Classifying small index subfactors
UC Berkeley extended probabilistic operator algebras seminar

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Where do subfactors come from?

Some examples include:

- Groups – from \( G \subset R \), we get \( R^G \subset R \) and \( R \subset R \rtimes G \).
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – \( \text{Rep}(U_q(\mathfrak{g})) \)
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- composites of known subfactors

However, there are certain possible infinite families without uniform constructions.

Remark

Just as von Neumann algebras come in pairs \((M, M')\), subfactors come in pairs \((A \subset B, B' \subset A')\).
Index for subfactors

Theorem (Jones [Jon83])
For a II$_1$-subfactor $A \subset B$,

$$[B : A] \in \left\{ 4 \cos^2 \left( \frac{\pi}{n} \right) \bigg| n = 3, 4, \ldots \right\} \cup [4, \infty].$$

Moreover, there exists a subfactor at each index.

Definition
The Jones tower of $A = A_0 \subset A_1 = B$ (finite index) is given by

$$A_0 \subset A_1 \subset A_2 \subset A_3 \subset \cdots$$

where $e_i$ is the projection in $B(L^2(A_i))$ with range $L^2(A_{i-1})$. 

Principal graphs

Let $\rho = A L^2(B)_B$.

**Definition**
The principal graph $\Gamma_+$ has one vertex for each isomorphism class of simple $A\alpha_A$ and $A\rho_B$. There are

$$\dim(\text{Hom}_{A-B}(\alpha\rho, \beta))$$

edges from $\alpha$ to $\beta$.
The dual principal graph $\Gamma_-$ is defined similarly using $B - B$ and $B - A$ bimodules.

- $\Gamma_{\pm}$ is pointed, where the base point is the trivial bimodule $A L^2(A)_A$, $B L^2(B)_B$ respectively.
- Duality is given by contragredient, which is always at the same depth, although duals at odd depths of $\Gamma_{\pm}$ are on $\Gamma_{\mp}$.

**Fact**
The dual graph of $A_0 \subset A_1$ is the principal graph of $A_1 \subset A_2$. 
Examples of principal graphs

- **index < 4**: $A_n, D_{2n}, E_6, E_8$. No $D_{odd}$ or $E_7$.
- **Graphs for $R \subset R \rtimes G$ obtained from $G$ and $\text{Rep}(G)$**.
  
  $$(\quad , \quad )$$

- **Haagerup 333**
  
  $$(\quad , \quad )$$

- **First graph is principal, second is dual principal**.
- **Leftmost vertex is the trivial bimodule**.
- **Red tags for duality (contragredient of bimodules)**.
- **Duality of odd vertices by depth and height**
The standard invariant: two towers of centralizer algebras

\[ P_{3,+} = A'_0 \cap A_3 \supset A'_1 \cap A_3 = P_{2,-} \]
\[ P_{2,+} = A'_0 \cap A_2 \supset A'_1 \cap A_2 = P_{1,-} \]
\[ P_{1,+} = A'_0 \cap A_1 \supset A'_1 \cap A_1 = P_{0,-} \]
\[ P_{0,+} = A'_0 \cap A_0 \]

These centralizer algebras are finite dimensional [Jon83], and they form a planar algebra [Jon99].
Popa’s reconstruction theorem

Popa axiomatized the standard invariant of a subfactor, and showed how to reconstruct a subfactor from an abstract standard invariant.

**Theorem (Popa [Pop94])**

Every (strongly) amenable standard invariant is realized by a unique subfactor of $R$ up to conjugacy.
Finite depth

**Definition**
If the principal graph is finite, then the subfactor and standard invariant are called **finite depth**.

**Example:** \( R \subset R \rtimes G \) for finite \( G \)
For \( G = S_3 \):

- Principal graph:

- Dual principal graph:

**Theorem (Ocneanu Rigidity)**
There are only finitely many standard invariants with the same finite principal graphs.
Supertransitivity

Definition
We say a principal graph is $n$-supertransitive if it begins with an initial segment consisting of the Coxeter-Dynkin diagram $A_{n+1}$, i.e., an initial segment with $n$ edges.

Examples

- ▶ is 1-supertransitive
- ▶ is 2-supertransitive
- ▶ is 3-supertransitive
Small index subfactor classification program

Steps of subfactor classifications:

1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])
For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

If we enumerate all graph pairs with norm at most $r$, we have found all principal graphs with index at most $r^2$. 
Known small index subfactors, 1994

- Haagerup’s partial classification to $3 + \sqrt{3}$
- Popa’s $A_\infty$ at all indices
Haagerup’s enumeration

**Theorem (Haagerup [Haa94])**

Any non $A_\infty$-standard invariant in the index range $(4, 3 + \sqrt{2})$ must have principal graphs a translation of one of

- $(\quad, \quad)$
- $(\quad, \quad)$
- $(\quad, \quad)$
- $(\quad, \quad)$

*Translation* means raising the supertransitivity of both graphs by the same even amount.

**Definition (Morrison-Snyder [MS12])**

A *vine* is a graph pair which represents an infinite family of graph pairs obtained by translation.
Main tools for Haagerup’s enumeration

Play associativity off of Ocneanu’s triple point obstruction.
- **Associativity**: graphs must be similar
- **Ocneanu’s triple point obstruction**: graphs must be different!

The consequence is a strong constraint.

**Example**

The following pairs are not allowed:

\[
\left( \begin{array}{c}
\begin{array}{c}
\text{graph 1}
\end{array}
\end{array} \right) \text{ and } \left( \begin{array}{c}
\begin{array}{c}
\text{graph 2}
\end{array}
\end{array} \right)
\]

They must be paired with each other:

\[
\left( \begin{array}{c}
\begin{array}{c}
\text{graph 1}
\end{array}
\end{array} \right) \text{ and } \left( \begin{array}{c}
\begin{array}{c}
\text{graph 2}
\end{array}
\end{array} \right)
\]
Known small index subfactors, 2007

- Asaeda-Yasuda eliminate Haagerup vine
- Bisch eliminates Hexagon vine
- Bisch-Nicoara-Popa’s continuous family with same standard invariant at index 6
Known small index subfactors, 2011

- Extended Haagerup
- Classification to index 5 (Izumi, Jones, Morrison, P, Peters, Snyder, Tener)
Weeds and vines

The classification to index 5 introduced weeds and vines.

Definition
A weed is a graph pair which represents an infinite family of graph pairs obtained by translation and extension.
An extension of a graph pair adds new vertices and edges at strictly greater depths than the maximum depth of any vertex in the original pair.

\[ \mathcal{F} = \left( \text{graph 1}, \text{graph 2} \right) \]

Using weeds allows us to bundle hard cases together, ensuring the enumerator terminates.
We can uniformly treat vines using number theory, based on the following theorem inspired by Asaeda-Yasuda [AY09]:

**Theorem (Calegari-Morrison-Snyder [CMS11])**

For a fixed vine $\mathcal{V}$, there is an effective (computable) constant $R(\mathcal{V})$ such that any $n$-translate with $n > R(\mathcal{V})$ has norm squared which is not a cyclotomic integer.

**Theorem [CG94, ENO05]**

The index of a finite depth subfactor (which is equal to the norm squared of the principal graph) must be a cyclotomic integer.
Why do we care about index $3 + \sqrt{5}$?

- Standard invariants at index 4 are completely classified.
  - $\mathbb{Z}/2 \ast \mathbb{Z}/2 = D_\infty$ is amenable
- Standard invariants at index 6 are wild.
  - There is (at least) one standard invariant for every normal subgroup of the modular group $\mathbb{Z}/2 \ast \mathbb{Z}/3 = \text{PSL}(2, \mathbb{Z})$
  - There are unclassifiably many distinct hyperfinite subfactors with standard invariant $A_3 \ast D_4$ (Brothier-Vaes [BV13])
- $4 = 2 \times 2$ and $6 = 2 \times 3$ are composite indices, as is $3 + \sqrt{5} = 2\tau^2$ where $\tau = \frac{1+\sqrt{5}}{2}$.
**Conjecture (Morrison-Peters (2012) [MP14b])**

There are exactly two non-$A_\infty$ standard invariants in the index range $(5, 3 + \sqrt{5})$:

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
<th>Index</th>
<th>Existence, Uniqueness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(2)_5$</td>
<td>((\begin{array}{c} \rightarrow \ \downarrow \end{array}), (\begin{array}{c} \rightarrow \ \downarrow \end{array}))</td>
<td>5.04892</td>
<td>[Wen90], [MP14b]</td>
</tr>
<tr>
<td>$SU(3)_4$</td>
<td>((\begin{array}{c} \rightarrow \ \downarrow \end{array}), (\begin{array}{c} \rightarrow \ \downarrow \end{array}))</td>
<td>5.04892</td>
<td>[Wen88], [MP14b]</td>
</tr>
</tbody>
</table>

**Theorem (Morrison-Peters (2012) [MP14b])**

There is exactly one 1-supertransitive subfactor in the index range $(5, 3 + \sqrt{5})$.
Brothier-Vaes unclassifiably many subfactors with standard invariant $A_3 \ast D_4$ at index 6

Liu classified composite standard invariants from $A_3$ and $A_4$

1-supertransitive to index $6\frac{1}{5}$ (Liu-Morrison-P) [LMP15]
1-supertransitive subfactors at index $3 + \sqrt{5}$

Theorem (Liu [Liu13], partial proof by [IMP13])

There are exactly seven 1-supertransitive standard invariants with index $3 + \sqrt{5}$:

- $(\begin{smallmatrix} \text{self-dual} \\ \end{smallmatrix})$
- $(\begin{smallmatrix} \text{and its dual} \\ \end{smallmatrix})$
- $(\begin{smallmatrix} \text{and its dual} \\ \end{smallmatrix})$
- $(\begin{smallmatrix} \text{and its dual (A}_3 \ast A_4) \\ \end{smallmatrix})$

These are all the standard invariants of composed inclusions of $A_3$ and $A_4$ subfactors.

Open question

How many hyperfinite subfactors have Bisch-Jones’ Fuss-Catalan $A_3 \ast A_4$ standard invariant at index $3 + \sqrt{5}$?

- $A_3 \ast A_4$ and $A_2 \ast T_2$ are not amenable [Pop94, HI98].
Standard invariants at index $3 + \sqrt{5}$

**Conjecture (Morrison-P (2012) [MP14a])**

At $3 + \sqrt{5}$, we have only the following standard invariants:

<table>
<thead>
<tr>
<th>name</th>
<th>Principal graphs</th>
<th>#</th>
<th>Existence/Uniqueness</th>
</tr>
</thead>
<tbody>
<tr>
<td>4442</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>1</td>
<td>[MP15, MP14a], Izumi</td>
</tr>
<tr>
<td>$3\mathbb{Z}/2 \times \mathbb{Z}/2$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>1</td>
<td>Izumi, [MP15]</td>
</tr>
<tr>
<td>$3\mathbb{Z}/4$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>2</td>
<td>Izumi, [PP13]</td>
</tr>
<tr>
<td>$2D2$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>2</td>
<td>Izumi, [MP14a]</td>
</tr>
<tr>
<td>$A_3 \otimes A_4$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>1</td>
<td>$\otimes$, [Liu13, IMP13]</td>
</tr>
<tr>
<td>fish 2</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>2</td>
<td>BH, [Liu13, IMP13]</td>
</tr>
<tr>
<td>fish 3</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>2</td>
<td>[IMP13, Liu13]</td>
</tr>
<tr>
<td>$A_3 \ast A_4$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>2</td>
<td>[BJ97],</td>
</tr>
<tr>
<td>$A_\infty$</td>
<td>$\begin{array}{c} \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \ \text{\textbullet}\text{\textbullet}\text{\textbullet}\text{\textbullet} \end{array}$</td>
<td>1</td>
<td>[Pop93]</td>
</tr>
</tbody>
</table>

- 1-supertransitive case known by [Liu13, IMP13, LMP15]
Methods to push classification results further

Enumeration:
- 1-supertransitive classification to $6\frac{1}{5}$ [LMP15]
- New high-tech graph pair enumerator, based on Brendan McKay’s isomorph free enumeration by canonical construction paths [McK98]. Two independent implementations, same results. (Afzaly and Morrison-P)
- Popa’s principal graph stability [Pop95, BP14]

Obstructions:
- Number theory for stable weeds (Calegari-Guo) [CG15]
- Morrison’s hexagon obstruction [Mor14]
- Souped up triple point obstruction [Pen15]
Why better combinatorics are needed

Three ways we produce redundant isomorphism classes of graphs:

1. Equivalent generating steps from same object give isomorphic results.

\[ \begin{array}{c}
\begin{array}{c}
\text{and}
\end{array}
\end{array} \]

2. Two inequivalent generating steps applied to the same object can yield isomorphic objects.

\[ \begin{array}{c}
\begin{array}{c}
\text{and}
\end{array}
\end{array} \]

3. Starting with two non-isomorphic objects and applying a generating step can result in isomorphic objects.

Problems fixed by McKay’s isomorph-free enumeration [McK98]!
Popa’s principal graph stability

Definition
We say $\Gamma_\pm$ is stable at depth $n$ if every vertex at depth $n$ connects to at most one vertex at depth $n + 1$, no two vertices at depth $n$ connect to the same vertex at depth $n + 1$, and all edges between depths $n$ and $n + 1$ are simple.

Theorem (Popa [Pop95], Bigelow-P [BP14])
Suppose $A \subset B$ (finite index) has principal graphs $(\Gamma_+, \Gamma_-)$. Suppose that the truncation $\Gamma_\pm(n + 1) \neq A_{n+2}$ and $\delta > 2$.

1. If $\Gamma_\pm$ are stable at depth $n$, then $\Gamma_\pm$ are stable at depth $k$ for all $k \geq n$, and $\Gamma_\pm$ are finite.

2. If $\Gamma_+$ is stable at depths $n$ and $n + 1$, then $\Gamma_\pm$ are stable at depth $n + 1$.

Part (2) uses the 1-click rotation in the planar algebra.
Stable weeds

Definition
A stable weed represents an infinite family of graph pairs obtained by translation and finite stable extension.

\[ C = \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) \]

Theorem (Guo)
Let \( S_M \) be the class of finite graphs satisfying:
1. all vertices have valence at most \( M \), and
2. at most \( M \) vertices have valence \( > 2 \).

Then ignoring \( A_n, D_n, A_n^{(1)}, \) and \( D_n^{(1)} \), only finitely many graphs in \( S_M \) have norm squared which is a cyclotomic integer.

- Result is effective for a given fixed stable weed [CG15].
- Calegari-Guo eliminate our troublesome cylinder \( C \) by hand.
Known small index subfactors, today

Theorem (Afzaly-Morrison-P)

The conjectures of Morrison-Peters (up to index $5\frac{1}{4} > 3 + \sqrt{5}$) and Morrison-P hold.
Thank you for listening!

Slides available at

Articles pushing from index 5 to index $5\frac{1}{4}$:

- Morrison and Peters - Index in $(5, 3 + \sqrt{5})$ - Int. J. Math. MR3254427
- Liu - Biprojections and virtual normalizers - Trans. AMS arXiv:1308.5656

My recent such articles:

- with Morrison - Constructing spokes with 1-strand jellyfish - Trans. AMS MR3314808
- with Liu and Morrison - 1-supertransitive below $6\frac{1}{5}$ - Comm. Math. Phys. MR3306607
- with Morrison - 2-supertransitive at index $3 + \sqrt{5}$ - Submitted arXiv:1406.3401
- with Afzaly and Morrison - The classification of subfactors with index less than $5\frac{1}{4}$ - Coming very soon!


Masaki Izumi, Scott Morrison, and David Penneys, *Quotients of \(A_2 \ast T_2\)*, 2013, accepted in the *Canadian Journal of Mathematics* March 2015, extended version available as “Fusion categories between \(C \boxtimes D\) and \(C \ast D\)” at arXiv:1308.5723.


