C^*-algebras from planar algebras

In honor of George Elliott's 70th birthday

David Penneys (UCLA)
joint work with Michael Hartglass

International Conference on C^*-algebras and Dynamical Systems

June 30, 2015

The subfactor – planar algebra correspondence

- Given a finite index II_1-subfactor $N \subset M$, its standard invariant forms a subfactor planar algebra.
- Conversely, given a subfactor planar algebra $\mathcal{P}_\bullet$, Popa showed how to reconstruct a II_1-subfactor whose standard invariant is $\mathcal{P}_\bullet$.

Theorem (Ocneanu, Popa)

If $N \subset M$ is a finite depth, finite index hyperfinite II_1-subfactor, its standard invariant is a complete invariant.

C^*-algebras from planar algebras

In a series of articles with Michael Hartglass, we study canonical C^*-algebras associated to planar algebras in order to develop a connection between subfactor theory, C^*-algebras, and non-commutative geometry.

- Part I, to appear **Trans. AMS** `arXiv:1401.2485`
- Part II, **J. Funct. Anal.** `arXiv:1401.2485`
- Part III, in preparation!

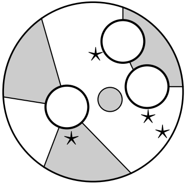
Main tools

Our main tools for Parts I and II are:

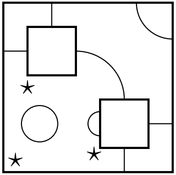
- Voiculescu's free Gaussian functor
- Pimsner's Fock space construction associated to a C^*-Hilbert bimodule
- Guionnet-Jones-Shlyakhtenko's diagrammatic reproof of Popa's reconstruction theorem

(Sub)factor planar algebras

- A shaded subfactor planar algebra is an axiomatization of the standard invariant of a finite index subfactor.



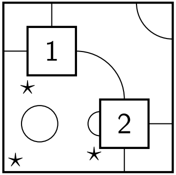
- We work with an unshaded factor planar algebras, which axiomatize rigid C*-tensor categories of bifinite bimodules over a single factor.



Planar algebras

Definition

A planar algebra is a sequence of finite dimensional complex vector spaces $\mathcal{P}_n$ for $n \geq 0$ together with an action by planar tangles.

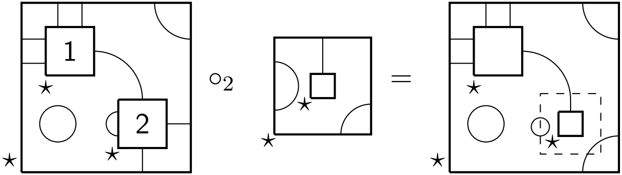


$: \mathcal{P}_3 \times \mathcal{P}_5 \to \mathcal{P}_5$

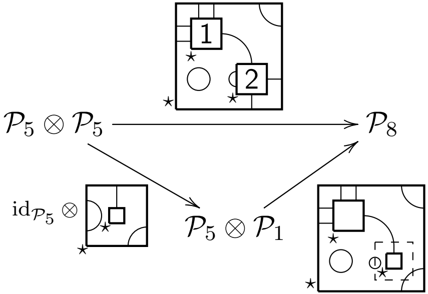
- The number of strings connected to the input disks tells you the domain.
- The number of strings connected to the output disk tells you the codomain.

Composition

There is a natural notion of tangle composition:

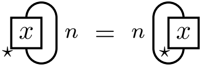


An action of planar tangles means that composition of tangles must correspond to composition of multilinear maps:

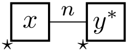


Factor planar algebras

- A planar algebra is a planar $*$-algebra if each $\mathcal{P}_n$ has an involution $*$ compatible with the reflection of planar tangles.
- A planar $*$-algebra is a factor planar algebra if
  - (Evaluable): $\mathcal{P}_0 \cong \mathbb{C}$ with the empty diagram identified with $1 \in \mathbb{C}$. Thus each closed loop is replaced by a scalar δ.
  - (Spherical): For all $n \geq 1$ and all $x \in \mathcal{P}_{2n}$, we have

$\mathrm{tr}(x) =$ 

  - (Positive): For all $n \geq 0$, we have a positive definite inner product on $\mathcal{P}_n$ given by

$\langle x, y \rangle =$  .

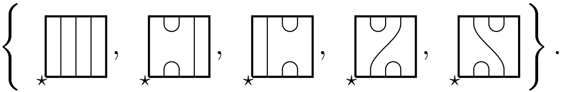
Jones' Index Rigidty Theorem

In a factor planar algebra $\mathcal{P}_\bullet$, $\delta \in \{2\cos(\pi/n) : n \geq 3\} \cup [2, \infty)$.

Temperley-Lieb

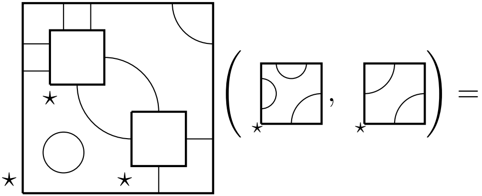
$\mathcal{TL}_\bullet(\delta)$ has $\delta \in \{2\cos(\pi/n) | n \geq 3\} \cup [2, \infty)$.

$\mathcal{TL}_k$ is the linear span of all planar string diagrams with no internal disks and k marked boundary points.

$TL_6 = \mathrm{span}_{\mathbb{C}}$ 

Adjoint is the conjugate-linear extension of reflection of tangles.

- This is a factor planar algebra if $\delta > 2$.
- If $\delta = 2\cos(\pi/n)$, must take quotient by zero length vectors.
- The action is as follows:



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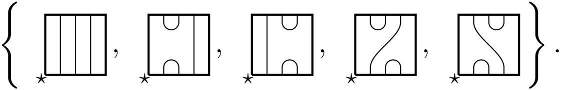
$TL_6 = \mathrm{span}_{\mathbb{C}}$ 

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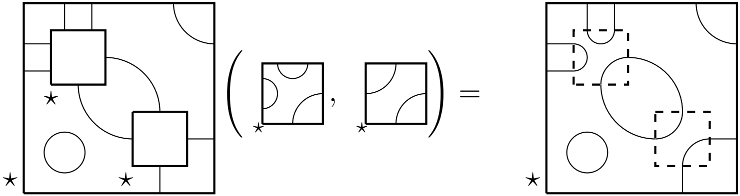

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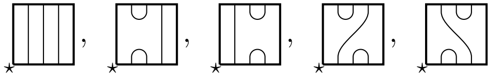
$$TL_6 = \text{span}_{\mathbb{C}} \left\{ \cdots \right\}.$$



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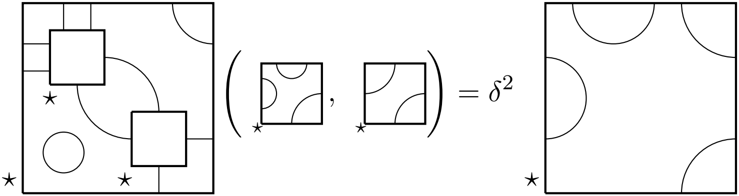
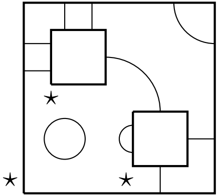


Non-commutative polynomials

$\mathcal{NC}_\bullet(n)$ is the factor planar algebra of non-commuting polynomials.

Take n self-adjoint non-commuting variables $X_1, \ldots, X_n$.

- $\mathcal{NC}_k$ is the $\mathbb{C}$-span of monomials of degree k.
- The involution is the conjugate-linear extension of reversing a monomial: $(X_{i_1} \cdots X_{i_k})^* = X_{i_k} \cdots X_{i_1}$.
- The action is as follows:



$(X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5}, X_{j_1} X_{j_2} X_{j_3} X_{j_4} X_{j_5})$

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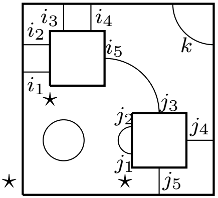

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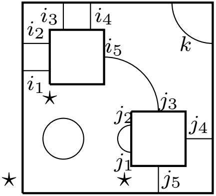


$(X_{i_1} X_{i_2} X_{i_3} X_{i_4} X_{i_5}, X_{j_1} X_{j_2} X_{j_3} X_{j_4} X_{j_5})$

$$= n \cdot \delta_{i_5, j_3} \cdot \delta_{j_1, j_2} \cdot X_{i_1} X_{i_2} X_{i_3} X_{i_4} \left(\sum_{k=1}^{n} X_k^2 \right) X_{j_4} X_{j_5}$$

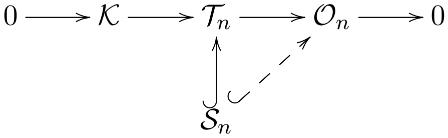
Voiculescu's free Gaussian functor

Begin with a real Hilbert space $H_{\mathbb{R}}$ with $\dim_{\mathbb{R}}(H_{\mathbb{R}}) = n < \infty$.

1. Take its complexification $H_{\mathbb{C}}$.
2. Form the full Fock space $\mathcal{F}(H_{\mathbb{C}}) = \bigoplus_{n\geq 0} \bigotimes^n \mathcal{H}_{\mathbb{C}}$.
3. We look at the left creation and annihilation operators:

$$L_\eta(\xi_1 \otimes \cdots \otimes \xi_n) = \eta \otimes \xi_1 \otimes \cdots \otimes \xi_n$$
$$L_\eta^*(\xi_1 \otimes \cdots \otimes \xi_n) = \langle \eta | \xi_1 \rangle \xi_2 \otimes \cdots \otimes \xi_n$$

4. Toeplitz algebra: $\mathcal{T}_n = \mathrm{C}^* \{ L(\eta), L(\eta)^* | \eta \in \mathbb{H}_{\mathbb{C}} \}$
5. Free semi-circular algebra: $\mathcal{S}_n = \mathrm{C}^* \{ L(\eta) + L(\eta)^* | \eta \in \mathbb{H}_{\mathbb{R}} \}$
6. Cuntz algebra: $\mathcal{O}_n = \mathcal{T}_n / \mathcal{K}$



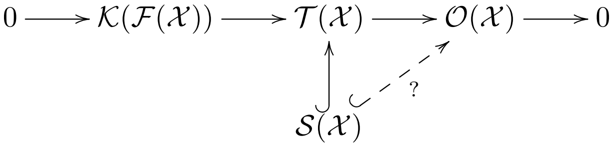
Pimsner's Fock space associated to a C*-Hilbert bimodule

Let $\mathcal{B}$ be the ground C*-algebra. Begin with a C*-Hilbert bimodule $\mathcal{X}$ with a distinguished real subspace $\mathcal{X}_{\mathbb{R}}$ such that $\mathcal{X}_{\mathbb{R}} \cdot \mathcal{B} = \mathcal{X}$.

1. Form the full Fock space $\mathcal{F}(\mathcal{X}) = \bigoplus_{n \geq 0} \bigotimes_{\mathcal{B}}^{n} \mathcal{X}$.
2. We look at the left creation and annihilation operators:

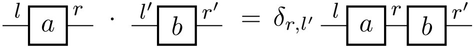
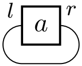
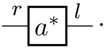
$$L_\eta(\xi_1 \otimes \cdots \otimes_n) = \eta \otimes \xi_1 \otimes \cdots \otimes \xi_n$$
$$L_\eta^*(\xi_1 \otimes \cdots \otimes_n) = \langle \eta | \xi_1 \rangle_{\mathcal{B}} \xi_2 \otimes \cdots \otimes \xi_n$$

3. Pimsner-Toeplitz algebra: $\mathcal{T}(\mathcal{X}) = \mathrm{C}^* \{L(\eta), L(\eta)^* | \eta \in \mathcal{X}\}$
4. Free semi-circular alg: $\mathcal{S}(\mathcal{X}) = \mathrm{C}^* \{L(\eta) + L(\eta)^* | \eta \in \mathcal{X}_{\mathbb{R}}\}$
5. Cuntz-Pimsner algebra: $\mathcal{O}(\mathcal{X}) = \mathcal{T}(\mathcal{X}) / \mathcal{K}(\mathcal{F}(\mathcal{X}))$



The ground C*-algebra $\mathcal{B}$

Let $\mathcal{P}_\bullet$ be a factor planar algebra.

- Set $\mathcal{B}_n = \bigoplus_{l,r=0}^{n} \mathcal{P}_{l,r}$
  - Multiplication: 
  - (Semi-finite) trace: $\mathrm{Tr}(a) = \delta_{l,r}$ 
  - Involution: $a^\dagger =$  .

  Each $\mathcal{B}_n$ is finite dimensional, and $\mathcal{B}_n \hookrightarrow \mathcal{B}_{n+1}$.

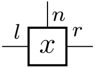
- $\mathcal{B} = \varinjlim \mathcal{B}_n = \overline{\bigcup_{n\geq 0} \mathcal{B}_n}^{\|\cdot\|} \cong \bigoplus_{\alpha \in V(\Gamma)} \mathcal{K}$.

  $\mathcal{B}$ is nonunital, AF, and generated by minimal projections.

The C*-Hilbert bimodule $\mathcal{X}$ and $\mathcal{F}(\mathcal{X})$

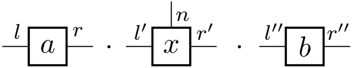
- $\mathcal{X}_n$ is the $\mathcal{B}-\mathcal{B}$ Hilbert bimodule generated by $\bigoplus_{l,r\geq 0} \mathcal{P}_{l,n,r}$.

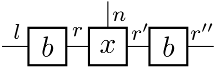
  Set $\mathcal{X} = \mathcal{X}_1$.

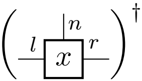
  $\in \mathcal{P}_{l,n,r}$

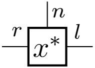
- The left and right $\bigcup_{n\geq 0} \mathcal{B}_n$ actions are given by:

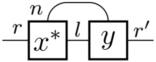
  $= \delta_{r,l'}\delta_{r',l''} \cdot$
  

  .

- $\mathcal{X}_n$ has an involution $\dagger$:
  

  $=$
  

  .

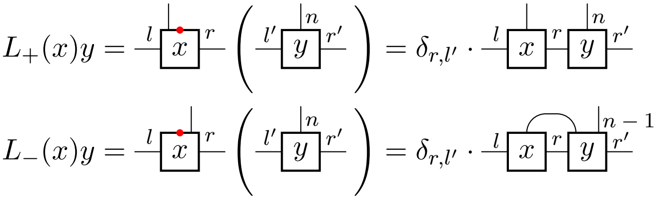
- Have a $\mathcal{B}$-valued inner product: $\langle x|y\rangle_{\mathcal{B}} = \delta_{l,l'}$
  

  .

- Full Fock space $\mathcal{F}(\mathcal{X}) = \bigoplus_{n=0}^{\infty} \mathcal{X}_n \cong \bigoplus_{n\geq 0} \bigotimes_{\mathcal{B}}^{n} \mathcal{X}$.

The Pimsner-Toeplitz and Cuntz-Pimsner algebras

- For $x \in \mathcal{X}$, we get creation and annihilation operators $L_{\pm}(x)$:



  Note $L_+(x)^* = L_-(x^\dagger)$.

- Pimsner-Toeplitz algebra $\mathcal{T}(\mathcal{P}_\bullet) = \mathrm{C}^*\{\mathcal{B}, L_{\pm}(x) | x \in \mathcal{X}\}$.
- Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{P}_\bullet)$ is $\mathcal{T}(\mathcal{P}_\bullet)/\mathcal{K}(\mathcal{P}_\bullet)$.

the free semicircular algebra of $\mathcal{P}_\bullet$

We have a distinguished real subspace $\mathcal{X}_\mathbb{R} = \{\xi \in \mathcal{X} | \xi = \xi^\dagger\} \subset \mathcal{X}$.

- Free semi-circular alg:
  $\mathcal{S}(\mathcal{P}_\bullet) = \mathrm{C}^*\{\mathcal{B}, L_+(\xi) + L_-(\xi) | \xi \in \mathcal{X}_\mathbb{R}\}$.

For a C*-Hilbert bimodule $\mathcal{Y}$ over $\mathcal{B}$, work of Germain and Pimsner, gives KK-equivalences $\mathcal{B} \hookrightarrow \mathcal{S}(\mathcal{Y}) \hookrightarrow \mathcal{T}(\mathcal{Y})$.

Theorem (Hartglass-P., part I)

$$K_0(\mathcal{S}(\mathcal{P}_\bullet)) = \mathbb{Z}\{\alpha | \alpha \in V(\Gamma)\} \text{ and } K_1(\mathcal{S}(\mathcal{P}_\bullet)) = (0).$$

Here, Γ is the so-called principal graph of $\mathcal{P}_\bullet$, a combinatorial invariant which encodes data about the minimal projections in $\mathcal{P}_{2n}$ and fusion with the strand.

Compressions

By taking various compressions of $\mathcal{A}(\mathcal{P}_\bullet)$ for $\mathcal{A} = \mathcal{O}, \mathcal{T}, \mathcal{S}$, we have the chart below:

	$\mathcal{A} = \mathcal{O}$	$\mathcal{A} = \mathcal{T}$	$\mathcal{A} = \mathcal{S}$
$\mathcal{A}(\mathcal{P}_\bullet)$	Cuntz-Pimsner	Pimsner-Toeplitz	semifinite GJS algebra
$\mathcal{A}(\Gamma)$	Cuntz-Krieger $\mathcal{O}_{\vec{\Gamma}}$	Toeplitz-Cuntz-Krieger $\mathcal{T}_{\vec{\Gamma}}$	free graph algebra $\mathcal{S}(\Gamma)$
$\mathcal{A}_0(\mathcal{P}_\bullet)$	Doplicher-Roberts $\mathcal{O}_\rho$	Toeplitz extension by $\mathcal{K}$	GJS algebra
$\mathcal{A}_0(\mathcal{NC}_\bullet)$	Cuntz $\mathcal{O}_n$	Toeplitz $\mathcal{T}_n$	Free semicircular system

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- To get $\mathcal{A}_0(\mathcal{P}_\bullet)$, we cut down by the empty diagram, which is a projection in $\mathcal{B}$.
- We have $\mathcal{A}(\mathcal{P}_\bullet) \cong \mathcal{A}_0(\mathcal{P}_\bullet) \otimes \mathcal{K}$.

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- We have $\mathcal{A}(\mathcal{P}_\bullet) \cong \mathcal{A}_0(\mathcal{P}_\bullet) \otimes \mathcal{K}$.
- When $\mathcal{P}_\bullet = \mathcal{NC}_\bullet$, we get the algebras from Voiculescu's free Gaussian functor.

Properties of the free graph algebra $\mathcal{S}(\Gamma)$

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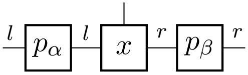
- Vertices $\alpha \in V(\Gamma)$ are equivalence classes of minimal projections in the ground algebra $\mathcal{B}$.
- There are

$\dim \Big\{$  $\Big| \, x \in \mathcal{P}_{l,1,r} \Big\}$

edges between α and β.

Theorem (Hartglass-P.)

- $\mathcal{S}(\Gamma)$ is simple, has unique trace, and has stable rank 1.
- $\mathcal{S}(\mathcal{P}_\bullet)(\cong \mathcal{S}(\Gamma) \otimes \mathcal{K})$ has comparison of projections: If $\mathrm{Tr}(p) > \mathrm{Tr}(q)$, then $\exists\, v \in \mathcal{S}(\mathcal{P}_\bullet)$ with $v^*v = q$ and $vv^* \leq p$.

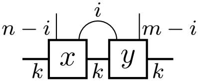
Corollary:

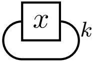
$$\{\mathrm{Tr}(p) : p \in P(\mathcal{S}(\mathcal{P}_\bullet))\} = \mathbb{R}_+ \cap \mathbb{Z}\{\dim(\alpha) | \alpha \in V(\Gamma))\}$$

The Guionnet-Jones-Shlyakhtenko C*-algebras

Our original motivation was to study C*-algebras arising from GJS's diagrammatic reproof of Popa's reconstruction theorem.

- Set $\mathrm{Gr}_k(\mathcal{P}_\bullet) = \bigoplus_{n\geq 0} \mathcal{P}_{2k+n}$ with multiplication and trace:

$$x \cdot y = \sum_{i=0}^{\min\{n,m\}}$$ 

and $\mathrm{tr}(x) = \frac{\delta_{n,0}}{\delta^k} \cdot$  .

  $\mathcal{A}_k$ is the GNS C*-completion of $\mathrm{Gr}_k(\mathcal{P}_\bullet)$ on $L^2(\mathrm{Gr}_k(\mathcal{P}_\bullet), \mathrm{tr})$.
- $\mathcal{M}_k = \mathcal{A}_k''$ is an interpolated free group II_1-factor.

Fact

$\mathcal{A}_k \cong 1_k \mathcal{S}(\mathcal{P}_\bullet) 1_k$ where $1_k =$ , which is full as $\mathcal{S}(\mathcal{P}_\bullet)$ is simple.

Properties of the GJS C*-algebras

Theorem (Hartglass-P., part II)

$\mathcal{A}_k$ is Morita equivalent to $\mathcal{S}(\mathcal{P}_\bullet)$. (In fact $\mathcal{S}(\mathcal{P}_\bullet) \cong \mathcal{A}_k \otimes \mathcal{K}$.)

- $K_0(\mathcal{A}_k) \cong \mathbb{Z}\{\alpha | \alpha \in V(\Gamma)\}$
- $K_1(\mathcal{A}_k) = (0)$.
- $\mathcal{A}_k$ is simple with unique trace and stable rank 1.

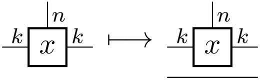
$\mathcal{A}_0$ is either projectionless, or $\{\mathrm{tr}(p) | p \in P(\mathcal{A}_0)\}$ is dense in $[0, 1]$.

Hence the $\mathcal{A}_k$ are quite different from Voiculescu's C*-algebras from free semi-circular families.

The GJS reconstruction reproof

Fact (Guionnet-Jones-Shlyakhtenko)

We have a Jones tower $\mathcal{M}_k \hookrightarrow \mathcal{M}_{k+1}$ by



The Jones index $[\mathcal{M}_1 : \mathcal{M}_0] = \delta^2$, and $\mathcal{M}_0' \cap \mathcal{M}_k = \mathcal{P}_{2k}$.

Theorem (Hartglass-P., part II)

The same diagram gives a Watatani tower $\mathcal{A}_k \hookrightarrow \mathcal{A}_{k+1}$.
The Watatani index $[\mathcal{A}_1 : \mathcal{A}_0] = \delta^2$, and $\mathcal{A}_0' \cap \mathcal{A}_k = \mathcal{P}_{2k}$.

Failure of Goldman's theorem

Example (Hartglass-P., part II)

Let $\mathcal{P} = \mathcal{TL}_\bullet(\sqrt{2})$. In this case, $[\mathcal{M}_1 : \mathcal{M}_0] = 2$ so by Goldman's theorem, $\mathcal{M}_1 \cong \mathcal{M}_0 \rtimes \mathbb{Z}/(2\mathbb{Z})$.

However, $\mathcal{A}_1 \cong \mathcal{A}_0 \rtimes \mathbb{Z}/(2\mathbb{Z})$. Otherwise, there would be a Pimsner-Popa basis $\{1, u\}$ of $\mathcal{A}_1$ over $\mathcal{A}_0$. In K-theory, this means that $[1_{\mathcal{A}_2}] = 2[1_{\mathcal{A}_0}]$ which is impossible since Γ is the A_3 Coxeter-Dynkin diagram.

A first spectral triple

Let $\mathcal{P}_\bullet$ be a factor planar algebra. Form the filtered algebra Gr_0.

Define the number operator N on Gr_0 by $N\left(\overset{\big|\, n}{\boxed{x}}\right) = n\,\overset{\big|\, n}{\boxed{x}}$.

Theorem (Hartglass-P., part III)

$(\mathrm{Gr}_0, L^2(\mathrm{Gr}_0), N)$ is a θ-summable spectral triple with compact resolvent.

Adapting results of Ozawa-Rieffel to amalgamated free products:

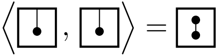
Theorem (Hartglass-P., part III)

$(\mathrm{Gr}_0, L^2(\mathrm{Gr}_0), N)$ is a compact quantum metric space in the sense of Rieffel. That is, the induced topology on the state space from

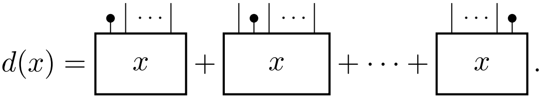
$$\rho(\mu,\nu) = \sup\left\{|\mu(a)-\nu(a)| \,\middle|\, a \in \mathrm{Gr}_0 \text{ with } \|[D,a]\| \leq 1\right\}$$

agrees with the weak-$*$ topology.

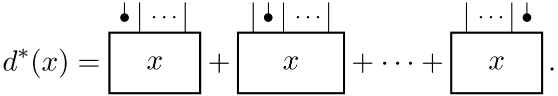
The cup derivative

Pick a special 'cup' element  $\in \mathcal{P}_1$ with  $= 1$.

On Gr_0 we have the <u>cup derivative</u>

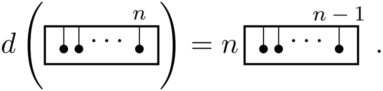


Note that d is closable with adjoint $d^*|_{\mathrm{Gr}_0}$ given by



Remark

We call d the cup derivative because



A second spectral triple

Define the cup Laplacian to be $L = dd^* + d^*d$ on Gr_0.

Theorem (Hartglass-P., part III)

$(\mathrm{Gr}_0, L^2(\mathrm{Gr}_0), L_\cup)$ is a θ-summable spectral triple with compact resolvent.

We're currently in the process of studying more properties of our spectral triples.

Thank you for listening!

Slides available at:
http://math.ucla.edu/~dpenneys/PenneysShijiazhuang2015.pdf

- Part I, to appear **Trans. AMS** arXiv:1401.2485
- Part II, **J. Funct. Anal.** arXiv:1401.2485
- Part III, in preparation!