Monoidal categories enriched in braided monoidal categories

AMS JMM Special Session on
Fusion Categories and Quantum Symmetries

David Penneys
joint with Scott Morrison

January 5, 2017

Linear categories - data

A linear (Vec enriched) category $\mathcal{C}$ has objects $a, b, c, \cdots \in \mathcal{C}$ and

- for every $a, b \in \mathcal{C}$, an object $\mathcal{C}(a \to b) \in \mathsf{Vec}$
- for every $a, b, c \in \mathcal{C}$, a composition morphism

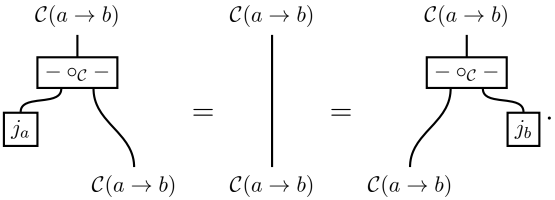
$$\mathcal{C}(a \to b)\mathcal{C}(b \to c) \xrightarrow{-\circ_{\mathcal{C}}-} \mathcal{C}(a \to c)$$

- for every $a \in \mathcal{C}$, an identity morphism $\mathrm{id}_a \in \mathcal{C}(a \to a)$, which one can think of as a morphism $j_a : 1_{\mathsf{Vec}} \to \mathcal{C}(a \to a)$.

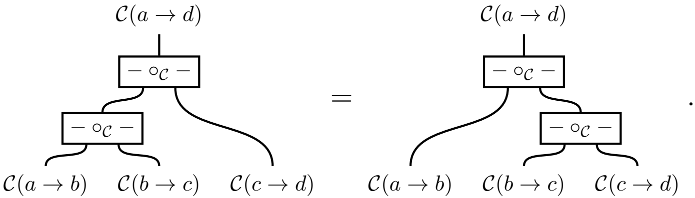
Linear categories - axioms

The composition and identity morphisms satisfy the axioms

- (identity)



- (associativity)



$\mathcal{V}$-enriched categories - data

Enriched categories were introduced by Eilenberg and Kelly [EK66]. (See also [Kel05].)

Let $\mathcal{V}$ be a monoidal category.
A $\mathcal{V}$-enriched category $\mathcal{C}$ has objects $a, b, c, \dots \in \mathcal{C}$ and

- for every $a, b \in \mathcal{C}$, an object $\mathcal{C}(a \to b) \in \mathcal{V}$
- for every $a, b, c \in \mathcal{C}$, a composition morphism

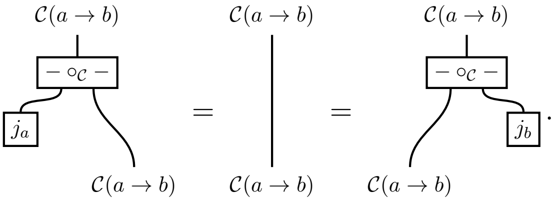
$$\mathcal{C}(a \to b)\mathcal{C}(b \to c) \xrightarrow{-\circ_{\mathcal{C}}-} \mathcal{C}(a \to c)$$

- for every $a \in \mathcal{C}$, an identity element $j_a \in \mathcal{V}(1_{\mathcal{V}} \to \mathcal{C}(a \to a))$.

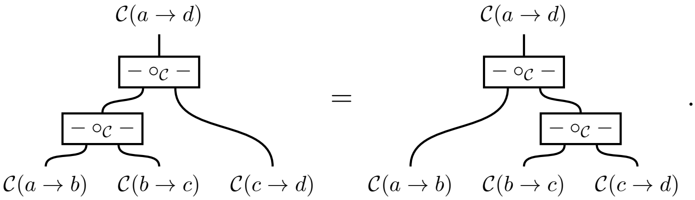
$\mathcal{V}$-enriched categories - axioms

The composition and identity morphisms satisfy the axioms

- (identity)



- (associativity)



$\mathcal{V}$-monoidal categories

To define a $\mathcal{V}$-*monoidal category*, we require $\mathcal{V}$ be braided.

Definition [MP17]

A (strict) $\mathcal{V}$-monoidal category $\mathcal{C}$ is a $\mathcal{V}$-enriched category with

- a distinguished unit object $1_{\mathcal{C}} \in \mathcal{C}$
- for every $a, b \in \mathcal{C}$, a tensor product object $ab \in \mathcal{C}$.
- for every $a, b, c, d \in \mathcal{C}$, a tensor product morphism

$$\mathcal{C}(a \to c)\mathcal{C}(b \to d) \xrightarrow{-\otimes_{\mathcal{C}}-} \mathcal{C}(ab \to cd)$$

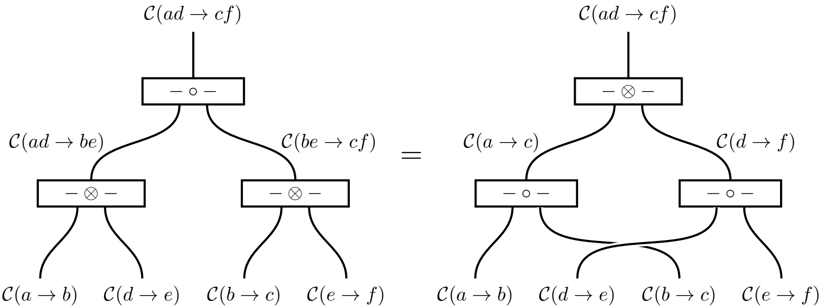
This data satisfies a variety of axioms, the most important being associativity of tensor product and the *braided interchange relation*.

The braided interchange relation

Morphisms in an ordinary monoidal category satisfy an exchange relation. If $f_1 \in \mathcal{C}(a \to b)$, $f_2 \in \mathcal{C}(b \to c)$, $g_1 \in \mathcal{C}(d \to e)$, and $g_2 \in \mathcal{C}(e \to f)$, we have

$$(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (f_1 \circ f_2) \otimes (g_1 \circ g_2).$$

In a $\mathcal{V}$-monoidal category, we replace the ordinary exchange relation with the braided interchange relation:

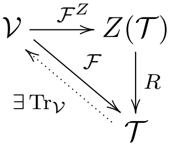


The main theorem

Theorem [MP17]

Let $\mathcal{V}$ be a braided monoidal category. There is a bijective correspondence between:

1. rigid $\mathcal{V}$-monoidal categories $\mathcal{C}$, such that $x \mapsto \mathcal{C}(1_{\mathcal{C}} \to x)$ admits a left adjoint
2. pairs $(\mathcal{T}, \mathcal{F}^Z)$ with $\mathcal{T}$ a rigid monoidal category and $\mathcal{F}^Z$ braided oplax monoidal $(\mu_{u,v} : \mathcal{F}^Z(uv) \to \mathcal{F}^Z(u)\mathcal{F}^Z(v))$ such that $\mathcal{F} := \mathcal{F}^Z \circ R$ admits a right adjoint.

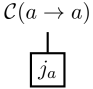


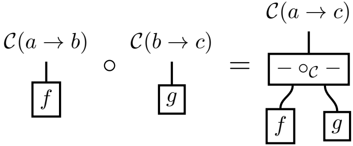
These pairs can also be called *oplax module tensor categories for* $\mathcal{V}$ in the spirit of [HPT16a].

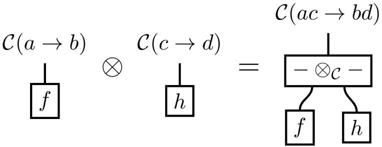
The underlying tensor category

Given a $\mathcal{V}$-monoidal category $\mathcal{C}$, the *underlying tensor category* $\mathcal{C}^{\mathcal{V}}$ has the same objects as $\mathcal{C}$, and the hom spaces are given by

$$\mathcal{C}^{\mathcal{V}}(a \to b) := \mathcal{V}(1_{\mathcal{V}} \to \mathcal{C}(a \to b)).$$

- Identity: 

- Composition: 

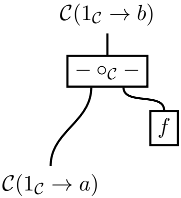
- Tensor product: 

- When $\mathcal{C}$ is rigid, so is $\mathcal{C}^{\mathcal{V}}$.

The categorified 'trace'

The functor $\mathcal{C}(1_{\mathcal{C}} \to -) : \mathcal{C}^{\mathcal{V}} \to \mathcal{V}$ is given by $a \mapsto \mathcal{C}(1_{\mathcal{C}} \to a)$ and

$$\mathcal{C}^{\mathcal{V}}(a \to b) \ni f \longmapsto$$



We only consider (rigid) $\mathcal{V}$-monoidal $\mathcal{C}$ such that $\mathcal{C}(1_{\mathcal{C}} \to -)$ has a left adjoint $\mathcal{F} : \mathcal{V} \to \mathcal{C}^{\mathcal{V}}$. We show that $\mathcal{F}$ lifts to a braided oplax monoidal functor $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{C}^{\mathcal{V}})$.

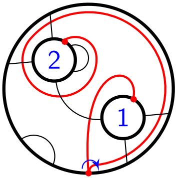
In [HPT16a], we showed that when $\mathcal{V}$ is braided pivotal, $\mathcal{T}$ is pivotal, and $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{T})$ is pivotal braided strong monoidal, a right adjoint of $\mathcal{F} = \mathcal{F}^Z \circ R$ is a *categorified trace* $\mathrm{Tr}_{\mathcal{V}} : \mathcal{T} \to \mathcal{V}$.

A related result

Theorem [HPT16b]

Let $\mathcal{V}$ be a braided pivotal monoidal category.
There is an equivalence of categories between:

1. The category of *anchored planar algebras* in $\mathcal{V}$



2. The category of *pointed module tensor categories* for $\mathcal{V}$.
   These are triples $(\mathcal{T}, \mathcal{F}^Z, t)$ such that
   - $\mathcal{T}$ is a pivotal monoidal category
   - $\mathcal{F}^Z : \mathcal{V} \to Z(\mathcal{T})$ is a braided pivotal strong monoidal functor such that $\mathcal{F} = \mathcal{F}^Z \circ R$ admits a right adjoint
   - $t \in \mathcal{T}$ is a symmetrically self-dual object which generates $\mathcal{T}$ as a module tensor category.

Example: de-equivariantization

Let G be a finite group and $\mathcal{T}$ be a rigid monoidal category. Suppose we have a fully faithful strong monoidal functor

$$\mathcal{F}^Z : \mathsf{Rep}(G) \to Z(\mathcal{T}).$$

Then $\mathcal{T}_{/\!\!/\mathcal{F}}$ is a $\mathsf{Rep}(G)$-enriched tensor category. We may apply the braided lax monoidal fiber functor $\mathsf{Rep}(G) \to \mathsf{Vec}$ to transport the $\mathsf{Rep}(G)$-enrichment back to Vec. This two step process recovers the usual notion of de-equivariantization.

There is a similar 'quotienting' procedure for fiber functors to sVec due to [BGH$^+$16]. This merits further study!

De-equivariantization and $SU(2)_k$

Consider $\mathcal{T} = SU(2)_k$, which has fusion graph A_{k+1}.

Let $\mathcal{V} = \langle 1_{\mathcal{T}}, g \rangle$, which embeds in $Z(\mathcal{T})$.

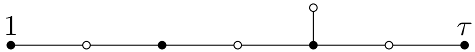
- When $k = 4n$, $\mathcal{V} \cong \mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$, and we may de-equivariantize to get a $D_{2(n+1)}$ category.
- When $k = 4n + 2$, g is a fermion ($\theta_g = -1$), and $\mathcal{V} \cong$ sVec.

Kevin Walker showed that the case $k = 4n + 2$ gives rise to the D_{odd} *spin planar algebras*.
(See also Jaffe-Liu's *para planar algebras* [JL16].)

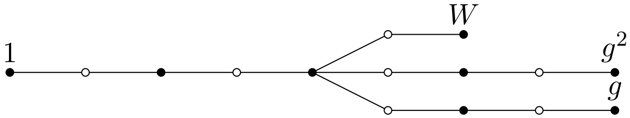
Other interesting examples

Here are two examples of fully faithful braided strong monoidal functors that we would like to investigate further.

- $Z(\mathrm{Ad}(E_8))$ contains a full copy of Fib [BEK01].



- $Z(\mathrm{Ad}(4442))$ contains a full copy of SU$(3)_3$ [GI15, Bru16].



Thank you for listening!

Monoidal categories enriched in braided monoidal categories.
Scott Morrison and David Penneys.
Preprint available at `arXiv:1701.00567`.

Planar algebras in braided tensor categories.
André Henriques, David Penneys, and James Tener.
Preprint available at `arXiv:1607.06041`.

Jens Böckenhauer, David E. Evans, and Yasuyuki Kawahigashi, *Longo-Rehren subfactors arising from α-induction*, Publ. Res. Inst. Math. Sci. **37** (2001), no. 1, 1–35, MR1815993 arXiv:math/0002154v1.

Paul Bruillard, Cesar Galindo, Tobias Hagge, Siu-Hung Ng, Julia Yael Plavnik, Eric C. Rowell, and Zhenghan Wang, *Fermionic modular categories and the 16-fold way*, 2016, arXiv:1603.09294.

Paul Bruillard, *Rank 4 premodular categories*, New York J. Math. **22** (2016), 775–800, With an Appendix by César Galindo, Siu-Hung Ng, Julia Plavnik, Eric Rowell and Zhenghan Wang. arXiv:1204.4836 MR3548123.

Samuel Eilenberg and G. Max Kelly, *Closed categories*, Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965), Springer, New York, 1966, MR0225841, pp. 421–562.

Pinhas Grossman and Masaki Izumi, *Quantum doubles of generalized Haagerup subfactors and their orbifolds*, 2015, arXiv:1501.07679.

André Henriques, David Penneys, and James E. Tener, *Categorified trace for module tensor categories over braided tensor categories*, Doc. Math. **21** (2016), 1089–1149, arXiv:1509.02937.

______, *Planar algebras in braided tensor categories*, 2016, arXiv:1607.06041.

Arthur Jaffe and Zhengwei Liu, *Planar para algebras, reflection positivity*, 2016, arXiv:1602.02662.

G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, MR2177301, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

Scott Morrison and David Penneys, *Monoidal categories enriched in braided monoidal categories*, 2017, arXiv:1701.00567.