Classifying small index subfactors

AMS JMM Special MRC Session on Quantum Information and Fusion Categories

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What is a subfactor?

Definition

A factor is a von Neumann algebra with trivial center.
A subfactor is an inclusion $A \subset B$ of factors.

Remark

Von Neumann algebras come in pairs (M, M').
Subfactors do too: $(A \subset B, B' \subset A')$.

Theorem (Jones [Jon83])

For a subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

We will restrict our attention to a finite index subfactor $A \subset B$.

Where do subfactors come from?

Some examples include:

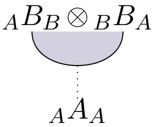
- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- tinkering with known subfactors (orbifolds, composites, ...)

However, there are certain possible infinite families without uniform constructions.

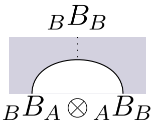
Finite index and the standard representation

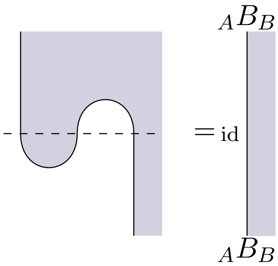
The bimodule ${}_AB_B$ is the standard representation of $A \subset B$.
A finite index subfactor $A \subset B$ comes with canonical maps:

Inclusion:



Evaluation:





Since A, B are analytical objects, these maps also have adjoints.

$\mathrm{Rep}(A \subset B)$

Definition

The representation 2-category of $A \subset B$ is given by

(0) 0-morphisms: $\{A, B\}$

(1) 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$

(2) 2-morphisms: bimodule intertwiners

- This 2-category is semi-simple, unitary, rigid, pivotal. It is spherical iff $A \subset B$ extremal.
- The $A - A$ bimodules form a rigid C*-tensor category called the 'principal even part'.
- The $B - B$ bimodules form the 'dual even part'.
- The principal even and dual even parts are Morita equivalent:

$$ {}_A\mathrm{Mod}_A \xleftrightarrow[{}_B\mathrm{Mod}_A]{{}_A\mathrm{Mod}_B} {}_B\mathrm{Mod}_B $$

Subfactor/representation 2-category correspondence

Theorem (Popa [Pop94])

There is a Tannaka-Krein like duality between (strongly) amenable subfactors and their representation 2-categories.

$$A \subset B \longleftrightarrow \mathrm{Rep}(A \subset B)$$

Theorem (many authors)

Subfactors correspond to Frobenius algebra objects in rigid C*-tensor categories.

- Finite depth subfactors correspond to Frobenius algebras in unitary fusion categories.

Fusion categories

Definition

$A \subset B$ has finite depth if $\mathrm{Rep}(A \subset B)$ has finitely many isomorphism classes of simple bimodules.

- Then both even parts are unitary fusion categories.
- Subfactors are a vital source of interesting fusion categories.

Suppose we have a Frobenius algebra object $\mathcal{A} \in \mathcal{C}$, a unitary fusion category.

- Get a subfactor representation 2-category from $\mathcal{C}$, the $\mathcal{C}$-module category $\mathcal{M} = \mathsf{Mod}_{\mathcal{A}}$, and the commutant:

$$\mathcal{C} \underset{\mathcal{M}^{\mathsf{op}}}{\overset{\mathcal{M}}{\longleftrightarrow}} \mathcal{C}'_{\mathcal{M}}$$

- Use Popa's theorem to recover a finite depth subfactor!

Examples of fusion categories

Let G be a finite group.

Example

$\mathrm{Rep}(G)$, category of finite dimensional $\mathbb{C}$-representations.

Example

$\mathrm{Vec}(G, \omega)$, G-graded vector spaces, $\omega \in H^3(G, \mathbb{C}^\times)$.

- Simple objects $V_g \cong \mathbb{C}$ for each $g \in G$.
- $V_g \otimes V_h = V_{gh}$
- The 3-cocycle gives the associator natural isomorphism:

$$\alpha_{g,h,k} : (V_g \otimes V_h) \otimes V_k \xrightarrow{\omega_{g,h,k}} V_g \otimes (V_h \otimes V_k).$$

The pentagon axiom is exactly the 3-cocycle condition.

$\mathrm{Rep}(R \subset R \rtimes G)$

From a finite group G, get the group subfactor $R \subset R \rtimes G$.

Example

- The principal even part ($R - R$ bimodules) is $\mathcal{C} = \mathrm{Vec}(G)$.
- $R \rtimes G$ corresponds to the algebra object $\mathbb{C}[G] \in \mathrm{Vec}(G)$.
- $\mathcal{M} = \mathsf{Mod}_{\mathbb{C}[G]} \subset \mathrm{Vec}(G)$ has one simple object: $\mathbb{C}[G]$.
- In this case, $\mathcal{C}'_{\mathcal{M}} = \mathrm{Rep}(G)$.

$$\mathrm{Vec}(G) \xleftrightarrow{\mathsf{Mod}_{\mathbb{C}[G]}} \mathrm{Rep}(G)$$

The Haagerup: an 'exotic' example

The Haagerup fusion category $\mathcal{H}$ has 6 simple objects $1, g, g^2, X, gX, g^2X$ satisfying the following fusion rules:

- $\langle g \rangle \cong \mathrm{Vec}(\mathbb{Z}/3\mathbb{Z})$, with trivial associator,
- $Xg \cong g^{-1}X$, and
- $X^2 \cong 1 \oplus X \oplus gX \oplus g^2X$ (the quadratic relation).

The algebra object $1 \oplus X$ gives an 'exotic' subfactor with index

$$\frac{5+\sqrt{13}}{2} \approx 4.30278.$$

$\mathcal{H}$ has only been constructed by brute force.

- It appears $\mathcal{H}$ belongs to an infinite family, but only examples up to $\mathbb{Z}/19$ have been constructed [EG11].

Classifying small index subfactors

- A finite group G gives a subfactor $R \subset R \rtimes G$ which remembers G.
- Classifying all subfactors is hopeless.

Restrict the search space: one way is to look at small index.

Reminder:

The representation 2-category of $A \subset B$ is given by

(0) 0-morphisms: $\{A, B\}$

(1) 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$

(2) 2-morphisms: bimodule intertwiners

Principal graphs

Definition

The principal (induction) graph Γ_+ has one vertex for each isomorphism class of simple ${}_AP_A$ and ${}_AQ_B$. There are

$$\dim(\mathrm{Hom}_{A-B}(P \otimes_A B, Q))$$

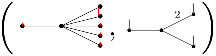
edges from P to Q.

The dual principal (restriction) graph Γ_- has a similar definition using $B - B$ and $B - A$ bimodules.

- $\Gamma_\pm$ is pointed, where the base point is ${}_AA_A$, ${}_BB_B$ respectively.
- The depth of a vertex is its distance to the base point.
- Duals always occur at the same depth, since B is a $*$-algebra. However, duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$.

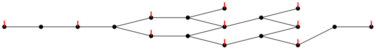
Examples of principal graphs

- index < 4: ADE classification, but no D_{odd} or E_7.
- index $= 4$: affine Dynkin diagrams
- Graphs for $R \subset R \rtimes G$ obtained from $\mathrm{Vec}(G)$ and $\mathrm{Rep}(G)$.



$G = S_3$

- Principal graph for $R^G \subset R^H$ is the induction-restriction graph for $H \subset G$:



$S_5 \subset S_6$

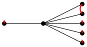
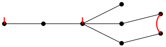
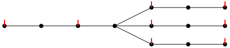
- First graph is principal, second is dual principal.
- Leftmost vertex corresponds to base points ${}_AA_A$, ${}_BB_B$.
- Red tags for duality (${}_AP_A \mapsto \overline{{}_AP_A}$) of even vertices.
- Duality of odd vertices by depth and height

Supertransitivity

Definition

A principal graph is n-supertransitive if has an initial segment with n edges before branching.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Small index subfactor classification program

Steps of subfactor classifications:

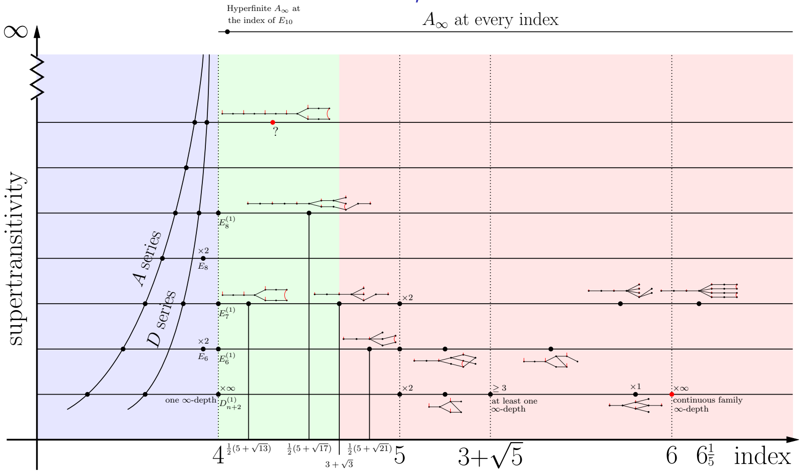
1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

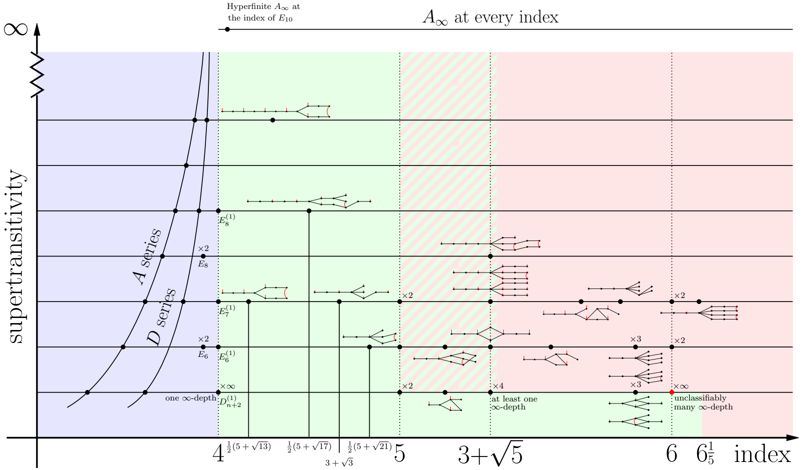
If we enumerate all graph pairs with norm at most r, we have found all principal graphs of subfactors with index at most r^2.

Known small index subfactors, 2009



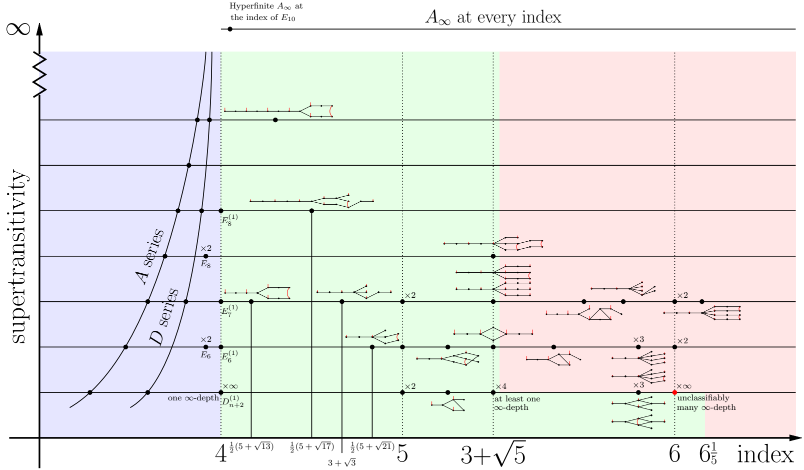
- Quantum groups and their quantum subgroups
- Composites
- Haagerup's exotic subfactor and classification to $3+\sqrt{3}$
- Izumi's Cuntz algebra examples (2221, 3^n)

Known small index subfactors, 2014



- Classification to 5 [MS12, MPPS12, IJMS12, PT12, IMP+14]
- Examples at $3+\sqrt{5}$ [MP13, PP13, IMP13, MP14]
- 1-supertransitive to $6\frac{1}{5}$ and examples at $3+2\sqrt{2}$ [LMP14]

Known small index subfactors, today



Theorem (Afzaly-Morrison-P, 2015)

We know all subfactor standard invariants up to index $5\frac{1}{4}$.

Thank you for listening!

Slides available at
`http://www.math.ucla.edu/~dpenneys/PenneysJMM2015.pdf`

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