Classifying small index subfactors

UC Berkeley extended probabilistic operator algebras seminar

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Where do subfactors come from?

Some examples include:

- Groups – from $G \curvearrowright R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – $\mathrm{Rep}(\mathcal{U}_q(\mathfrak{g}))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- composites of known subfactors

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs (M, M'), subfactors come in pairs $(A \subset B, B' \subset A')$.

Index for subfactors

Theorem (Jones [Jon83])

For a II_1-subfactor $A \subset B$,

$$[B\colon A] \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

Definition

The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \overset{e_1}{\subset} A_2 \overset{e_2}{\subset} A_3 \overset{e_3}{\subset} \cdots$$

where e_i is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$.

Principal graphs

Let $\rho =_A L^2(B)_B$.

Definition

The principal graph Γ_+ has one vertex for each isomorphism class of simple ${}_A\alpha_A$ and ${}_A\rho_B$. There are

$$\dim(\mathrm{Hom}_{A-B}(\alpha\rho, \beta))$$

edges from α to β.

The dual principal graph Γ_- is defined similarly using $B - B$ and $B - A$ bimodules.

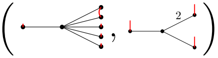
- $\Gamma_\pm$ is pointed, where the base point is the trivial bimodule ${}_A L^2(A)_A$, ${}_B L^2(B)_B$ respectively.
- Duality is given by contragredient, which is always at the same depth, although duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$.

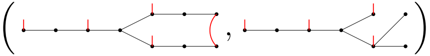
Fact

The dual graph of $A_0 \subset A_1$ is the principal graph of $A_1 \subset A_2$.

Examples of principal graphs

- index < 4: A_n, D_{2n}, E_6, E_8. No D_{odd} or E_7.
- Graphs for $R \subset R \rtimes G$ obtained from G and $\mathrm{Rep}(G)$.



- Haagerup 333 

- First graph is principal, second is dual principal.
- Leftmost vertex is the trivial bimodule.
- Red tags for duality (contragredient of bimodules).
- Duality of odd vertices by depth and height

The standard invariant: two towers of centralizer algebras

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \\
P_{3,+} & = & A_0' \cap A_3 & \supset & A_1' \cap A_3 & = & P_{2,-} \\
 & & \cup & & \cup & & \\
P_{2,+} & = & A_0' \cap A_2 & \supset & A_1' \cap A_2 & = & P_{1,-} \\
 & & \cup & & \cup & & \\
P_{1,+} & = & A_0' \cap A_1 & \supset & A_1' \cap A_1 & = & P_{0,-} \\
 & & \cup & & & & \\
P_{0,+} & = & A_0' \cap A_0 & & & &
\end{array}
$$

These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].

Popa's reconstruction theorem

Popa axiomatized the standard invariant of a subfactor, and showed how to reconstruct a subfactor from an abstract standard invariant.

Theorem (Popa [Pop94])

Every (strongly) amenable standard invariant is realized by a unique subfactor of R up to conjugacy.

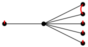
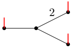
Finite depth

Definition

If the principal graph is finite, then the subfactor and standard invariant are called <u>finite depth</u>.

Example: $R \subset R \rtimes G$ for finite G

For $G = S_3$:

- Principal graph: 
- Dual principal graph: 

Theorem (Ocneanu Rigidity)

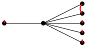
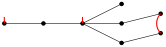
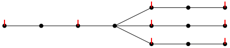
There are only finitely many standard invariants with the same finite principal graphs.

Supertransitivity

Definition

We say a principal graph is $\underline{n\text{-supertransitive}}$ if it begins with an initial segment consisting of the Coxeter-Dynkin diagram A_{n+1}, i.e., an initial segment with n edges.

Examples

-  is 1-supertransitive
-  is 2-supertransitive
-  is 3-supertransitive

Small index subfactor classification program

Steps of subfactor classifications:

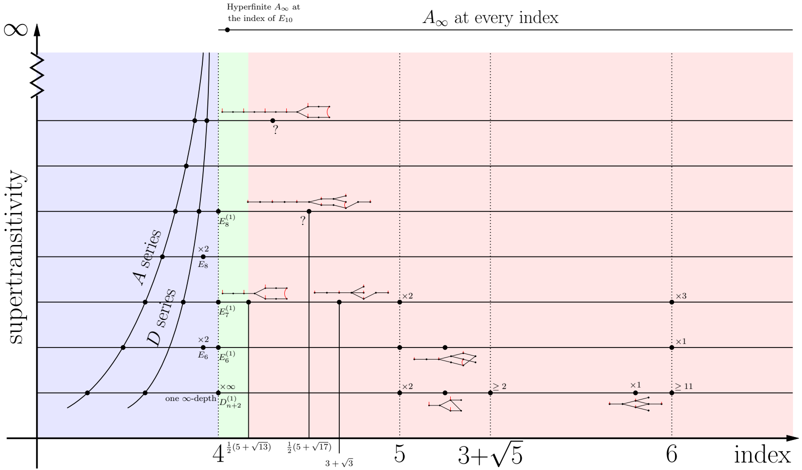
1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

If we enumerate all graph pairs with norm at most r, we have found all principal graphs with index at most r^2.

Known small index subfactors, 1994

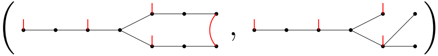
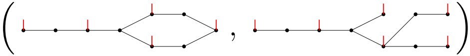


- Haagerup's partial classification to $3+\sqrt{3}$
- Popa's A_∞ at all indices

Haagerup's enumeration

Theorem (Haagerup [Haa94])

Any non A_∞-standard invariant in the index range $(4, 3+\sqrt{2})$ must have principal graphs a translation of one of

- 
- 
- 

Translation means raising the supertransitivity of both graphs by the same even amount.

Definition (Morrison-Snyder [MS12])

A vine is a graph pair which represents an infinite family of graph pairs obtained by translation.

Main tools for Haagerup's enumeration

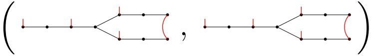
Play associativity off of Ocneanu's triple point obstruction.

- Associativity: graphs must be similar
- Ocneanu's triple point obstruction: graphs must be different!

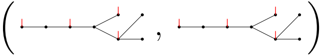
The consequence is a strong constraint.

Example

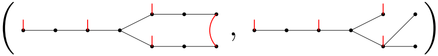
The following pairs are not allowed:



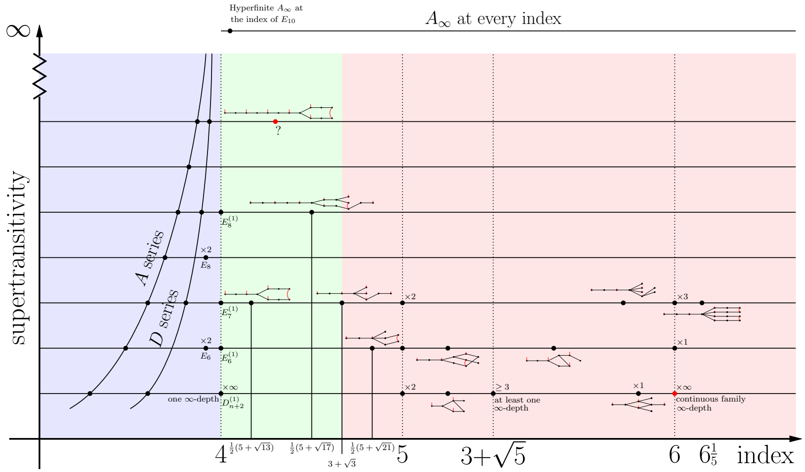
and



They must be paired with each other:

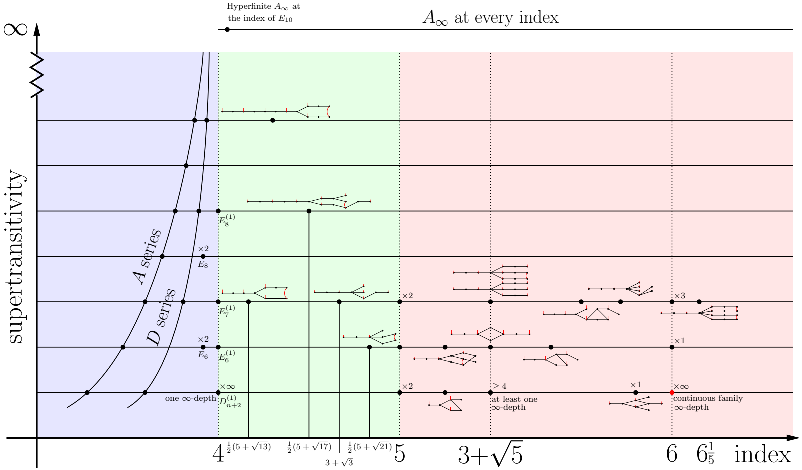


Known small index subfactors, 2007



- Asaeda-Yasuda eliminate Haagerup vine
- Bisch eliminates Hexagon vine
- Bisch-Nicoara-Popa's continuous family with same standard invariant at index 6

Known small index subfactors, 2011



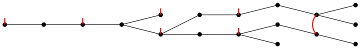
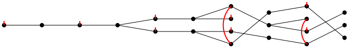
- Extended Haagerup
- Classification to index 5 (Izumi, Jones, Morrison, P, Peters, Snyder, Tener)

Weeds and vines

The classification to index 5 introduced weeds and vines.

Definition

A <u>weed</u> is a graph pair which represents an infinite family of graph pairs obtained by translation and extension.
An <u>extension</u> of a graph pair adds new vertices and edges at strictly greater depths than the maximum depth of any vertex in the original pair.

$\mathcal{F} = \Big($  ,  $\Big)$

Using weeds allows us to bundle hard cases together, ensuring the enumerator terminates.

Eliminating vines with number theory

We can uniformly treat vines using number theory, based on the following theorem inspired by Asaeda-Yasuda [AY09]:

Theorem (Calegari-Morrison-Snyder [CMS11])

For a fixed vine $\mathcal{V}$, there is an effective (computable) constant $\mathcal{R}(\mathcal{V})$ such that any n-translate with $n > \mathcal{R}(\mathcal{V})$ has norm squared which is not a cyclotomic integer.

Theorem [CG94, ENO05]

The index of a finite depth subfactor (which is equal to the norm squared of the principal graph) must be a cyclotomic integer.

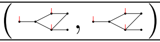
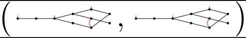
Why do we care about index $3+\sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 * \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 * \mathbb{Z}/3 = PSL(2,\mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with standard invariant $A_3 * D_4$ (Brothier-Vaes [BV13])
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3+\sqrt{5} = 2\tau^2$ where $\tau = \frac{1+\sqrt{5}}{2}$.

Index $(5, 3+\sqrt{5})$

Conjecture (Morrison-Peters (2012) [MP14b])

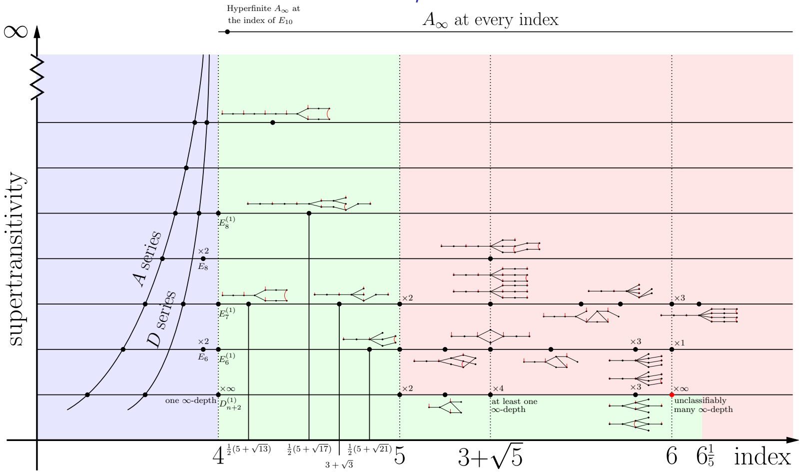
There are exactly two non-A_∞ standard invariants in the index range $(5, 3+\sqrt{5})$:

name	Principal graphs	Index	Existence, Uniqueness
$SU(2)_5$		5.04892	[Wen90], [MP14b]
$SU(3)_4$		5.04892	[Wen88], [MP14b]

Theorem (Morrison-Peters (2012) [MP14b])

There is exactly one 1-supertransitive subfactor in the index range $(5, 3+\sqrt{5})$

Known small index subfactors, 2013

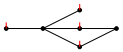
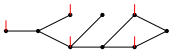
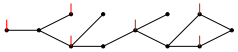
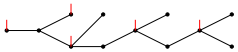


- Brothier-Vaes unclassifiably many subfactors with standard invariant $A_3 * D_4$ at index 6
- Liu classified composite standard invariants from A_3 and A_4
- 1-supertransitive to index $6\frac{1}{5}$ (Liu-Morrison-P) [LMP15]

1-supertransitive subfactors at index $3+\sqrt{5}$

Theorem (Liu [Liu13], partial proof by [IMP13])

There are exactly seven 1-supertransitive standard invariants with index $3+\sqrt{5}$:

- ( , ) self-dual
- ( , ) and its dual
- ( , ) and its dual
- ( $\cdots$,  $\cdots$) and its dual ($A_3 * A_4$)

These are all the standard invariants of composed inclusions of A_3 and A_4 subfactors.

Open question

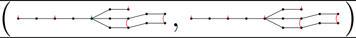
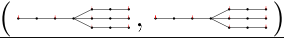
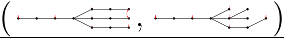
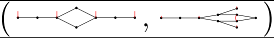
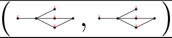
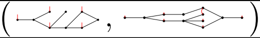
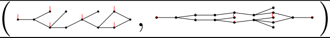
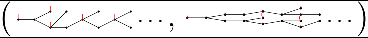
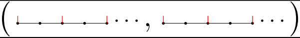
How many hyperfinite subfactors have Bisch-Jones' Fuss-Catalan $A_3 * A_4$ standard invariant at index $3+\sqrt{5}$?

- $A_3 * A_4$ and $A_2 * T_2$ are not amenable [Pop94, HI98].

Standard invariants at index $3+\sqrt{5}$

Conjecture (Morrison-P (2012) [MP14a])

At $3+\sqrt{5}$, we have only the following standard invariants:

name	Principal graphs	#	Existence/Uniqueness
4442		1	[MP15, MP14a], Izumi
$3^{\mathbb{Z}/2\times\mathbb{Z}/2}$		1	Izumi, [MP15]
$3^{\mathbb{Z}/4}$		2	Izumi, [PP13]
$2D2$		2	Izumi, [MP14a]
$A_3 \otimes A_4$		1	$\otimes$, [Liu13, IMP13]
fish 2		2	BH, [Liu13, IMP13]
fish 3		2	[IMP13, Liu13]
$A_3 * A_4$		2	[BJ97],
A_∞		1	[Pop93]

- 1-supertransitive case known by [Liu13, IMP13, LMP15]

Methods to push classification results further

Enumeration:

- 1-supertransitive classification to $6\frac{1}{5}$ [LMP15]
- New high-tech graph pair enumerator, based on Brendan McKay's isomorph free enumeration by canonical construction paths [McK98]. Two independent implementations, same results. (Afzaly and Morrison-P)
- Popa's principal graph stability [Pop95, BP14]

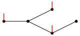
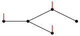
Obstructions:

- Number theory for stable weeds (Calegari-Guo) [CG15]
- Morrison's hexagon obstruction [Mor14]
- Souped up triple point obstruction [Pen15]

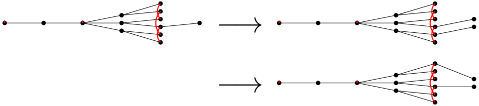
Why better combinatorics are needed

Three ways we produce redundant isomorphism classes of graphs:

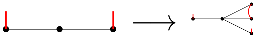
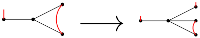
(1) Equivalent generating steps from same object give isomorphic results.

 and 

(2) Two inequivalent generating steps applied to the same object can yield isomorphic objects.



(3) Starting with two non-isomorphic objects and applying a generating step can result in isomorphic objects.

 and 

Problems fixed by McKay's isomorph-free enumeration [McK98]!

Popa's principal graph stability

Definition

We say $\Gamma_\pm$ is stable at depth n if every vertex at depth n connects to at most one vertex at depth $n+1$, no two vertices at depth n connect to the same vertex at depth $n+1$, and all edges between depths n and $n+1$ are simple.

Theorem (Popa [Pop95], Bigelow-P [BP14])

Suppose $A \subset B$ (finite index) has principal graphs (Γ_+, Γ_-).
Suppose that the truncation $\Gamma_\pm(n+1) \neq A_{n+2}$ and $\delta > 2$.

(1) If $\Gamma_\pm$ are stable at depth n, then $\Gamma_\pm$ are stable at depth k for all $k \geq n$, and $\Gamma_\pm$ are finite.

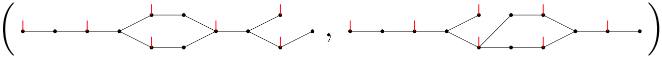
(2) If Γ_+ is stable at depths n and $n+1$, then $\Gamma_\pm$ are stable at depth $n+1$.

Part (2) uses the 1-click rotation in the planar algebra.

Stable weeds

Definition

A stable weed represents an infinite family of graph pairs obtained by translation and finite stable extension.

$\mathcal{C} =$ 

Theorem (Guo)

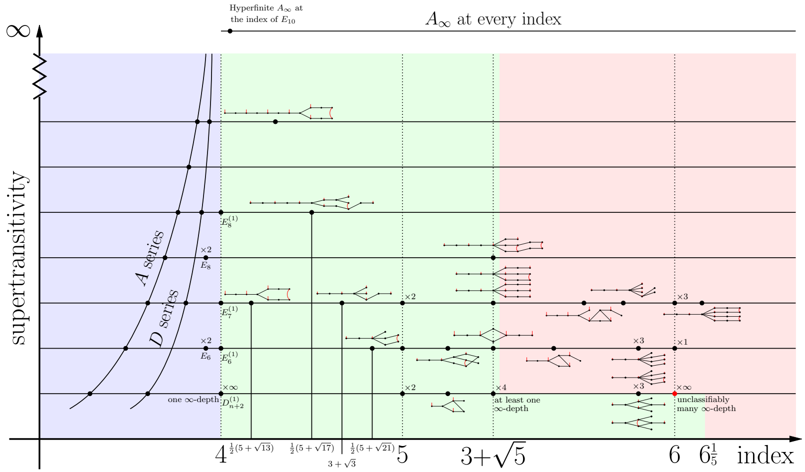
Let $\mathcal{S}_M$ be the class of finite graphs satisfying:

1. all vertices have valence at most M, and
2. at most M vertices have valence > 2.

Then ignoring A_n, D_n, $A_n^{(1)}$, and $D_n^{(1)}$, only finitely many graphs in $\mathcal{S}_M$ have norm squared which is a cyclotomic integer.

- Result is effective for a given fixed stable weed [CG15].
- Calegari-Guo eliminate our troublesome cylinder $\mathcal{C}$ by hand.

Known small index subfactors, today



Theorem (Afzaly-Morrison-P)

The conjectures of Morrison-Peters (up to index $5\frac{1}{4} > 3+\sqrt{5}$) and Morrison-P hold.

Thank you for listening!

Slides available at

`http://www.math.ucla.edu/~dpenneys/PenneysUCB2015.pdf`

Articles pushing from index 5 to index $5\frac{1}{4}$:

- Morrison and Peters - Index in $(5, 3+\sqrt{5})$ - **Int. J. Math.** `MR3254427`
- Liu - Composites of A_3 and A_4 - **Adv. Math.** `arXiv:1308.5691`
- Liu - biprojections and virtual normalizers - **Trans. AMS** `arXiv:1308.5656`
- Morrison - Hexagon obstruction - **Bull. Lond. Math. Soc.** `MR3210716`
- Calegari and Guo - Number theory for stable weeds - `arXiv:1502.00035`

My recent such articles:

- with Bigelow - Spokes and jellyfish - **Math. Ann.** `MR3157990`
- with Morrison - Constructing spokes with 1-strand jellyfish - **Trans. AMS** `MR3314808`
- with Peters - Constructing spokes with 2-strand jellyfish - **Pacific Math. J.** `arXiv:1308.5197`
- with Izumi and Morrison - 1-supertransitive at $3+\sqrt{5}$ - **Canad. J. Math.** `arXiv:1308.5723`
- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - **Comm. Math. Phys.** `MR3306607`
- new obstruction - **Adv. Math.** `MR3311757`
- with Izumi, Morrison, Peters, and Snyder - index exactly 5 - **Bul. Lond. Math. Soc.** `arXiv:1406.2389`
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