Infinite index subfactors and the GICAR algebras

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II_1-factors

Defintion

A factor is a von Neumann algebra with trivial center.
A factor A is a II_1-factor if it is infinite dimensional and it has a tracial state $\mathrm{tr}_A\colon A \to \mathbb{C}$.

- GNS representation of A on $L^2(A)$: $\langle \widehat{x}, \widehat{y} \rangle = \mathrm{tr}_A(y^* x)$.
- Conjugate-linear unitary $J\colon L^2(A) \to L^2(A)$ by $\widehat{x} \mapsto \widehat{x^*}$.

Theorem

$A' \cap B(L^2(A)) = JAJ = \{JaJ | a \in A\}$.

Jones' index

Definition

Let $A \subset B$ be a II_1-subfactor. $[B\colon A] = \dim_A(L^2(B))$.

Observation [PP86]

$A \subset B$ has finite index if and only if B is a finitely-generated projective A-module.

Theorem [Jon83]

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

The basic construction

Definition

On $L^2(B)$, we define the Jones projection e_A as the projection with range $L^2(A) \subset L^2(B)$.

Definition

The basic construction of $A \subset B$ is the von Neumann algebra $\langle B, e_A \rangle \subset B(L^2(B))$.
Equivalently, $\langle B, e_A \rangle = JA'J$.

$$\begin{array}{ccccc} & & & & A' \\ & & & \swarrow & \\ B & \to & L^2(B) & \leftarrow & JBJ \\ & \nearrow & & \nwarrow & \\ A & & & & JAJ \end{array}$$

The basic construction

Definition

On $L^2(B)$, we define the Jones projection e_A by $e_A(\widehat{b}) = \widehat{E_A(b)}$.
e_A is the projection with range $L^2(A) \subset L^2(B)$.

Definition

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$$\begin{array}{ccccc} \langle B, e_A \rangle & & & & A' \\ & \searrow & & \swarrow & \\ B & \to & L^2(B) & \leftarrow & JBJ \\ & \nearrow & & \nwarrow & \\ A & & & & JAJ \end{array}$$

Iteration

Theorem [Jon83]

If $[B\colon A] < \infty$, then $\langle B, e_A\rangle$ is a II_1-factor, and

$$[\langle B, e_A\rangle : B] = [B\colon A].$$

- Markov trace property:

$$\operatorname{tr}_{\langle B, e_A\rangle}(e_A x) = [B\colon A]^{-1} \operatorname{tr}_B(x)$$

for all $x \in B$.

The Jones tower

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

The Temperley-Lieb algebras

Definition

The Jones projections e_i satisfy the following relations:

(1) $e_i = e_i^* = e_i^2$,

(2) $e_i e_j = e_j e_i$ for $|i-j| > 1$, and

(3) $e_i e_{i\pm 1} e_i = [B\colon A]^{-1} e_i$

Renormalize: $d^2 = [B\colon A]$, $E_i = de_i$. This gives:

(1) $dE_i = dE_i^* = E_i^2$,

(2) $E_i E_j = E_j E_i$ for $|i-j| > 1$, and

(3) $E_i E_{i\pm 1} E_i = E_i$

The Temperley-Lieb algebras

Definition

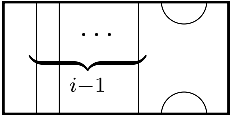
Let $TL_n(d)$ be the complex $*$-algebra generated by $1, E_1, \ldots, E_{n-1}$ satisfying the relations:

(1) $dE_i = dE_i^* = E_i^2$,

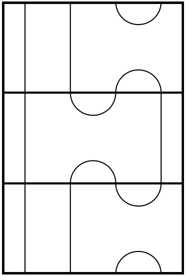
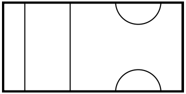
(2) $E_iE_j = E_jE_i$ for $|i-j| > 1$, and

(3) $E_iE_{i\pm 1}E_i = E_i$

Kauffman diagrams

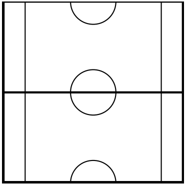
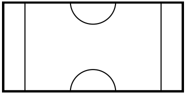
- String diagram for E_i: 

- Multiplication is stacking:

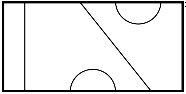
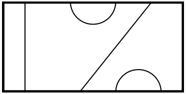
$E_3E_2E_3 =$  $=$  $= E_3$

Kauffman diagrams

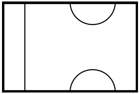
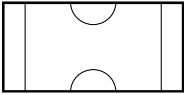
- Closed loops count for a multiplicative factor of d:

$E_2^2 =$  $= d$  $= dE_2$

- The adjoint is reflection about x-axis:

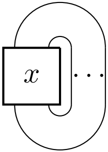
 $^* =$ 

- $TL_n(d) \hookrightarrow TL_{n+1}(d)$ by adding a string to the right:

$TL_2(d) \ni$  $\longmapsto$  $\in TL_3(d)$

The trace

We define a tracial linear functional tr_n on $TL_n(d)$:

$$\mathrm{tr}_n(x) = \frac{1}{d^n}$$


This trace satisfies the Markov property!

Theorem [Jon83]

$\langle x, y\rangle = \mathrm{tr}_n(y^* x)$ is positive semi-definite on $TL_n(d)$ for all $n \geq 0$ if and only if

$$d \in \{2\cos(\pi/k) | k \geq 3\} \cup [2, \infty).$$

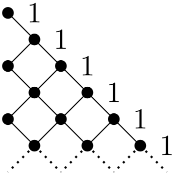
Standard invariant

Two towers of centralizer algebras: $(A_0' \cap A_n)_{n\geq 0}$, $(A_1' \cap A_{n+1})_{n\geq 0}$

- Markov trace
- Finite dimensional C^*-algberas (semi-simple)
- Bratteli diagram
- $TL_n(d) \subset A_0' \cap A_n$ where $d^2 = [A_1 : A_0]$ when $d \geq 2$

Example Bratteli diagram

The Bratteli diagram for the algebras $TL_n(d)$ for $d \geq 2$ is half of Pascal's triangle:



Infinite index

What if $[B\colon A] = \infty$?

II_∞-factors

Definition

A II_∞-factor A is a factor such that:

- $1 \in A$ is an infinite projection, and
- There is a weight $\mathrm{Tr}_A \colon A^+ \to [0,\infty]$ which is:

Tracial: $\mathrm{Tr}_A(x^*x) = \mathrm{Tr}_A(xx^*)$ for all $x \in A$,
Faithful: $\mathrm{Tr}_A(x^*x) = 0 \Longleftrightarrow x = 0$,
Normal: $x_i \nearrow x \Longrightarrow \mathrm{Tr}_A(x_i) \nearrow \mathrm{Tr}_A(x)$ where $x_i, x \in A^+$,
Semifinite: for all $x \in A^+$, there is a $0 \leq y \leq x$ such that $\mathrm{Tr}_A(y) < \infty$.

Caution

$\mathrm{Tr}_A(1_A) = \infty$!

GNS representation

We can still define the GNS-representation. Define

- $\mathfrak{n}_{\mathrm{Tr}_A} = \{x \in A | \mathrm{Tr}_A(x^*x) < \infty\}$,
- $\langle \widehat{x}, \widehat{y} \rangle = \mathrm{Tr}_A(y^*x)$ for $\widehat{x}, \widehat{y} \in \mathfrak{n}_{\mathrm{Tr}_A}$ by polarization,
- $L^2(A) = \overline{\mathfrak{n}_{\mathrm{Tr}_A}}^{\|\cdot\|_2}$.

We still get:

- Conjugate-linear unitary $J \colon L^2(A) \to L^2(A)$ by $\widehat{x} \mapsto \widehat{x^*}$.

Theorem

$A' \cap B(L^2(A)) = JAJ = \{JaJ | a \in A\}$.

The basic construction

Suppose $A \subset B$ is a II_1-subfactor with $[B\colon A] = \infty$.

Facts

(1) $\langle B, e_A\rangle$ is a II_∞-factor.

- There is a canonical trace $\mathrm{Tr}_{\langle B,e_A\rangle}$ on $\langle B, e_A\rangle^+$ satisfying

$$\mathrm{Tr}_{\langle B,e_A\rangle}(xe_Ay) = \mathrm{tr}_B(xy) \text{ for all } x, y \in B.$$

(2) $\mathrm{Tr}_{\langle B,e_A\rangle}|_{B^+} = \infty$, so $L^2(B) \not\subseteq L^2\langle B, e_A\rangle$.

(3) There is no Jones projection $e_B\colon L^2\langle B, e_A\rangle \to L^2(B)$.

Caution

$\mathrm{Tr}_{\langle B,e_A\rangle}(e_A) = 1$, so $\mathrm{Tr}_{\langle B,e_A\rangle}(1 - e_A) = \infty$.

Iteration

We can still iterate the basic construction. We use the formula:

$$\langle B, e_A \rangle = JA'J \subset B(L^2(B)).$$

We iteratively define

$$A_{n+1} = J_n A'_{n-1} J_{n-1} \subset B(L^2(A_n)).$$

Fact

There is a canonical semifinite trace Tr_n on A_n for all $n \geq 2$, and A_n is a II_∞-factor.

Odd Jones projections

Theorem [Bur03]

(1) $\mathrm{Tr}_{2n+1}|_{A_{2n}^+} = \mathrm{Tr}_{2n}$ and

(2) $\mathrm{Tr}_{2n+2}|_{A_{2n+1}^+} = \infty$.

This implies:

- $L^2(A_{2n}) \subset L^2(A_{2n+1})$, so there is a Jones projection $e_{2n+1} \in A_{2n+2} \subset B(L^2(A_{2n+1}))$ with range $L^2(A_{2n})$.
- $L^2(A_{2n+1}) \not\subseteq L^2(A_{2n+2})$

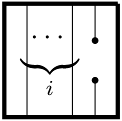
Fact

The odd Jones projections e_{2n+1} satisfy:

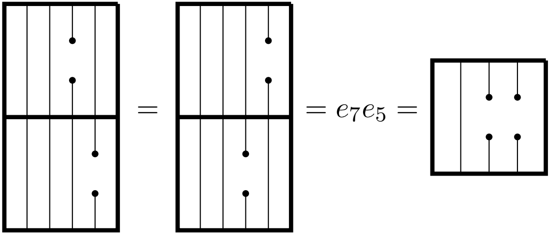
(1) $e_i = e_i^* = e_i^2$,

(2) $e_i e_j = e_j e_i$ for all i, j.

String diagrams

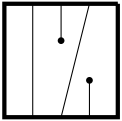
String diagram for e_{2i+1} : 

Multiplication is stacking:

$e_5 e_7 =$ 

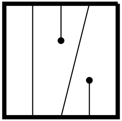
We remove closed strings (dots at both sides).

Higher relative commutants

Idea: we can make sense of diagrams like  .

Theorem

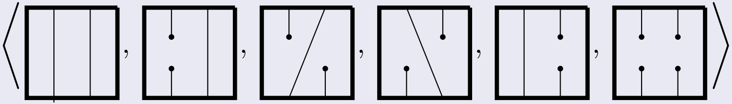
For $i, j \leq n$, $e_{2i-1} \sim e_{2j-1}$ in $A_0' \cap A_{2n}$.
Hence $A_0' \cap A_{2n}$ is not abelian when $n \geq 2$.

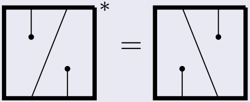
 is a partial isometry witnessing $e_3 \sim e_5 \in A_0' \cap A_6$.

GICAR algebras

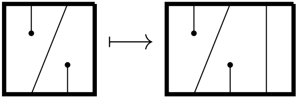
Definition

For $n \geq 0$, let G_n be the complex $*$-algebra generated by these diagrams. For example:

$G_2 =$ 

- Adjoint is reflection: 

- $G_n \hookrightarrow G_{n+1}$ by adding a string to the right:

$G_2 \ni$  $\in G_3$

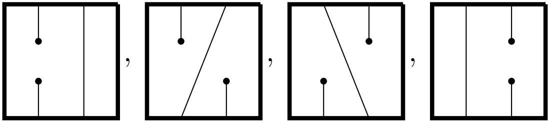
GICAR algberas

Theorem

The complex $*$-algebras G_n are isomorphic to the GICAR algebras

$$G_n \cong \bigoplus_{k=1}^{n} M_{\binom{n}{k}}(\mathbb{C}).$$

For example, the $*$-algebra generated by the diagrams

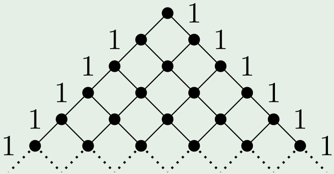


modulo diagrams with fewer through strings is isomorphic to $M_2(\mathbb{C})$.

Bratteli diagram

Theorem

The Bratteli diagram for the algebras G_n is given by Pascal's triangle:

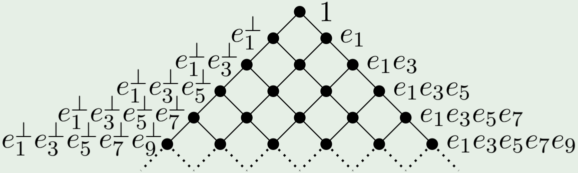


GICAR in $A_0' \cap A_{2n}$

Theorem

Given a subfactor $A_0 \subset A_1$ of arbitrary index, $G_n \hookrightarrow A_0' \cap A_{2n}$.

Proof



For $k \leq n$, the (semi-finite) traces of $e_1 e_3 \cdots e_{2k-1}$ and $e_1^\perp e_3^\perp \cdots e_{2k-1}^\perp$ are non-zero.

Can we only get GICAR?

Question

Is there an infinite index II_1-subfactor $A_0 \subset A_1$ such that $A_0' \cap A_{2n} = G_n$ for all n?

Theorem

If $G_0 = \mathrm{Stab}(1) \subset S_\infty = G_1$, then if $A_0 = R \rtimes G_0 \subset R \rtimes G_1 = A_1$, $\dim(A_0' \cap A_{2n}) < \infty$ for all n. However, $\dim(A_0' \cap A_{2n})$ grows too quickly.

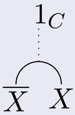
The group-subgroup subfactor won't work.

unitary tensor categories

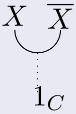
Properties

A unitary tensor category has the following properties (and more):

- for each object $X \in C$, there is an object $\overline{X} \in C$ and evaluation and coevaluation morphisms



$\in \mathrm{Hom}(\overline{X} \otimes X, 1_C)$,



$\in \mathrm{Hom}(1_C, X \otimes \overline{X})$

  satisfying the zigzag relations,
- for each $X, Y \in C$, an involution $*\colon \mathrm{Hom}(X,Y) \to \mathrm{Hom}(Y,X)$ such that if $f \in \mathrm{Hom}(X,Y)$ and $g \in \mathrm{Hom}(Y,Z)$, then $(g \circ f)^* = g^* \circ f^*$, and
- for each $X, Y \in C$, $\mathrm{Hom}(X,Y)$ is a Banach space, and $\mathrm{Hom}(X,X)$ is a C^*-algebra.

Temperley-Lieb

Fact

In a unitary tensor category, picking a distinguished object X, we get canonical maps

$$TL_{2n}(\mathrm{tr}(X)) \to \mathrm{End}(X \otimes \overline{X} \otimes \cdots X \otimes \overline{X})$$
$$TL_{2n+1}(\mathrm{tr}(X)) \to \mathrm{End}(X \otimes \overline{X} \otimes \cdots X \otimes \overline{X} \otimes X)$$

by using the evaluation and coevaluation maps.

Finite index subfactors give unitary 2-categories

Definition

The paragroup of $A \subset B$ is the 2-category given by

- 0-morphisms: $\{A, B\}$
- 1-morphisms: bimodule summands of $L^2(A_k)$ for some $k \geq 0$
- 2-morphisms: intertwiners (live inside $A_0' \cap A_k$, $A_1' \cap A_{k+1}$)

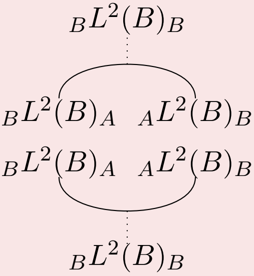
Fact

The Temperley-Lieb algebras arise by looking at the distinguished 1-morphism $X =_A L^2(B)_B$.

Infinite index subfactors do not give unitary 2-categories

Caution

There are no evaluation and coevaluation maps



$$\in \operatorname{Hom}(\underbrace{_BL^2(B) \otimes_A L^2(B)_B}_{\cong {_BL^2\langle B, e_A\rangle_B}}, {_BL^2(B)_B})$$

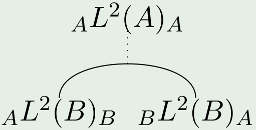
$$\in \operatorname{Hom}({_BL^2(B)_B}, \underbrace{_BL^2(B) \otimes_A L^2(B)_B}_{\cong {_BL^2\langle B, e_A\rangle_B}})$$

Since $L^2(B) \not\subseteq L^2\langle B, e_A\rangle$, and there is no Jones projection e_B.

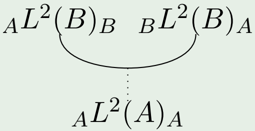
GICAR

Fact

The GICAR algebras arise by looking at the evaluation and coevalutation maps



$$= e_A \in \operatorname{Hom}(\underbrace{{}_A L^2(B) \otimes_B L^2(B)_A}_{\cong {}_A L^2(B)_A}, {}_A L^2(A)_A)$$



$$= i_A \in \operatorname{Hom}({}_A L^2(A)_A, \underbrace{{}_A L^2(B) \otimes_B L^2(B)_A}_{\cong {}_A L^2(B)_A})$$

where e_A is the Jones projection and i_A is the inclusion.

Thank you for listening!

Slides available at:

http://math.berkeley.edu/~dpenneys

Preprint available at:

http://math.berkeley.edu/~dpenneys/GICAR.pdf

Michael Burns, *Subfactors, planar algebras, and rotations*, 2003, Ph.D. Thesis at the University of California Berkeley.

Vaughan F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25, MR696688, DOI:10.1007/BF01389127.

Mihai Pimsner and Sorin Popa, *Entropy and index for subfactors*, Ann. Sci. École Norm. Sup. (4) **19** (1986), no. 1, 57–106, MR860811.