}
\text{circle} & = 1 \\
\begin{array}{c}
\text{vertical line}
\end{array} & = \begin{array}{c}
\text{horizontal line}
\end{array}
\end{align*}
\]
Example: $T_2$

Generated by a trivalent vertex:

Skein relations:

\[
\begin{align*}
\text{circle} & = \text{fork} = \tau = \frac{1 + \sqrt{5}}{2} \\
\text{fork} & = 0 \\
\text{double circle} & = \text{straight line} \\
\text{double straight line} & = \frac{1}{\tau} \text{fork} + \text{fork}^* = \text{Y}
\end{align*}
\]
Example: Free product $A_2 \ast T_2$

All non-crossing string diagrams with red and blue strings satisfying the previous relations.

Example: Tensor product $A_2 \boxtimes T_2$

All crossing string diagrams with red and blue strings satisfying the previous relations, and a Reidemeister two relation
Given a rigid $C^*$-tensor category, e.g., a unitary fusion category, and a ‘nice’ object $X$, we can construct a planar algebra.

\[ \mathcal{PA}(C, X)_n = \text{Hom}(1, X^\otimes n): \]

- $\text{ev}_X = X \cdot X$ and $\text{coev}_X = X \cdot X$

zig-zag relation: $X = X \cdot X \cdot X$. 

Fusion categories between $C \boxtimes D$ and $C \ast D$
Tensor categories to planar algebras (cont.)

- unitary implies positive and spherical:

  \[ \langle f, g \rangle = \begin{array}{c}
  f \\
  \star \\
  \end{array} \rightarrow_n \begin{array}{c}
  \star \\
  g^* \\
  \end{array} \]

  \[ \text{tr}(f) = \begin{array}{c}
  f \\
  n \\
  \end{array} = \begin{array}{c}
  f \\
  n \\
  \end{array} \]

- spherical implies pivotal:

  \[ \begin{array}{c}
  f \\
  n \\
  \end{array} = \begin{array}{c}
  f \\
  n \\
  \end{array} \]
Planar algebras to tensor categories

Given a factor planar algebra, can construct its rigid $C^*$-tensor category of projections.

- Objects are (formal direct sums of) projections
- Tensoring is horizontal concatenation $p \otimes q = \begin{array}{c} p \\ m \\ n \end{array} \begin{array}{c} q \\ m \\ n \end{array}$
- $\text{Hom}(p, q) = \{ x \mid x = qxp \}$, i.e., $\begin{array}{c} x \\ m \\ n \end{array} = \begin{array}{c} x \\ m \\ n \end{array}$.
- Composition of morphisms is vertical stacking.
Duality is rotation by $\pi$

\[
\begin{array}{c}
\bar{p} = \rule{2cm}{0.5mm} p = \rule{2cm}{0.5mm} p \\ \ .
\end{array}
\]

The adjoint $\ast : \text{Hom}(p, q) \to \text{Hom}(q, p)$ is the adjoint in $P_\bullet$.

**Theorem**

- $P_\bullet \to \text{Pro}(P_\bullet) \to \mathcal{PA}(\text{Pro}(P_\bullet), \|)$ is the identity.
- $(\mathcal{C}, X) \to \mathcal{PA}(\mathcal{C}, X) \to \text{Pro}(\mathcal{PA}(\mathcal{C}, X))$ is an equivalence.
Fusion graphs

Definition

Given a rigid $C^*$ tensor category $C$ and a ‘nice’ object $X$, we define $\Gamma_X$, the fusion graph with respect to $X$, as follows:

- Vertices: equivalence classes of simple objects
- Edges: If $P$ is simple, $P \otimes X = \bigoplus_{Q \text{ simple}} N_{P,X}^Q Q$.
  There are $N_{P,X}^Q$ edges between simples $P, Q \in C$.

Example: $A_2$

Two simples $1, \theta$, and $\theta \otimes \theta = 1$, so $\Gamma_\theta = 1 \xrightarrow{\theta}$.

Example: $T_2$

Two simples $1, \tau$, and $\tau \otimes \tau = 1 \oplus \tau$, so $\Gamma_\tau = 1 \xrightarrow{\tau}$.
Planar algebras with $\delta < 2$

**Theorem**

The factor planar algebras with $\delta < 2$ are as follows:

<table>
<thead>
<tr>
<th>name</th>
<th>principal graph</th>
<th>#</th>
<th>constructed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td><img src="https://example.com/graph.png" alt="Graph" /></td>
<td>1</td>
<td>[Jon83]</td>
</tr>
<tr>
<td>$D_{2n}$</td>
<td><img src="https://example.com/graph.png" alt="Graph" /></td>
<td>1</td>
<td>[Ocn88, Kaw95]</td>
</tr>
<tr>
<td>$T_n$</td>
<td><img src="https://example.com/graph.png" alt="Graph" /></td>
<td>1</td>
<td>[KO02, EO04]</td>
</tr>
<tr>
<td>$E_6$</td>
<td><img src="https://example.com/graph.png" alt="Graph" /></td>
<td>2</td>
<td>[Ocn88, BN91]</td>
</tr>
<tr>
<td>$E_8$</td>
<td><img src="https://example.com/graph.png" alt="Graph" /></td>
<td>2</td>
<td>[Ocn88, Izu94]</td>
</tr>
</tbody>
</table>
Composing fusion categories

Interpolating between tensor products and free products of fusion categories.

Simplest examples of fusion categories have 2 objects.

- $A_2 - A_2$ (Warmup)
- $A_2 - T_2$ (Main motivation - Bisch-Haagerup 1994)
- $T_2 - T_2$ (Bonus!)
Two copies of $A_2$

Take two copies of $A_2$:

$$\alpha = \quad \text{and} \quad \theta = \quad $$

where $\alpha \otimes \alpha \cong 1$ and $\theta \otimes \theta \cong 1$. We have the following skein relations:

$$
\begin{align*}
\text{[Diagram 1]} &= 1 \\
\text{[Diagram 2]} &= 1 \\
\text{[Diagram 3]} &= 1 \\
\text{[Diagram 4]} &= 1
\end{align*}
$$
Simple objects

**Proposition**
Suppose $C$ is generated by $\alpha, \theta$. Then either $C$ is the free product $A_2 \ast A_2$, or there is an $n \in \mathbb{N}$ such that $(\alpha \theta)^n \cong 1$, but $(\alpha \theta)^{n-1} \not\cong 1$. Any word in $\alpha, \theta$ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

**Example**
If $n = 3$, then (representatives for) the simple objects are

$1$, $\alpha$, $\theta$, $\alpha \theta$, $\theta \alpha$, $\alpha \theta \alpha$

Even though $\theta \alpha \theta$ is simple, it is isomorphic to $\alpha \theta \alpha$. 
Another generator

- Free product $A_2 \ast A_2$ has no extra relations.
- In the tensor product $A_2 \boxtimes A_2$, $\mid$ and $\mid$ commute:

- If there is such an $n \in \mathbb{N}$, then we have an isomorphism $U : (\mid\mid\mid)^n \to 1$.
- For $n = 3$: $\ast U : (\mid\mid\mid) \xrightarrow{\sim} (\mid\mid\mid)$.
Relations for $U$

**Proposition**

$U$ satisfies the following skein relations:

- $UU^* = ||$ and $U^*U = ||$
- Rotation relation:
  \[
  *U* = *U*^* = \omega_U^{-1} *U
  \]
  for some $n$-th root of unity $\omega_U$.
- Jellyfish relations:
  \[
  *U*_{2n} = *U*^*_n \quad \text{and} \quad *U*_{2n} = \omega_U *U^*_n
  \]
Bigelow-Morrison-Peters-Snyder [BMPS12]

The Haagerup and extended Haagerup subfactor planar algebras have a generator \( S \in P_{n,+} \) where \( n = 4, 8 \) respectively satisfying:

- \( S \star f(2n+2) = i \sqrt{n(n+2)} \frac{[n][n+2]}{[n+1]} \star f(2n+2) \),

- \( S \star f(2n+4) = \frac{[2][2n+4]}{[n+1][n+2]} \star f(2n+4) \),

- Capping \( S \) gives zero, and

- (Absorption) \( S^2 = f(n) \).
The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations.

2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.
Theorem

These relations are consistent and sufficient to evaluate all closed diagrams. Hence there are exactly $n$ distinct categories satisfying $(||)^n \cong 1$. These are $\text{Vec}_D^{\omega}$. 

Remark

If we draw a black string for $X = \alpha \oplus \theta$, 

$$|| = | + |,$$

then the fusion graph $\Gamma_X$ is $A_{2n-1}^{(1)}$.

Equivariantization ($| \leftrightarrow |$) gives $D_{n+2}^{(1)}$. 

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Fusion categories between $C \boxtimes D$ and $C \ast D$
What about $A_2$ and $T_2$?

- Can we interpolate between tensor and free product for $A_2$ and $T_2$?
- This question was asked by Bisch and Haagerup in 1994.
Possible subfactors $A_3 \boxtimes A_4 \leq \mathcal{BHF}_n \leq A_3 \ast A_4$.

Possible fusion categories $A_2 \boxtimes T_2 \leq \frac{1}{2} \mathcal{BHF}_n \leq A_2 \ast T_2$.

$\mathcal{BHF}_1 = A_3 \boxtimes A_4 = \left( \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right)$

$\mathcal{BHF}_2 = \left( \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right)$

$\mathcal{BHF}_3 = \left( \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right)$

$\mathcal{BHF}_n = \left( \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right)$

$\mathcal{BHF}_\infty = A_3 \ast A_4 = \left( \begin{array}{c} \begin{array}{c} \end{array} \end{array} \right)$
Suppose \( \mathcal{C} \) is generated by \( \theta, \rho \) with \( \theta \otimes \theta \cong 1 \) and \( \rho \otimes \rho \cong 1 \oplus \rho \).

\[
\theta = \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array}
\quad \text{and} \quad \rho = \begin{array}{c|c|c}
\hline
& & \\
& Y & \\
& & \\
\hline
\end{array}
\]

We have the following skein relations:

\[
\begin{align*}
\begin{array}{c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} &= 1 \\
\begin{array}{c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} &= \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} = \tau = \frac{1 + \sqrt{5}}{2} \\
\begin{array}{c}
\hline
& & \\
& & \\
& & \\
\hline
\end{array} &= 0 \\
\begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} &= \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} \\
\begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} &= \frac{1}{\tau} \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} + \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} \\
\begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array} &= \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array}^* = \begin{array}{c|c|c}
\hline
& & \\
& X & \\
& & \\
\hline
\end{array}
\end{align*}
\]
Simple objects

**Proposition**

Suppose $C$ is generated by $\rho$, $\theta$. Then either $C$ is the free product $A_2 \ast T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho\theta)^n \cong (\theta\rho)^n$, but $(\rho\theta)^{n-1} \not\cong (\theta\rho)^{n-1}$. Any word in $\rho$, $\theta$ of length $\leq 2n$ is a simple object. Words of length $< 2n$ give distinct simples.

**Example**

If $n = 2$, then (representatives for) the simple objects are

1, $\rho$, $\theta$, $\rho\theta$, $\theta\rho$, $\rho\theta\rho$, $\theta\rho\theta$, $\rho\theta\rho\theta$

Even though $\theta\rho\theta\rho$ is simple, it is isomorphic to $\rho\theta\rho\theta$. 
Another generator

- Free product $A_2 * T_2$ has no extra relations.
- In the tensor product $A_2 \boxtimes T_2$, $\mid$ and $\mid$ commute:

\[
\begin{array}{c}
\begin{array}{c}
\text{X} \\
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\mid
\end{array}
\end{array}
\]

- For $1 < n \leq \infty$, we have $(\mid\mid)^n \cong (\mid\mid)^n$:

\[
\[
\begin{array}{c}
\begin{array}{c}
\ast
\end{array}
\end{array}
\rightarrow
\begin{array}{c}
\begin{array}{c}
\mid
\end{array}
\end{array}
\]

where we draw $\mid$ for $(\mid\mid)^{n-1}$.
Relations for $U$

- **Reidemeister relations**: 
  \[
  \begin{array}{c}
  = \quad U^* \quad U \quad \text{and} \quad = \quad U \quad U^*.
  \end{array}
  \]

- **Rotation relations**: 
  \[
  \begin{array}{c}
  = U^* = \omega_U^{-1} U.
  \end{array}
  \]

  where $\omega_U$ is a $2n$-th root of unity.
Theorem

$U$ satisfies the following jellyfish relations:

1. $U^* = U^*$
2. $U = \sigma_U^{-1}$
3. $U^* = \frac{\omega_U}{\tau} U^* + \sigma_U^{-1}$

Here $\sigma_U^2 = \omega_U$. Switching $U$ with $-U$ switches the sign of $\sigma_U$. 
Existence and uniqueness for $n = 1, 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Liu 2013]

$BHF_n$ exists and is unique for $n = 1, 2, 3, \infty$.
$BHF_n$ does not exist for $4 \leq n < \infty$.

Theorem [Izumi-Morrison-Penneys 2013]

$\frac{1}{2}BHF_n$ exists and is unique for $n = 1, 2, 3, \infty$.
$\frac{1}{2}BHF_n$ does not exist for $4 \leq n \leq 10$.

Both proofs discovered simultaneously and independently.

- IMP’s method - construction for $n = 1, 2, 3$ ad hoc, only eliminates $4 \leq n \leq 10$. Conjecturally eliminates all $4 \leq n < \infty$.
- Liu’s method - uniform construction, eliminates all $4 \leq n < \infty$. 

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Fusion categories between $C \boxtimes D$ and $C \ast D$
Uniqueness and nonexistence

- For $n = \infty$, no more relations, so planar algebra is unique.
- When $n < \infty$, we consider the following diagram:

\[
\begin{array}{ccc}
\ast & = & \ast \\
U & & U \\
\ast & & \ast \\
U^* & & U^*
\end{array}
\]

- First, we pull the $U$ upward using the jellyfish relations.
  Then we compare the results.
- For $n = 1, 2, 3$, we get $\omega_U = 1$, so planar algebra is unique.
- For $4 \leq n \leq 10$, the results are inconsistent. We conjecture the results are inconsistent for all $4 \leq n < \infty$. 
What about $T_2$ and $T_2$?

Suppose $C$ is generated by $\rho, \mu$ with $\rho \otimes \rho \cong 1 \oplus \rho$ and $\mu \otimes \mu \cong 1 \oplus \mu$.

$$\rho = \quad \text{and} \quad \mu = \quad$$

We have the following skein relations:

\[
\begin{align*}
\begin{array}{c}
\includegraphics{image1.png} \\
\includegraphics{image2.png} \\
\includegraphics{image3.png} \\
\includegraphics{image4.png} \\
\includegraphics{image5.png}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\includegraphics{image6.png} \\
\includegraphics{image7.png} \\
\includegraphics{image8.png} \\
\includegraphics{image9.png} \\
\includegraphics{image10.png}
\end{array}
\end{align*}
\]
Simple objects

**Proposition**

Suppose $C$ is generated by $\rho, \mu$. Then either $C$ is the free product $T_2 \ast T_2$, or there is an $n \in \mathbb{N}$ such that $(\rho \mu)^n \cong 1$, but $(\rho \mu)^{n-1} \not\cong 1$. Any word in $\rho, \mu$ of length $\leq n$ is a simple object. Words of length $< n$ give distinct simples.

**Example**

If $n = 3$, then (representatives for) the simple objects are

1, $\rho$, $\mu$, $\rho \mu$, $\mu \rho$, $\rho \mu \rho$

Even though $\mu \rho \mu$ is simple, it is isomorphic to $\rho \mu \rho$. 
Again, we have another generator $U$ when $2 \leq n < \infty$.

**Proposition**

$U$ satisfies the following skein relations:

- $UU^* = \|\|$ and $U^*U = \|\|
- Rotation relation:

\[
\star U \star = \star U^* \star = \omega_U^{-1} \star U
\]

for some $n$-th root of unity $\omega_U$. 
Theorem

$U$ satisfies the following jellyfish relations:

1. $U = \sigma_U^{-1}$
2. $U = \frac{\omega_U}{\tau} U^* + \sigma_U^{-1}$
3. $U^* = U^* U$
4. $U^* = \frac{1}{\tau} U + U^* U$
Existence and uniqueness for $n = 2, 3, \infty$, nonexistence for $4 \leq n < \infty$

Theorem [Izumi-Morrison-Penneys 2013]
This $T_2 - T_2$ category exists and is unique for $n = 2, 3, \infty$. Does not exist for $4 \leq n \leq 10$.

Theorem [Liu 2013]
A similar, but much better result for subfactors. Existence and uniqueness for $n = 2, 3, \infty$. Non-existence for $4 \leq n < \infty$.

- Again, IMP’s method only eliminates $4 \leq n \leq 10$. Conjecturally eliminates all $4 \leq n < \infty$.
- Liu’s method is uniform, eliminates all $4 \leq n < \infty$. 
What about

- Subfactors between $A_3 \boxtimes A_5$ and $A_3 \ast A_5$
- Fusion categories between $A_2 \boxtimes \frac{1}{2} A_5$ and $A_2 \ast \frac{1}{2} A_5$

($\frac{1}{2} A_5 = \text{Rep}(S_3)$)

The situation is much harder since $\frac{1}{2} A_5$ has three objects $1, \rho, \alpha$ with $\rho \otimes \rho \cong 1 \oplus \rho \oplus \alpha$.

I am exploring certain cases of these in joint work with Liu:
Thank you for listening!
(Preprints coming soon)


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