Bicommutant categories from fusion categories

David Penneys, UCLA
joint with André Henriques

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Categorical analogies

Tensor categories categorify algebras.

algebra A finite dimensional algebra center $Z(A)$ commutant $Z_B(A)$ of A in B	tensor category $\mathcal{C}$ fusion category Drinfel'd center $\mathcal{Z}(\mathcal{C})$ commutant $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{D}$
$B(H)$ commutant $A' := Z_{B(H)}(A)$ von Neumann algebra $A = A''$	$\mathrm{Bim}(R)$, all bimodules commutant $\mathcal{C}' := \mathcal{Z}_{\mathrm{Bim}(R)}(\mathcal{C})$ bicommutant category $\mathcal{C} \cong \mathcal{C}''$

Bicommutant categories categorify von Neumann algebras.

Today, we will prove the *categorified finite dimensional bicommutant theorem*.

Unitary fusion categories

We start with a unitary fusion category $\mathcal{C} \subset \mathrm{Bim}(R)$.

- Have a system of bifinite bimodules and intertwiners
- The system is closed under
  - Finite direct sum: $x, y \in \mathcal{C} \Rightarrow x \oplus y \in \mathcal{C}$
  - Connes fusion: $x, y \in \mathcal{C} \Rightarrow x \boxtimes y \in \mathcal{C}$
  - Contragredient: $x \in \mathcal{C} \Rightarrow \overline{x} \in \mathcal{C}$
  - Taking sub-bimodules: $x \subset y \in \mathcal{C} \Rightarrow x \in \mathcal{C}$
- Finitely many isomorphism classes of irreducible bimodules.

Examples

- Start with a finite group G, and form $R \subset R \rtimes G$. The $R - R$ bimodules generated by $L^2(R \rtimes G)$ form Vec(G).
- Given a finite index, finite depth subfactor $R \subset M$, the $R - R$ bimodules generated by $L^2(M)$ form a unitary fusion category.

Graphical calculus

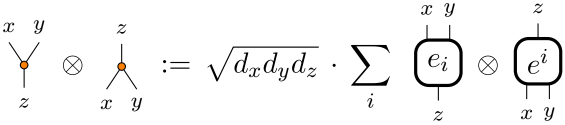
Fix a finite set $\mathrm{Irr}(\mathcal{C})$ of representatives of irreducibles.

- Morphisms $f : x \otimes y \to z$ are represented by coupons.
- For all $x, y, z \in \mathrm{Irr}(\mathcal{C})$, $\mathrm{Hom}(1, x \otimes y \otimes z)$ is a finite dimensional Hilbert space with inner product $\langle f, g \rangle = g^* \circ f$.

Choose dual bases:

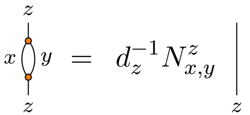
$$e_i \in \mathrm{Hom}(1, x \otimes y \otimes z) \text{ and } e^i \in \mathrm{Hom}(1, \overline{z} \otimes \overline{y} \otimes \overline{x})$$

We represent the canonical element by colored nodes

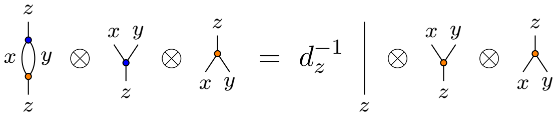


The canonical element is independent of choice of basis.

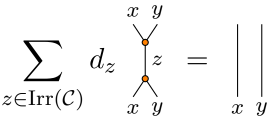
Important relations



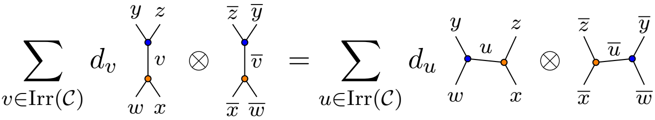
(Bigon 1)



(Bigon 2)



(Fusion)



(I=H)

We'll use Snyder convention and ignore all scalars.

Commutant $\mathcal{C}'$ of $\mathcal{C}$ in $\mathrm{Bim}(R)$

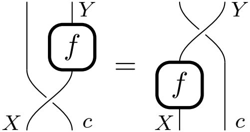
The commutant $\mathcal{C}' \subset \mathrm{Bim}(R)$ of $\mathcal{C} \subset \mathrm{Bim}(R)$ has:

- Objects are pairs (X, e_X) where $X \in \mathrm{Bim}(R)$, and e_X is a unitary half braiding with $\mathcal{C}$

  $e_{X,c} =$ 

  $: X \boxtimes c \to c \boxtimes X$

  These half braidings must satisfy compatibility conditions.
- Morphisms $f : (X, e_X) \to (Y, e_Y)$ are bimodule maps $f : X \to Y$ which commute with the half braidings:



$\mathcal{C}'$ is a tensor category, but it is usually not braided.

Functor $\mathrm{Bim}(R) \to \mathcal{C}'$

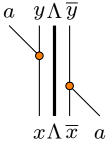
We have a way to construct lots of objects in $\mathcal{C}'$.

$$\underline{\Delta} : \mathrm{Bim}(R) \to \mathcal{C}' \qquad \underline{\Delta}(\Lambda) = (\Delta(\Lambda), e_{\Delta(\Lambda)}) = (\Delta, e_\Delta).$$

$\Delta \in \mathrm{Bim}(R)$ with unitary half braiding $e_{\Delta,a} : \Delta \boxtimes a \to a \boxtimes \Delta$.

$$\Delta := \bigoplus_{x \in \mathrm{Irr}(\mathcal{C})} x \boxtimes \Lambda \boxtimes \overline{x}\,.$$

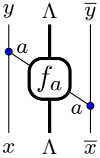
$$e_{\Delta,a} := \sum_{x,y \in \mathrm{Irr}(\mathcal{C})}$$



Description of $\mathrm{End}_{\mathcal{C}'}(\Delta)$

The map that sends $f = \left(f_a : \Lambda \boxtimes a \to a \boxtimes \Lambda\right)_{a \in \mathrm{Irr}(\mathcal{C})}$ to

$$T_f := \sum_{a,x,y \in \mathrm{Irr}(\mathcal{C})}$$



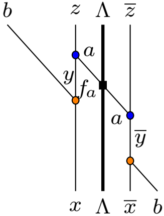
$$: \Delta \to \Delta$$

induces an isomorphism

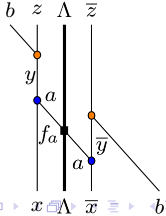
$$\bigoplus_{a \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathrm{Bim}(R)}(\Lambda \boxtimes a, a \boxtimes \Lambda) \;\cong\; \mathrm{End}_{\mathcal{C}'}(\underline{\Delta}(\Lambda)).$$

Note $T_f \in \mathrm{End}_{\mathcal{C}'}(\Delta)$ using the (I=H) Relation:

$$\sum_{a,x,y,z \in \mathrm{Irr}(\mathcal{C})}$$



$$= \sum_{a,x,y,z \in \mathrm{Irr}(\mathcal{C})}$$



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Absorbing objects

Definition

An object Ω in a tensor category $\mathcal{T}$ is *absorbing* if $\Omega \otimes t \cong \Omega \cong t \otimes \Omega$ for all $t \in \mathcal{T}$.

- Isomorphisms are not required to be natural or canonical.
- Absorbing objects are unique up to isomorphism if they exist.
- Taking $t = 1 \oplus 1$, we have $\Omega \cong \Omega \otimes (1 \oplus 1) \cong \Omega \oplus \Omega$.

Examples

- $\ell^2(\mathbb{N})$ is absorbing in the category of separable Hilbert spaces.
- $\ell^2(G) \otimes \ell^2(\mathbb{N})$ is absorbing in Rep(G), G a countable group
- ${}_R L^2(R) \otimes \ell^2(\mathbb{N}) \otimes L^2(R)_R$ is absorbing in $\mathrm{Bim}(R)$.

Absorbing objects of $\mathcal{C}'$

Absorbing objects of $\mathcal{T} \subset \mathrm{Bim}(R)$ *control* half braidings of $\mathcal{T}' \subset \mathrm{Bim}(R)$.

Theorem

If $\Omega \in \mathcal{T}$ is absorbing and $(X, e_X) \in \mathcal{T}'$, then e_X is completely determined by $e_{X,\Omega}$.

When $\mathcal{C}$ is a unitary fusion category, $\mathcal{C}'$ has absorbing objects.

Theorem

If $\Lambda \in \mathrm{Bim}(R)$ is absorbing, then $\Delta \in \mathcal{C}'$ is absorbing.

- If $\Lambda \in \mathrm{Bim}(R)$ is absorbing, then $\mathrm{End}_{\mathcal{C}'}(\Delta)$ is a factor.
- $\mathrm{End}_{\mathrm{Bim}(R)}(\Lambda) \hookrightarrow \mathrm{End}_{\mathcal{C}'}(\Delta)$ is a subfactor!

The main theorem

Recall $\mathcal{C}$ is a unitary fusion category. The bicommutant $\mathcal{C}''$ allows infinite direct sums, so $\mathcal{C}$ is not a bicommutant category.
Let $\mathcal{C} \otimes_{\mathsf{Vec}} \mathsf{Hilb}$ be the category obtained from $\mathcal{C}$ by allowing infinite direct sums. (This is sometimes called the ind-category of $\mathcal{C}$.)

Theorem

$\mathcal{C} \otimes_{\mathsf{Vec}} \mathsf{Hilb}$ is a bicommutant category.

This theorem categorifies the following well-known result:

- A finite dimensional $*$-algebra that can be faithfully represented on a Hilbert space is in fact a von Neumann algebra.

Corollary

$\mathcal{C}'$ is also a bicommutant category.

Outline of the proof

There is an obvious fully faithful embedding $\mathcal{C} \otimes_{\mathsf{Vec}} \mathsf{Hilb} \hookrightarrow \mathcal{C}''$.

The proof of essential surjectivity has 3 main steps:

1. The underlying object X of an object $(X, e_X) \in \mathcal{C}''$ is of the form $X \cong \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c \otimes H_c$ for $H_c \in \mathsf{Hilb}$.
2. Two objects (X, e_X^1) and (X, e_X^2) have the same half braiding with an absorbing object $\Omega \in \mathcal{C}'$, i.e., $e^1_{X,\Omega} = e^2_{X,\Omega}$.
3. Absorbing objects uniquely determine half braidings.

Proof of 1.

- Start with $(X, e_X) \in \mathcal{C}''$.
- Take $\Lambda = L^2(R) \otimes L^2(R)$ and form $\Delta = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c \otimes \overline{c} \in \mathcal{C}'$.
- Have a bimodule isomorphism $e : X \boxtimes \Delta \to \Delta \boxtimes X$.

$$e : \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} X \boxtimes c \otimes \overline{c} \longrightarrow \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c \otimes \overline{c} \boxtimes X$$

- Isomorphism is R-linear for *four* distinct R-actions!
- Apply functor $\mathrm{Hom}_{\text{3rd}-R, \text{4th}-R}(L^2(R), -)$ to see

$$\begin{aligned} X &\cong \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\text{3rd}-R, \text{4th}-R}(L^2(R), X \boxtimes c \otimes \overline{c}) \\ &\cong \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\text{3rd}-R, \text{4th}-R}(L^2(R), c \otimes \overline{c} \boxtimes X) \\ &\cong \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c \otimes \underbrace{\mathrm{Hom}_{R-R}(L^2(R), \overline{c} \boxtimes X)}_{H_c} \end{aligned}$$

Thank you for listening!

Slides available at:
http://www.math.ucla.edu/~dpenneys/PenneysJMM2016.pdf

Article available at:
http://arxiv.org/abs/1511.05226