Classifying small index subfactors
Great Plains Operator Theory Symposium

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In celebration of the 60th birthday of Vaughan Jones
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Where do subfactors come from?

Some examples include:

- Groups – from $G \rtimes R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – $\text{Rep}(U_q(\mathfrak{g}))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- composites of known subfactors

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs $(M, M')$, subfactors come in pairs $(A \subset B, B' \subset A')$. 
Theorem [Jon83]
For a II$_1$-subfactor $A \subset B$,

$$[B : A] \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \mid n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

Definition
The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$$

where $e_i$ is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$. 

Two towers of centralizer algebras

\[
P_{3,+} = A_0' \cap A_3 \supset A_1' \cap A_3 = P_{2,-} \\
\quad \cup \quad \cup \\
P_{2,+} = A_0' \cap A_2 \supset A_1' \cap A_2 = P_{1,-} \\
\quad \cup \quad \cup \\
P_{1,+} = A_0' \cap A_1 \supset A_1' \cap A_1 = P_{0,-} \\
\quad \cup \\
P_{0,+} = A_0' \cap A_0
\]

These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].
Planar algebras [Jon99]

Definition
A shaded planar tangle has

- a finite number of inner boundary disks
- an outer boundary disk
- non-intersecting strings
- a marked interval $\star$ on each boundary disk
Composition of tangles

We can compose planar tangles by insertion of one into another if the number of strings matches up:

Definition
The shaded planar operad consists of all shaded planar tangles (up to isotopy) with the operation of composition.
Definition

A *planar algebra* is a family of vector spaces $P_{k,\pm}$, $k = 0, 1, 2, \ldots$ and an action of the shaded planar operad.

$$P_{2,-} \times P_{1,+} \times P_{1,+} \rightarrow P_{3,+}$$
Example: Temperley-Lieb

$TL_{n, \pm}(\delta)$ is the complex span of non-crossing pairings of $2n$ points arranged around a circle, with formal addition and scalar multiplication.

$TL_{3, +}(\delta) = \text{Span}_\mathbb{C} \{ \begin{array}{c} \star \otimes, \star \otimes, \star \otimes, \star \otimes, \star \otimes \end{array} \}.$

Planar tangles act on $TL$ by inserting diagrams into empty disks, smoothing strings, and trading closed loops for factors of $\delta$.

\[
\begin{array}{c}
\star \\
\otimes \otimes \\
\end{array}
\begin{array}{c}
\star \\
\otimes \otimes \\
\end{array} = \begin{array}{c}
\star \\
\otimes \otimes \\
\end{array} = \delta^2 \begin{array}{c}
\star \\
\otimes \otimes \\
\end{array}
\]
Subfactor planar algebras

Definition
A planar ∗-algebra $P_\bullet$ is a subfactor planar algebra if it is:
- Finite dimensional: $\dim(P_k,\pm) < \infty$ for all $k$
- Evaluable: $P_{0,\pm} \cong \mathbb{C}$ by sending the empty diagram to $1_\mathbb{C}$
- Sphericality: $\text{Tr}(x) = \frac{1}{2}x \star = \star x$
- Positivity: each $P_{k,\pm}$ has an adjoint $\star$ such that the sesquilinear form $\langle x, y \rangle := \text{Tr}(y^* x)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant $\delta \in \{2 \cos(\pi/n) | n \geq 3\} \cup [2, \infty)$. 
Principal graphs

The complex $\ast$-algebras $P_{n,\pm}$ are all finite dimensional. The tower

$$P_{0,+} \subset P_{1,+} \subset P_{2,+} \subset \cdots$$

where the inclusion is given by

is described by its Bratteli diagram (and the trace).
Principal graphs

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where the inclusion is given by

![Bratteli diagram](image)

is described by its Bratteli diagram (and the trace).

- The non-reflected part is the principal graph $\Gamma_+$.  
- Get the dual principal graph $\Gamma_-$ by looking at the Bratteli diagram for the tower $(P_{n,-})$.  

Examples of principal graphs

- **index < 4**: $A_n, D_{2n}, E_6, E_8$. No $D_{odd}$ or $E_7$.
- **index = 4**: $A_{2n-1}^{(1)}, D_{n+2}^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, A_\infty, A_\infty^{(1)}, D_\infty$
- **Graphs for $R \subset R \rtimes G$ obtained from $G$ and $\text{Rep}(G)$.**

  \[
  \begin{array}{c}
  (\quad, 2) \\
  \end{array}
  \]

- **Haagerup 333**

  \[
  \begin{array}{c}
  (\quad, \quad) \\
  \end{array}
  \]

- **extended Haagerup 733**

  \[
  \begin{array}{c}
  (\quad, \quad) \\
  \end{array}
  \]

- **First graph is principal, second is dual principal.**
- **Leftmost vertex corresponds to $P_{0,\pm} \cong \mathbb{C}$.**
- **Red tags for duality of even vertices.**
- **Duality of odd vertices by depth and height**
Finite depth

Definition
If the principal graph is finite, then the subfactor and standard invariant/planar algebra are called finite depth.

Example: \( R \subset R \rtimes G \) for finite \( G \)
For \( G = S_3 \):

- Principal graph:

- Dual principal graph:
Supertransitivity

Definition
We say a principal graph is $n$-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram $A_{n+1}$, i.e., an initial segment with $n$ edges.

Examples

- is 1-supertransitive
- is 2-supertransitive
- is 3-supertransitive
Invariants of subfactors

\[ A \subset B \]

\[ (P_+, P_-) \]

\[ (\Gamma_+, \Gamma_-) \]
Known small index subfactors

- Map of known small index subfactors modified from Jones-Morrison-Snyder Bulletin AMS survey [JMS14].
The extended Haagerup subfactor

[Bigelow-Morrison-Peters-Snyder [BMPS12]]

The extended Haagerup subfactor is the unique subfactor with principal graphs

\[ \begin{array}{c}
\quad \\
( , )
\end{array} \]

- Last remaining possible graph in Haagerup’s classification to \( 3 + \sqrt{3} \) [Haa94] by work of Asaeda-Yasuda [AY09].
- Largest known supertransitivity outside the \( A \) and \( D \) series. High supertransitivity is exceedingly rare!
- Planar algebra constructed using Bigelow’s jellyfish algorithm.
Jellyfish relations

**Theorem [Bigelow-Morrison-Peters-Snyder [BMPS12]]**

The Haagerup and extended Haagerup subfactor planar algebras have a generator $S \in P_{n,+}$ where $n = 4, 8$ respectively satisfying:

$\begin{align*}
&\star \quad 2n - 1 \quad \star \\
&\quad f(2n+2) = i \frac{\sqrt{[n][n+2]}}{[n+1]} \\
\end{align*}$

$\begin{align*}
&\star \quad 2n \quad \star \\
&\quad f(2n+4) = \frac{[2][2n+4]}{[n+1][n+2]} \\
\end{align*}$

(Absorption) capping $S$ gives zero and $S^2 = f(n) \in TL_{n,+}$. 
The jellyfish algorithm

We can evaluate all closed diagrams as follows:

1. First, pull all generators to the outside using the jellyfish relations

2. Second, reduce the number of generators using the capping and absorption (multiplication) relations.
Consistency and positivity

Theorem [Jones-Penneys [JP11], Morrison-Walker]
Every subfactor planar algebra embeds in the graph planar algebra
of its principal graph.

This serves two purposes:

1. To show the planar algebra is non-zero, give a representation.
2. Graph planar algebras are always finite dimensional, spherical,
   and positive. Only need to check evaluable.
Spoke graphs

Examples of spoke principal graphs

- $A_n, D_{2n}, E_6, E_8$,
- $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$,
- $A_{\infty}, A_{\infty}^{(1)}, D_{\infty}$
- Principal graphs for $R \subset R \rtimes G$, $G$ finite (Diagram: $\rightarrow\leftarrow$, $\leftarrow\rightarrow$)
- 2221
- Haagerup 333
- 3311
- 3333
- 4442
- extended Haagerup 733
Spokes and jellyfish

Assume all generators of $P_\bullet$ are at the same depth $n$.

**Theorem [Bigelow-Penneys [BP14]]**

- $P_\bullet$ has 2-strand jellyfish relations $\iff$ one graph is a spoke.

- $P_\bullet$ has 1-strand jellyfish relations $\iff$ both graphs are spokes.
Constructing spoke subfactors with jellyfish

**Theorem [Morrison-Penneys [MP12a]]**
We automate finding 1-strand relations for these subfactors:

- Izumi-Xu 2221 [Han10]
- [GdlHJ89] 3311
- Izumi $3 \mathbb{Z}/2 \times \mathbb{Z}/2$ (index $3 + \sqrt{5}$)
- 4442 (index $3 + \sqrt{5}$)

For the above, both principal graphs are the same spoke graph.

**Theorem [Penneys-Peters [PP13]]**
We give explicit 2-strand relations for Izumi’s $3 \mathbb{Z}/4$ subfactor

- $(\quad, \quad)$ (index $3 + \sqrt{5}$)
Small index subfactor classification program

Focuses of the classification program:

- Enumerate graph pairs and apply obstructions.
- Construct examples when graphs survive.
- Place exotic examples into families.
Why do we care about index $3 + \sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 \ast \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 \ast \mathbb{Z}/3 = PSL(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with the same standard invariant [BNP07, BV13]

- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3 + \sqrt{5} = 2\tau^2$ where $\tau = \frac{1 + \sqrt{5}}{2}$. 
1-supertransitive subfactors at index $3 + \sqrt{5}$

Theorem [Liu [Liu13a]], partial proof by [IMP13]
There are exactly seven 1-supertransitive subfactor planar algebras with index $3 + \sqrt{5}$:

- $(\quad , \quad)$ self-dual
- $(\quad , \quad)$ and its dual
- $(\quad , \quad)$ and its dual
- $(\quad \ldots , \quad \ldots)$ and its dual ($A_3 \ast A_4$)

These are all the standard invariants of composed inclusions of $A_3$ and $A_4$ subfactors.

Open question
How many hyperfinite subfactors have Bisch-Jones’ Fuss-Catalan $A_3 \ast A_4$ standard invariant at index $3 + \sqrt{5}$?

- $A_3 \ast A_4$ and $A_2 \ast T_2$ are not amenable [Pop94, HI98].
Theorem [Liu-Morrison-Penneys [LMP13]]

An exactly 1-supertransitive subfactor planar algebra with index at most $6 \frac{1}{5}$ either comes from a composed inclusion (and has index $3 + \sqrt{5}$ or 6), or is one of 3 self-dual planar algebras at index $3 + 2\sqrt{2}$:

- $(\quad, \quad)$
- $(\quad, \quad)$ two complex conjugate

- Can push classification results above index 6!
- Could hope that the only wildness at index 6 is “group-like”
Index \((5, 3 + \sqrt{5})\)

**Conjecture [Morrison-Peters [MP12b]]**

There are exactly two non Temperley-Lieb subfactor planar algebras in the index range \((5, 3 + \sqrt{5})\):

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
<th>Index</th>
<th>Constructed</th>
</tr>
</thead>
</table>
| $SU(2)_5$  | (\begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,0.5);
  \draw (-0.5,0.5) -- (-1,0.5);
  \draw (-1,0) -- (-1,0.5);
\end{tikzpicture}) | 5.04892 | [Wen90], [MP12b]     |
| $SU(3)_4$  | (\begin{tikzpicture}
  \draw (-0.5,0) -- (-0.5,0.5);
  \draw (-0.5,0.5) -- (-1,0.5);
  \draw (-1,0) -- (-1,0.5);
\end{tikzpicture}) | 5.04892 | [Wen88], [MP12b]     |

**Theorem [Morrison-Peters [MP12b]]**

There is exactly one 1-supertransitive subfactor in the index range \((5, 3 + \sqrt{5})\)
Subfactor planar algebras at index $3 + \sqrt{5}$

Conjecture [Morrison-Penneys]

At $3 + \sqrt{5}$, we have only the following subfactor planar algebras:

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
<th>#</th>
<th>Constructed</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4442$</td>
<td>(                    ,            )</td>
<td>1</td>
<td>[MP12a], Izumi</td>
</tr>
<tr>
<td>$3\mathbb{Z}/2\times\mathbb{Z}/2$</td>
<td>(                  ,            )</td>
<td>1</td>
<td>Izumi, [MP12a]</td>
</tr>
<tr>
<td>$3\mathbb{Z}/4$</td>
<td>(                  ,            )</td>
<td>2</td>
<td>Izumi, [PP13]</td>
</tr>
<tr>
<td>$2D2$</td>
<td>(                  ,            )</td>
<td>2</td>
<td>Izumi, [MPP]</td>
</tr>
<tr>
<td>$A_3 \otimes A_4$</td>
<td>(                      ,      )</td>
<td>1</td>
<td>$\otimes$</td>
</tr>
<tr>
<td>fish 2</td>
<td>(        ,                  )</td>
<td>2</td>
<td>BH</td>
</tr>
<tr>
<td>fish 3</td>
<td>(        ,                  )</td>
<td>2</td>
<td>[IMP13]</td>
</tr>
<tr>
<td>$A_3 \ast A_4$</td>
<td>(          ,                )</td>
<td>2</td>
<td>[BJ97]</td>
</tr>
<tr>
<td>$A_\infty$</td>
<td>(          ,                )</td>
<td>1</td>
<td>[Pop93]</td>
</tr>
</tbody>
</table>

▶ 1-supertransitive case known by [Liu13a, IMP13, LMP13]
Methods to push classification results further

- The non-initial triple point obstruction
- Popa’s principal graph stability [Pop95, BP14]
- Liu’s virtual normalizers for 1-supertransitive subfactors [Liu13b] (pushed 1-supertransitive classification to $6\frac{1}{5}$ [LMP13])
- Afzaly’s principal graph enumerator, based on Brendan McKay’s isomorph free enumeration by canonical construction paths
- New general initial triple point obstruction [Pen13]

Theorem [Afzaly-Morrison-Penneys]
The conjectures of Morrison-Peters and Morrison-Penneys hold with at most finitely many exceptions.
Thank you for listening!
Recent articles:


▶ with Izumi and Morrison - 1-supertransitive at $3 + \sqrt{5}$ - Submitted - arXiv:1308.5723

▶ new obstruction - Submitted - arXiv:1307.5890


Vaughan F. R. Jones, Scott Morrison, and Noah Snyder, *The classification of subfactors of index at most 5*, Bull. Amer. Math. Soc. (N.S.) 51


Scott Morrison, David Penneys, and Emily Peters, *Equivariantizations and 3333 spoke subfactors at index* $3 + \sqrt{5}$, In preparation.


