**II\(_1\)-factors**

**Definition**

A factor is a von Neumann algebra with trivial center. A factor \(A\) is a II\(_1\)-factor if it is infinite dimensional and it has a tracial state \(\text{tr}_A : A \to \mathbb{C}\).

- GNS representation of \(A\) on \(L^2(A)\): \(\langle \hat{x}, \hat{y} \rangle = \text{tr}_A(y^*x)\).
- Conjugate-linear unitary \(J : L^2(A) \to L^2(A)\) by \(\hat{x} \mapsto \hat{x}^*\).

**Theorem**

\[A' \cap B(L^2(A)) = JAJ = \{Jaj| a \in A\}\].
Jones’ index

Definition

Let $A \subset B$ be a $II_1$-subfactor. $[B : A] = \dim_A(L^2(B))$.

Observation [PP86]

$A \subset B$ has finite index if and only if $B$ is a finitely-generated projective $A$-module.

Theorem [Jon83]

For a $II_1$-subfactor $A \subset B$,

$$[B : A] \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \left| n = 3, 4, \ldots \right. \right\} \cup [4, \infty] .$$
The basic construction

**Definition**

On $L^2(B)$, we define the Jones projection $e_A$ as the projection with range $L^2(A) \subset L^2(B)$.

**Definition**

The basic construction of $A \subset B$ is the von Neumann algebra $\langle B, e_A \rangle \subset B(L^2(B))$. Equivalently, $\langle B, e_A \rangle = JA'J$. 

\[ A' \]
\[ \leftrightarrow \]
\[ B \rightarrow L^2(B) \leftarrow JBJ \]
\[ \uparrow \]
\[ \downarrow \]
\[ A \rightarrow JAJ \]
**The basic construction**

**Definition**

On $L^2(B)$, we define the Jones projection $e_A$ by $e_A(\hat{b}) = \hat{E}_A(\hat{b})$. $e_A$ is the projection with range $L^2(A) \subset L^2(B)$.

**Definition**

The basic construction of $A \subset B$ is the von Neumann algebra $\langle B, e_A \rangle \subset B(L^2(B))$.

Equivalently, $\langle B, e_A \rangle = JA'J$.

\[
\begin{align*}
\langle B, e_A \rangle & \quad \quad A' \\
B & \rightarrow L^2(B) \leftarrow JBJ \\
A & \quad \quad JAJ
\end{align*}
\]
Theorem [Jon83]

If $[B : A] < \infty$, then $\langle B, e_A \rangle$ is a $II_1$-factor, and

$$[\langle B, e_A \rangle : B] = [B : A].$$

- Markov trace property:

$$\text{tr}_{\langle B, e_A \rangle}(e_A x) = [B : A]^{-1} \text{tr}_B(x)$$

for all $x \in B$. 
The Jones tower

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ is given by

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$$

where $e_i$ is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$. 
The Temperley-Lieb algebras

**Definition**

The Jones projections $e_i$ satisfy the following relations:

1. $e_i = e_i^* = e_i^2$,
2. $e_ie_j = ejej$ for $|i - j| > 1$, and
3. $e_i e_i \pm 1 e_i = [B : A]^{-1} e_i$

Renormalize: $d^2 = [B : A]$, $E_i = de_i$. This gives:

1. $dE_i = dE_i^* = E_i^2$,
2. $E_iE_j = E_jE_i$ for $|i - j| > 1$, and
3. $E_iE_i \pm 1 E_i = E_i$
The Temperley-Lieb algebras

Definition

Let $TL_n(d)$ be the complex $\ast$-algebra generated by $1, E_1, \ldots, E_{n-1}$ satisfying the relations:

1. $dE_i = dE_i^* = E_i^2$,
2. $E_i E_j = E_j E_i$ for $|i - j| > 1$, and
3. $E_i E_{i \pm 1} E_i = E_i$
Kauffman diagrams

- String diagram for $E_i$:

- Multiplication is stacking:

$$E_3 E_2 E_3 = E_3$$
Kauffman diagrams

- Closed loops count for a multiplicative factor of $d$:
  \[ E_2^2 = d = dE_2 \]

- The adjoint is reflection about $x$-axis:
  \[ \text{adjoint} = \]

- $TL_n(d) \hookrightarrow TL_{n+1}(d)$ by adding a string to the right:
  \[ TL_2(d) \ni \rightarrow \in TL_3(d) \]
We define a tracial linear functional $\text{tr}_n$ on $TL_n(d)$:

$$\text{tr}_n(x) = \frac{1}{d^n} x \ldots$$

This trace satisfies the Markov property!

**Theorem [Jon83]**

$\langle x, y \rangle = \text{tr}_n(y^*x)$ is positive semi-definite on $TL_n(d)$ for all $n \geq 0$ if and only if

$$d \in \{2 \cos(\pi/k) | k \geq 3\} \cup [2, \infty).$$
Standard invariant

Two towers of centralizer algebras: \((A'_0 \cap A_n)_{n \geq 0}, (A'_1 \cap A_{n+1})_{n \geq 0}\)

- Markov trace
- Finite dimensional C*-algberas (semi-simple)
- Bratteli diagram
- \(TL_n(d) \subset A'_0 \cap A_n \) where \(d^2 = [A_1 : A_0] \) when \(d \geq 2\)
The Bratteli diagram for the algebras $TL_n(d)$ for $d \geq 2$ is half of Pascal’s triangle:
What if $[B : A] = \infty$?
\( \mathcal{II}_\infty \)-factors

**Definition**

A \( \mathcal{II}_\infty \)-factor \( A \) is a factor such that:

- \( 1 \in A \) is an infinite projection, and
- There is a weight \( \text{Tr}_A : A^+ \to [0, \infty] \) which is:
  
  - **Tracial**: \( \text{Tr}_A(x^*x) = \text{Tr}_A(xx^*) \) for all \( x \in A \),
  
  - **Faithful**: \( \text{Tr}_A(x^*x) = 0 \iff x = 0 \),
  
  - **Normal**: \( x_i \uparrow x \iff \text{Tr}_A(x_i) \uparrow \text{Tr}_A(x) \) where \( x_i, x \in A^+ \),
  
  - **Semifinite**: for all \( x \in A^+ \), there is a \( 0 \leq y \leq x \) such that \( \text{Tr}_A(y) < \infty \).

**Caution**

\( \text{Tr}_A(1_A) = \infty \)!
We can still define the GNS-representation. Define

- \( \mathfrak{n}_{\text{Tr}_A} = \{ x \in A | \text{Tr}_A(x^*x) < \infty \} \),
- \( \langle \hat{x}, \hat{y} \rangle = \text{Tr}_A(y^*x) \) for \( \hat{x}, \hat{y} \in \mathfrak{n}_{\text{Tr}_A} \) by polarization,
- \( L^2(A) = \overline{\mathfrak{n}_{\text{Tr}_A}} \cdot \cdot \cdot \cdot 2 \).

We still get:

- Conjugate-linear unitary \( J : L^2(A) \to L^2(A) \) by \( \hat{x} \mapsto \hat{x}^* \).

**Theorem**

\( A' \cap B(L^2(A)) = JAJ = \{ JaJ | a \in A \} \).
Suppose $A \subset B$ is a $II_1$-subfactor with $[B:A] = \infty$.

**Facts**

1. $\langle B, e_A \rangle$ is a $II_\infty$-factor.
   - There is a canonical trace $\text{Tr}_{\langle B, e_A \rangle}$ on $\langle B, e_A \rangle^+$ satisfying
     \[
     \text{Tr}_{\langle B, e_A \rangle}(xe_Ay) = \text{tr}_B(xy) \text{ for all } x, y \in B.
     \]
2. $\left. \text{Tr}_{\langle B, e_A \rangle} \right|_{B^+} = \infty$, so $L^2(B) \nsubseteq L^2\langle B, e_A \rangle$.
3. There is no Jones projection $e_B : L^2\langle B, e_A \rangle \to L^2(B)$.

**Caution**

$\text{Tr}_{\langle B, e_A \rangle}(e_A) = 1$, so $\text{Tr}_{\langle B, e_A \rangle}(1 - e_A) = \infty$. 
We can still iterate the basic construction. We use the formula:

$$\langle B, e_A \rangle = JA'J \subset B(L^2(B)).$$

We iteratively define

$$A_{n+1} = J_n A'_{n-1} J_{n-1} \subset B(L^2(A_n)).$$

**Fact**

There is a canonical semifinite trace $\text{Tr}_n$ on $A_n$ for all $n \geq 2$, and $A_n$ is a $II_{\infty}$-factor.
Odd Jones projections

**Theorem [Bur03]**

(1) \( \text{Tr}_{2n+1}|_{A_{2n}^+} = \text{Tr}_{2n} \) and

(2) \( \text{Tr}_{2n+2}|_{A_{2n+1}^+} = \infty. \)

This implies:

- \( L^2(A_{2n}) \subset L^2(A_{2n+1}) \), so there is a Jones projection \( e_{2n+1} \in A_{2n+2} \subset B(L^2(A_{2n+1})) \) with range \( L^2(A_{2n}) \).
- \( L^2(A_{2n+1}) \not\subset L^2(A_{2n+2}) \)

**Fact**

The odd Jones projections \( e_{2n+1} \) satisfy:

(1) \( e_i = e_i^* = e_i^2 \),

(2) \( e_i e_j = e_j e_i \) for all \( i, j \).
String diagrams

String diagram for $e_{2i+1}$:

Multiplication is stacking:

$e_5 e_7 = e_7 e_5 = \ldots$

We remove closed strings (dots at both sides).
Higher relative commutants

Idea: we can make sense of diagrams like \( \cdots \).

**Theorem**

For \( i, j \leq n \), \( e_{2i-1} \sim e_{2j-1} \) in \( A'_0 \cap A_{2n} \).

Hence \( A'_0 \cap A_{2n} \) is not abelian when \( n \geq 2 \).

\[ \cdots \]

is a partial isometry witnessing \( e_3 \sim e_5 \in A'_0 \cap A_6 \).
**Definition**

For $n \geq 0$, let $G_n$ be the complex $\ast$-algebra generated by these diagrams. For example:

$$G_2 = \langle \begin{array}{cccc}
\begin{array}{c}
\vdots
\end{array}
, & \begin{array}{c}
\vdots
\end{array}
, & \begin{array}{c}
\vdots
\end{array}
, & \begin{array}{c}
\vdots
\end{array}
, & \begin{array}{c}
\vdots
\end{array}
, & \begin{array}{c}
\vdots
\end{array}
\end{array} \rangle$$

- Adjoint is reflection:

$$
\begin{array}{c}
\vdots
\end{array}
\ast
\begin{array}{c}
\vdots
\end{array}
= 
\begin{array}{c}
\vdots
\end{array}
\begin{array}{c}
\vdots
\end{array}
$$

- $G_n \hookrightarrow G_{n+1}$ by adding a string to the right:

$$G_2 \ni 
\begin{array}{c}
\vdots
\end{array}
\begin{array}{c}
\vdots
\end{array}
\rightarrow 
\begin{array}{c}
\vdots
\end{array}
\begin{array}{c}
\vdots
\end{array}
\in G_3$$
The complex $\ast$-algebras $G_n$ are isomorphic to the GICAR algebras

$$G_n \cong \bigoplus_{k=1}^{n} M_{(n)}^{(k)}(\mathbb{C}).$$

For example, the $\ast$-algebra generated by the diagrams

modulo diagrams with fewer through strings is isomorphic to $M_2(\mathbb{C})$. 
Theorem

The Bratteli diagram for the algebras $G_n$ is given by Pascal’s triangle:

```
  1
  1 1
  1 1 1
  1 1 1 1
  1 1 1 1 1
  1 1 1 1 1 1
```

This diagram represents the entries in Pascal's triangle, with each entry corresponding to the number of ways to choose elements from a set.

David Penneys

Infinite index subfactors and the GICAR algebras
Theorem

Given a subfactor $A_0 \subset A_1$ of arbitrary index, $G_n \hookrightarrow A'_0 \cap A_{2n}$.

Proof

For $k \leq n$, the (semi-finite) traces of $e_1 e_3 \cdots e_{2k-1}$ and $e_1 \perp e_3 \cdots e_{2k-1}$ are non-zero.
Can we only get GICAR?

**Question**

Is there an infinite index $\text{II}_1$-subfactor $A_0 \subset A_1$ such that $A'_0 \cap A_{2n} = G_n$ for all $n$?

**Theorem**

If $G_0 = \text{Stab}(1) \subset S_\infty = G_1$, then if $A_0 = R \rtimes G_0 \subset R \rtimes G_1 = A_1$, $\dim(A'_0 \cap A_{2n}) < \infty$ for all $n$. However, $\dim(A'_0 \cap A_{2n})$ grows too quickly.

The group-subgroup subfactor won't work.
A unitary tensor category has the following properties (and more):

- for each object $X \in C$, there is an object $\overline{X} \in C$ and evaluation and coevaluation morphisms

$$
\begin{align*}
\begin{array}{c}
1_C \\
\overline{X} \otimes X
\end{array} &\in \text{Hom}(\overline{X} \otimes X, 1_C), \\
\begin{array}{c}
X \\
\overline{X} \otimes 1_C
\end{array} &\in \text{Hom}(1_C, X \otimes \overline{X})
\end{align*}
$$

satisfying the zigzag relations,

- for each $X, Y \in C$, an involution

$\ast : \text{Hom}(X, Y) \to \text{Hom}(Y, X)$ such that if $f \in \text{Hom}(X, Y)$ and $g \in \text{Hom}(Y, Z)$, then $(g \circ f)^* = g^* \circ f^*$, and

- for each $X, Y \in C$, $\text{Hom}(X, Y)$ is a Banach space, and $\text{Hom}(X, X)$ is a $C^*$-algebra.
Fact

In a unitary tensor category, picking a distinguished object $X$, we get canonical maps

$$TL_{2n}(\text{tr}(X)) \rightarrow \text{End}(X \otimes X \otimes \cdots X \otimes X)$$

$$TL_{2n+1}(\text{tr}(X)) \rightarrow \text{End}(X \otimes X \otimes \cdots X \otimes X \otimes X)$$

by using the evaluation and coevaluation maps.
Finite index subfactors give unitary 2-categories

Definition

The paragroup of $A \subset B$ is the 2-category given by

- 0-morphisms: $\{A, B\}$
- 1-morphisms: bimodule summands of $L^2(A_k)$ for some $k \geq 0$
- 2-morphisms: intertwiners (live inside $A'_0 \cap A_k$, $A'_1 \cap A_{k+1}$)

Fact

The Temperley-Lieb algebras arise by looking at the distinguished 1-morphism $X =_A L^2(B)_B$. 
Infinite index subfactors do not give unitary 2-categories

Caution

There are no evaluation and coevaluation maps

\[
\begin{align*}
    BL^2(B)_B & \in \text{Hom}\left( BL^2(B) \otimes_A L^2(B)_B, BL^2(B)_B \right) \\
    \cong_{BL^2(B, e_A)_B} & \\
    BL^2(B)_A & \in \text{Hom}(BL^2(B)_B, BL^2(B) \otimes_A L^2(B)_B) \\
    \cong_{BL^2(B, e_A)_B} & \\
\end{align*}
\]

Since \( L^2(B) \not\subseteq L^2(B, e_A) \), and there is no Jones projection \( e_B \).
**Fact**

The GICAR algebras arise by looking at the evaluation and coevaluation maps

\[
AL^2(A)_A = e_A \in \text{Hom}(AL^2(B) \otimes_B L^2(B)_A, AL^2(A)_A) \\
\cong_{AL^2(B)_A} \\
AL^2(B)_B \quad BL^2(B)_A \\
\cong_{AL^2(B)_A} \\
AL^2(A)_A
\]

where \(e_A\) is the Jones projection and \(i_A\) is the inclusion.
Thank you for listening!

Slides available at:
http://math.berkeley.edu/~dpenneys

Preprint available at:
http://math.berkeley.edu/~dpenneys/GICAR.pdf
