Embedding subfactor planar algebras in graph planar algebras

NCGOA, Vanderbilt University

D. Penneys (joint work with V.F.R. Jones)

University of California, Berkeley

May 11, 2010

Main Theorem

Theorem

A finite depth subfactor planar algebra embeds in the graph planar algebra of its principal graph.

Uses

- Constructing subfactors, e.g., Haagerup [Peters '08], extended Haagerup [BMPS '10], groups [Gupta '08]
- Obstructions, e.g., classification to index 5 [JMPPS...]

1. The canonical relative commutant planar algebra
   - The basic construction for strongly Markov inclusions
   - The canonical planar algebra
   - Burns' treatment of rotation
   - Uniqueness
2. The canonical planar algebra is isomorphic to the graph planar algebra
   - Loop algebras
3. The embedding theorem for finite depth, subfactor planar algebras

Strongly Markov inclusions

Let $M_0 \subset (M_1, \mathrm{tr}_1)$ be an inclusion of finite von Neumann algebras. Let $M_2 = \langle M_1, e_1 \rangle$ be the basic construction, and let Tr_2 be the canonical trace on M_2, i.e., the unique extension of

$$xe_1y \mapsto \mathrm{tr}_1(xy) \text{ for all } x, y \in M_1.$$

Definition

An inclusion of finite von Neumann algebras $M_0 \subset (M_1, \mathrm{tr}_1)$ is called strongly Markov if

- Tr_2 is finite with $\mathrm{Tr}_2(1)^{-1} \mathrm{Tr}_2 |_M = \mathrm{tr}$, and
- $M_2 = M_1 e_1 M_1$.

In this case, we define $[M_1 : M_0] = \mathrm{Tr}_2(1)$.

Iterating the basic construction

Theorem

Suppose $M_0 \subset (M_1, \mathrm{tr}_1)$ is strongly Markov, and let $\mathrm{tr}_2 = \mathrm{Tr}_2(1)^{-1}\,\mathrm{Tr}_2$. Then $M_1 \subset (M_2, \mathrm{tr}_2)$ is strongly Markov and

$$[M_2 \colon M_1] = [M_1 \colon M_0].$$

From here on, $M_0 \subset (M_1, \mathrm{tr}_1)$ is a strongly Markov inclusion.

The set up

For $n \geq 1$,

- Iteratively define $M_{n+1} = \langle M_n, e_n \rangle$ with normalized tr_n.
- Set $d = [M_1 : M_0]^{1/2}$ and $E_n = d e_n$.
- Set $v_n = E_n E_{n-1} \cdots E_1$.

Fact

For all $n \in \mathbb{N}$, $\bigotimes_{M_0}^{n} M_1 \cong M_n$ and $\bigotimes_{M_0}^{n} L^2(M_1, \operatorname{tr}_1) \cong L^2(M_n, \operatorname{tr}_n)$ via the isomorphism

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \longleftrightarrow x_1 v_1 x_2 v_2 \cdots v_{n-1} x_n.$$

We will identify these spaces from now on.

The planar algebra

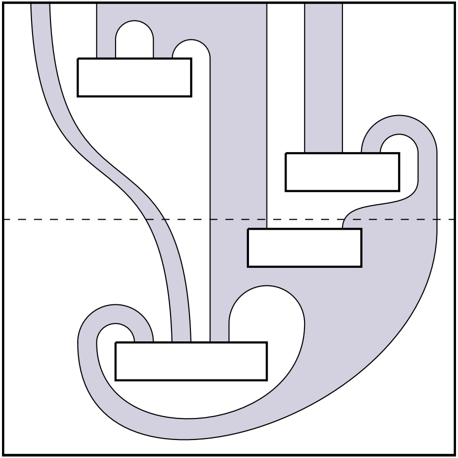
For $n \geq 0$, we set

$$P_{n,+} = M_0' \cap L^2(M_n, \mathrm{tr}_n) \cong M_0' \cap M_n$$
$$P_{n,-} = M_1' \cap L^2(M_{n+1}, \mathrm{tr}_n) \cong M_1' \cap M_{n+1}$$

Then we define an action of the planar operad on these vector spaces.

Tangle action

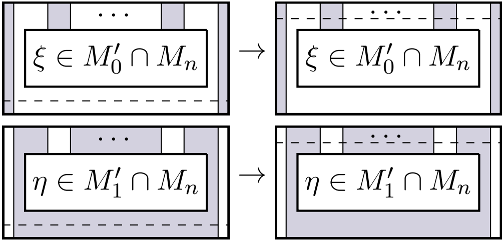
Step 1: Isotope tangle into a standard form:



Note: We allow $*$'s in shaded regions!

The planar algebra

Step 2: Read from bottom to top using rules locally. Think of shaded regions as elements of M_1 and unshaded regions as $\otimes$'s. Labelled boxes correspond to insertion of central vectors.

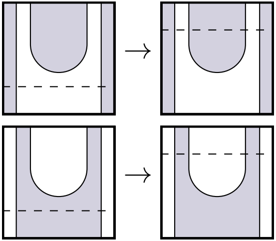


$$x \otimes y \mapsto x \otimes \xi \otimes y$$

$$x \mapsto x\eta = \eta x$$

Tangle action

Step 2: Read from bottom to top using rules locally. Think of shaded regions as elements of M_1 and unshaded regions as $\otimes$'s.



$$x \otimes y \longmapsto x \otimes 1 \otimes y$$

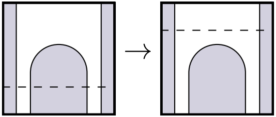
$$x \longmapsto d^{-1} \sum_{b \in B} xb \otimes b^* = d^{-1} \sum_{b \in B} b \otimes b^* x$$

Here $B = \{b\}$ is a Pimnser-Popa basis for M_1 over M_0, i.e.,

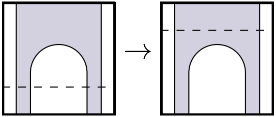
$$\sum_{b \in B} b e_1 b^* = 1.$$

Tangle action

Step 2: Read from bottom to top using rules locally. Think of shaded regions as elements of M_1 and unshaded regions as $\otimes$'s.



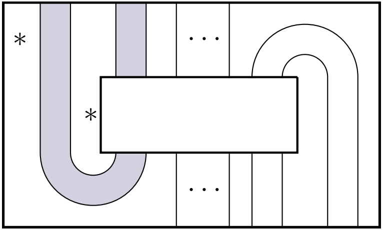
$$x \otimes y \otimes z \longmapsto dxE_{M_0}(y) \otimes z = dx \otimes E_{M_0}(y)z$$



$$x \otimes y \longmapsto xy.$$

Example

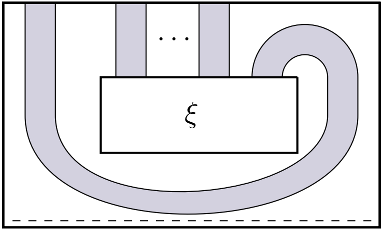
We will compute the action of



on $\xi = \sum_{i=1}^{k} x_1^i \otimes \cdots \otimes x_n^i \in M_0' \cap M_n$.

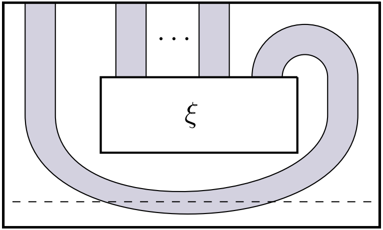
Example

We start at $1_{\mathbb{C}}$:



Example

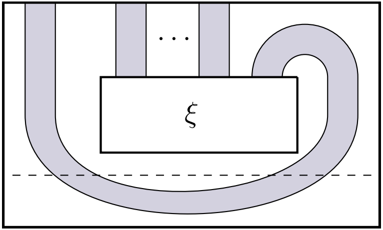
Passing the first critical point, we have



$$1_{\mathbb{C}} \mapsto 1_M.$$

Example

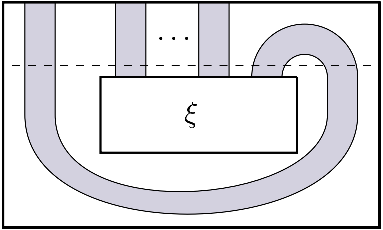
Passing the second critical point, we have



$$1_{\mathbb{C}} \mapsto 1_M \mapsto \sum_{b \in B} b \otimes b^*$$

Example

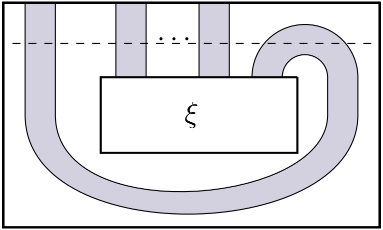
Passing the internal box, we have



$$1_{\mathbb{C}} \mapsto 1_M \mapsto \sum_{b \in B} b \otimes b^* \mapsto \sum_{b \in B} b \otimes \xi \otimes b^*$$

Example

Passing the third critical point, we have

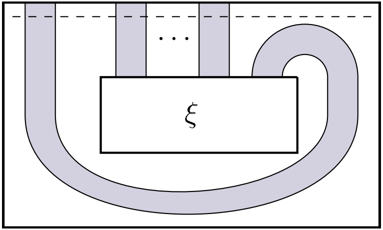


$$1_{\mathbb{C}} \mapsto 1_M \mapsto \sum_{b\in B} b \otimes b^* \mapsto \sum_{b\in B} b \otimes \xi \otimes b^*$$

$$\mapsto \sum_{b\in B} \sum_{i=1}^{k} b \otimes x_1^i \otimes \cdots \otimes x_{n-1}^i \otimes x_n^i b^*$$

Example

Passing the last critical point, we have



$$1_{\mathbb{C}} \mapsto 1_M \mapsto \sum_{b\in B} b\otimes b^* \mapsto \sum_{b\in B} b\otimes \xi \otimes b^*$$

$$\mapsto \sum_{b\in B}\sum_{i=1}^{k} b\otimes x_1^i \otimes \cdots \otimes x_{n-1}^i \otimes x_n^i b^*$$

$$\mapsto \sum_{b\in B}\sum_{i=1}^{k} b\otimes x_1^i \otimes \cdots \otimes x_{n-1}^i E_{M_0}(x_n^i b^*)$$

Burns' definition of rotation

Fact

For all $n \in \mathbb{N}$, $M_0' \cap M_n$ is equal to

$$M_0' \cap L^2(M_n, \mathrm{tr}_n) := \left\{ \xi \in L^2(M_n, \mathrm{tr}_n) \middle| x\xi = \xi x \text{ for all } x \in M_0 \right\}.$$

Definition

The rotation is given by

$$\rho(\xi) = \sum_{b \in B} \sum_{i=1}^{k} b \otimes x_1^i \otimes \cdots x_{n-1}^i E_{M_0}(x_n^i b^*) = \sum_{b \in B} L_b R_b^*(\xi).$$

Burns' proof of periodicity

Theorem (Burns)

The rotation is periodic.

Note: we don't require extremality!

Proof.

For all $\xi \in M_0' \cap M_n$ and all $\eta = y_1 \otimes \cdots \otimes y_n \in L^2(M_n, \mathrm{tr}_n)$,

$$\begin{aligned}\langle \rho(\xi), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, E_{M_0}(b^* y_1) y_2 \otimes \cdots \otimes y_n \otimes b \rangle\end{aligned}$$

Burns' proof of periodicity

Theorem (Burns)

The rotation is periodic.

Note: we don't require extremality!

Proof.

For all $\xi \in M_0' \cap M_n$ and all $\eta = y_1 \otimes \cdots \otimes y_n \in L^2(M_n, \mathrm{tr}_n)$,

$$\begin{aligned} \langle \rho(\xi), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle E_{M_0}(b^* y_1)^* \xi, y_2 \otimes \cdots \otimes y_n \otimes b \rangle \end{aligned}$$

Burns' proof of periodicity

Theorem (Burns)

The rotation is periodic.

Note: we don't require extremality!

Proof.

For all $\xi \in M_0' \cap M_n$ and all $\eta = y_1 \otimes \cdots \otimes y_n \in L^2(M_n, \mathrm{tr}_n)$,

$$\begin{aligned}\langle \rho(\xi), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi E_{M_0}(b^* y_1)^*, y_2 \otimes \cdots \otimes y_n \otimes b \rangle\end{aligned}$$

Burns' proof of periodicity

Theorem (Burns)

The rotation is periodic.

Note: we don't require extremality!

Proof.

For all $\xi \in M_0' \cap M_n$ and all $\eta = y_1 \otimes \cdots \otimes y_n \in L^2(M_n, \mathrm{tr}_n)$,

$$\begin{aligned}\langle \rho(\xi), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, y_2 \otimes \cdots \otimes y_n \otimes b E_{M_0}(b^* y_1) \rangle\end{aligned}$$

Burns' proof of periodicity

Theorem (Burns)

The rotation is periodic.

Note: we don't require extremality!

Proof.

For all $\xi \in M_0' \cap M_n$ and all $\eta = y_1 \otimes \cdots \otimes y_n \in L^2(M_n, \mathrm{tr}_n)$,

$$\begin{aligned}\langle \rho(\xi), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle \xi, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \langle \xi, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle\end{aligned}$$

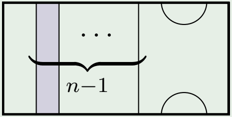
Hence $\rho^n = \mathrm{id}$. □

Uniqueness

Key Lemma

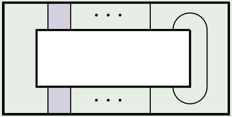
Suppose $P_\bullet$ is a planar algebra with modulus $d \neq 0$ and $Q_{n,\pm} \subset P_{n,\pm}$ are subalgebras which are closed under the following operations:

(1) left and right multiplication by tangles

$E_n =$ 

$\in P_{n+1,+}$ for $n \in \mathbb{N}$,

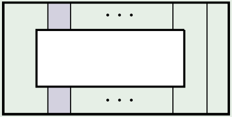
(2) The maps from $P_{n,+}$ as follows:

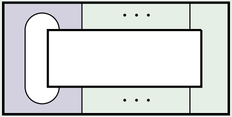
(i) $\alpha_n =$ 

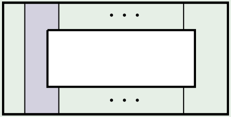
$: P_{n,+} \to P_{n-1,+}$,

Uniqueness

Key Lemma

(ii) $\beta_{n+1} =$  $: P_{n,+} \to P_{n+1,+}$,

(iii) $\gamma_n^+ =$  $: P_{n,+} \to P_{n-1,-}$, and

(3) the map $i_n^- =$  $: P_{n,-} \to P_{n+1,+}$.

Then the $Q_{n,\pm}$ define a planar subalgebra $Q_\bullet \subset P_\bullet$.

Proof.

Check that $Q_\bullet$ is closed under all annular maps. □

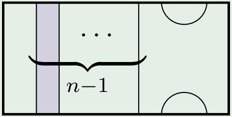
Uniqueness

Theorem

Given a strongly Markov inclusion $M_0 \subset (M_1, \mathrm{tr}_1)$, there is a unique planar algebra $P_\bullet$ of modulus $d = [M_1 : M_0]^{1/2}$ where

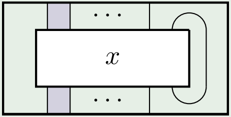
$$P_{n,+} = M_0' \cap M_n \text{ and}$$
$$P_{n,-} = M_1' \cap M_{n+1}$$

such that

(1) for $n \in \mathbb{N}$, $E_n =$ 

$\in P_{n+1,+}$,

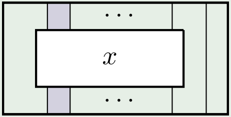
(2) for $x \in P_{n,+}$,

(i) 

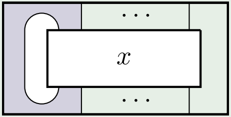
$= dE_{M_{n-1}}(x)$,

Uniqueness

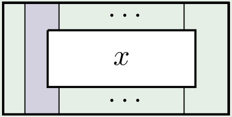
Theorem

(ii) 

$= x \in P_{n+1}$,

(iii) 

$= dE^{M_0'}_{M_1'}(x) = \frac{1}{d}\sum_{b \in B} bxb^*$, and

(3) for $x \in P_{n,-}$, 

$= x \in P_{n+1,+}$.

Isomorphism with the graph planar algebra

Theorem

Suppose $M_0 \subset (M_1, \mathrm{tr}_1)$ is a connected inclusion of finite dimensional von Neumann algebras with the Markov trace. Let Γ be the Bratteli diagram of the inclusion. Then the canonical relative commutant planar algbera is isomorphic to the graph planar algebra of Γ.

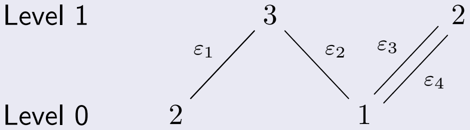
Proof.

We inductively define isomorphisms of M_n with algebras of loops on an augmented Bratteli diagram. These isomorphisms identify the relative commutants with algebras of loops on the original Bratteli diagram. □

Loop algebras

Example

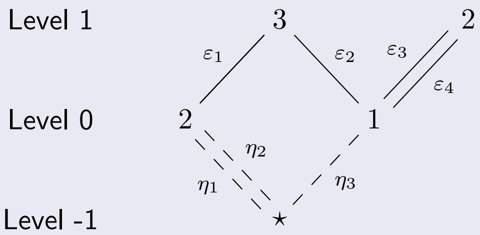
Let $M_0 = \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C} \subset \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}) = M_1$ with Bratteli diagram



Loop algebras

Example

Let $M_0 = \mathrm{Mat}_2(\mathbb{C}) \oplus \mathbb{C} \subset \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}) = M_1$ with Bratteli diagram

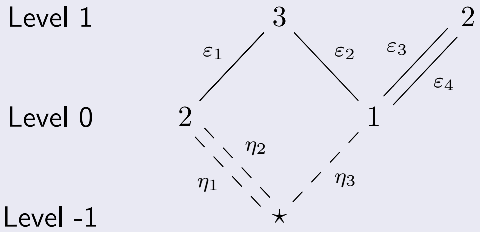


Loop algebras

Example

M_n is isomorphic to loops of length $2n$ starting at $\star$ and passing through level $n = 0, 1$. For $i, j, k, l \in \{1, 2\}$,

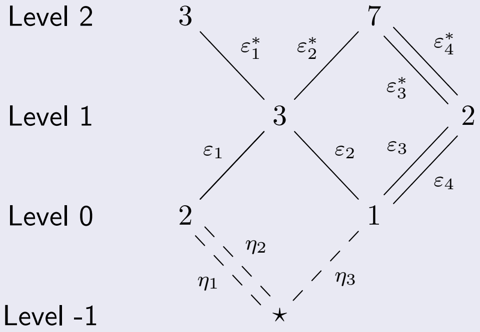
$$[\eta_i \eta_j^*] \cdot [\eta_k \eta_l^*] = \delta_{j,k} [\eta_i \eta_l^*].$$



Loop algebras

Example

To get isomorphisms with higher M_n's, we reflect Γ to get more levels. Then we take loops starting at $\star$ passing through level n.



The set up

Let $Q_\bullet$ be a finite depth, subfactor planar algebra. Let $s = 2r$ be minimal such that

$$Q_{s,+} \subset Q_{s+1,+} \subset Q_{s+2,+}$$

is standard ($Q_{s+2,+}$ is the basic construction). Then the (graph underlying the) Bratteli diagram for

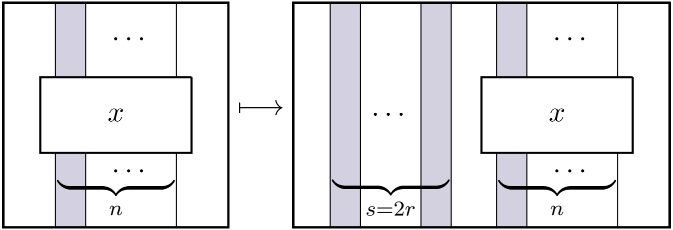
$$M_0 = Q_{s,+} \subset Q_{s+1,+} = M_1$$

is the principal graph of $Q_\bullet$. Set

$$P_{n,+} = M_0' \cap M_n = Q_{s,+}' \cap Q_{s+n,+} \text{ and}$$
$$P_{n,-} = M_1' \cap M_{n+1} = Q_{s+1,+}' \cap Q_{s+n+1,+}.$$

The embedding map

Define $\Phi\colon Q_\bullet \to P_\bullet$ by adding $s = 2r$ strings to the left for $x \in Q_{n,+}$ and adding $s+1$ strings to the left for $x \in Q_{n,-}$. For example, $\Phi\colon Q_{n,+} \to P_{n,+}$ is given by



Theorem

Φ is an inclusion of planar algebras.

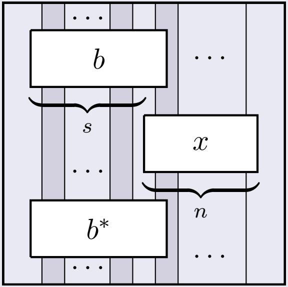
The proof

Proof.

We use the Key Lemma. The only tricky part is capping off on the left.

Let $B = \{b\}$ be a Pimsner-Popa basis for $M_1 = Q_{s+1,+}$ over $M_0 = Q_{s,+}$. Then each $b \in B$ is an $(s+1,+)$-box in $Q_{s+1,+}$, so for all $x \in Q_{n,+}$, we have

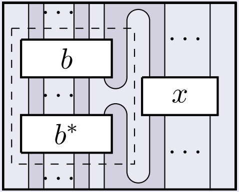
$$\gamma_n^+(\Phi(x)) = \frac{1}{d}\sum_{b\in B} b\Phi(x)b^* = \frac{1}{d}\sum_{b\in B}$$



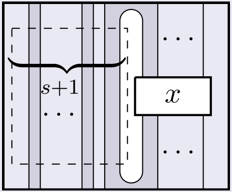
The proof

Proof.

$= \frac{1}{d} \sum_{b \in B}$



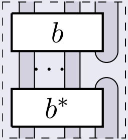
$=$



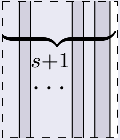
$= \Phi(\gamma_n^+(x))$

as

$\frac{1}{d} \sum_{b \in B}$



$= \sum_{b \in B} b e_{s+1} b^* = 1_{P_{s+2}} =$



.

□

Thank you for listening!

Slides and preprint available at:

http://math.berkeley.edu/~dpenneys/grad.html