Planar structure for inclusions of finite von Neumann algebras

by

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Abstract

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This dissertation consists of three self-contained papers from my graduate work at UC Berkeley. The chapters increase in complexity from the annular Temperley-Lieb category to strongly Markov inclusions of finite von Neumann algebras to infinite index II_1-subfactors.

In Chapter 2, we discuss how two copies of the cyclic category generate the annular Temperley-Lieb category. In the process, we give a presentation of the annular Temperley-Lieb category via generators and relations, and we see the cyclic category evolve from the simplicial and semi-simplicial categories.

Chapter 3 is joint work with Vaughan F. R. Jones. First, we define a canonical planar $*$-algebra associated to a strongly Markov inclusion of finite von Neumann algebras (the notion of such an inclusion is defined within). Second, we show for an inclusion of finite dimensional C^*-algebras with the Markov trace, the canonical planar algebra is isomorphic to the graph planar algebra of the Bratteli diagram of the inclusion. We use this fact to show that a subfactor planar algebra embeds into the graph planar algebra of its principal graph.

In Chapter 4, we expand upon Burns' work on rotations for infinite index II_1-subfactors. We start with a II_1-factor bimodule, and we construct a tower of centralizer algebras and a sequence of central L^2-vectors. In the finite index setting, the centralizer algebras and central L^2-vectors agree, but in the infinite index setting, these spaces can differ dramatically. We develop planar calculi for both sequences which are compatible. Interestingly, we obtain planar structure without Jones' basic construction or the resulting Jones projections! We also generalize Burns work on extremality and the existence of rotations to the bimodule setting, and we recover his main theorem. Along the way, we prove some results about relative tensor products of extended positive cones, and we give an example of an infinite index subfactor with finite dimensional higher relative commutants.

Contents

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Chapter 1

Introduction

Finite index subfactors

Mathematicians are taxonomists; we classify species of mathematical objects into types. Herein, the species are factors, von Neumann algebras with trivial centers, first defined by von Neumann in his study of quantum mechanics. Murray and von Neumann classified factors into three types, and constructed examples of each. All factors in this subsection are type II_1.

Sometimes distinct species share common traits. Fields and II_1-factors are algebraically simple, so we study maps in these categories by studying inclusions, i.e., subfields or subfactors. Nakamura and Takeda strengthened this connection with their Galois correspondence for the intermediate subfactor lattice for $M \subset M \rtimes G$ for a finite group G [NT60a, NT60b]. Hence some refer to subfactor theory as "noncommutative Galois theory."

In his pioneering paper [Jon83], Jones defined an index for a subfactor $M_0 \subset M_1$, showed

$$[M_1 \colon M_0] \in \left\{4\cos^2(\pi/n)\middle| n = 3, 4, 5, \dots\right\} \cup [4, \infty],$$

and constructed an example with each allowed index. To do so, he used the "basic construction" which constructs a tower of factors $M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$.The subfactors in this subsection are assumed to be finite index.

Just as topologists study a complicated topological space by its homology groups, we study a subfactor by its standard invariant, two sequences of finite dimensional C^*-algebras $P_{n,+} = M_0' \cap M_n$ and $P_{n,-} = M_1' \cap M_{n+1}$ [Jon83, Jon86]. The standard invariant has been axiomatized in three similar ways, each emphasizing slightly different structure: Ocneanu's paragroups [Ocn88, EK98], Popa's λ-lattices [Pop95], and Jones' planar algebras [Jon99]. Given a standard invariant $P_\bullet$, one can construct a subfactor whose standard invariant is $P_\bullet$ [Pop95, GJS07].

The rich structure of a planar algebra provides connections between subfactor theory, combinatorics, quantum algebra, and tensor categories. Given a subfactor $N \subset M$, its planar algebra $P_\bullet$ encodes two simpler invariants: the index, and the principal graphs, which are bipartite induction-restriction graphs associated to the representation theory of the subfactor. The two "even parts" of $P_\bullet$ form two C^*-tensor categories of $N-N$ bimodules and $M-M$ bimodules respectively. If there are only finitely many isomorphism classes of such bimodules, the subfactor is called finite depth, and the "even parts" are fusion categories [ENO05]. In this case, the two fusion categories are Morita equivalent [Müg03] via the two "odd parts" of $P_\bullet$, which are module categories of $N-M$ and $M-N$ bimodules.

Subfactors and groups also share traits. For an outer action of a finite group G on a factor M and a subgroup $H \subset G$, the planar algebra of the fixed point subfactor $M^G \subset M^H$ encodes the induction-restriction data of $H \subset G$. If H is trivial, one "even part" of $P_\bullet$ is the fusion category of representations of G. This also works for actions of quantum groups.

Jones proved that every finite group has a unique outer action on the hyperfinite II_1-factor R [Jon80]. Popa extended this result in his classification of amenable subfactors [Pop94] where he shows that each amenable standard invariant has a unique "action" on R.

Infinite index subfactors

Some finite index results generalize to infinite index subfactors, such as discrete, irreducible, "depth 2" subfactors correspond to outer (cocycle) actions of Kac algebras [HO89, EN96], and the classical Galois correspondence still holds for outer actions of infinite discrete groups and minimal actions of compact groups [ILP98]. We ask:

Question. *What is a suitable standard invariant for infinite index subfactors?*

There are several candidates for the standard invariant, each with its pros and cons. For example, we could take the towers $P_{n,\pm}$ as in the introduction since Enock and Nest showed

$$M_i' \cap M_j \cong M_{i+2}' \cap M_{j+2} \text{ for all } i, j \geq 0$$

in [EN96]. In his Ph.D. thesis [Bur03], Burns studied rotations and extremality for infinite index subfactors, and he initiated the search for planar structure. He crucially observed that for finite index, the centralizer algebras $M_0' \cap M_n$ and the central L^2-vectors

$$M_0' \cap L^2(M_n) = \left\{\xi \in L^2(M_n) \middle| x\xi = \xi x \text{ for all } x \in M_0\right\}$$

coincide. As this is no longer true for infinite index, he focused on the spaces $M_0' \cap L^2(M_n)$, and he showed $M_0 \subset M_1$ is (approximately) extremal if and only if a (non-)unitary rotation operator exists on the $M_0' \cap L^2(M_n)$.

1.1 Chapter synopses

This dissertation consists of three self-contained papers from my graduate work at UC Berkeley. The chapters increase in complexity from the annular Temperley-Lieb category to inclusions of finite von Neumann algebras to infinite index II_1-subfactors.

Chapter 2: A cyclic approach to the annular Temperley-Lieb category

This paper was published in J. Knot Theory and its Ramifications [Pen12a]. Its abstract is as follows:

In [Jon00], Jones found two copies of the cyclic category $\mathsf{c}\Delta$ in the annular Temperley-Lieb category Atl. We give an abstract presentation of Atl to discuss how these two copies of $\mathsf{c}\Delta$ generate Atl together with the coupling constants and the coupling relations. We then discuss modules over the annular category and homologies of such modules, the latter of which arises from the cyclic viewpoint.

Chapter 3: The embedding theorem for finite depth subfactor planar algebras

This joint paper with Vaughan F. R. Jones was published in Quantum Topology [JP11]. Its abstract is as follows:

We define a canonical planar $*$-algebra from a strongly Markov inclusion of finite von Neumann algebras. In the case of a connected unital inclusion of finite dimensional C*-algebras with the Markov trace, we show this planar algebra is isomorphic to the bipartite graph planar algebra of the Bratteli diagram of the inclusion. Finally, we show that a finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.

Chapter 4: A planar calculus for infinite index subfactors

This paper was accepted to Communications in Mathematical Physics on May 8, 2012; it can be found at `arXiv:1110.3504` [Pen12b]. Its abstract is as follows:

We develop an analog of Jones' planar calculus for II_1-factor bimodules with arbitrary left and right von Neumann dimension. We generalize to bimodules Burns' results on rotations and extremality for infinite index subfactors. These results are obtained without Jones' basic construction and the resulting Jones projections.

Chapter 2

A cyclic approach to the annular Temperley-Lieb category

2.1 Introduction

The Temperley-Lieb algebras have been studied extensively beginning with Temperley and Lieb's first paper in statistical mechanics regarding hydrogen bonds in ice-type lattices [TL71]. Since, these algebras have been instrumental in many areas of mathematics, including subfactors [Jon83] and knot theory [Jon85]. The well known diagrammatic representation of these algebras was introduced by Kauffman in [Kau87] in his skein theoretic definition of the Jones polynomial. From these diagrams, we get the Temperley-Lieb category whose objects are n points on a line, morphisms are diagrams with non-intersecting strings, and composition is stacking tangles vertically (we read bottom to top).

Historically, the (affine/annular) Temperley-Lieb algebras have been presented as quotients of the (affine) Hecke algegras [Jon94]. Graham and Lehrer define cellular structures for these algebras in [GL96], and they give the representation theory for affine Temperley-Lieb in [GL98]. Jones' definition of the annular Temperley-Lieb category (see [Jon99], [Jon01]), which we will denote Atl, differs slightly Graham and Lehrer's. First, Atl-tangles have a checkerboard shading, so each disk has an even number of boundary points. Second, the rotation is periodic in Atl, similar to the rotation in Connes' cyclic category $\mathsf{c}\Delta$, studied by Connes [Con83], [Con94], Loday and Quillen [LQ83], [Lod98], and Tsygan [Tsy83]. Jones found a connection between Atl and $\mathsf{c}\Delta$ in [Jon00], and raised the question we now address: how does Atl arise from the interaction of two copies of the cyclic category?

In answering this question, we see Atl evolve from simple categories. The opposite of the simplicial category $\mathsf{s}\Delta^{\mathsf{op}}$ (see 2.5.4) has a well known pictorial representation much like the Temperley-Lieb category: objects are $2n+2$ points on a line, morphisms are rectangular planar tangles with only shaded caps and unshaded cups, and composition is stacking. In fact, these diagrams closely resemble the string diagrams arising from an adjoint functor pair.

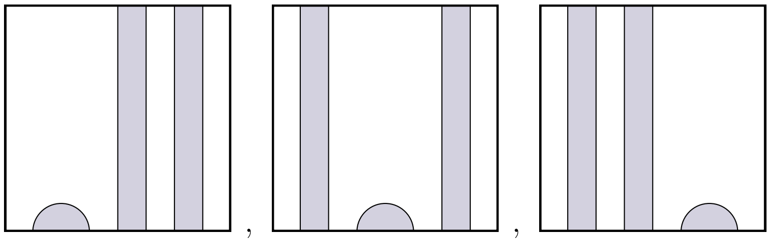

Figure 2.1: Face maps $d_0, d_1, d_2 \colon [2] \to [1]$

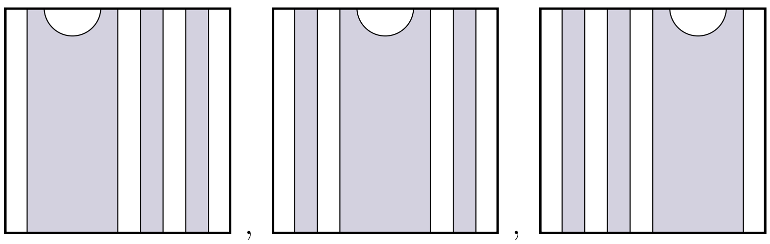

Figure 2.2: Degeneracies $s_0, s_1, s_2 \colon [2] \to [3]$

An asymmetry is present in the above tangles: all shaded regions can be "capped" by applying a face map, but not every unshaded region can be "cupped" by applying a degeneracy. This asymmetry can be corrected by closing the rectangular tangles into annuli, still enforcing the same shading requirements. Jones showed the resulting category is isomorphic to $\mathsf{c}\Delta^{\mathsf{op}}$ in [Jon00]. Of course the category with the reverse shading is also isomorphic to $\mathsf{c}\Delta$ (and $\mathsf{c}\Delta^{\mathsf{op}}$), and these two subcategories generate Atl.

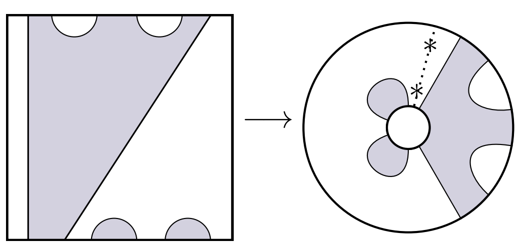

Figure 2.3: Closing up rectangular tangles into annuli

Outline

In Section 2.2, we will define Atl and offer candidates for generators and relations. We will then prove some uniqueness results which will be crucial to our approach. In Section 2.3, we will take these candidates and define an abstract category $\mathsf{a}\Delta$, the annular category, via generators and relations. We then prove existence of a standard form for words. In Section 2.4, we prove Theorem 2.4.8, which says there is an isomorphism of involutive categories $\mathsf{Atl} \cong \mathsf{a}\Delta$ (the isomorphism preserves an involution).

After we have our description of Atl in terms of abstract generators and relations, we recover the result of Jones in [Jon00] in 2.5, i.e. two isomorphisms from $\mathsf{c}\Delta^{\mathsf{op}}$ to subcategories $\mathsf{cAtl}^{\pm}$ of Atl. After a note on augmentation of the cyclic category in 2.5, we prove the main result of the paper, Theorem 2.5.27, which shows Atl is a quotient of the pushout of augmented copies of $\mathsf{c}\Delta$ and $\mathsf{c}\Delta^{\mathsf{op}}$ over a groupoid T of finite cyclic groups:

$$\begin{array}{ccccc} \mathsf{T} & \longrightarrow & \widetilde{\mathsf{c}\Delta^{\mathsf{op}}} & & \\ \downarrow & & \downarrow & & \\ \widetilde{\mathsf{c}\Delta} & \longrightarrow & \mathsf{PO} & & \\ & & & \searrow & \\ & & & & \mathsf{Atl}. \end{array}$$

In Section 2.6, we define the notion of an annular object in a category C. As $\mathsf{c}\Delta^{\mathsf{op}}$ lives inside $\mathsf{a}\Delta$ (in two ways), we will have notions of Hochschild and cyclic homology of annular objects in abelian categories. We define these notions and give some easy results in 2.6.

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2.2 The Category Atl

Notation 2.2.1. All categories will be denoted by capital letters in the following sans-serif font: ABC... The categories we discuss will be small, and we will write $X \in \mathsf{A}$ to denote that $X \in \mathrm{Ob}(\mathsf{A})$, the set of objects of A. We will write $\mathsf{A}(X, Y)$ to denote the set of morphisms $\varphi\colon X \to Y$ where $X, Y \in \mathsf{A}$, and we will write $\mathrm{Mor}(\mathsf{A})$ to denote the collection of all morphisms in A. In the sequel, objects of our categories will be the symbols $[n]$ for $n \in \mathbb{Z}_{\geq 0} \cup \{0\pm, \pm\}$. For simplicity and aesthetics, we will write $\mathsf{A}(m, n)$ instead of $\mathsf{A}([m], [n])$.

Definition 2.2.2. A category A is called involutive if for all $X, Y \in \mathsf{A}$, there is a map $*\colon \mathsf{A}(X, Y) \to \mathsf{A}(Y, X)$ called the involution such that

(1) $\mathrm{id}_X^* = \mathrm{id}_X$ for all $X \in \mathsf{A}$,

(2) $(T^*)^* = T$ for all $T \in \mathsf{A}(X, Y)$, and

(3) for all $X, Y, Z \in \mathsf{A}$ and all $T \in \mathsf{A}(X, Y)$ and $S \in \mathsf{A}(Y, Z)$, $(S \circ T)^* = T^* \circ S^*$.

In other words, there is a contravariant functor $*\colon \mathsf{A} \to \mathsf{A}$ of period two which fixes all objects.

Definition 2.2.3. Suppose A and B are categories and $F\colon \mathsf{A} \to \mathsf{B}$ is a functor.

(1) F is called an isomorphism of categories if there is a functor $G\colon \mathsf{B} \to \mathsf{A}$ such that $F \circ G = \mathrm{id}_{\mathsf{B}}$ and $G \circ F = \mathrm{id}_{\mathsf{A}}$, the identity functors. In this case, we say categories A and B are isomorphic, denoted $\mathsf{A} \cong \mathsf{B}$.

(2) If A and B are involutive, we say F is involutive if it preserves the involution, i.e. $F(\varphi^*) = \varphi^*$ for all $\varphi \in \mathsf{A}(X, Y)$ for all $X, Y \in \mathsf{A}$.

(3) An isomorphism of involutive categories is an involutive isomorphism of said categories.

Remark 2.2.4. It is clear that if A is involutive, then $\mathsf{A} \cong \mathsf{A}^{\mathsf{op}}$.

Annular Tangles

We provide a definition of an annular (m, n)-tangle which is a fusion of the ideas in [Jon99] and [KS04].

Definition 2.2.5. An annular (m, n)-pretangle for $m, n \in \mathbb{Z}_{\geq 0}$ consists of the following data:

(1) The closed unit disk D in $\mathbb{C}$,

(2) The skeleton of T, denoted $S(T)$, consisting of:

  (a) the boundary of D, denoted $D_0(T)$,

  (b) the closed disk D_1 of radius $1/4$ in $\mathbb{C}$, whose boundary is denoted $D_1(T)$,

  (c) $2m$, respectively $2n$, distinct marked points on $D_1(T)$, respectively $D_0(T)$, called the boundary points of $D_i(T)$ for $i = 0, 1$. Usually we will call the boundary points of $D_0(T)$ external boundary points of T and the boundary points of $D_1(T)$ internal boundary points.

  (d) inside D, but outside D_1, there is a finite set of disjointly smoothly embedded curves called strings which are either closed curves, called loops, or whose boundaries are marked points of the $D_i(T)$'s and the strings meet each $D_i(T)$ transversally, $i = 0, 1$. Each marked point on $D_i(T)$, $i = 0, 1$ meets exactly one string.

(3) The connected components of $D \backslash S(T)$ are called the regions of T and are either shaded or unshaded so that regions whose closures meet have different shadings.

Definition: If there are boundary points of $D_i(T)$, then an interval of $D_i(T)$, $i = 0, 1$, is a connected arc on $D_i(T)$ between two boundary points of $D_i(T)$. A simple interval of $D_i(T)$, $i = 0, 1$, is an interval of $D_i(T)$ in T which touches only two (adjacent) boundary points. If there are no boundary points of $D_i(T)$, then a (simple) interval of $D_i(T)$ is $D_i(T)$ itself.

(4) For each $D_i(T)$, $i = 0, 1$, there is a distinguished simple interval of $D_i(T)$ denoted $*_i(T)$ whose interior meets an unshaded region. Starting at $*_i(T)$ on $D_i(T)$, we order the marked points of $D_i(T)$ clockwise. This numbering, along with the shading, induces an orientation on the pre-tangle.

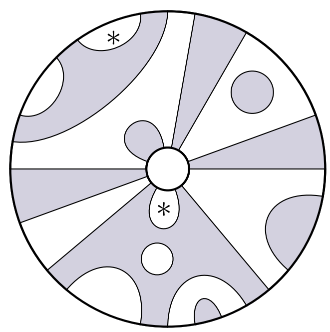


Figure 2.4: An example of an annular tangle

Remarks 2.2.6. (1) If $m = 0$, there are two kinds of annular $(0, n)$-pretangles depending on whether the region meeting $D_1(T)$ is unshaded or shaded. If the region meeting $D_1(T)$ is unshaded, we call T an annular $(0+, n)$-pretangle, and if the region is shaded, we call T an annular $(0-, n)$-pretangle. Likewise, when $n = 0$, there are two kinds of annular $(m, 0)$-pretangles. If the region meeting $D_0(T)$ is unshaded, we call T an annular $(m, 0+)$-pretangle, and if the region is shaded, we call T an annular $(m, 0-)$-pretangle. Additionally, we have annular $(0\pm, 0\pm)$-pretangles and annular $(0\pm, 0\mp)$-pretangles.

(2) Loops may be shaded or unshaded.

Definition 2.2.7. An annular (m, n)-tangle is an orientation-preserving diffeomorphism class of an annular (m, n)-pretangle for $m, n \in \mathbb{N} \cup \{0\pm\}$. The diffeomorphisms preserve (but do not necessarily fix!) D_0 and D_1.

Definition 2.2.8. Given an annular (m, n)-tangle T, and an annular (l, m)-tangle S, we define the annular (l, m)-tangle $T \circ S$ by isotoping S so that $D_0(S)$, the marked points of $D_0(S)$, and $*_0(S)$, coincide with $D_1(T)$, the marked points of $D_1(T)$, and $*_1(T)$ respectively. The strings may then be joined at $D_1(T)$ and smoothed, and $D_1(T)$ is removed to obtain $T \circ S$ whose diffeomorphism class only depends on those of T and S.

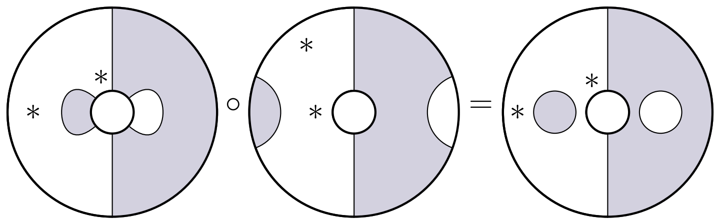


Figure 2.5: An example of composition of annular tangles

Definition 2.2.9. If T is an annular (m, n)-tangle, we define T^* to be the annular (n, m)-tangle obtained by reflecting T about the circle of radius $3/4$, which switches $D_i(T)$ and $*_i(T)$, $i = 0, 1$. Clearly $(T^*)^* = T$ and $(T \circ S)^* = S^* \circ T^*$ for composable S and T.

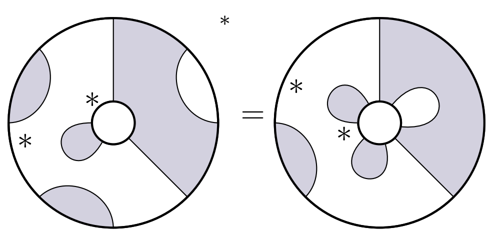


Figure 2.6: An example of the adjoint of an annular tangle

Definition 2.2.10. Let T be an annular (m, n)-tangle.

Caps: A cap of T is a string that connects two internal boundary points. The set of caps of T will be denoted $\text{caps}(T)$.

$\partial\Lambda$: If $\Lambda \in \text{caps}(T)$, there is a unique interval of $D_1(T)$, denoted $\partial\Lambda$, such that $\Lambda \cup \partial\Lambda$ is a closed loop (which is not smooth at two points) which does not contain D_1 in its interior. Using $\partial\Lambda$, the cap Λ inherits an orientation as $D_1(T)$ is oriented clockwise. Denote this orientation by an arrow on Λ.

Index: We define the cap index of Λ, denoted $\text{ind}(\Lambda)$, to be the number of the marked point to which the arrow points. The set of cap indices of T forms an increasing sequence, which we denote $\text{capind}(T)$.

$B(\Lambda)$: For $\Lambda \in \text{caps}(T)$, we let $B(\Lambda) = \{\Lambda' \in \text{caps}(T) | \partial\Lambda' \subseteq \partial\Lambda\}$, and we say an element $\Lambda' \in B(\Lambda)$ is bounded by Λ or that Λ bounds Λ'.

Definition 2.2.11. Let T be an annular (m, n)-tangle.

Cups: A cup V of T is a string that connects two external boundary points. The set of cups of T will be denoted $\text{cups}(T)$.

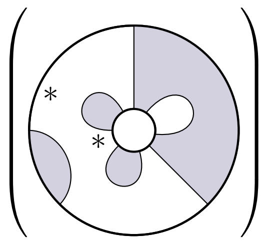
capind  $= \{1, 4, 7\}$

Figure 2.7: An example of cap indices

$\underline{\partial V}$: If $V \in \text{cups}(T)$, there is a unique interval of $D_0(T)$, denoted ∂V, such that $V \cup \partial V$ is a closed loop (which is not smooth at two points) which does not contain D_1 in its interior. Using ∂V, the cup V inherits an orientation as $D_0(T)$ is oriented clockwise. Denote this orientation by an arrow on V.

Index: We define the cup index of V, denoted $\text{ind}(V)$, to be the number of the marked point to which the arrow points. The set of cup indices of T forms an increasing sequence, which we denote $\text{cupind}(T)$.

$\underline{B(V)}$: For $V \in \text{cups}(T)$, we let $B(V) = \{V' \in \text{cups}(T) | \partial V' \subseteq \partial V\}$, and we say an element $V' \in B(V)$ is bounded by V or that V bounds V'.

Remark 2.2.12. Note $\text{capind}(T) = \text{cupind}(T^*)$ for all annular tangles T.

Definition 2.2.13. Suppose T is an annular (m, n)-tangle.

$\underline{\text{ts}(T)}$: A through string is a string of T which connects an internal boundary point of T to an external boundary point of T. The set of through strings is denoted $\text{ts}(T)$. Note that $|\text{ts}(T)| \in 2\mathbb{Z}_{\geq 0}$. We order $\text{ts}(T)$ clockwise starting at $*_0(T)$, so each through string of T has a number.

$\underline{\text{ts}_0(T)}$: Suppose T has a through string. Using $*_0(T)$ as our reference, we go counterclockwise along $D_0(T)$ to the first through string, which is denoted $\text{ts}_0(T)$. Note the number of $\text{ts}_0(T)$ is $|\text{ts}(T)|$.

$\underline{\text{ts}_1(T)}$: Suppose T has a through string. Using $*_1(T)$ as our reference, we go counterclockwise along $D_1(T)$ to the first through string, which is denoted $\text{ts}_1(T)$. We denote the number of $\text{ts}_1(T)$ by $\#\,\text{ts}_1(T)$.

$\underline{\text{rel}_*(T)}$: We define the relative star position of T, denoted $\text{rel}_*(T)$, as follows:

(1) Suppose T has an odd number k of non-contractible loops. Then there is a unique region R which touches both a non-contractible loop and $D_1(T)$. If R is unshaded, we define $\text{rel}_*(T)$ to be the symbol $\pm(k)$, and if R is shaded, we define $\text{rel}_*(T)$ to be the symbol $\mp(k)$. This notation signifies the shading switches from unshaded to shaded, respectively shaded to unshaded, as we read T from $D_1(T)$ to $D_0(T)$.

(2) Suppose T has an even number k of non-contractible loops. If $k = 0$, then there is a unique region R which touches both $D_0(T)$ and $D_1(T)$. If $k \geq 1$, then there is a unique region R which touches both a non-contractible loop and $D_1(T)$. If R is unshaded, we define $\mathrm{rel}_*(T)$ to be the symbol $+(k)$, and if R is shaded, we define $\mathrm{rel}_*(T)$ to be the symbol $-(k)$.

(3) Suppose T has a through string. We define

$$\mathrm{rel}_*(T) = \left\lfloor \frac{\#\, \mathrm{ts}_1(T)}{2} \right\rfloor \mod \left(\frac{|\, \mathrm{ts}(T)|}{2} \right) \in \left\{ 0, 1, \ldots, \frac{|\, \mathrm{ts}(T)|}{2} - 1 \right\}.$$

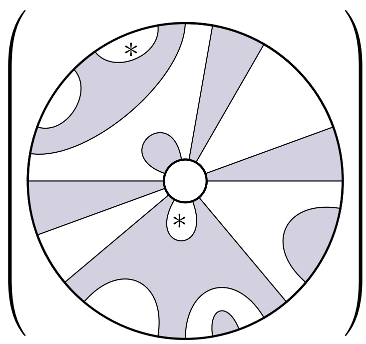
rel_*  $= 2$

Figure 2.8: An example of relative star position

"Generators and Relations" of Atl

Definition 2.2.14. Suppose T is an annular tangle. A loop of T is called contractible if it is contractible in $D \setminus D_1$. Otherwise it is called non-contractible.

Definition 2.2.15 (Atl Tangle)**.** An annular (m, n)-tangle T is called an Atl (m, n)-tangle if T has no contractible loops.

Definition 2.2.16. Let Atl be the following small category:

Objects: $[n]$ for $n \in \mathbb{N} \cup \{0\pm\}$

Morphisms: Given $m, n \in \mathbb{N} \cup \{0\pm\}$, $\mathsf{Atl}(m, n)$ is the set of all triples (T, c_+, c_-) where T is an Atl (m, n)-tangle and $c_+, c_- \in \mathbb{Z}_{\geq 0}$.

Composition: Given $(S, a_+, a_-) \in \mathsf{Atl}(m, n)$ and $(T, b_+, b_-) \in \mathsf{Atl}(l, m)$, we define $(S, a_+, a_-) \circ (T, b_+, b_-) \in \mathsf{Atl}(l, n)$ to be the triple (R, c_+, c_-) obtained as follows: let R_0 be the annular (l, n)-tangle $S \circ T$. Let d_+, respectively d_-, be the number of shaded, respectively unshaded, contractible loops. Let R be the Atl (l, n)-tangle obtained from R_0 by removing all contractible loops, and set $c_\pm = a_\pm + b_\pm + d_\pm$.

Remark 2.2.17. For simplicity and aesthetics, we write T for the morphism $(T, 0, 0) \in$ Mor(Atl).

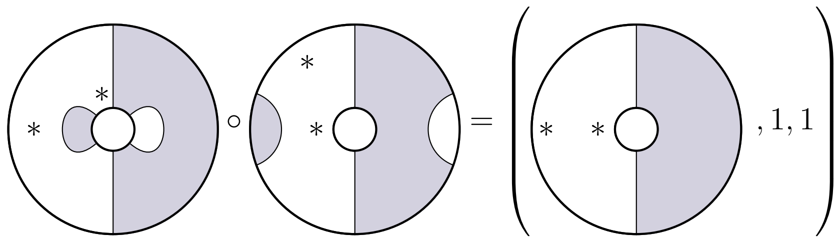


Figure 2.9: An example of composition in Atl

Definition 2.2.18. We give the following names to the following distinguished Atl (n, m)-tangles:

(A) Let a_1 be the only Atl $(1, 0+)$-tangle with no loops, and let a_2 be the only Atl $(1, 0-)$-tangle with no loops. For $n \geq 2$ and $i \in \{1, \ldots, 2n\}$, let a_i be the Atl $(n, n-1)$-tangle

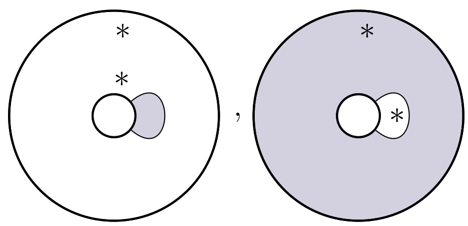


Figure 2.10: $a_1 \in \mathsf{Atl}(1, 0+)$ and $a_2 \in \mathsf{Atl}(1, 0-)$

whose i^{th} and $(i+1)^{\text{th}}$ (modulo $2n$) internal boundary point are joined by a string and all other internal boundary points are connected to external boundary points such that

(i) If $i = 1$, then the first external point is connected to the third internal point.

(ii) If $1 < i < 2n$, then the first external point is connected to the first internal point.

(iii) If $i = 2n$, then the first external point is connected to the $(2n-1)^{\text{th}}$ internal point.

(B) Let b_1 be the only Atl $(0+, 1)$-tangle with no loops, and let b_2 be the only Atl $(0-, 1)$-tangle with no loops. For $n \geq 1$ and $i \in \{1, \ldots, 2n+2\}$, let b_i be the Atl $(n, n+1)$-tangle whose i^{th} and $(i+1)^{\text{th}}$ (modulo $2n+2$) external boundary point are joined by a string and all other internal boundary points are connected to external boundary points such that

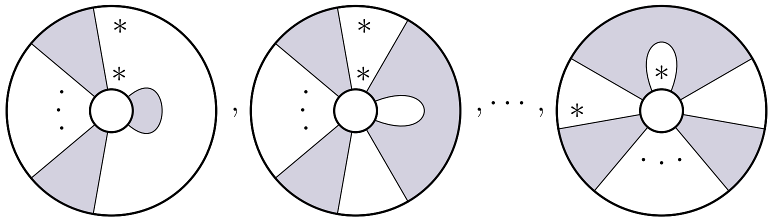


Figure 2.11: $a_1, a_2, \cdots, a_{2n} \in \mathsf{Atl}(n, n-1)$. (without the dots, $n = 3$)

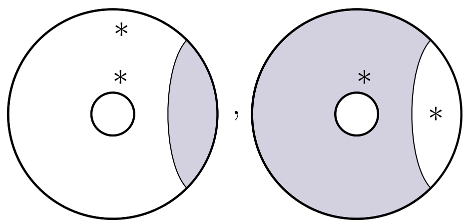


Figure 2.12: $b_1 \in \mathsf{Atl}(0+, 1)$ and $b_2 \in \mathsf{Atl}(0-, 1)$

(i) If $i = 1$, then the third external point is connected to the first internal point.

(ii) If $1 < i$, then the first external point is connected to the first internal point.

(iii) If $i = 2n + 2$, then the first internal point is connected to the $(2n + 1)^{\text{th}}$ external point.

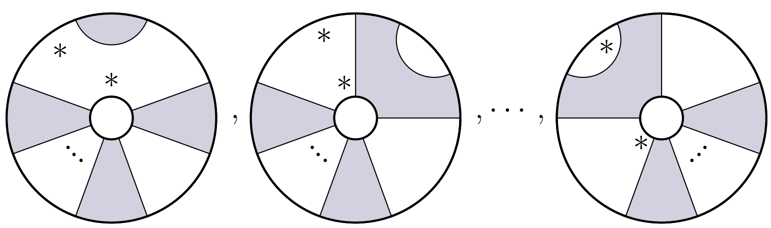


Figure 2.13: $b_1, b_2, \cdots, b_{2n+2} \in \mathsf{Atl}(n, n+1)$ (without the dots, $n = 3$)

(T) For $n = 1$, let t be the identity $(1, 1)$-tangle. For $n \geq 2$, let t be the Atl (n, n)-tangle where all internal points are connected to external point such that the third external point is connected to the first internal point.

Theorem 2.2.19. *The following relations hold in* Atl*:*

(1) $a_i a_j = a_{j-2} a_i$ *for* $i < j - 1$ *and* $(i, j) \neq (1, 2n)$*,*

(2) $b_i b_j = b_{j+2} b_i$ *for* $i \leq j$ *and* $(i, j) \neq (1, 2n + 2)$*,*

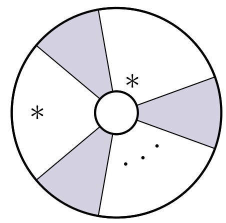


Figure 2.14: $t \in \mathsf{Atl}(n,n)$ (without the dots, $n = 3$)

(3) $t^n = \mathrm{id}_{[n]}$,

(4) $a_i t = t a_{i-2}$ *for* $i \geq 3$,

(5) $b_i t = t b_{i-2}$ *for* $i \geq 3$,

(6) $(\mathrm{id}_{[0+]}, 1, 0) = a_1 b_1 \in \mathsf{Atl}(0+, 0+)$ *and* $(\mathrm{id}_{[0+]}, 0, 1) = a_2 b_2 \in \mathsf{Atl}(0-, 0-)$. *If* $a_i b_j \in \mathsf{Atl}(n,n)$ *with* $n \geq 1$, *then*

$$a_i b_j = \begin{cases} t^{-1} & \text{if } (i,j) = (1, 2n+2) \\ b_{j-2} a_i & \text{if } i < j-1, (i,j) \neq (1, 2n+2) \\ \mathrm{id}_{[n]} & \text{if } i = j-1 \\ (\mathrm{id}_{[n]}, 1, 0) & \text{if } i = j \text{ and } i \text{ is odd} \\ (\mathrm{id}_{[n]}, 0, 1) & \text{if } i = j \text{ and } i \text{ is even} \\ \mathrm{id}_{[n]} & \text{if } i = j+1 \\ b_j a_{i-2} & \text{if } i > j+1, (i,j) \neq (2n+2, 1) \\ t & \text{if } (i,j) = (2n+2, 1) \end{cases}$$

(7) $(\mathrm{id}_{[n]}, 1, 0)$ *and* $(\mathrm{id}_{[n]}, 0, 1)$ *commute with all* $(T, c_+, c_-) \in \mathsf{Atl}(n,n)$ *where* $n \in \mathbb{N} \cup \{0\pm\}$.

Proof. These relations can be easily verified by drawing pictures. □

Involution and Tangle Type

Proposition 2.2.20. *The map* $*\colon \mathsf{Atl} \to \mathsf{Atl}$ *given by* $[n]^* = [n]$ *for all* $n \in \mathbb{N} \cup \{0\pm\}$ *and* $(T, c_+, c_-)^* = (T^*, c_+, c_-)$ *defines an involution on* Atl.

Corollary 2.2.21. *We have an isomorphism of categories* $\mathsf{Atl} \cong \mathsf{Atl}^{op}$.

Proposition 2.2.22. *The involution on* Atl *satisfies*

A/B: $a_i^* = b_i$ *for* $i = 1, 2$ *if* $a_1 \in \mathsf{Atl}(1, 0+)$ *and* $a_2 \in \mathsf{Atl}(1, 0-)$. *For* $n \geq 2$ *and* $a_i \in \mathsf{Atl}(n, n-1)$, *so* $i \in \{1, \ldots, 2n\}$, $a_i^* = b_i \in \mathsf{Atl}(n-1, n)$.

T: For $n \in \mathbb{N}$ *and* $t \in \mathsf{Atl}(n,n)$, $t^* = t^{-1}$.

D: For $n \in \mathbb{N} \cup \{0\pm\}$, $(\mathrm{id}_{[n]}, 1, 0)^* = (\mathrm{id}_{[n]}, 1, 0)$ *and* $(\mathrm{id}_{[n]}, 0, 1)^* = (\mathrm{id}_{[n]}, 0, 1)$.

Proof. Obvious. □

Definition 2.2.23. An Atl (m, n)-tangle T is said to be of

Type I: if T is either $\mathrm{id}_{[n]}$ for some $n \in \mathbb{N} \cup \{\pm 0\}$, or T has no cups, at least one cap, and no non-contractible loops, with the limitation on $*_0(T)$ that exactly one of the following occurs:

I-1: There are no through strings, so $*_0(T)$ is uniquely determined. Note that if $n = 0-$, then there is no $*_0(T)$.

I-2: There are through strings. Using $*_1(T)$ as our reference, we go counterclockwise to the first through string, and travel outward until we reach a marked point p of $D_0(T)$. The simple interval meeting p whose interior touches an unshaded region is $*_0(T)$.

Type II: if T has no cups or caps, so T is a power of the rotation (including the identity tangle) or an annular $(0, 0)$-tangle with k non-contractible loops (here we do not specify $0\pm$).

Type III: if T is either $\mathrm{id}_{[n]}$ for some $n \in \mathbb{N} \cup \{\pm 0\}$, or T has no caps, at least one cup, and no non-contractible loops, with the limitation on $*_1(T)$, that exactly one of the following occurs:

III-1: There are no through strings, so $*_1(T)$ is uniquely determined. Note that if $m = 0-$, then there is no $*_1(T)$.

III-2: There are through strings. Using $*_0(T)$ as our reference, we go counterclockwise to the first through string, and travel outward until we reach a marked point p of $D_1(T)$. The simple interval meeting p whose interior touches an unshaded region is $*_1(T)$.

Denote the set of all tangles of Type i by $\mathcal{T}_i$, and denote the set of all (m, n)-tangles of Type i by $\mathcal{T}_i(m, n)$ for $i \in \{I, II, III\}$.

Remark 2.2.24. Note that

(1) the a_i's are all Type I, and

(2) the b_i's are all Type III.

Notation 2.2.25. We will use the notation $s_+ = a_2 b_1 \in \mathsf{Atl}(0+, 0-)$ and $s_- = a_1 b_2 \in \mathsf{Atl}(0-, 0+)$.

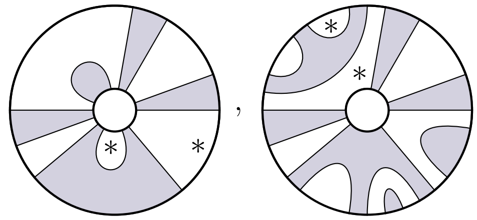


Figure 2.15: Examples of tangles of Types I and III

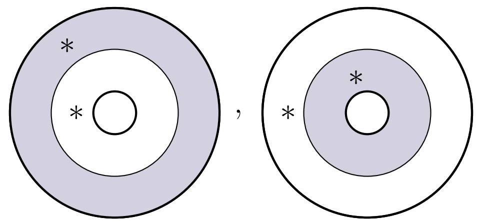


Figure 2.16: Type II tangles s_+, s_-

Remark 2.2.26. For the case $a_ib_j \colon [0] \to [0]$ (where we do not specify $\pm$), a suitable version of relation (6) reads

$$a_ib_j = \begin{cases} s_- & \text{if } (i,j) = (1,2) \\ (\mathrm{id}_{[0+]}, 1, 0) & \text{if } i = j = 1 \\ (\mathrm{id}_{[0-]}, 0, 1) & \text{if } i = j = 2 \\ s_+ & \text{if } (i,j) = (2,1). \end{cases}$$

Note that we replace $t^{\pm 1}$ with $s_\pm$, which supports Graham and Lehrer's reasoning that the rotation converges to the non-contractible loop as $n \to 0$ in [GL98].

Lemma 2.2.27. *Let $m, n \in \mathbb{N} \cup \{0\pm\}$. Types are related to the involution as follows:*

(1) $T \in \mathcal{T}_I(m,n)$ if and only if $T^* \in \mathcal{T}_{III}(m,n)$, and

(2) If $T \in \mathcal{T}_{II}(n,n)$, then $T^* \in \mathcal{T}_{II}(n,n)$.

Proof. Obvious. □

Proposition 2.2.28. *Let $m, n \in \mathbb{N} \cup \{0\pm\}$.*

Type I: Any $T \in \mathcal{T}_I(m,n)$ is uniquetly determined by $\mathrm{capind}(T)$. *Moreover,* $\mathrm{rel}_*(T) \in \{0, +(0), -(0)\}$.

Type II: Suppose $m = n \in \mathbb{N}$ or $m, n \in \{0+, 0-\}$. Any $T \in \mathcal{T}_{II}(m,n)$ is uniquely determined by $\mathrm{rel}_*(T)$.

Type III: Any $T \in \mathcal{T}_{III}(m,n)$ is uniquely determined by $\operatorname{cupind}(T)$. *Moreover,* $\operatorname{rel}_*(T) \in \{0, +(0), -(0)\}$.

Proof.

Type I: Suppose $T_1, T_2 \in \mathcal{T}_1(m,n)$ with $\operatorname{capind}(T_1) = \operatorname{capind}(T_2)$. If $\Lambda_i \in \operatorname{caps}(T_i)$ for $i = 1, 2$ with $\operatorname{ind}(\Lambda_1) = \operatorname{ind}(\Lambda_2)$, note that $|B(\Lambda_1)| = |B(\Lambda_2)|$, so the Λ_i's must end at the same points. Hence all caps of T_i start and end at the same points for $i = 1, 2$. Now note that all other points on $D_1(T_i)$ for $i = 1, 2$ (if there are any) are connected to through strings, and recall $*_0(T_i)$ is uniquely determined by $*_1(T_i)$ for $i = 1, 2$. Hence $T_1 = T_2$. The statement about $\operatorname{rel}_*(T)$ follows immediately from conditions (I-1) and (I-2).

Type II: Note that exactly one of the following occurs:

(1) $m = n$ and $T = \operatorname{id}_{[n]}$, in which case $\operatorname{rel}_*(T) \in \{0, +(0), -(0)\}$,

(2) $m = n$ and $T = t^k$ where $0 < k < n$, in which case $\operatorname{rel}_*(T) = k$,

(3) $m = n = 0\pm$ and $T = (s_\mp s_\pm)^k$ for some $k \in \mathbb{N}$, in which case $\operatorname{rel}_*(T) = \pm(2k)$, or

(4) $m = 0\pm$ and $n = 0\mp$ and $T = (s_\pm s_\mp)^k s_\pm$ for some $k \in \mathbb{Z}_{\geq 0}$, in which case $\operatorname{rel}_*(T) = \pm(2k+1)$.

Type III: This follows immediately from the Type I case and Lemma 2.2.27. □

Lemma 2.2.29. *Tangle type is preserved under tangle composition for tangles.*

Proof.

Type I: Suppose $S, T \in \mathcal{T}_I$ such that $R = S \circ T$ makes sense. Certainly R has no cups or loops. It remains to verify that $*_0(R)$ is in the right place. A problem could only arise in the case where both S and T have through strings, but we see that if S and T both satisfy condition (I-2), then so does R.

Type II: Obvious.

Type III: Suppose $S, T \in \mathcal{T}_{III}$ such that $R = S \circ T$ makes sense. Then by Lemma 2.2.27, we have $T^*, S^* \in \mathcal{T}_I$ and $R^* = T^* \circ S^*$ makes sense, so by the Type I case, $R^* \in \mathcal{T}_I$, and once more by 2.2.27, $R \in \mathcal{T}_{III}$. □

Corollary 2.2.30. *By 2.2.24 and Proposition 2.2.29,*

(1) any composite of a_i's is in $\mathcal{T}_I$, and

(2) any composite of b_i's is in $\mathcal{T}_{III}$.

Unique Tangle Decompositions

For this section, we use the convention that if $n = 0\pm$ and $z \in \mathbb{Z}$, then $n + z = z$.

Definition 2.2.31. A tangle $T \in \mathcal{T}_I$ is called irreducible if there is at most one cap bounding $*_1(T)$, and if there is a cap Λ bounding $*_1(T)$, then all other caps of T are bounded by Λ.

Remark 2.2.32. If $T \in \mathcal{T}_I(m, n)$ for $m \geq 1$ is irreducible, then T has a unique representation as follows:

Case 1: if there is no cap bounding $*_1(T)$, then $T = a_{i_k} \cdots a_{i_1}$ with $i_j > i_{j+1}$ for all $j \in \{1, \ldots, k-1\}$ and $i_j < 2(m-j)+2$ for all $j \in \{1, \ldots, k\}$.

Case 2: If there is a cap bounding $*_1(T)$, then $T = a_q a_{i_k} \cdots a_{i_1} a_{j_l} \cdots a_{j_1}$ where $k, l \geq 0$ and

(i) $q = 2n + 2$,

(ii) $i_r > i_{r+1}$ for all $r \in \{1, \ldots, k-1\}$, $i_1 < j_l$, and $j_s > j_{s+1}$ for all $s \in \{1, \ldots, l-1\}$, and

(iii) $i_r \leq 2(k-r)+1$ for all $r \in \{1, \ldots, k\}$ and $j_s \geq 2(m-s)+1$ for all $s \in \{1, \ldots, l\}$.

Uniqueness follows by looking at the cap indices which are given as follows:

Case 1: If there is no cap bounding $*_1(T)$, then $\text{capind}(T) = \{i_k, \cdots, i_1\}$.

Case 2: If there is a cap Λ bounding $*_1(T)$, then $\text{ind}(\Lambda) = 2(m-l)$ and $\text{capind}(T) = \{i_k, \cdots, i_1, 2(m-l), j_l, \cdots, j_1\}$.

Remarks 2.2.33. Suppose $T \in \mathcal{T}_I(m, n-1)$ with $m > n-1 \geq 1$ is irreducible such that $*_1(T)$ is bounded. Let $T = a_q a_{i_k} \cdots a_{i_1} a_{j_l} \cdots a_{j_1}$ be the representation afforded by the above remark. If $S \in \mathcal{T}_I(n-1, p)$ and $R = S \circ T$, then

(1) there is a cap Λ of R bounding $*_1(R)$, of index $2(m-l)$. All other caps of R bounding $*_1(R)$ have smaller index than Λ.

(2) $|B(\Lambda)| = k + l + 1$.

(3) $\text{capind}(R) = \{i_k, \cdots, i_1, c_1, \cdots, c_s, 2(m-l), j_l, \ldots, j_1\}$ for some $c_1, \ldots, c_s \in \mathbb{N}$ and $s = m - p - k - l - 1$.

Lemma 2.2.34. *Suppose $T_1 \in \mathcal{T}_I(m, m-u-1)$ and $T_2 \in \mathcal{T}_I(m, m-v-1)$ with $m-u, m-v \geq 2$ are irreducible and each has one cap bounding $*_1$. Suppose $S_1 \in \mathcal{T}_I(m-u-1, w)$ and $S_2 \in \mathcal{T}_I(m-v-1, w)$ such that $S_1 \circ T_1 = S_2 \circ T_2$. Then $T_1 = T_2$.*

Proof. Set $R = S_1 \circ T_1 = S_0 \circ T_0$. We have that $*_1(R)$ is bounded by a cap Λ with index $2(m-u) = 2(m-v)$, so $u = v$. Now we have unique irreducible decompositions

$$T_1 = a_p a_{i_k} \cdots a_{i_1} a_{j_l} \cdots a_{j_1} \text{ and}$$
$$T_2 = a_q a_{g_r} \cdots a_{g_1} a_{h_s} \cdots a_{h_1},$$

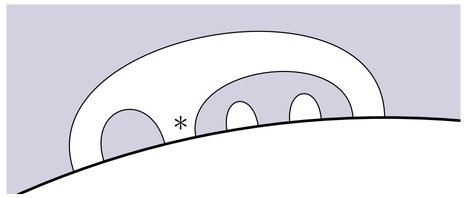


Figure 2.17: $R = S \circ T$, zoomed in near $*_1(R)$ where $T = a_{2n+2}a_1a_2a_4a_{2m-1} \in \mathcal{T}_I(m,n)$ is irreducible

and as the cap indices of R are unique, we have

$$\begin{aligned}\text{capind}(R) &= \{i_k, \cdots, i_1, c_1, \cdots, c_s, 2(m-u), j_l \cdots, j_1\} \\ &= \{g_r, \cdots, g_1, c_1, \ldots, c_s, 2(m-v), h_s, \cdots, h_1\}.\end{aligned}$$

Hence we must have equality of the two sequences:

$$\{i_k, \cdots, i_1, 2(m-u), j_l \cdots, j_1\} = \{g_r, \cdots, g_1, 2(m-v), h_s, \cdots, h_1\},$$

and $T_1 = T_2$ by Proposition 2.2.28. □

Proposition 2.2.35. *Each $T \in \mathcal{T}_I(m,n)$ where $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\pm\}$ has a unique decomposition $T = W_r \cdots W_1$ such that W_i is irreducible for all $i = 1, \ldots, r$.*

Proof.

Existence: The existence of such a decomposition will follow from Algorithm 3.2 below.

Uniqueness: We induct on r. Suppose $r = 1$. Then uniqueness follows from Remark 2.2.32. Suppose now that $r > 1$ and the result holds for all concatenations of fewer irreducible words. Suppose we have another decomposition

$$T = W_r \cdots W_1 = U_s \cdots U_1.$$

Then by the induction hypothesis, we must have $s \geq r$. As W_1 and U_1 are irreducible, we apply Lemma 2.2.34 with

(1) $T_1 = W_1$ and $S_1 = W_r \cdots W_2$, and

(2) $T_2 = U_1$ and $S_2 = U_s \cdots U_2$

to see that $W_1 = U_1$. We may now apply appropriate b_i's to T (on the right) to get rid of $W_1 = U_1$ to get

$$W' = W_r \cdots W_2 = U_s \cdots U_2.$$

where W' is equal to a concatenation of fewer irreducible words. By the induction hypothesis, we can conclude $r = s$ and $U_i = W_i$ for all $i = 2, \ldots, r$. We are finished. □

Algorithm 2.2.36. The following algorithm expresses a Type I tangle $T \in \mathcal{T}_I(m,n)$ where $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\pm\}$ as a composite of a_i's in the form required by Proposition 2.2.35. Set $T_0 = T$, $m_0 = m$, and $r = 1$.

Step 1: Let $S_1 = \{\Lambda \in \text{caps}(T_0) | *_1(T_0) \subset \partial\Lambda \text{ and } \text{ind}(\Lambda) \in 2\mathbb{N}\}$. Let S_0 be the set of all caps that are not in $B(\Lambda)$ for some $\Lambda \in S_1$. If $S_1 = \emptyset$, proceed to Step 4.

Step 2: Suppose $|S_1| \geq 1$. Select the cap $\Lambda \in S_1$ with the largest index. There are two cases:

> Case 1: $B(\Lambda) = \{\Lambda\}$. Set $W_r = \alpha_{\text{ind}(\Lambda)}$. Proceed to Step 3.
>
> Case 2: $B(\Lambda) \setminus \{\Lambda\} \neq \emptyset$. List the cap indices for all caps $\Lambda' \in B(\Lambda) \setminus \{\Lambda\}$ in decreasing order from right to left, $i_k, \cdots, i_1$ where $i_j > i_{j+1}$ for all $j \in \{1, \ldots, k-1\}$. where $k = |B(\Lambda) \setminus \{\Lambda\}|$. Set $q = \text{ind}(\Lambda) - 2k$ and $W_r = a_q a_{i_k} \cdots a_{i_1}$.

Step 3: Note that W_r is irreducible. Move $*_1(T_0)$ counterclockwise to the closest simple interval outside of Λ whose interior touches an unshaded region (which is necessarily 2 regions counterclockwise), and remove all caps in $B(\Lambda)$ from T_0 to get a new tangle, called T_1. Note that $T_0 = T_1 W_r$. Set m_1 equal to half the number of internal boundary points of T_1, and set $r_1 = r$. Now set $T_0 = T_1$, $m_0 = m_1$, and $r = r_1 + 1$. Go back to Step 1.

Step 4: List the cap indices for all caps $\Lambda \in S_0$ in decreasing order from right to left, $i_k, \cdots, i_1$ where $i_j > i_{j+1}$ for all $j \in \{1, \ldots, k-1\}$. There are two cases:

(i) There are fewer than m_0 caps. Set $W_r = a_{i_k} \cdots a_{i_1}$. Note that W_r is irreducible and $T_0 = W_r$. We are finished.

(ii) There are m_0 caps. Proceed to Step 5.

Step 5: There are two cases:

(i) If the region touching $D_0(T_0)$ is unshaded, set $W_r = a_1 a_{i_{k-1}} \cdots a_{i_1}$. Note that W_r is irreducible and $T_0 = W_r$. We are finished.

(ii) If the region touching $D_0(T_0)$ is shaded, set $W_r = a_2 a_{i_{k-1}} \cdots a_{i_1}$. Note that W_r is irreducible and $T_0 = W_r$. We are finished.

Note that $T = W_r \cdots W_1$ satisfies the conditions of Proposition 2.2.35.

The following Theorem is merely a strengthening of Corollary 1.16 in [Jon94].

Theorem 2.2.37 (Atl Tangle Decomposition)**.** *Each Atl (m,n)-tangle T can be written uniquely as a composite $T = T_{III} \circ T_{II} \circ T_I$ where $T_i \in \mathcal{T}_i$ for all $i \in \{I, II, III\}$.*

Proof. We begin by proving the uniqueness of such a decomposition as it will tell us how to find such a decomposition.

Uniqueness: Suppose we have a decomposition $T = T_{III} \circ T_{II} \circ T_I$ where $T_I \in \mathcal{T}_I(m, l)$, $T_{II} \in \mathcal{T}_{II}(l, k)$, and $T_{III} \in \mathcal{T}_{III}(k, n)$ for some $l, k \in \mathbb{N} \cup \{0\pm\}$. Note that l, k are uniquely determined by $|\operatorname{ts}(T)|$ and the shading of T. Note further that $\operatorname{capind}(T_I) = \operatorname{capind}(T)$, $\operatorname{rel}_*(T_{II}) = \operatorname{rel}_*(T)$, and $\operatorname{cupind}(T_{III}) = \operatorname{cupind}(T)$. Hence T_i is uniquely determined for $i \in \{I, II, III\}$ by Proposition 2.2.28.

Existence: Let $l = k$ be the number of through strings of T. If $l = k = 0$, set $l = 0+$, respectively $l = 0-$ if the region meeting $D_1(T)$ is unshaded, respectively shaded, and set $k = 0+$, respectively $k = 0-$ if the region meeting $D_0(T)$ is unshaded, respectively shaded. Let $T_I \in \mathcal{T}_I(m, l)$ be the unique tangle with $\operatorname{capind}(T_I) = \operatorname{capind}(T)$. Let $T_{II} \in \mathcal{T}_{II}(l, k)$ be the unique tangle with $\operatorname{rel}_*(T_{II}) = \operatorname{rel}_*(T)$. Let $T_{III} \in \mathcal{T}_{III}(k, n)$ be the unique tangle with $\operatorname{cupind}(T_{III}) = \operatorname{cupind}(T)$. It is now obvious that $T = T_{III} \circ T_{II} \circ T_I$. $\square$

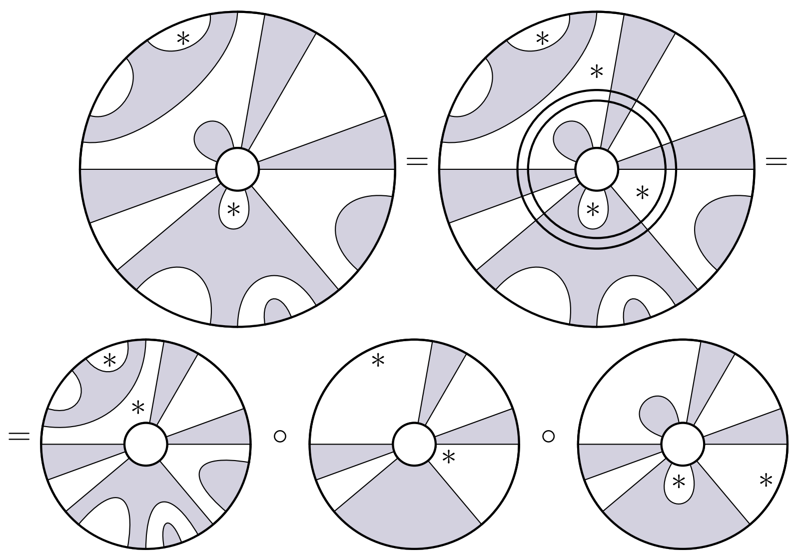


Figure 2.18: Decomposition of an ATL tangle into $T_{III} \circ T_{II} \circ T_I$

2.3 The Category aΔ

Generators and Relations

Definition 2.3.1. Let aΔ, the annular category, be the following small category:

Objects: $[n]$ for $n \in \mathbb{N} \cup \{0\pm\}$, and

Morphisms: generated by

$$\begin{aligned}
\alpha_1\colon [1] \to [0+],\ \ \alpha_2\colon [1] \to [0-],\ \text{ and}\\
\alpha_i\colon [n] \longrightarrow [n-1] \text{ for } i = 1,\dots,2n \text{ and } n \geq 2;\\
\beta_1\colon [0+] \to [1],\ \ \beta_2\colon [0-] \to [1],\ \text{ and}\\
\beta_i\colon [n] \longrightarrow [n+1] \text{ for } i = 1,\dots,2n+2 \text{ and } n \geq 1;\\
\tau\colon [n] \longrightarrow [n] \text{ for all } n \in \mathbb{N};\ \text{ and}\\
\delta_\pm\colon [n] \longrightarrow [n] \text{ for all } n \in \mathbb{N} \cup \{0\pm\}
\end{aligned}$$

subject to the following relations:

(1) $\alpha_i\alpha_j = \alpha_{j-2}\alpha_i$ for $i < j-1$ and $(i,j) \neq (1,2n)$,

(2) $\beta_i\beta_j = \beta_{j+2}\beta_i$ for $i \leq j$ and $(i,j) \neq (1,2n+2)$,

(3) $\tau^n = \mathrm{id}_{[n]}$,

(4) $\alpha_i\tau = \tau\alpha_{i-2}$ for $i \geq 3$,

(5) $\beta_i\tau = \tau\beta_{i-2}$ for $i \geq 3$,

(6) $\delta_+ = \alpha_1\beta_1 \in \mathsf{a}\Delta(0+,0+)$ and $\delta_- = \alpha_2\beta_2 \in \mathsf{a}\Delta(0-,0-)$. If $\alpha_i\beta_j\colon [n] \to [n]$ with $n \geq 1$, then

$$\alpha_i\beta_j = \begin{cases} \tau^{-1} & \text{if } (i,j) = (1,2n+2)\\ \beta_{j-2}\alpha_i & \text{if } i < j-1, (i,j) \neq (1,2n+2)\\ \mathrm{id}_{[n]} & \text{if } i = j-1\\ \delta_+ & \text{if } i = j \text{ and } i \text{ is odd}\\ \delta_- & \text{if } i = j \text{ and } i \text{ is even}\\ \mathrm{id}_{[n]} & \text{if } i = j+1\\ \beta_j\alpha_{i-2} & \text{if } i > j+1, (i,j) \neq (2n+2,1)\\ \tau & \text{if } (i,j) = (2n+2,1) \end{cases}$$

(7) $\delta_\pm$ commutes with all other generators (including $\delta_\mp$).

Involution and Word Type

Definition 2.3.2. A morphism $h \in \mathrm{Mor}(\mathsf{a}\Delta)$ will be called primitive if h is equal to α_i, β_i, t, $\delta_\pm$, or $\mathrm{id}_{[n]}$ for $n \in \mathbb{N} \cup \{0\pm\}$. A word on $\mathsf{a}\Delta$ is a sequence $h_r \cdots h_1$ with $r \geq 1$ of primitive morphisms in $\mathsf{a}\Delta$. We say the length of such a word is $r \in \mathbb{N}$. By convention, we will say a word has length zero if and only if $r = 1$ and $h_1 = \mathrm{id}_{[n]}$ for some $n \in \mathbb{N} \cup \{0\pm\}$.

Definition 2.3.3. We define a map $*$ on $\mathrm{Ob}(\mathsf{a}\Delta)$ and on primitive morphisms in $\mathrm{Mor}(\mathsf{a}\Delta)$:

(Ob) For $n \in \mathbb{N} \cup \{0\pm\}$, define $[n]^* = [n]$.

(I) For all $n \in \mathbb{N} \cup \{0\pm\}$, define $\mathrm{id}_{[n]}^* = \mathrm{id}_{[n]}$.

(A) For $\alpha_1 \in \mathsf{a}\Delta(1, 0+)$, define $\alpha_1^* = \beta_1 \in \mathsf{a}\Delta(0+, 1)$. For $\alpha_2 \in \mathsf{a}\Delta(1, 0-)$, define $\alpha_2^* = \beta_2 \in \mathsf{a}\Delta(0-, 1)$. For $n \geq 2$ and $\alpha_i \in \mathsf{a}\Delta(n, n-1)$, so $i \in \{1, \ldots, 2n\}$, define $\alpha_i^* = \beta_i \in \mathsf{a}\Delta(n-1, n)$.

(B) For $\beta_1 \in \mathsf{a}\Delta(0+, 1)$, define $\beta_1^* = \alpha_1 \in \mathsf{a}\Delta(1, 0+)$. For $\beta_2 \in \mathsf{a}\Delta(0-, 1)$, define $\beta_2^* = \alpha_2 \in \mathsf{a}\Delta(1, 0-)$. For $n \geq 1$ and $\beta_i \in \mathsf{a}\Delta(n, n+1)$, so $i \in \{1, \ldots, 2n+2\}$, define $\beta_i^* = \alpha_i \in \mathsf{a}\Delta(n+1, n)$.

(T) For $n \in \mathbb{N}$ and $\tau \in \mathsf{a}\Delta(n, n)$, define $\tau^* = \tau^{-1}$.

(D) For $n \in \mathbb{N} \cup \{0\pm\}$ and $\delta_\pm \in \mathsf{a}\Delta(n, n)$, define $\delta_\pm^* = \delta_\pm$.

Proposition 2.3.4. *The following extension of $*$ to* $\mathrm{Mor}(\mathsf{a}\Delta)$ *is well defined:*

- *If $h_r \cdots h_1$ is a word on $\mathsf{a}\Delta$, then we define $(h_r \cdots h_1)^* = h_1^* \cdots h_r^*$.*

Hence $*$ extends uniquely to an involution on $\mathsf{a}\Delta$.

Proof. We must check that $*$ preserves the relations of $\mathsf{a}\Delta$. Note that relations (3), (6), and (7) are preserved by $*$, and the following pairs are switched: (1) & (2) and (4) & (5). □

Corollary 2.3.5. *We have an isomorphism of categories $\mathsf{a}\Delta \cong \mathsf{a}\Delta^{op}$.*

Proposition 2.3.6. *The following additional relations hold in $\mathsf{a}\Delta$:*

(1) $\alpha_1\tau = \alpha_{2n-1}$ and $\alpha_2\tau = \alpha_{2n}$,

(2) $\tau\beta_{2n+1} = \beta_1$, $\tau\beta_{2n+2} = \beta_2$, and

(3) $\beta_1\tau = \tau^2\beta_{2n-1}$ and $\beta_2\tau = \tau^2\beta_{2n}$.

Proof. (1) By relations (4) and (5), we have

$$\alpha_{2n-1} = \alpha_{2n-1}\tau^n = \tau\alpha_{2n-3}\tau^{n-1} = \cdots = \tau^{n-1}\alpha_3\tau = \tau^n\alpha_1 = \alpha_1.$$

The proof of the other relation is similar.

(2) These relations are merely $*$ applied to (1).

(3) By relations (4) and (6), we have

$$\tau^2\beta_{2n-1} = \tau^2\beta_{2n-1}\tau^n = \tau^2\tau\beta_{2n-3}\tau^{n-1} = \cdots = \tau^2\tau^{n-1}\beta_1\tau = \tau^{n+1}\beta_1\tau = \beta_1\tau.$$

The proof of the other relation is similar.

□

Notation 2.3.7. (1) If $h \in \text{Mor}(\mathsf{a}\Delta)$, we write $h \in \mathcal{A}_1$ if $h = \alpha_i \in \mathsf{a}\Delta(1, 0\pm)$ where $i \in \{1, 2\}$. We write $h \in \mathcal{A}_n$ where $n \geq 2$ if $h = \alpha_i \in \mathsf{a}\Delta(n, n-1)$ for some $i \in \{1, \dots, 2n\}$. We write $h \in \mathcal{A}$ if $h \in \mathcal{A}_n$ for some $n \geq 1$. Similarly we define $\mathbb{B}_n$ for $n \in \mathbb{N} \cup \{0\pm\}$ and $\mathbb{B}$.

(2) For convenience, we will use the notation $\sigma_+ = \alpha_2\beta_1 \in \mathsf{a}\Delta(0+, 0-)$ and $\sigma_- = \alpha_1\beta_2 \in \mathsf{a}\Delta(0-, 0+)$.

Definition 2.3.8. A word $w = h_r \cdots h_1$ on $\mathsf{a}\Delta$ is called

Type I: if w has length zero or if $h_i \in \mathcal{A}$ for all $i \in \{1, \dots, r\}$.

Type II: if either

(1) w has length zero,

(2) $r > 0$ and $h_i = \tau$ for all $i \in \{1, \dots, r\}$, or

(3) $r = 2s$ for some $s > 0$ and $h_i h_{i+1} = \sigma_\pm$ for all odd i so that

$$w = \begin{cases} (\sigma_\pm\sigma_\mp)^k\sigma_\pm & \text{if } s = 2k+1 \text{ is odd, or} \\ (\sigma_\pm\sigma_\mp)^k & \text{if } s = 2k \text{ is even.} \end{cases}$$

Type III: if w has length zero or if $h_i \in \mathbb{B}$ for all $i \in \{1, \dots, r\}$.

Denote the set of all words of Type i by $\mathcal{W}_i$, and denote the set of all words of Type i with domain $[m]$ and codomain $[n]$ by $\mathcal{W}_i(m, n)$ for $i \in \{I, II, III\}$.

Lemma 2.3.9. *Let $m, n \in \mathbb{N} \cup \{0\pm\}$. Types are related to the involution as follows:*

(1) $w \in \mathcal{W}_I(m, n)$ if and only if $w^* \in \mathcal{W}_{III}(n, m)$, and

(2) If $w \in \mathcal{W}_{II}(n, n)$, then $w^* \in \mathcal{W}_{II}(n, n)$.

Proof. Obvious. □

Standard Forms

Notation 2.3.10. if we replace j with $j + 2$ in the statement of relation (1), we get the equivalent relation

(1') $\alpha_j\alpha_i = \alpha_i\alpha_{j+2}$ for all $j \geq i$ with $(j, i) \neq (2n, 1)$

as maps $[n + 1] \to [n - 1]$.

Definition 2.3.11. A word $w \in \mathcal{W}_I(m, n)$ with $m \geq 1$ is called irreducible if either

(1) $w = \alpha_{i_k} \cdots \alpha_{i_1}$ where $i_r > i_{r+1}$ for all $r \in \{1, \dots, k-1\}$ and $i_r < 2(m - r) + 2$ for all $r \in \{1, \dots, k\}$, in which case we also say w is ordered, or

(2) $w = \alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_1} \in \mathcal{W}_I(m,n)$ where $m \geq 1$ and $l, k \geq 0$ such that

  (i) $q = 2n+2$,

  (ii) $i_r > i_{r+1}$ for all $r \in \{1, \ldots, k-1\}$, $i_1 < j_l$, and $j_s > j_{s+1}$ for all $s \in \{1, \ldots, l-1\}$, and

  (iii) $i_r \leq 2(k-r)+1$ for all $r \in \{1, \ldots, k\}$ and $j_s \geq 2(m-s)+1$ for all $s \in \{1, \ldots, l\}$.

Remark 2.3.12. If $\alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_1}$ is irreducible as in (2) of 2.3.11, then so are

$$\alpha_q \alpha_{i_k} \cdots \alpha_{i_1} \alpha_{j_l} \cdots \alpha_{j_r} \text{ and } \alpha_q \alpha_{i_k} \cdots \alpha_{i_s}$$

for all $r \in \{1, \ldots, l\}$ and $s \in \{1, \ldots, k\}$. In particular, if $l > 0$, then $j_l = 2(m-l)+1$, and if $k > 0$, then $i_k = 1$.

Algorithm 2.3.13. Suppose $w = \alpha_{i_k} \cdots \alpha_{i_1} \in \mathcal{W}_1(m, n-1)$ is ordered where $n-1 > 0$. The following algorithm gives words u_1, u_2 where u_1 is irreducible and $\alpha_{2n} w = u_2 u_1$. Set $u_1 = \alpha_{2n} w$ and $u_3 = \mathrm{id}_{[n-1]}$.

Step 1: If u_1 is irreducible, set $u_2 = u_3$. We are finished. Otherwise, proceed to Step 2.

Step 2: There is a $j \in \{1, \ldots, k\}$ such that $2(k-j)+1 < i_j < 2(m-j)+1$. Pick j minimal with this property. Use relation (1) to push $\alpha_{i_k} \cdots \alpha_{i_{j+1}}$ past α_{i_j} to get

$$w = \alpha_{2n} \alpha_{i_j - 2(k-j)+2} \alpha_{i_{k-1}} \cdots \alpha_{i_{j+1}} \alpha_{i_{j-1}} \cdots \alpha_{i_1}.$$

Note that

$$1 < i_j - 2(k-j) < 2(m-j)+1-2(k-j) = 2(m-k)+1 = 2n+1,$$

as $m-k=n$, so we may use relation (1') to get

$$\alpha_{i_j - 2(k-j)} \alpha_{2n+2} \alpha_{i_{k-1}} \cdots \alpha_{j+1} \alpha_{j-1} \cdots \alpha_{i_1}.$$

Set $u_2 = \alpha_{i_j - 2(k-j)+2} u_3$. Now set $u_3 = u_2$. Set

$$u_1 = \alpha_{2n+2} \alpha_{i_{k-1}} \cdots \alpha_{j+1} \alpha_{j-1} \cdots \alpha_{i_1}.$$

Go back to Step 1.

Proof. We need only prove the above algorithm terminates. Note one of the α_i's increases in index each iteration, which cannot happen indefinitely. □

Proposition 2.3.14. *Suppose $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\pm\}$. Each $w \in \mathcal{W}_I(m,n)$ has a decomposition $w = w_r \cdots w_1$ where each $w_i \in W_I$ is irreducible. Such a decomposition of w is called a standard decomposition of w.*

Proof. We induct on the length of w. If the length of w is 1, then we are finished. Suppose w has length greater than 1 and the result holds for all words of shorter length. Use relation (1') to get $w' = w_r \cdots w_1$ where each w_i is ordered and for each $s \in \{1, \dots, r-1\}$. If $w_s = \alpha_{i_a} \cdots \alpha_{i_1}$ and $w_{s+1} = \alpha_{j_b} \cdots \alpha_{j_1}$, then $i_a = 1$, $j_1 = 2k$, so $\alpha_{j_1}\alpha_{i_a} = \alpha_{2k}\alpha_1 \in \mathsf{a}\Delta(k+1, k-1)$ for some $k \geq 2$. There are two cases.

Case 1: $r = 1$. Then $w = w_1$ is ordered, hence irreducible, and we are finished.

Case 2: Suppose $r > 1$. As $w_2 = \alpha_{i_a} \cdots \alpha_{i_1}$ where $\alpha_{i_1} = \alpha_{2k} \in \mathsf{a}\Delta(k, k-1)$, we apply Algorithm 2.3.13 to the word $\alpha_{2k}w_1$ to obtain u_1, u_2 with u_1 irreducible such that $u_2u_1 = \alpha_{2k}w_1$. We now note that $w = w'u_1$ where

$$w' = w_r \cdots w_3 \alpha_{i_a} \cdots \alpha_{i_2}$$

is a word of strictly smaller length. Applying the induction hypothesis to w' gives us the desired result. □

Theorem 2.3.15 (Standard Forms)**.** *Suppose $w = h_r \cdots h_1$ is a word on $\mathsf{a}\Delta$ in $\mathsf{a}\Delta(m, n)$ for $m, n \in \mathbb{N} \cup \{0\pm\}$. Then there is a decomposition $w = \delta_+^{c_+}\delta_-^{c_-}w_{III}w_{II}w_I$ where $w_i \in \mathcal{W}_i$ for all $i \in \{I, II, III\}$, $c_\pm \geq 0$, and w_I and w_{III}^* are in the form afforded by Proposition 2.3.14.*

Proof. Note that it suffices to find $v_i \in \mathcal{W}_i$ for $i \in \{I, II, III\}$ and $c_\pm \geq 0$ such that $w = \delta_+^{c_+}c_-^{c_-}v_{III}v_{II}v_I$, as we may then set $w_{II} = v_{II}$ and apply Proposition 2.3.14 to v_I and v_{III}^* to get w_I and w_{III}^* respectively. We induct on r. The case $r = 1$ is trivial. Suppose $r > 1$ and the result holds for all words of shorter length. Apply the induction hypothesis to $w' = h_{r-1} \cdots h_1$ to get

$$w' = \delta_+^{c'_+}\delta_-^{c'_-}u_{III}u_{II}u_I.$$

There are 4 cases.

(D) Suppose $h_r = \delta_\pm$. Set $c_\pm = c'_\pm + 1$, $c_\mp = c'_\mp$, and $v_i = u_i$ for all $i \in \{I, II, III\}$. We are finished.

(B) Suppose $h_r \in B$. Set $c_\pm = c'_\pm$ and $v_i = u_i$ for $i \in \{I, II, III\}$. We are finished.

(T) Suppose $h_r = \tau$. Set $c_\pm = c'_\pm$ and $w_I = u_I$. As we push τ right using relation (5) and Proposition 2.3.6, only two extraordinary possibilities occur:

Case 1: τ meets β_{2n+1} or β_{2n+2} in $\mathsf{a}\Delta(n, n+1)$, so τ disappears when using Proposition 2.3.6, or

Case 2: τ meets $\beta_1 \in \mathsf{a}\Delta(0+, 1)$ or $\beta_2 \in \mathsf{a}\Delta(0-, 1)$, so τ disappears as $\mathrm{id}_{[1]} = \tau \in \mathsf{a}\Delta(1, 1)$.

Hence we get that $w = v'_{III}\tau^s$ where $v_{III} \in \mathcal{W}_{III}$ and $s \in \{0, 1\}$. If $s = 0$, set $v_{II} = u_{II}$, and if $s = 1$, set $v_{II} = \tau u_{II}$. We are finished.

(A) Suppose $h_r = \alpha_q$ for some $q \in \mathbb{N}$. Use relation (6) to push α_q to the right of the β's. There are five cases.

Case 1: We use the relation $\alpha_i\beta_j = \tau^{\pm 1}$. Arguing as in Case (T) we are finished.

Case 2: We use the relation $\alpha_i\beta_{i\pm 1} = \mathrm{id}_{[k]}$ for some $k \in \mathbb{N} \cup \{0\pm\}$, so $\alpha_q u_{III} = v_{III}$ for some $v_{III} \in \mathcal{W}_{III}$. Set $c_\pm = c'_\pm$ and $v_i = u_i$ for $i \in \{I, II\}$. We are finished.

Case 3: We use the relation $\alpha_i\beta_i = \delta_\pm$, so $\alpha_q u_{III} = \delta_\pm v_{III}$ for some $v_{III} \in \mathcal{W}_{III}$. Set $c_\pm = c'_\pm + 1$, $c_\mp = c'_\mp$, and $v_i = u_i$ for $i \in \{I, II\}$. We are finished.

Case 4: α_q can be pushed all the way to the right of u_{III} to obtain $\alpha_q u_{III} = v_{III}\alpha_p$ for some $p \in \mathbb{N}$ and $v_{III} \in \mathcal{W}_{III}$. Then necessarily $u_{II} = \tau^s$ for some $s \in \mathbb{Z}_{\geq 0}$, so we use relation (4) and 2.3.6 to push α_p to the right of the τ's. Hence we obtain $\alpha_p u_{II} = v_{II}\alpha_k$ for some $k \in \mathbb{N}$ and $v_{II} \in \mathcal{W}_{II}$. Set $c_\pm = c'_\pm$ and $v_I = \alpha_k u_I$. We are finished.

Case 5: α_q can be pushed all the way to the right except for the last β_i. This means $\alpha_q u_{III} = v_{III}\alpha_i\beta_j$ for some $v_{III} \in \mathcal{W}_{III}$ where $\alpha_i\beta_j = \sigma_\pm$. Set $v_{II} = \sigma_\pm u_{II}$, $c_\pm = c'_\pm$, and $v_I = u_I$. We are finished.

□

Definition 2.3.16. If $w \in \mathrm{Mor}(\mathsf{a}\Delta)$, a decomposition of w as in Theorem 2.3.15 is called a standard form of w.

Remark 2.3.17. It will be a consequence of Theorem 2.4.8 that a word $w \in \mathsf{a}\Delta$ has a unique standard form.

2.4 The Isomorphism of Categories $\mathsf{a}\Delta \cong \mathsf{Atl}$

Proposition 2.4.1. *The following defines an involutive functor* $F\colon \mathsf{a}\Delta \to \mathsf{Atl}$*:*

Objects: $F([n]) = [n]$ *for all* $n \in \mathbb{N} \cup \{0\pm\}$,

Morphisms:

(A) Set $F(\alpha_i) = a_i$,

(B) Set $F(\beta_i) = b_i$,

(T) Set $F(\tau) = t$, *and*

(D) Set $F(\delta_+ \in \mathsf{a}\Delta(n,n)) = (\mathrm{id}_{[n]}, 1, 0)$ *and* $F(\delta_- \in \mathsf{a}\Delta(n,n)) = (\mathrm{id}_{[n]}, 0, 1)$ *for* $n \in \mathbb{N} \cup \{0\pm\}$.

Proof. We must check that $F(\mathrm{id}_{[n]}) = \mathrm{id}_{[n]}$ for all $n \in \mathbb{N} \cup \{0\pm\}$ and that F preserves composition, but both these conditions follow from Theorem 2.2.19. It is clear $*$ preserves the involution by Proposition 2.2.22. □

Remark 2.4.2. We construct a functor $G\colon \mathsf{Atl} \to \mathsf{a}\Delta$ as follows: we create a function $G\colon \mathsf{Atl} \to \mathsf{a}\Delta$ taking objects to objects (this part is easy as objects in both categories have the same names) and $\mathsf{Atl}(m,n) \to \mathsf{a}\Delta(m,n)$ bijectively such that $F \circ G = \mathrm{id}_{\mathsf{Atl}}$. It will follow immediately that G is a functor and $G \circ F = \mathrm{id}_{\mathsf{a}\Delta}$.

Theorem 2.4.3. *Let $m,n \in \mathbb{N} \cup \{0\pm\}$. Then $F_i = F|_{\mathcal{W}_i(m,n)}\colon \mathcal{W}_i(m,n) \to \mathcal{T}_i(m,n)$ is bijective for all $i \in \{I, II, III\}$, i.e. there is a bijective correspondence between words of Type i and Atl tangles of Type i for all $i \in \{I, II, III\}$.*

Proof.

Type I: Note that $\mathrm{im}(F_I) \subset \mathcal{T}_I(m,n)$. We construct the inverse G_I for F_I. Note that by Proposition 2.2.35, each $T \in \mathcal{T}_I(m,n)$ can be written uniquely as $T = W_r \cdots W_1$, which can further be expanded as

$$T = \underbrace{a_{i_p} \cdots a_{i_1}}_{W_r} \underbrace{a_{j_q} \cdots a_{j_1}}_{W_2} \cdots \underbrace{a_{k_r} \cdots a_{k_1}}_{W_1}$$

satisfying 2.2.35. Set

$$G_I(T) = \alpha_{i_p} \cdots \alpha_{i_1} \cdots \alpha_{j_q} \cdots \alpha_{j_1} \alpha_{k_r} \cdots \alpha_{k_1}.$$

It follows $F_I \circ G_I = \mathrm{id}$. Now by Proposition 2.3.14, every word of Type I can be written in this form. Hence we see G_I is in fact the inverse of F_I

Type II: Obvious.

Type III:. From the Type I case and the involutions in $\mathsf{a}\Delta$ and Atl, we have the following bijections:

$$\mathcal{T}_{III}(m,n) \longleftrightarrow \mathcal{T}_I(n,m) \longleftrightarrow \mathcal{W}_I(n,m) \longleftrightarrow \mathcal{W}_{III}(m,n).$$

□

Definition 2.4.4. We define $G\colon \mathsf{Atl} \to \mathsf{a}\Delta$ as follows:

Objects: $G([n]) = [n]$ for all $n \in \mathbb{N} \cup \{0\pm\}$.

Morphisms: First define $G(T,0,0)$ for a $T \in \mathcal{T}_i$ for $i \in \{I, II, III\}$ by the bijective correspondences given in Theorem 2.4.3. Then for an arbitrary Atl (m,n)-tangle T, define $G(T,0,0)$ by

$$G(T,0,0) = G(T_{III},0,0) \circ G(T_{II},0,0) \circ G(T_I,0,0)$$

where T_i for $i \in \{I, II, III\}$ is defined for T as in the Atl Decomposition Theorem 2.2.37. Finally, define $G(T,c_+,c_-) = \delta_+^{c_+}\delta_-^{c_-} G(T,0,0)$ for an arbitrary morphism $(T,c_+,c_-) \in \mathrm{Mor}(\mathsf{Atl})$. Note that G is well defined by the uniqueness part of 2.2.37.

Proposition 2.4.5. *If T is an Atl (m,n)-tangle of Type i for $i \in \{I, II, III\}$, then $F \circ G(T) = T$.*

Proof. This is immediate from the definition of G and Theorem 2.4.3. □

Corollary 2.4.6. *$F \circ G = \mathrm{id}_{\mathsf{Atl}}$, so G restricted to $\mathsf{Atl}(m,n)$ is injective into $\mathsf{a}\Delta(m,n)$ for all $m,n \in \mathbb{N} \cup \{0\pm\}$.*

Proof. This follows immediately from Theorem 2.2.37 and the definition of G as F is a functor. □

Proposition 2.4.7. *G restricted to $\mathsf{Atl}(m,n)$ is surjective onto $\mathsf{a}\Delta(m,n)$.*

Proof. We have that every word $w \in \mathrm{Mor}(\mathsf{a}\Delta)$ is equal to a word $\delta_+^{c_+}\delta_-^{c_-}w_{III}w_{II}w_I$ in standard form where w_i is of Type i for $i \in \{I, II, III\}$. By 2.4.3 there are unique Atl tangles T_i of Type i such that $w_i = G(T_i)$ for all $i \in \{I, II, III\}$. Set $T = T_{III} \circ T_{II} \circ T_I$, and note this decomposition into a composite of Atl tangles of Types I, II, and III is unique by 2.2.37. It follows that

$$G(T, c_+, c_-) = \delta_+^{c_+}\delta_-^{c_-}w_{III}w_{II}w_I = w$$

by the definition of G. □

Theorem 2.4.8. *$F\colon \mathsf{a}\Delta \to \mathsf{Atl}$ is an isomorphism of involutive categories. Hence $\mathsf{a}\Delta$ is a presentation of Atl via generators and relations.*

Proof. Obvious from Corollary 2.4.6 and Proposition 2.4.7. □

Corollary 2.4.9. *Each word $w \in \mathrm{Mor}(\mathsf{a}\Delta)$ has a unique standard form.*

Proof. Each Atl tangle has a unique decomposition as $T_{III} \circ T_{II} \circ T_I$. Note T_{III}^* and T_I have unique decompositions as in Proposition 2.2.35 which correspond under the isomorphism of categories to decompositions as in Proposition 2.3.14. We are finished. □

2.5 The Annular Category from Two Cyclic Categories

The Cyclic Category

In this subsection, we recover Jones' result in [Jon00] that there are two copies of (the opposite of) the cyclic category $\mathsf{c}\Delta^{\mathsf{op}}$ in $\mathsf{a}\Delta \cong \mathsf{Atl}$. We will recycle the notation t from Section 1. The definitions from this section are adapted from [Lod98].

Definition 2.5.1. Let cAtl^+ be the subcategory of Atl with objects $[n]$ for $n \in \mathbb{N}$ such that for $m,n \in \mathbb{N}$, $\mathsf{cAtl}(m,n)$ is the set of annular (m,n)-tangles with no loops, only shaded caps, and only unshaded cups. Let cAtl^- be the image of cAtl^+ under the involution of Atl, i.e. $\mathsf{cAtl}^-(m,n)$ is the set of annular (m,n)-tangles with no loops, only unshaded caps, and only shaded cups.

Remark 2.5.2. Clearly $\mathsf{cAtl}^+ \cong \mathsf{cAtl}^-$.

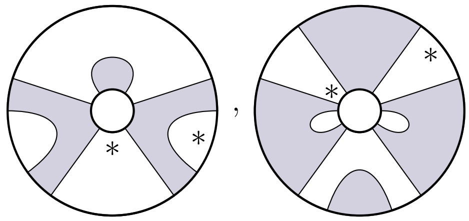


Figure 2.19: Examples of morphisms in cAtl^+ and cAtl^- respectively.

Definition 2.5.3. The opposite of the cyclic category $\mathsf{c}\Delta^{\mathsf{op}}$ is given by

Objects: $[n]$ for $n \in \mathbb{Z}_{\geq 0}$ and

Morphisms: generated by

$$\begin{aligned} d_i &: [n] \longrightarrow [n-1] \text{ for } i = 0, \ldots, n \text{ where } n \geq 1 \\ s_i &: [n] \longrightarrow [n+1] \text{ for } i = 0, \ldots, n \text{ where } n \geq 0 \\ t &: [n] \longrightarrow [n] \text{ where } n \geq 0 \end{aligned}$$

subject to the relations

(1) $d_i d_j = d_{j-1} d_i$ for $i < j$.

(2) $s_i s_j = s_{j+1} s_i$ for $i \leq j$,

(3) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1, \end{cases}$

(4) $t^{n+1} = \mathrm{id}_{[n]}$,

(5) $d_i t = t d_{i-1}$ for $1 \leq i \leq n$, and

(6) $s_i t = t s_{i-1}$ for $1 \leq i \leq n$.

Remark 2.5.4. The opposite of the simplicial category $\mathsf{s}\Delta^{\mathsf{op}}$ is the subcategory of $\mathsf{c}\Delta^{\mathsf{op}}$ generated by the d_i's and the s_i's subject to relations (1)-(3).

Remark 2.5.5. Similar to Proposition 2.3.6, we have the additional relations in $\mathsf{c}\Delta^{\mathsf{op}}$ that $d_0 t = d_n$ and $s_0 t = t^2 s_n$.

Definition 2.5.6. For $n \in \mathbb{Z}_{\geq 0}$, we define $s_{-1} \colon [n] \to [n+1]$ by $s_{-1} = t s_n$. This map is called the extra degeneracy.

Remark 2.5.7. In [Lod98], Loday names this map s_{n+1}. However, we will use the name s_{-1} considering Proposition 2.5.8, Corollary 2.5.16, and the fact that if R is a unital commutative ring, A is a unital R-algebra, and $C_\bullet$ is the cyclic R-module (see Section 2.6) arising from the Hochschild complex with coefficients in A, then $C_n = A^{\otimes n+1}$, and

$$\begin{aligned} s_{-1}(a_0 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes \cdots \otimes a_n, \\ s_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \text{ for } 0 \le i \le n-1, \text{ and} \\ s_n(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_n \otimes 1. \end{aligned}$$

Proposition 2.5.8. *The following additional relations hold for $s_{-1} \in \mathsf{c}\Delta^{\mathsf{op}}(n, n+1)$:*

(1) $s_{-1}s_i = s_{i+1}s_{-1}$ *for all* $i \ge 0$,

(2) $d_i s_{-1} = \begin{cases} \mathrm{id}_{[n]} & \textit{if } i = 0 \\ s_{-1}d_{i-1} & \textit{if } 1 \le i \le n \\ t & \textit{if } i = n+1, \textit{and} \end{cases}$

(3) $s_0 t = t s_{-1}$.

Proof. (1) Using relations (2) and (6), we get

$$s_{-1}s_i = ts_{n+1}s_i = ts_i s_n = s_{i+1}ts_n = s_{i+1}s_{-1}.$$

(2) Using Remark 2.5.5, we have $d_0 s_{-1} = d_0 t s_n = d_n s_n = \mathrm{id}_{[n]}$. If $1 \le i \le n$, then using relations (3) and (5), we have

$$d_i s_{-1} = d_i t s_n = t d_{i-1} s_n = t s_{n-1} d_{i-1} = s_{-1} d_{i-1}.$$

Finally, $d_{n+1}s_{-1} = d_{n+1}ts_n = td_n s_n = t\,\mathrm{id}_{[n]} = t$.

(3) Using Remark 2.5.5, we have $s_0 t = t^2 s_n = t s_{-1}$.

□

Remark 2.5.9. We may now add s_{-1} to the list of generators of $\mathsf{c}\Delta^{\mathsf{op}}$ after appropriately altering relations (3) and (6).

Proposition 2.5.10. *Suppose $w = h_r \cdots h_1$ is a word on $\mathsf{c}\Delta^{\mathsf{op}}$ in $\mathsf{c}\Delta^{\mathsf{op}}(m, n)$ for $m, n \in \mathbb{Z}_{\ge 0}$. Then there is a decomposition $w = w_{III}w_{II}w_I$ such that*

(D) $w_I = d_{i_a} \cdots d_{i_1}$ *with* $i_j > i_{j+1}$ *for all* $j \in \{1, \dots, a-1\}$.

(T) $w_{II} = t^k$ *for some* $k \ge 0$, *and*

(S) $w_{III} = s_{i_b} \cdots s_{i_1}$ *with* $i_j < i_{j+1}$ *for all* $j \in \{1, \dots, b-1\}$,

Proof. The proof is similar to Theorem 2.3.15, but much easier. We proceed by induction on r. If $r = 1$, the result is trivial. Suppose $r > 1$ and the result holds for all words of shorter length. Apply the induction hypothesis to $w' = h_{r-1} \cdots h_1$ to get

$$w' = u_{III} u_{II} u_I$$

satisfying (1)-(3). There are three cases.

(T) Suppose $h_r = t$. Set $w_I = u_I$. Use relation (6) and Remark 2.5.5 to push t to the right of the s_i's. Either it makes it all the way, or it disappears in the process. Define w_{II} accordingly. Order the s_i's using relation (2) to get w_{III}. We are finished.

(D) Suppose $h_r = d_i$. Use relation (3) to push d_i to the right of the s_j's. One of three possibilities occurs:

 (1) We only use the relation $d_i s_j = s_k d_l$. Thus we can push d_i all the way to the right. Now push d_i right of the t's using relation (5) and Remark 2.5.5. Order the s_j's using relation (2) to get w_{III}, define w_{II} in the obvious way, and reorder the d_i's using relation (1) to get w_I. We are finished.

 (2) We use the relation $d_i s_j = \mathrm{id}$, and d_i disappears. Set $w_i = u_i$ for $i \in \{I, II\}$, and order the s_j's using relation (2) to get w_{III}. We are finished.

 (3) We use the relation $d_{n+1} s_{-1} = t$. We are now argue as in Case (T). We are finished.

(S) Suppose $h_r = s_i$. Order $s_i u_{III}$ using relation (2) to get w_{III}, and set $w_i = u_i$ for $i \in \{I, II\}$. We are finished.

□

Theorem 2.5.11. *The following defines an injective functor $H^+ \colon \mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{a}\Delta$:*

Objects: $H^+([n]) = [n+1]$ *for* $n \in \mathbb{Z}_{\geq 0}$*, and*

Morphisms: Let $n \in \mathbb{Z}_{\geq 0}$*.*

(D) For $d_j \in \mathsf{c}\Delta^{\mathsf{op}}(n, n-1)$*, set* $H^+(d_j) = \alpha_{2j+1} \in \mathsf{a}\Delta(n+1, n)$*.*

(T) For $t \in \mathsf{c}\Delta^{\mathsf{op}}(n, n)$*, set* $H^+(t) = \tau \in \mathsf{a}\Delta(n+1, n+1)$*.*

(S) For $s_j \in \mathsf{c}\Delta^{\mathsf{op}}(n, n+1)$*, set* $H^+(s_j) = \beta_{2j+2} \in \mathsf{a}\Delta(n+1, n+2)$*.*

Proof. Clearly H^+ is a functor as the relations are satisfied. Injectivity follows immediately from Corollary 2.4.9 and Proposition 2.5.10. □

Remark 2.5.12. Note that $H^+(s_{-1}) = H^+(ts_n) = H^+(t)H^+(s_n) = \tau\beta_{2n+2} = \beta_{2n+4}\tau$.

Corollary 2.5.13. *The image of* $F \circ H^+ \colon \mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{Atl}$ *is* cAtl^+*. Hence* $\mathsf{c}\Delta^{\mathsf{op}} \cong \mathsf{cAtl}^+$*.*

Proof. It is clear $F \circ H^+$ is injective and lands in cAtl^+ as all generators of $\mathsf{c}\Delta^{\mathsf{op}}$ land in cAtl^+. Surjectivity follows from Theorem 2.2.37. □

Corollary 2.5.14. *A decomposition $w = w_{III}w_{II}w_I$ as in Proposition 2.5.10 is unique.*

Theorem 2.5.15. *The following defines an injective functor $H^-\colon \mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{a}\Delta$:*

Objects: $H^-([n]) = [n+1]$ for $n \in \mathbb{Z}_{\geq 0}$, and

Morphisms: Let $n \in \mathbb{Z}_{\geq 0}$.

(D) For $d_j \in \mathsf{c}\Delta^{\mathsf{op}}(n, n-1)$, set $H^-(d_j) = \alpha_{2j+2} \in \mathsf{a}\Delta(n+1, n)$.

(T) For $t \in \mathsf{c}\Delta^{\mathsf{op}}(n, n)$, set $H^-(t) = \tau \in \mathsf{a}\Delta([n+1], [n+1])$.

(S) For $s_j \in \mathsf{c}\Delta^{\mathsf{op}}(n, n+1)$, set $H^-(s_j) = \beta_{2j+3} \in \mathsf{a}\Delta(n+1, n+2)$.

Proof. Clearly H^- is a functor as the relations are satisfied. Injectivity follows immediately from Corollary 2.4.9 and Proposition 2.5.10. □

Remark 2.5.16. Note that $H^-(s_{-1}) = H^-(ts_n) = H^-(t)H^-(s_n) = \tau\beta_{2n+3} = \beta_1$.

Corollary 2.5.17. *The image of $F \circ H^-\colon \mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{Atl}$ is cAtl^-. Hence $\mathsf{c}\Delta^{\mathsf{op}} \cong \mathsf{cAtl}^-$.*

Remark 2.5.18. cAtl^+ and cAtl^- are exactly the two copies of $\mathsf{c}\Delta^{\mathsf{op}}$ in Atl found by Jones in [Jon00].

Corollary 2.5.19. *There is an isomorphism $\mathsf{c}\Delta \cong \mathsf{c}\Delta^{\mathsf{op}}$.*

Proof. We have $\mathsf{cAtl}^- \cong \mathsf{c}\Delta^{\mathsf{op}} \cong \mathsf{cAtl}^+$. Note the involution in Atl is an isomorphism $\mathsf{cAtl}^+ \cong (\mathsf{cAtl}^-)^{\mathsf{op}}$. The result follows. □

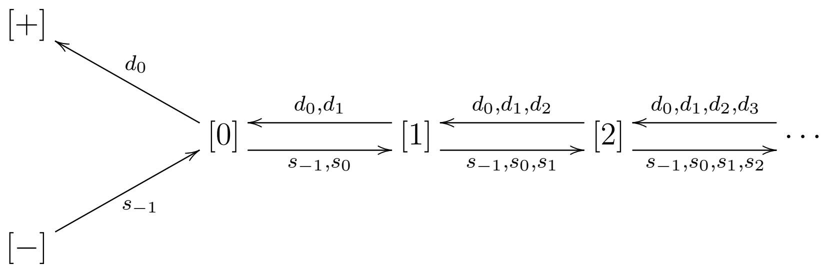
Augmenting the Cyclic Category

Recall from algebraic topology that the reduced (singular, simplicial, cellular) homology of a space X is obtained by inserting an augmentation map $\varepsilon\colon C_0(X) \to \mathbb{Z}$ where $C_0(X)$ denotes the appropriate zero chains. In the language of the semi-simplicial category, we see that this is the same thing as looking at an augmented semi-simplicial abelian group, i.e., a functor from the opposite of the augmented semi-simplicial category, which is obtained from the opposite of the semi-simplicial category (see 2.5.4) by adding an object $[-1]$ and the generator $d_0\colon [0] \to [-1]$ subject to the relation $d_id_j = d_{j-1}d_i$ for $i < j$.

$$[-1] \xleftarrow{d_0} [0] \xleftarrow{d_0, d_1} [1] \xleftarrow{d_0, d_1, d_2} [2] \xleftarrow{d_0, d_1, d_2, d_3} \cdots$$

This immediately raises the question of how one should augment the opposite of the cyclic category. The surprising answer comes from the symmetry arising from the extra degeneracy

s_{-1}. We should add two objects, $[+]$ and $[-]$, and maps $d_0 \colon [0] \to [+]$ and $s_{-1} \colon [-] \to [0]$ subject to the relations $d_i d_j = d_{j-1} d_i$ for $i < j$ and $s_i s_j = s_{j+1} s_i$ for $i \leq j$:



As $t \colon [0] \to [0]$ is the identity, we need not worry about the other relations. Under the isomorphism $\mathsf{c}\Delta^{\mathsf{op}} \cong \mathsf{cAtl}^{+}$ described in the previous subsection, these maps should be represented by the following diagrams:

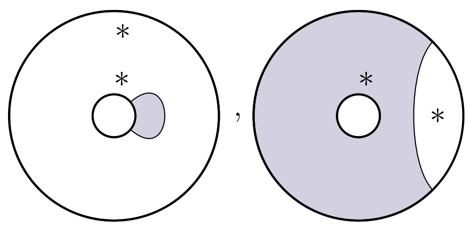


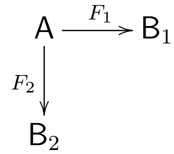
Figure 2.20: Maps $d_0 \colon [0] \to [+]$ and $s_{-1} \colon [-] \to [0]$

Note that these morphisms satisfy the shading convention of cAtl^{+} once we add $[0\pm]$ to the objects of cAtl^{+}. We cannot use just one object as we would then violate the shading convention and closed loops would arise. We will denote the augmented opposite of the cyclic category by $\widetilde{\mathsf{c}\Delta^{\mathsf{op}}}$. For our main result, we will also need to consider the augmented cyclic category $\widetilde{\mathsf{c}\Delta}$, which is just the category $\widetilde{\mathsf{c}\Delta^{\mathsf{op}}}$ with the arrows switched.

Pushouts of Small Categories

Let Cat be the category of small categories. Note that pushouts exist in Cat.

Definition 2.5.20. Suppose $\mathsf{A}, \mathsf{B}_1, \mathsf{B}_2$ are small categories and $F_i \colon \mathsf{A} \to \mathsf{B}_i$ for $i = 1, 2$ are functors. Then the pushout of the diagram



is the small category C defined as follows:

<u>Objects:</u> $\mathrm{Ob}(\mathsf{C})$ is the pushout in Set of the diagram

$$\begin{array}{ccc} \mathrm{Ob}(\mathsf{A}) & \xrightarrow{F_1} & \mathrm{Ob}(\mathsf{B}_1) \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle G_1} \\ \mathrm{Ob}(\mathsf{B}_2) & \xrightarrow[G_2]{} & \mathrm{Ob}(\mathsf{C}) \end{array}$$

This defines maps $G_i\colon \mathrm{Ob}(\mathsf{B}_i) \to \mathrm{Ob}(\mathsf{C})$ for $i = 1, 2$.

<u>Morphisms:</u> For $X, Y \in \mathrm{Ob}(\mathsf{C})$, $\mathrm{Mor}(X, Y)$ is the set of all words of the form $\varphi_n \circ \cdots \circ \varphi_1$ such that

(1) $\varphi_i \in \mathrm{Mor}(\mathsf{B}_1) \cup \mathrm{Mor}(\mathsf{B}_2)$ for all $i = 1, \ldots, n$,

(2) the source of φ_1 is in $G_1^{-1}(X) \cup G_2^{-1}(X)$ and the target of φ_n is in $G_1^{-1}(Y) \cup G_2^{-1}(Y)$,

(3) for all $i = 1, \ldots, n-1$, either

  (i) the target of φ_i is the source of φ_{i+1}, or

  (ii) the target of φ_i is $Z_i \in \mathrm{im}(F_j) \subseteq \mathsf{B}_j$ for some $j \in \{1, 2\}$, and the source of φ_{i+1} is in $F_k(F_j^{-1}(Z_i))$ where $k \neq j$.

subject to the relation $F_1(\psi) = F_2(\psi)$ for every morphism $\psi \in \mathrm{Mor}(\mathsf{A})$.

Notation 2.5.21. In the sequel, we will need to discuss $\widetilde{\mathsf{c}\Delta}$, the augmented cyclic category. In order that no confusion can arise, we will add a $*$ to morphisms to emphasize the fact that they compose in the opposite order. For example, we have generators d_i^* satisfying the relation $d_j^* d_i^* = d_i^* d_{j-1}^*$ for $i < j$.

Definition 2.5.22. Define the small category/groupoid T by

<u>Objects:</u> $[n]$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\pm\}$

<u>Morphisms:</u> Generated by $t\colon [n] \to [n]$ subject to the relation $t^{n+1} = \mathrm{id}_{[n]}$ for $n \in \mathbb{Z}_{\geq 0}$.

Definition 2.5.23. Let PO be the pushout in Cat of the following diagram:

$$\begin{array}{ccc} \mathsf{T} & \xrightarrow{F_1} & \widetilde{\mathsf{c}\Delta^{\mathsf{op}}} \\ {\scriptstyle F_2}\downarrow & & \\ \widetilde{\mathsf{c}\Delta} & & \end{array}$$

where $F_i([n]) = [n]$ for $n \in \mathbb{Z}_{\geq 0} \cup [\pm]$ for $i = 1, 2$ and $F_1(t) = t$ and $F_2(t) = (t^*)^{-1} = (t^{-1})^*$. Note that if $\mathsf{c}\Delta^{\mathsf{op}}$ has generators d_i, s_i, t and $\mathsf{c}\Delta$ has generators d_i^*, s_i^*, t^*, then PO is the category given by

<u>Objects:</u> $[n]$ for $n \in \mathbb{Z}_{\geq 0} \cup \{\pm\}$ and

Morphisms: generated by

$$\begin{aligned} d_0\colon [0] &\to [+] \text{ and } s^*_{-1}\colon [0] \to [-] \\ s_{-1}\colon [+] &\to [0] \text{ and } d^*_0\colon [-] \to [0] \\ d_i, s^*_{i-1}\colon [n] &\longrightarrow [n-1] \text{ for } i = 0, \dots, n \text{ where } n \geq 1 \\ s_i, d^*_{i+1}\colon [n] &\longrightarrow [n+1] \text{ for } i = -1, \dots, n \text{ where } n \geq 0 \\ t\colon [n] &\longrightarrow [n] \text{ where } n \geq 0 \end{aligned}$$

subject to the relations

(1) $d_i d_j = d_{j-1} d_i$ and $s^*_i s^*_j = s^*_{j-1} s^*_i$ for $i < j$,

(2) $s_i s_j = s_{j+1} s_i$ and $d^*_i d^*_j = d^*_{j+1} d^*_i$ for $i \leq j$,

(3) $d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j, j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$ and $s^*_{i-1} d^*_j = \begin{cases} d^*_{j-1} s^*_{i-1} & \text{if } i < j-1 \\ \mathrm{id}_{[n]} & \text{if } i = j, j+1 \\ d^*_j s^*_{i-2} & \text{if } i > j+1, \end{cases}$

(4) $t^{n+1} = \mathrm{id}_{[n]}$,

(5) $d_i t = t d_{i-1}$ for $1 \leq i \leq n$ and $s^*_i t = t s^*_{i-1}$ for $0 \leq i \leq n$, and

(6) $s_i t = t s_{i-1}$ for $0 \leq i \leq n$ and $d^*_i t = t d^*_{i-1}$ for $1 \leq i \leq n$.

Note that $t = (t^*)^{-1}$ as PO is the pushout, so t^* does not appear in the above list.

Remark 2.5.24. Note that PO is involutive using the obvious involution as hinted by the $*$-notation.

Definition 2.5.25. Let $\mathsf{PO}(\delta_+, \delta_-)$ be the small category obtained from PO by adding generating morphisms $\delta\pm\colon [n] \to [n]$ for all $n \in \mathbb{Z}_{\geq 0} \cup \{\pm\}$ which commute with all other morphisms. The maps $\delta_\pm$ are called the coupling constants.

Remark 2.5.26. Note that $\mathsf{PO}(\delta_+, \delta_-)$ is involutive if we define $(\delta_\pm)^* = \delta_\pm$.

Theorem 2.5.27. *$\mathsf{a}\Delta$ is isomorphic to the category Q obtained from $\mathsf{PO}(\delta_+, \delta_-)$ with the additional relations*

(1) $d_i s^*_j = \begin{cases} s^*_{j-1} d_i & \textit{if } i < j \\ s^*_j d^*_{i+1} & \textit{if } j > i \end{cases}$

(2) $d_i d^*_j = \begin{cases} d^*_{j-1} d_i & \textit{if } i < j \\ \delta_- & \textit{if } i = j \end{cases}$

(3) $s_i^* s_j = \begin{cases} s_{j-1}s_i^* & \text{if } i < j \\ \delta_+ & \text{if } i = j \end{cases}$

Proof. Define a map $\Psi \colon \mathsf{a}\Delta \to \mathsf{Q}$ by

Objects: Define $\Psi([0\pm]) = [\pm]$. For $n \geq 1$, define $\Psi([n]) = [n-1]$.

Morphisms: We define Ψ on primitive morphisms:

(A) Define $\Psi(\alpha_1 \in \mathsf{a}\Delta(1, 0+)) = d_0 \in \mathsf{Q}(0, +)$ and $\Psi(\alpha_2 \in \mathsf{a}\Delta(1, 0-)) = s_{-1}^* \in \mathsf{Q}(0, -)$. For $n \geq 2$, define

$$\Psi(\alpha_i \in \mathsf{a}\Delta(n, n-1)) = \begin{cases} s_{(i-3)/2}^* \in \mathsf{Q}(n-1, n-2) & \text{if } i \text{ is odd} \\ d_{(i-2)/2} \in \mathsf{Q}(n-1, n-2) & \text{if } i \text{ is even.} \end{cases}$$

(B) Define $\Psi(\beta_1 \in \mathsf{a}\Delta(0+, 1)) = s_{-1} \in \mathsf{Q}(-, 0)$ and $\Psi(\beta_2 \in \mathsf{a}\Delta(0-, 1)) = d_0^* \in \mathsf{Q}(+, 0)$. For $n \geq 1$, define

$$\Psi(\beta_i \in \mathsf{a}\Delta(n, n+1)) = \begin{cases} s_{(i-3)/2} \in \mathsf{Q}(n-1, n) & \text{if } i \text{ is odd} \\ d_{(i-2)/2}^* \in \mathsf{Q}(n-1, n) & \text{if } i \text{ is even.} \end{cases}$$

(T) For $n \geq 1$, define $\Psi(\tau \in \mathsf{a}\Delta(n, n)) = t \in \mathsf{Q}(n-1, n-1)$.

(D) Define $\Psi(\delta_\pm) = \delta_\pm$.

One checks Ψ is a well defined isomorphism by showing the relations match up. □

Remarks 2.5.28. (1) The above relations are called the coupling relations.

(2) Usually we study representations of $\mathsf{c}\Delta$ and $\mathsf{a}\Delta$ in abelian categories and the coupling constants are multiplication by scalars. These scalars can be built into the coupling relations in our abelian category without first defining $\mathsf{PO}(\delta_+, \delta_-)$. Hence an annular object in an abelian category (see Section 2.6) is obtained from the pushout of two cyclic objects over a T-object and then quotienting out by the coupling relations.

(3) Another way to skip passing to $\mathsf{PO}(\delta_+, \delta_-)$ is to take the linearization of all our categories over some unital commutative ring R (make the morphism sets R-modules) and choose scalars $\delta_\pm$ for the coupling relations.

2.6 Annular Objects

Definition 2.6.1. An annular object in an arbitrary category C is a functor $\mathsf{a}\Delta \to \mathsf{C}$. A cyclic object is a functor $\mathsf{c}\Delta^{\mathsf{op}} \to \mathsf{C}$. If C is an abelian category and $X_\bullet$ is an annular, respectively cyclic, object, we replace $X_\bullet(\tau \in \mathsf{a}\Delta(n, n))$ with $(-1)^{n-1}X_\bullet(\tau)$, respectively we replace $X_\bullet(t \in \mathsf{c}\Delta^{\mathsf{op}}(n, n))$ with $(-1)^n X_\bullet(t)$, to account for the sign of the cyclic permutation.

Remarks 2.6.2. Each annular object has two restrictions to cyclic objects.

Notation 2.6.3. Usually such a functor is denoted with a bullet subscript, e.g. $X_\bullet$. If $X_\bullet$ is such a functor, we will use the following standard notation:

(1) $X_\bullet([n]) = X_n$ for $n \in \mathbb{Z}_{\geq 0}$ and $X_\bullet([0\pm]) = X_\pm$ where applicable.

(2) $X_\bullet(\varphi) = \varphi$, i.e. we will use the same notation for the images of the morphisms in the category C.

Note 2.6.4. For an annular object in an abelian category, relations (4), (5), and (6) become

(4') $\alpha_i \tau = -\tau \alpha_{i-2}$ for $i \geq 3$,

(5') $\beta_i \tau = -\tau \beta_{i-2}$ for $i \geq 3$, and

(6') if $\alpha_i \beta_j \colon [n] \to [n]$ with $n \geq 2$ and $(i,j) = (1, 2n+2), (2n+2, 1)$, then $\alpha_1 \beta_{2n+2} = (-1)^{n-1} \tau^{-1}$ and $\alpha_{2n+2} \beta_1 = (-1)^{n-1} \tau$.

Proposition 2.3.6 becomes

(1') $\alpha_1 \tau = (-1)^{n-1} \alpha_{2n-1}$ and $\alpha_2 \tau = (-1)^{n-1} \alpha_{2n}$

(2') $\tau \beta_{2n+1} = (-1)^n \beta_1$, $\tau \beta_{2n+2} = (-1)^n \beta_2$, and

(3') $\beta_1 \tau = (-1)^{n-1} \tau^2 \beta_{2n-1}$ and $\beta_2 \tau = (-1)^{n-1} \tau^2 \beta_{2n}$.

Note 2.6.5. For a cyclic object in an abelian category, relations (5) and (6) become

(5') $d_i t = -t d_{i-1}$ for $i \geq 1$ and

(6') $s_i t = -t s_{i-1}$ for $i \geq 1$.

Following Remark 2.5.5, we have

(i) $d_0 t = (-1)^n d_n$ and

(ii) $s_0 t = (-1)^n t^2 s_n$.

Definition 2.5.6 becomes $s_{-1} = (-1)^{n+1} t s_n$. Parts (2) and (3) of Proposition 2.5.8 become

(2') $d_{n+1} s_{-1} = (-1)^n t$ and

(3') $s_0 t = -t s_{-1}$.

Remark 2.6.6. The necessity of this sign convention becomes apparent in calculations with Connes' boundary map (see 2.6.19 and 2.6.20).

Definition 2.6.7. Let C be an involutive category and suppose $X_\bullet \colon \mathsf{a}\Delta \to \mathsf{C}$ is an annular object in C. Then $X_\bullet^* \colon \mathsf{a}\Delta \to \mathsf{C}$ is also an annular object in C where

Objects: $X^*_\bullet([n]) = X_n$ for all $n \in \mathbb{N} \cup \{0\pm\}$, and

Morphisms: $X^*_\bullet(w) = X_\bullet(w^*)^*$ for all $w \in \mathrm{Mor}(\mathsf{a}\Delta)$.

If C is abelian, then $X^*_\bullet$ still satisfies the sign convention.

Remark 2.6.8. The representation theory of Atl was studied extensively by Graham and Lehrer in [GL98] and Jones in [Jon01]. In Definition/Theorem 2.2 in [Pet10], Peters gives a good summary of the case of an annular C^*-Hilbert module where $\delta_\pm$ is given by multiplication by $\delta > 2$.

Homologies of Annular Modules

As the semi-simplicial, simiplicial, and cyclic categories live inside $\mathsf{a}\Delta$, we can define Hochschild and cyclic homologies of annular objects in abelian categories. We will focus on annular modules and leave the generalization to an arbitrary abelian category to the reader. Fix a unital commutative ring R.

Definition 2.6.9. Given a semi-simplicial R-module $M_\bullet$, define the Hochschild boundary $b\colon M_n \to M_{n-1}$ for $n \geq 1$ by

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

The Hochscild homology of $M_\bullet$ is

$$HH_n(M_\bullet, b) = \ker(b)/\operatorname{im}(b)$$

for $n \geq 0$, where we set $M_{-1} = 0$, and $b\colon M_0 \to M_{-1}$ is the zero map.

Remark 2.6.10. As an annular R-module is a semi-simplicial R-module in two ways, we will have two Hochschild boundaries.

Definition 2.6.11. Suppose $X_\bullet$ is an annular R-module. Let $X^\pm_\bullet$ be the cyclic object obtained from $X_\bullet$ by restricting $X_\bullet$ to $G(\mathsf{cAtl}^\pm)$. For $n \geq 1$, define

$$HH^\pm_n(X_\bullet) = HH^\pm_{n-1}(X^\pm_\bullet).$$

Remark 2.6.12. The Hochschild boundaries of $X^\pm_\bullet$ for $n \geq 2$ are

$$b_+ = \sum_{i=0}^{n-1} (-1)^i \alpha_{2i+1} \text{ and } b_- = \sum_{i=0}^{n-1} (-1)^i \alpha_{2i+2}.$$

Definition 2.6.13. The above definition does not take into account $X_\pm$. We may define the reduced Hochschild homology by looking at the corresponding augmented cyclic objects (see Subsection 2.5). Define $b_\pm \colon X_1 \to X_\pm$ by $b_+ = \alpha_1 \colon X_1 \to X_+$ and $b_- = \alpha_2 \colon X_1 \to X_-$. Define the reduced Hochschild homology of $X_\bullet$ as follows:

$$\begin{aligned} \widetilde{HH}_n^\pm(X_\bullet) &= HH_n^\pm(X_\bullet) \text{ for } n \geq 2, \\ \widetilde{HH}_1^\pm(X_\bullet) &= \ker(b_\pm)/\operatorname{im}(b_\pm), \text{ and} \\ \widetilde{HH}_0^\pm(X_\bullet) &= \operatorname{coker}(b_\pm) \end{aligned}$$

Remark 2.6.14. The content of the next proposition was found by Jones in [Jon00].

Proposition 2.6.15. *Let $X_\bullet$ be an annular R-module. Then for all $n \geq 1$,*

$$\begin{aligned} \beta_1 b_+ + b_+\beta_1 &= \delta_+ \operatorname{id}_{X_n} \quad \textit{and} \\ \beta_2 b_- + b_-\beta_2 &= \delta_- \operatorname{id}_{X_n}, \end{aligned}$$

and when $n = \pm$,

$$\begin{aligned} b_+\beta_1 &= \delta_+ \operatorname{id}_{X_+} \quad \textit{and} \\ b_-\beta_2 &= \delta_- \operatorname{id}_{X_-}. \end{aligned}$$

Proof. This follows immediately from relation 6. □

Corollary 2.6.16. *If $\delta_\pm$ is multiplication by an element of $R^\times$, the group of units of R, then $\widetilde{HH}_n^\pm(X_\bullet) = 0$ for all $n \geq 0$.*

Corollary 2.6.17. *Let $N \subset M$ be an extremal, finite index II_1-subfactor, and let $X_\bullet$ be the annular $\mathbb{C}$-module given by its tower of relative commutants (see [Jon99], [Jon01]). Then $\widetilde{HH}_n^\pm(X_\bullet) = 0$ for all $n \geq 0$.*

Example 2.6.18 ($TL_\bullet(\mathbb{Z}, 0)$)**.** When $\delta_\pm \notin R^\times$, we can have non-trivial homology. For example, for $n \in \mathbb{N} \cup \{0\pm\}$, let $TL_n(\mathbb{Z}, 0)$ be the set of $\mathbb{Z}$-linear combinations of planar n-tangles with no input disks and no loops (adjust the definition of an annular n-tangle so that there is no D_1). The action of $T \in \mathsf{a}\Delta(m, n)$ on $S \in TL_m(\mathbb{Z}, 0)$ is given by tangle composition $F(T) \circ S$ with the additional requirement that if there are any closed loops, we get zero. We then extend this action $\mathbb{Z}$-linearly. Then $HH_n^\pm(TL_\bullet(\mathbb{Z}, 0)) \neq 0$ for all $n \geq 0$. In fact, the class of the planar n-tangle with only shaded, respectively unshaded, cups is a nontrivial element in $HH_n^\pm(TL_\bullet(\mathbb{Z}, 0))$ respectively. Clearly all such tangles are in $\ker(b_\pm)$. However, it is only possible to get an even multiple of this tangle in $\operatorname{im}(b_\pm)$. If a shaded region is capped off by an α_i to make a cup, there must be two ways of doing so. Using MAGMA [BCP97], the author has calculated the first few (+) reduced Hochschild homology groups of $TL_\bullet(\mathbb{Z}, 0)$ to be

$$\begin{aligned} \widetilde{HH}_0^+ &= \widetilde{HH}_1^+ = \mathbb{Z}, \\ HH_2^+ &= HH_3^+ = \mathbb{Z}/2, \\ HH_4^+ &= HH_5^+ = \mathbb{Z}/6, \text{ and} \\ HH_6^+ &= HH_7^+ = \mathbb{Z}/2 \oplus \mathbb{Z}/2. \end{aligned}$$

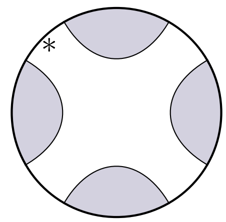


Figure 2.21: Representative for a nontrivial element in $HH_4^+(TL_\bullet(\mathbb{Z}, 0))$

For $n \geq 2$, the class of the tangle described above contributes a copy of $\mathbb{Z}/2\mathbb{Z}$. The question still remains whether this parity continues.

Definition 2.6.19. Given a cyclic R-module $C_\bullet$, define the cyclic bicomplex $BC_{**}(C_\bullet)$ be the bicomplex

$$\begin{array}{ccccccc}
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
C_3 & \xleftarrow{B} & C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
C_2 & \xleftarrow{B} & C_1 & \xleftarrow{B} & C_0 & & \\
\downarrow b & & \downarrow b & & & & \\
C_1 & \xleftarrow{B} & C_0 & & & & \\
\downarrow b & & & & & & \\
C_0 & & & & & &
\end{array}$$

where b is the Hochschild boundary obtained by looking at the corresponding semi-simplicial R-module, $B = (1-t)s_{-1}N \colon C_n \to C_{n+1}$ is Connes' boundary map, and

$$N = \sum_{i=0}^{n} t^i.$$

Recall $s_{-1} = (-1)^{n+1}ts_n$ is the extra degeneracy. The cyclic homology of $C_\bullet$ is given by $HC_n(C_\bullet) = H_n(\mathrm{Tot}(BC_{**}(C_\bullet)))$.

Remark 2.6.20. In order for $BC_{**}(C_\bullet)$ to be a bicomplex, we need b^2, B^2, and $bB + Bb$ to equal zero. While the first two are trivial, we must use Loday's sign convention to get the third. Setting

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i \colon C_n \to C_{n-1},$$

we have $b(1-t) = (1-t)b'$, $b's_{-1} + s_{-1}b' = \mathrm{id}$, and $b'N = Nb$, so

$$bB + Bb = b(1-t)s_{-1}N + (1-t)s_{-1}Nb = (1-t)(b's_{-1} + s_{-1}b')N = (1-t)N = 0.$$

Without this sign convention, we no longer have $bB + Bb = 0$.

Definition 2.6.21. Suppose $X_\bullet$ is an annular R-module. Then $X_\bullet$ becomes a cyclic module in two ways, so we have two cyclic homologies to study. For $n \geq 1$, define $HC_n^\pm(X_\bullet) = HC_{n-1}(X_\bullet^\pm)$.

Remark 2.6.22. For $n \geq 1$, $B_\pm \colon X_n \to X_{n+1}$ is given by

$$\begin{aligned} B_+ &= (-1)^n(1-\tau)(\tau\beta_{2n})\sum_{i=0}^{n-1}\tau^i = (-1)^n(1-\tau)(\beta_{2n+2}\tau)\sum_{i=0}^{n-1}\tau^i \\ &= (-1)^n(1-\tau)\beta_{2n+2}\sum_{i=0}^{n-1}\tau^i \text{ and} \\ B_- &= (-1)^n(1-\tau)\beta_1\sum_{i=0}^{n-1}\tau^i \end{aligned}$$

as the two extra degeneracies for $G(\mathsf{cAtl}^\pm)$ are $(-1)^n\tau\beta_{2n}$ and $(-1)^n\beta_1$ respectively.

Corollary 2.6.23. *If $\delta_\pm$ is multiplication by an element of $R^\times$, the group of units of R, then $HC_n^\pm(X_\bullet) = R$ for all odd $n \geq 1$ and $HC_n^\pm(X_\bullet) = 0$ for all even $n \geq 2$.*

Corollary 2.6.24. *Let $N \subset M$ be an extremal, finite index II_1-subfactor, and let $X_\bullet$ be the annular $\mathbb{C}$-module given by its tower of relative commutants. Then $HC_n^\pm(X_\bullet) = \mathbb{C}$ for all odd $n \geq 1$ and $HC_n^\pm(X_\bullet) = 0$ for all even $n \geq 2$.*

Example 2.6.25. Once again using MAGMA [BCP97], the author has calculated the first few (+) cyclic homology groups of $TL_\bullet(\mathbb{Z}, 0)$ to be

$$\begin{aligned} HC_1^+ &= \mathbb{Z} \\ HC_2^+ &= \mathbb{Z}/2 \\ HC_3^+ &= \mathbb{Z}/2 \oplus \mathbb{Z} \\ HC_4^+ &= \mathbb{Z}/2 \oplus \mathbb{Z}/6 \\ HC_5^+ &= \mathbb{Z}/2 \oplus \mathbb{Z}/6 \oplus \mathbb{Z}, \text{ and} \\ HC_6^+ &= \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/6. \end{aligned}$$

Chapter 3

The embedding theorem for finite depth subfactor planar algebras

3.1 Introduction

A powerful method of construction of subfactors is the use of commuting squares, which are systems of four finite dimensional von Neumann algebras

$$\begin{array}{ccc} A_{1,0} & \subset & A_{1,1} \\ \cup & & \cup \\ A_{0,0} & \subset & A_{0,1} \end{array}$$

included as above, with a faithful trace on $A_{1,1}$ so that $A_{1,0}$ and $A_{0,1}$ are orthogonal modulo their intersection $A_{0,0}$.

One iterates the basic construction of [Jon83] for the inclusions $A_{i,j} \subset A_{i,j+1}$ and $A_{i,j} \subset A_{i+1,j}$ to obtain a tower of inclusions $A_{0,n} \subset A_{1,n}$. By a lovely compactness argument of Ocneanu [JS97],[EK98], the standard invariant, or higher relative commutants, of the inductive limit inclusion $A_{0,\infty} \subset A_{1,\infty}$ are the algebras $A'_{0,1} \cap A_{n,0}$. Thus once bases have been chosen, the calculation of the relative commutants is a matter of elementary linear algebra.

It was to formalise this calculation that planar algebras were first introduced [Jon99]. Finite dimensional inclusions are given by certain graphs (Bratteli diagrams), and in [Jon00], a planar algebra associated purely combinatorially to a bipartite graph was introduced so that it is rather obviously the tower of relative commutants for an inclusion $B_0 \subset B_1$ having the graph as its Bratteli diagram. But because Ocneanu's notion of connection was never completely formalised in [Jon99], it was NOT proved that the planar algebra coming from a commuting square via Ocneanu compactness is a planar subalgebra of the one defined in [Jon00] for the graph of the inclusion $A_{0,0} \subset A_{1,0}$.

Meanwhile the theory of planar algebras grew in its own right and a new method of constructing subfactors evolved by looking at planar subalgebras of a given planar algebra

[Pet10],[BMPS09]. Now if a subfactor is of finite depth, then by [Pop90], there is a commuting square that constructs a hyperfinite model of it. Moreover the inclusion $A_{0,0} \subset A_{1,0}$ for this canonical commuting square has Bratteli diagram given by the so-called principal graph, which is a powerful subfactor invariant. Thus if the the result of the previous paragraph had been proved, it would have implied the following theorem, which is the main result of this paper:

Theorem. *A finite depth subfactor planar algebra is a planar subalgebra of the bipartite graph planar algebra of its principal graph.*

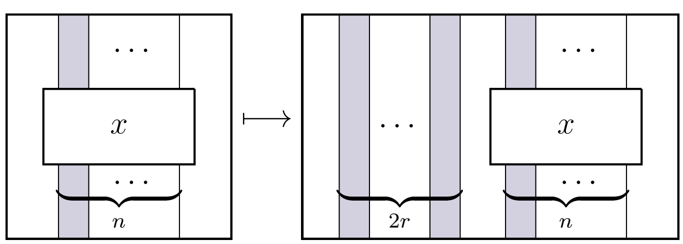
(See [MPS10] for the definition of the principal graph of a planar algebra.)

We prove this result with the interesting twist of not using connections. In particular, our proof does not invoke the dual principal graph, which is perhaps rather surprising.

There are three steps to our proof. The first step, Section 3.2, is to define a canonical planar $*$-algebra structure on the tower of relative commutants from a connected unital inclusion of finite dimensional C*-algebras whose Bratteli diagram is a given graph. We call this the canonical planar $*$-algebra associated to the inclusion. We do this in more generality, replacing finite dimensionality by a strong Markov property (see Definition 3.2.8), because it is no harder and should have applications.

The second step, Section 3.3, is to identify the canonical planar $*$-algebra with the bipartite graph planar algebra of [Jon00] in the finite dimensional case. Loops on the Bratteli diagram for the inclusion give bases for the relative commutants, so the isomorphism is constructed by choosing bases for the vector spaces in the canonical planar $*$-algebra.

Finally, in Section 3.4, we construct the embedding map as follows: given a finite depth subfactor planar algebra $Q_\bullet$, pick $2r$ suitably large so that the inclusion $Q_{2r,+} \subset Q_{2r+1,+} \subset (Q_{2r+2,+}, e_{2r+1})$ is standard, i.e., isomorphic to the basic construction. Set $M_0 = Q_{2r,+}$ and $M_1 = Q_{2r+1,+}$, and let $P_\bullet$ be the canonical planar $*$-algebra $P_\bullet$ associated to the inclusion $M_0 \subset M_1$. We prove in Theorem 3.4.1 that the map $Q_\bullet \to P_\bullet$ given by adding $2r$ or $2r+1$ strings on the left, depending on whether we are in $Q_{n,+}$ or $Q_{n,-}$ respectively, is an inclusion of planar algebras.



While this paper was being written, Morrison and Walker in [MW10] produced a totally different proof which constructs an embedding directly from the planar algebra $Q_\bullet$ without the use of algebra towers and centralisers. Their method also has the advantage that it applies to infinite depth subfactor planar algebras without alteration!

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3.2 The canonical planar $*$-algebra of a strongly Markov inclusion of finite von Neumann algebras

After defining the notion of a strongly Markov inclusion of finite von Neumann algebras, we show the basic construction is also strongly Markov with the same (Watatani) index. We then define the canonical planar $*$-algebra associated to a strongly Markov inclusion.

Many results of this section can be found in [Jon83], [PP86], [Wat90], [Jol90], [Pop94], [Bis97], and [Bur03], but our treatment differs slightly, so we provide some proofs for the reader's convenience.

Bases, traces, and strongly Markov inclusions

Notation 3.2.1. Throughout this paper, a trace on a finite von Neumann algebra means a faithful, normal, tracial state unless otherwise specified. We will write $M_0 \subset (M_1, \mathrm{tr}_1)$ to mean $M_0 \subset M_1$ is an inclusion of finite von Neumann algebras where tr_1 is a trace on M_1. We set $\mathrm{tr}_0 = \mathrm{tr}_1 |_{M_0}$.

Let $M_0 \subset (M_1, \mathrm{tr}_1)$. Let $M_2 = \langle M_1, e_1 \rangle = JM_0'J \subset B(L^2(M_1, \mathrm{tr}_1))$ be the basic construction, where e_1 is the Jones projection with range $L^2(M_0, \mathrm{tr}_0)$, and $J\colon L^2(M_1, \mathrm{tr}_1) \to L^2(M_1, \mathrm{tr}_1)$ is the antilinear unitary given by the antilinear extension of $x\Omega \mapsto x^*\Omega$, where $\Omega \in L^2(M_1, \mathrm{tr}_1)$ is the image of $1 \in M_1$.

Recall that there is a unique trace-preserving conditional expectation $E_{M_0}\colon M_1 \to M_0$ determined by $\mathrm{tr}_1(xy) = \mathrm{tr}_0(E_{M_0}(x)y)$ for all $x \in M_1$ and $y \in M_0$, i.e., E_{M_0} is the (Banach) adjoint of the inclusion of preduals $(M_0)_* \to (M_1)_*$ [Tak02]. The conditional expectation satisfies $e_1(x\Omega) = E_{M_0}(x)\Omega$ for all $x \in M_1$.

The following proposition is straightforward:

Proposition 3.2.2. *The following are equivalent for a finite subset $B = \{b\} \subset M_1$:*

(i) $1 = \sum_{b \in B} be_1b^*$,

(ii) $x = \sum_{b \in B} bE_{M_0}(b^*x)$ *for all* $x \in M_1$, *and*

(iii) $x = \sum_{b \in B} E_{M_0}(xb)b^*$ *for all* $x \in M_1$.

Definition 3.2.3. A Pimsner-Popa basis for M_1 over M_0 is a finite subset $B = \{b\} \subset M_1$ for which the conditions in Proposition 3.2.2 hold.

We refer the reader to [Wat90] for the proof of the following:

Proposition 3.2.4. *The following are equivalent:*

(i) There is a Pimsner-Popa basis for M_1 over M_0,

(ii) $M_1 \otimes_{M_0} M_1 \to M_2$ by $x \otimes y \mapsto xe_1y$ is an $M_1 - M_1$ bimodule isomorphism, and

(iii) $M_2 = M_1e_1M_1$.

Remark 3.2.5. $M_1 \otimes_{M_0} M_1$ is a $*$-algebra with multiplication $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 \otimes E_{M_0}(y_1x_2)y_2$ and adjoint $(x \otimes y)^* = y^* \otimes x^*$. If there is a Pimsner-Popa basis for M_1 over M_0, the sum $\sum_{b\in B} b \otimes b^*$ is independent of the choice of Pimsner-Popa basis B, as it is the identity. (We will renormalize in Proposition 3.2.25.)

Definition 3.2.6 ([Wat90])**.** If there is a Pimsner-Popa basis $B = \{b\}$ for M_1 over M_0, then we define the (Watatani) index

$$[M_1 \colon M_0] = \sum_{b\in B} bb^*,$$

which is independent of the choice of basis.

Definition 3.2.7. Recall from [Pop94] that M_2 has a canonical faithful, normal, semifinite trace Tr_2 which is the extension of the map $xe_1y \mapsto \operatorname{tr}_1(xy)$ for $x, y \in M_1$.

Definition 3.2.8. An inclusion $M_0 \subset (M_1, \operatorname{tr}_1)$ of finite von Neumann algebras is called Markov if it satisfies the Markov property:

(1) Tr_2 is finite with $\operatorname{Tr}_2(1)^{-1} \operatorname{Tr}_2|_{M_1} = \operatorname{tr}_1$.

A Markov inclusion is called strongly Markov if

(2) there is a Pimsner-Popa basis for M_1 over M_0.

Remark 3.2.9. Markov inclusions have been studied by Jolissaint [Jol90], Pimsner, Popa [PP86], [Pop94], and more. In [Jol90], Jolissaint showed that condition (1) implies condition (2) when the centers are atomic and the inclusion is connected, i.e., $Z(M_0)\cap Z(M_1) = M_1'\cap M_0$ is one dimensional. It is unknown to the authors at this point whether condition (1) implies condition (2) for connected inclusions with diffuse centers.

The adjective "strongly" in the term "strongly Markov" comes from Definition 3.6 in [BDH88], where they define the notion of "fortement d'indice fini" for a conditional expectation. This notion translates as the existence of a finite Pimsner-Popa basis.

Remark 3.2.10. Recall from [Pop94] that $\operatorname{Tr}_2(1)^{-1} \operatorname{Tr}_2$ extends tr_1 if and only if $\operatorname{Tr}_2(1) = [M_1 \colon M_0] \in [1, \infty)$.

Examples 3.2.11. (1) A finite Jones index inclusion of II_1-factors with the unique trace is strongly Markov, and the Watatani index is equal to the Jones index.

(2) A connected, unital inclusion of finite dimensional C*-algebras with the Markov trace is strongly Markov, and the index is equal to $\|\Lambda^T\Lambda\|$ where Λ is the bipartite adjacency matrix for the Bratteli diagram of the inclusion.

Suppose $M_0 \subset (M_1, \mathrm{tr}_1)$ is strongly Markov. Then M_2 is finite and $\mathrm{tr}_2 = [M_1\colon M_0]^{-1}\,\mathrm{Tr}_2$ extends tr_1, so we may iterate the basic construction for $M_1 \subset (M_2, \mathrm{tr}_2)$. Let $M_3 = \langle M_2, e_2\rangle \subset B(L^2(M_2, \mathrm{tr}_2))$, where e_2 is the Jones projection with range $L^2(M_1, \mathrm{tr}_1)$. Let Tr_3 be the canonical faithful, normal, semifinite trace on M_3 (see Definition 3.2.7). The following lemma is straightforward:

Lemma 3.2.12. *(1) The conditional expectation $E_{M_1}\colon M_2 \to M_1$ is given by $E_{M_1}(xe_1y) = xy$,*

(2) $e_1e_2e_1 = [M_1\colon M_0]^{-1}e_1$ and $e_2e_1e_2 = [M_1\colon M_0]^{-1}e_2$, and

(3) if B is a Pimsner-Popa basis for M_1 over M_0, then $\left\{[M_1\colon M_0]^{1/2}be_1 \middle| b \in B\right\}$ is a Pimsner-Popa basis for M_2 over M_1.

Theorem 3.2.13. *$M_1 \subset (M_2, \mathrm{tr}_2)$ is strongly Markov and $[M_2\colon M_1] = [M_1\colon M_0]$.*

Proof. Note $M_3 = M_2e_2M_2$ by Proposition 3.2.4 and Lemma 3.2.12, so the canonical trace Tr_3 on M_3 is finite. By Definition 3.2.7 and Lemma 3.2.12, if $x \in M_2$,

$$\mathrm{Tr}_3(x) = [M_1\colon M_0]\sum_{b\in B}\mathrm{Tr}_3(xbe_1e_2e_1b^*) = [M_1\colon M_0]\sum_{b\in B}\mathrm{tr}_2(xbe_1b^*) = [M_1\colon M_0]\,\mathrm{tr}_2(x).$$

Hence $[M_2\colon M_1] = \mathrm{Tr}_3(1) = [M_1\colon M_0]$, and $\mathrm{tr}_3 = [M_1\colon M_0]^{-1}\,\mathrm{Tr}_3$ extends tr_2. □

Definition 3.2.14. Suppose $P \subset B(L^2(M_1, \mathrm{tr}_1))$ is a von Neumann algebra containing M_1, tr_P is a trace on P extending tr_1, and p is a projection in P. We say the inclusion $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ is <u>standard</u> if there is an isomorphism of von Neumann algebras $\varphi\colon P \to M_2$ such that $\varphi|_{M_1} = \mathrm{id}_{M_1}$, $\mathrm{tr}_P = \mathrm{tr}_2 \circ\varphi$, and $\varphi(p) = e_1$.

The following lemma, which is an alteration of Lemma 5.8 of [Jol90] and uses ideas from Lemma 5.3.1 in [JS97], allows us to identify when inclusions are standard:

Lemma 3.2.15. *Suppose $M_0 \subset M_1 \subset (P, \mathrm{tr}_P, p)$ such that*

(1) $pmp = E_{M_0}(m)p$ for all $m \in M_1$, and

(2) $E_{M_1}(p) = [M_1\colon M_0]^{-1}$.

Then $\psi\colon M_1 \otimes_{M_0} M_1 \to M_1pM_1$ by $x\otimes y \mapsto xpy$ is an M_1-bilinear isomorphism of $*$-algebras. Hence $\varphi\colon M_1e_1M_1 \to M_1pM_1$ by $xe_1y \mapsto xpy$ is an isomorphism of $*$-algebras. Moreover, if

(3) $P = M_1pM_1$,

then $M_0 \subset M_1 \subset (P, \operatorname{tr}_P, p)$ *is standard via* φ. *Conversely, if* $M_0 \subset M_1 \subset (P, \operatorname{tr}_P, p)$ *is standard, then (1), (2), and (3) hold.*

Proof. First, note that $px = xp$ for all $x \in M_0$ by (1), and the map $M_1 \to M_1 p$ by $y \mapsto yp$ is injective by (2). Clearly ψ is surjective and preserves the $*$-algebra structure. Suppose

$$\psi\left(\sum_{i=1}^{k} x_i \otimes y_i\right) = \sum_{i=1}^{k} x_i p y_i = 0.$$

Then for all $x, y \in M_1$,

$$px\left(\sum_{i=1}^{k} x_i p y_i\right) yp = \left(\sum_{i=1}^{k} E_{M_0}(xx_i)E_{M_0}(y_i y)\right) p = 0 \Longrightarrow \sum_{i=1}^{k} E_{M_0}(xx_i)E_{M_0}(y_i y) = 0.$$

If $B = \{b\}$ is a Pimsner-Popa basis for M_1 over M_0, by Remark 3.2.5,

$$\sum_{i=1}^{k} x_i \otimes y_i = \sum_{a\in B} a \otimes a^* \left(\sum_{i=1}^{k} x_i \otimes y_i\right) \sum_{b \in B} b \otimes b^* = \sum_{a,b\in B} \sum_{i=1}^{k} a \otimes E_{M_0}(a^* x_i) E_{M_0}(y_i b) b^* = 0.$$

The remaining claims follow as in [Jol90]. □

The Jones tower and tensor products

We give the background necessary to define the canonical planar $*$-algebra associated to a Markov inclusion and to prove its uniqueness. Many facts stated without proof in Subsection 3.2 rely on the results of this subsection. In particular, the multistep basic construction described in this subsection helps us understand tangles which cap off on the left (see Proposition 3.2.47), which are crucial to the proof of Theorem 3.4.1, the main result of this paper.

For the rest of this section, let $M_0 \subset (M_1, \operatorname{tr}_1)$ be a strongly Markov inclusion of finite von Neumann algebras, and set $d = [M_1\colon M_0]^{1/2}$. For $n \in \mathbb{N}$, inductively define the basic construction

$$M_{n+1} = \langle M_n, e_n\rangle = M_n e_n M_n = J_n M_{n-1}' J_n \subset B(L^2(M_n, \operatorname{tr}_n))$$

with canonical trace tr_{n+1} extending tr_n and satisfying $\operatorname{tr}_{n+1}(xe_n) = d^{-2}\operatorname{tr}_n(x)$ for all $x \in M_n$ where $e_n \in B(L^2(M_n, \operatorname{tr}_n))$ is the Jones projection with range $L^2(M_{n-1}, \operatorname{tr}_{n-1})$. For $n \in \mathbb{N}$, set $E_n = de_n$.

Fact 3.2.16. *The* E_i*'s satisfy the Temperley-Lieb relations:*

(i) $E_i^2 = dE_i = dE_i^*$,

(ii) $E_iE_j = E_jE_i$ *for* $|i - j| > 1$, *and*

(iii) $E_i E_{i\pm 1} E_i = E_i$.

Proposition 3.2.17. *Suppose $N \subset (M, \mathrm{tr}_M)$ and $M \subset (P, \mathrm{tr}_P)$ such that $\mathrm{tr}_P|_M = \mathrm{tr}_M$. Suppose $A = \{a\}$ is a Pimsner-Popa basis for P over M and $B = \{b\}$ is a Pimsner-Popa basis for M over N. Then*

(1) $AB = \{ab | a \in A \text{ and } b \in B\}$ is a Pimsner-Popa basis for P over N,

(2) $[P\colon N] = [P\colon M][M\colon N]$, and

(3) $\sum_{b\in B} b e_N^P b^* = e_M^P \in B(L^2(P, \mathrm{tr}_P))$, where e_N^P is the projection $L^2(P, \mathrm{tr}_P) \to L^2(N, \mathrm{tr}_N)$ and e_M^P is the projection $L^2(P, \mathrm{tr}_P) \to L^2(M, \mathrm{tr}_M)$.

Proof. (1) For all $x \in P$,

$$\sum_{ab\in AB} abE_N^P(b^*a^*x) = \sum_{a,b} abE_N^M(E_M^P(b^*a^*x)) = \sum_{a,b} abE_N^M(b^*E_M^P(a^*x)) = \sum_{a} aE_M^P(a^*x) = x.$$

(2) Immediate from (1).

(3) If $p \in P$ and $\Omega \in L^2(P, \mathrm{tr}_P)$ is the image of $1 \in P$, then

$$\sum_{b\in B} be_N^P b^* p\Omega = \sum_{b\in B} bE_N^P(b^*p)\Omega = \sum_{b\in B} bE_N^M(b^*(E_M^P(p))\Omega = E_M^P(p)\Omega = e_M^P p\Omega.$$

□

Corollary 3.2.18. *$M_k \subset (M_n, \mathrm{tr}_n)$ is strongly Markov for all $0 \leq k \leq n$.*

The following technical lemma will be used to define the multistep basic construction in Proposition 3.2.20.

Lemma 3.2.19. *For all $0 \leq k \leq n$, let*

$$f_{n-k}^n = d^{k(k-1)}(e_n e_{n-1} \cdots e_{n-k+1})(e_{n+1} e_n \cdots e_{n-k+2}) \cdots (e_{n+k-1} e_{n+k-2} \cdots e_n) \in M_{n+k}.$$

If $0 \leq j \leq k \leq n$ and B is a Pimsner-Popa basis for M_{n-j} over M_{n-k}, then $\sum_{b\in B} bf_{n-k}^n b^* = f_{n-j}^n$.

Proof. For $j+1 \leq i \leq k$, let A_i be a Pimsner-Popa basis for M_{n-i+1} over M_{n-i}. Then $A = A_{j+1} \cdots A_k$ is a Pimsner-Popa basis for M_{n-j} over M_{n-k} by Proposition 3.2.17, and

$$\begin{aligned}\sum_{\substack{a_i\in A_i\\ j+1\leq i\leq k}} a_{j+1}\cdots a_k f_{n-k}^n a_k^* \cdots a_{j+1}^* &= \sum_{\substack{a_i\in A_i\\ j+1\leq i\leq k-1}} a_{j+1}\cdots a_{k-1} f_{n-k+1}^n a_{k-1}^* \cdots a_{j+1}^* \\ &= \cdots = \sum_{a_{j+1}\in A_{j+1}} a_{j+1} f_{n-j-1}^n a_{j+1}^* = f_{n-j}^n.\end{aligned}$$

For B another Pimsner-Popa basis for M_{n-j} over M_{n-k}, define $U \in \mathrm{Mat}_{|A|\times|B|}(M_{n-k})$ by $U_{a,b} = E^{M_{n-j}}_{M_{n-k}}(a^*b)$. If we consider A as a row vector in $\mathrm{Mat}_{1\times|A|}(M_{n-j})$, then $B = AU$ and $A = BU^*$. For $\ell \in \mathbb{N}$, let $F_\ell = f^n_{n-k} I_\ell \in \mathrm{Mat}_{\ell\times\ell}(M_{n+k})$, i.e., F_ℓ is the $\ell \times \ell$ diagonal matrix with all diagonal entries equal to f^n_{n-k}. Then since f^n_{n-k} commutes with M_{n-k}, we have

$$\sum_{b\in B} b f^n_{n-k} b^* = BF_{|B|}B^* = AUF_{|B|}U^*A^* = AUU^*F_{|A|}A^* = AF_{|A|}A^* = \sum_{a\in A} a f^n_{n-k} a^* = f^n_{n-j}.$$

□

Forms of the next proposition appear in [PP88], [Jol90], and [Bis97].

Proposition 3.2.20 (Multistep Basic Construction). *The inclusion*

$$M_{n-k} \subset M_n \subset (M_{n+k}, \mathrm{tr}_{n+k}, f^n_{n-k})$$

is standard. (See Remark 3.2.45).

Proof. Let B be a Pimsner-Popa basis for M_n over M_{n-k}. Then by Lemma 3.2.19,

$$\sum_{b\in B} b f^n_{n-k} b^* = 1,$$

so $M_n f^n_{n-k} M_n = M_{n+k}$. It is straightforward to check $f^n_{n-k} x f^n_{n-k} = E_{M_{n-k}}(x) f^n_{n-k}$ for all $x \in M_n$ and $E_{M_n}(f^n_{n-k}) = d^{-2k}$, and the result follows by Lemma 3.2.15. □

Remark 3.2.21. Note that $L^2(M_n, \mathrm{tr}_n)$ has left and right actions of $M_0, \dots, M_{2n}$, where as usual, the right action of M_i is the left action of $J_n M_i J_n \cong M_i^{\mathrm{op}}$. Note that $M_i' = J_n M_{2n-i} J_n$, so we define a canonical trace on $M_i' \cap B(L^2(M_n, \mathrm{tr}_n))$ by $\mathrm{tr}_i'(x) = \mathrm{tr}_{2n-i}(J_n x^* J_n)$ for all $x \in M_i' \cap B(L^2(M_n, \mathrm{tr}_n))$.

Proposition 3.2.22 (Shifts). *For all $0 \le k \le n$, there is a canonical isomorphism $M_k' \cap M_n \cong M_{k+2}' \cap M_{n+2}$.*

Proof. On $B(L^2(M_n, \mathrm{tr}_n))$, the map $x \mapsto J_n x^* J_n$ gives an anti-isomorphism $M_k' \cap M_n \cong M_n' \cap M_{2n-k}$. On $B(L^2(M_{n+1}, \mathrm{tr}_{n+1}))$, the map $y \mapsto J_{n+1} y^* J_{n+1}$ gives an anti-isomorphism $M_n' \cap M_{2n-k} \cong M_{k+2}' \cap M_{n+2}$. □

Proposition 3.2.23. *The canonical trace-preserving conditional expectation $M_{n+k} \to M_{n+k-i}$ is given by $x f^n_{n-k} y \mapsto d^{-2i} x f^n_{n-k+i} y$ where $x, y \in M_n$. The canonical trace-preserving conditional expectation $M_{n-k}' = J_n M_{n+k} J_n \to J_n M_{n+k-i} J_n = M_{n-k+i}'$ is given by the same formula, only with $x, y \in M_n' = J_n M_n J_n$.*

Proof. We prove the first statement, as the second is similar. By the Markov property, for all $x, y \in M_n$,

$$\mathrm{tr}_{n+k}(x f^n_{n-k} y) = d^{-2k}\,\mathrm{tr}_n(xy) = d^{-2i}\,\mathrm{tr}_{n+k-i}(x f^n_{n-k+i} y),$$

so the map is trace-preserving. Now M_{n+k-i}-bilinearity follows from the following two facts:

(i) for all $1 \leq i \leq k$, $M_{n-k} \subset M_{n-k+i}$, so $f^n_{n-k+i} f^n_{n-k} = f^n_{n-k}$, and

(ii) $E^{M_{n+k}}_{M_{n+k-i}}(f^n_{n-k}) = d^{-2i} f^n_{n-k+i}$.

□

We can now strengthen Proposition 2.7 from [Bis97], versions of which also appear in [Bur03]. This is the main proposition describing left-capping tangles.

Proposition 3.2.24. *Let $0 \leq k \leq \ell \leq n$, and let B be a Pimsner-Popa basis for M_ℓ over M_k. The conditional expectation $E^{M'_k}_{M'_\ell} \colon (M'_k \cap B(L^2(M_n, \mathrm{tr}_n)), \mathrm{tr}'_k) \to (M'_\ell \cap B(L^2(M_n, \mathrm{tr}_n)), \mathrm{tr}'_\ell)$ is given by*

$$E^{M'_k}_{M'_\ell}(x) = \frac{1}{d^{2(\ell-k)}} \sum_{b \in B} b x b^*.$$

In particular, this map is independent of n and the choice of basis.

Proof. The result follows from Lemma 3.2.19 and Proposition 3.2.23, since for $x, y \in J_n M_n J_n \subset M'_\ell$,

$$\sum_{b \in B} b x f^n_k y b^* = \sum_{b \in B} x b f^n_k b^* y = x f^n_\ell y.$$

□

To define our planar $*$-algebra in Subsection 3.2, we need the following fact, which follows from Proposition 3.2.4 and a simple induction argument.

Proposition 3.2.25. *For $k \in \mathbb{N}$, let $v_k = E_k E_{k-1} \cdots E_1$. For all $n \in \mathbb{N}$, there are isomorphisms of $M_1 - M_1$ bimodules*

$$\theta_n \colon \bigotimes_{M_0}^{n} M_1 \longrightarrow M_n \ \text{ by}$$
$$x_1 \otimes \cdots \otimes x_n \longmapsto x_1 v_1 x_2 v_2 \cdots v_{n-1} x_n.$$

Remark 3.2.26. Recall that $L^2(M_n, \mathrm{tr}_n)$ is the completion of M_n with inner product $\langle x, y \rangle = \mathrm{tr}_n(y^* x)$. As usual, θ_n gives an isomorphism of Hilbert-bimodules

$$\bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow L^2(M_n, \mathrm{tr}_n)$$

where the tensor product on the left is Connes' relative tensor product with inner product given inductively by

$$\langle x_1 \otimes u, y_1 \otimes v \rangle_n = \langle E_{M_0}(y_1^* x_1) u, v \rangle_{n-1}$$
$$\langle u \otimes x_n, v \otimes y_n \rangle_n = \langle u, v E_{M_0}(y_n x_n^*) \rangle_{n-1}.$$

The following operators will be useful in the definition of the rotation operators in Subsections 3.2 and 3.2.

Definition 3.2.27. Given $x \in M_1$, we get

(1) left and right multiplication operators

$$L(x), R(x)\colon \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1)$$

by $L(x)(v) = xv$ and $R(x)(v) = vx$, and

(2) left and right creation operators

$$L_x, R_x\colon \bigotimes_{M_0}^{n} L^2(M_1, \mathrm{tr}_1) \longrightarrow \bigotimes_{M_0}^{n+1} L^2(M_1, \mathrm{tr}_1)$$

by $L_x(v) = x \otimes v$ and $R_x(v) = v \otimes x$.

Fact 3.2.28. *For $x \in M_1$, we have*

$$L_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = E_{M_0}(x^* y_1) y_2 \otimes \cdots \otimes y_{n+1} \text{ and}$$
$$R_x^*(y_1 \otimes \cdots \otimes y_{n+1}) = y_1 \otimes \cdots \otimes y_n E_{M_0}(y_{n+1} x^*).$$

The following lemma will be instrumental in defining the action of tangles.

Lemma 3.2.29. *If A is a $\mathbb{C}$-algebra, V_1 is a right A-module, V_2 is an $A-A$ bimodule, and V_3 is a left A-module, then for each A-invariant $v_2 \in V_2$, the map*

$$v_1 \otimes v_3 \longmapsto v_1 \otimes v_2 \otimes v_3$$

defines a linear map $\phi_{v_2}\colon V_1 \otimes_A V_3 \to V_1 \otimes_A V_2 \otimes_A V_3$. Moreover, the map $v \mapsto \phi_v$ on $A' \cap V_2 = \{v \in V_2 | av = va \text{ for all } a \in A\}$ is $\mathbb{C}$-linear.

Proof. Middle A-linearity is satisfied as v_2 is A-invariant. □

Remark 3.2.30. This lemma gives an alternate proof that the map $E_{M_0'}^{M_1'}$ is well defined in Proposition 3.2.24. By Remark 3.2.5, $d^{-2}\sum_{b\in B} b \otimes b^*$ is independent of the choice of Pimsner-Popa basis B, so the composite map

$$x \longmapsto \phi_x \longmapsto \phi_x\left(d^{-2}\sum_{b\in B} b \otimes b^*\right) = d^{-2}\sum_{b\in B} b \otimes x \otimes b^* \longmapsto d^{-2}\sum_{b\in B} bxb^*$$

on $M_0' \cap B(L^2(M_n, \mathrm{tr}_n))$ is independent of the choice. Moreover, the result is M_1-invariant, since for any unitary $u \in M_1$, $\{ub | b \in B\}$ is another Pimsner-Popa basis for M_1 over M_0.

Definition of the canonical planar $*$-algebra

The definition of a planar $*$-algebra has evolved since its inception in [Jon99]. We use the definition of [Jon10] (see also [Pet10]), but we do not reproduce it here.

In [Jon99], it was shown how to endow the tower of relative commutants of an extremal, finite index II_1-subfactor with the structure of a subfactor planar algebra, i.e., a planar $*$-algebra $Q_\bullet = \{Q_{n,\pm}\}$ with $\dim(Q_{n,\pm}) < \infty$ for all $n \geq 0$ which is

- Spherical: $\dim(Q_{0,\pm}) = 1$ and any fully labelled 0-tangle is invariant under spherical isotopy. This implies shaded and unshaded contractible loops count for the same multiplicative factor of d, called the modulus of $Q_\bullet$.

- Positive-definite: The bilinear form on $Q_{n,\pm}$ given by $\langle a, b\rangle = d^{-n}\operatorname{tr}(b^*a)$ is positive definite.

The only essential ingredient to the construction of [Jon99] is a Pimsner-Popa basis, so the same construction applies to a strongly Markov inclusion $M_0 \subset (M_1, \operatorname{tr}_1)$. As we do not require the algebras to be factors or the inclusion to be extremal, the resulting planar algebra need not be spherical nor positive-definite nor have finite dimensional n-box spaces.

Below, we define a planar $*$-algebra structure on the vector spaces $P_{n,\pm}$ $(n \geq 0)$ given by $P_{n,+} = \theta_n^{-1}(M_0' \cap M_n)$ and $P_{n,-} = \theta_n^{-1}(M_1' \cap M_{n+1})$. This planar algebra is independent of any choices, so we call it the canonical planar $*$-algebra associated to $M_0 \subset (M_1, \operatorname{tr}_1)$.

We define the action of a planar tangle in standard form:

(1) all the input and output disks are horizontal rectangles with all strings (that are not closed loops) emanating from the top edges of the rectangles,

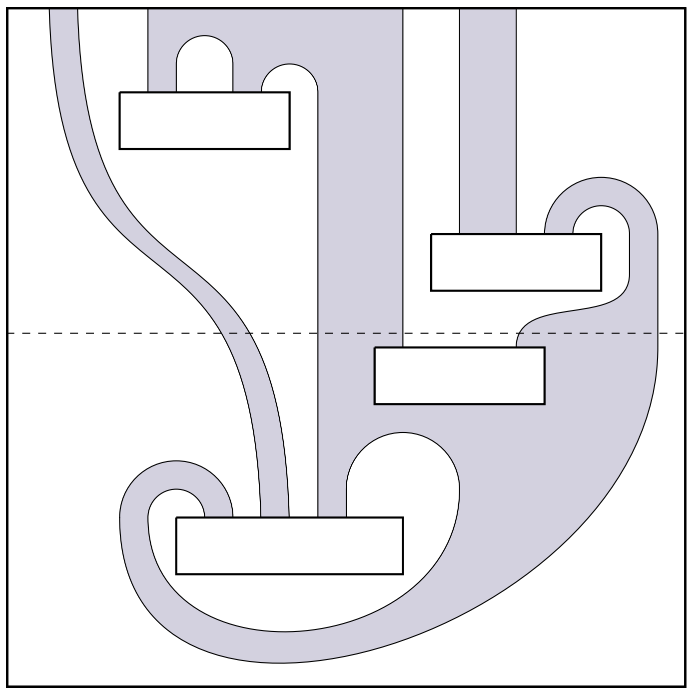
(2) all the input disks are in disjoint horizontal bands and all maxima and minima of strings are at different vertical levels, and not in the horizonal bands defined by the input disks, and

(3) the distinguished (starred) intervals of all the disks are at the left edges of the rectangles. (In the sequel, we will assume this convention and omit the $*$'s.)

We do not provide the proof of isotopy invariance, i.e., that the action is independent of the choice of standard form, as this proof is identical to that in [Jon99]. However, in Subsection 3.2, we provide Burns' elegant proof that the rotation operator is well-defined.

Suppose we have a $(k, \pm)$-tangle T in standard form with s input rectangles, and input rectangle j has $2r_j$ strings emanating from the top. We define the action of T on an s-tuple $\xi = (\xi_1, \cdots, \xi_s)$ where $\xi_j \in P_{r_j,\pm_j}$ and $\pm_j = \pm$ if the region just below input rectangle j is unshaded or shaded respectively.

We read the action of T on ξ by sliding a horizontal line through the tangle from bottom to top. For a fixed vertical y-value, off the input disks' horizontal bands and away from the relative extrema of the strings, the horizontal line will meet n_y shaded regions from left to right. One should think of the shaded regions along this line as elements of M_1 and the unshaded regions between shaded regions as the symbols $\otimes_{M_0}$. Near the top, the line will meet k or $k+1$ shaded regions depending on whether the left-most region of T is unshaded or shaded respectively. We illustrate a typical $(3,+)$-tangle with the horizontal line about half way through its travel:



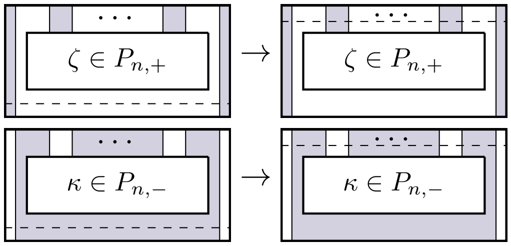
For each y coordinate of the horizontal line, one reads off an M_i-invariant element $\eta_y \in \bigotimes_{M_0}^{n_y} M_1$, where $i=0$ if T is a $(k,+)$-tangle and $i=1$ if T is a $(k,-)$-tangle.

The element η_y begins as $1 \in M_i$ near the bottom, and it remains constant as long as the horizontal line meets neither maxima, minima, nor rectangles. If the horizontal line passes input rectangle j for which exactly t shaded regions sit to the left, then we insert ξ_j into η_y as in Figure 3.1 by applying Lemma 3.2.29 with $v_2 = \xi_j$,

$$V_1 = \bigotimes_{M_0}^{t} M_1, \;\; V_2 = P_{r_j, \pm_j}, \;\; \text{and } \; V_3 = \bigotimes_{M_0}^{n_y - t} M_1.$$

Note that V_1, V_3 are considered as M_ℓ-modules and $P_{r_j,\pm_j}$ is an $M_\ell - M_\ell$ bimodule, where $\ell = 0$ if $\pm_j = +$ and $\ell = 1$ if $\pm_j = -$. Note that inserting ξ_j into η_y gives an M_i-invariant vector.

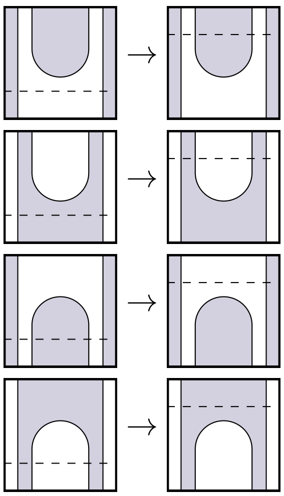
As the horizontal line passes a maximum or minimum, η_y changes according to Figure 3.2 where the changes indicated on the tensors are to be inserted into the position indicated by the shaded regions on the horizontal (dashed) line. With the exception of one case, each of these maps is an $M_1 - M_1$ bimodule map, so it will preserve M_i-invariant elements. The remaining case to consider is when the left-most or right-most shaded region is capped off by applying the third map pictured above, which is an $M_0 - M_0$ bimodule map. But this will



$$x \otimes y \mapsto x \otimes \zeta \otimes y$$

$$x \mapsto x\kappa = \kappa x$$

Figure 3.1: Inserting central vectors



$$x \otimes y \longmapsto x \otimes 1 \otimes y$$

$$x \longmapsto d^{-1} \sum_{b \in B} xb \otimes b^* = d^{-1} \sum_{b \in B} b \otimes b^* x$$

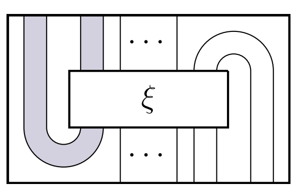
$$x \otimes y \otimes z \longmapsto dxE_{M_0}(y) \otimes z = dx \otimes E_{M_0}(y)z$$

$$x \otimes y \longmapsto xy.$$

Figure 3.2: Reading maxima and minima of planar tangles in standard form

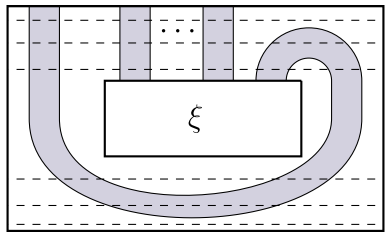
only occur when the distinguished (starred) interval of the external disk meets an unshaded region, so i would have to be 0 from the beginning.

The action of the tangle on ξ is the element $\eta_y \in P_{k,\pm}$ read for horizontal lines sufficiently close to the top. The $*$-structure is the same as that of [Jon99].

Example 3.2.31. To calculate  for

$$\xi = \sum_{i=1}^{k} x_1^i \otimes \cdots \otimes x_n^i \in \theta_n^{-1}(M_0' \cap M_n),$$

we first isotope the tangle into a standard form. The horizontal line travels upward as shown:



,

which we read as:

$$1_{\mathbb{C}} \mapsto 1_M \mapsto d^{-1} \sum_{b \in B} b \otimes b^* \mapsto d^{-1} \sum_{b \in B} b \otimes \xi \otimes b^* \mapsto d^{-1} \sum_{b \in B} \sum_{i=1}^{k} b \otimes x_1^i \otimes \cdots \otimes x_{n-1}^i \otimes x_n^i b^*$$
$$\mapsto \sum_{b \in B} \sum_{i=1}^{k} b \otimes x_1^i \otimes \cdots \otimes x_{n-1}^i E_{M_0}(x_n^i b^*),$$

the last line giving the output of the tangle applied to ξ.

Burns' treatment of the rotation operator on $P_{n,+}$

The key to showing that the $P_{n,\pm}$'s define a planar algebra is isotopy invariance, which relies on the existence of the rotation on $P_{n,\pm}$. A particularly elegant treatment of this is due to Michael Burns, but it only appears in his thesis [Bur03], so we include a proof below for the reader's convenience.

Definition 3.2.32. Let B be a Pimsner-Popa basis of M_1 over M_0. For

$$x = x_1 \otimes \cdots \otimes x_n \in \bigotimes_{M_0}^{n} M_1,$$

define $\rho(x) = \sum_{b \in B} L_b R_b^*(x) = \sum_{b \in B} b \otimes x_1 \otimes \cdots \otimes x_{n-1} E_{M_0}(x_n b^*)$ (see Example 3.2.31).

Proposition 3.2.33. *The map ρ preserves $P_{n,+}$, and its restriction to $P_{n,+}$ is independent of the choice of B.*

Proof. Middle linearity is respected by ρ, so it is well defined, though it may depend on B. By Lemma 3.2.29 and Remark 3.2.5, for M_0-invariant x, the sum

$$\sum_{b \in B} b \otimes x \otimes b^*$$

is independent of B. We obtain ρ by applying an $M_0 - M_0$ bilinear map which does not involve B, so the restriction of ρ is M_0-invariant and independent of B. $\square$

Theorem 3.2.34 ([Bur03])**.** *For $x \in P_{n,+}$ and $y_1, \ldots, y_n \in M_1$,*

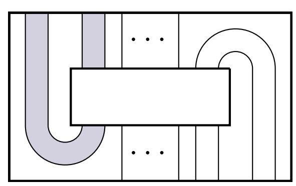
$$\langle \rho(x), y_1 \otimes \cdots \otimes y_n \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle,$$

so $\rho^n = \mathrm{id}$ on $P_{n,+}$.

Proof. As $\rho(x) = \sum_{b \in B} L_b R_b^*(x)$, we have

$$\begin{aligned}\langle \rho(x), y_1 \otimes \cdots \otimes y_n \rangle &= \sum_{b \in B} \langle L_b R_b^* x, y_1 \otimes \cdots \otimes y_n \rangle = \sum_{b \in B} \langle x, R_b L_b^* y_1 \otimes \cdots \otimes y_n \rangle \\ &= \sum_{b \in B} \langle x, E_{M_0}(b^* y_1) y_2 \otimes \cdots \otimes y_n \otimes b \rangle = \sum_{b \in B} \langle E_{M_0}(b^* y_1)^* x, y_2 \otimes \cdots \otimes y_n \otimes b \rangle \\ &= \sum_{b \in B} \langle x E_{M_0}(b^* y_1)^*, y_2 \otimes \cdots \otimes y_n \otimes b \rangle = \sum_{b \in B} \langle x, y_2 \otimes \cdots \otimes y_n \otimes b E_{M_0}(b^* y_1) \rangle \\ &= \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_1 \rangle.\end{aligned}$$

□

Corollary 3.2.35. *The rotation*  *on $P_{n,+}$ is well defined.*

The rotation on $P_{n,-}$

We mimic Burns' treatment of the rotation on $P_{n,+}$ to define the rotation on $P_{n,-}$.

Definition 3.2.36. Let B be a Pimsner-Popa basis of M_1 over M_0. For

$$x = x_1 \otimes \cdots \otimes x_{n+1} \in \bigotimes_{M_0}^{n+1} M_1,$$

define $\sigma(x) = \sum_{b \in B} R(b^*) R_1^* L_b(x) = \sum_{b \in B} b \otimes x_1 \otimes \cdots \otimes x_n E_{M_0}(x_{n+1}) b^*$.

Proposition 3.2.37. *The map σ preserves $P_{n,-}$, and its restriction to $P_{n,-}$ is independent of the choice of B.*

Proof. Similar to Proposition 3.2.33. □

Theorem 3.2.38. *For $x \in P_{n,-}$ and $y_1, \ldots, y_{n+1} \in M_1$,*

$$\langle \sigma(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle = \langle x, y_2 \otimes \cdots \otimes y_n \otimes y_{n+1} y_1 \otimes 1 \rangle.$$

Proof. Similar to Theorem 3.2.34. □

Corollary 3.2.39. $\sigma^n = \mathrm{id}$ *on* $P_{n,-}$.

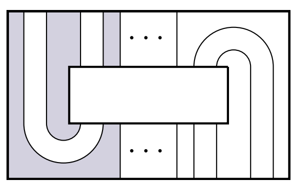
Proof. As σ preserves $P_{n,-}$, we repeatedly apply Theorem 3.2.38 for $x \in P_{n,-}$ to get

$$\begin{aligned}\langle \sigma^n(x), y_1 \otimes \cdots \otimes y_{n+1} \rangle &= \langle \sigma^{n-1}(x), y_2 \otimes \cdots \otimes y_n \otimes y_{n+1}y_1 \otimes 1 \rangle \\ &= \langle \sigma^{n-2}(x), y_3 \otimes \cdots \otimes y_n \otimes y_{n+1}y_1 \otimes y_2 \otimes 1 \rangle \\ &= \cdots = \langle x, y_{n+1}y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle.\end{aligned}$$

We then invoke Burns' trick again to get

$$\begin{aligned}\langle x, y_{n+1}y_1 \otimes y_2 \otimes \cdots \otimes y_n \otimes 1 \rangle &= \langle y_{n+1}^* x, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle \\ &= \langle x y_{n+1}^*, y_1 \otimes \cdots \otimes y_n \otimes 1 \rangle \\ &= \langle x, y_1 \otimes \cdots \otimes y_n \otimes y_{n+1} \rangle.\end{aligned}$$

□

Corollary 3.2.40. *The rotation*  *on* $P_{n,-}$ *is well defined.*

Uniqueness of the canonical planar ∗-algebra

We have the following facts whose proofs are similar to those in [Jon99] and will be omitted (they are straightforward from the results in Subsections 3.2 and 3.2). We shade tangles as much as possible, but sometimes we will not have enough information.

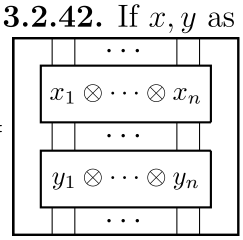
Proposition 3.2.41 (Multiplication)**.** *Suppose* $x, y \in M_n$ *such that*

$$\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n \text{ and } \theta_n^{-1}(y) = y_1 \otimes \cdots \otimes y_n$$

Then

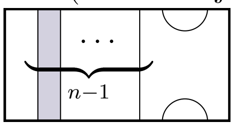
$$\theta_n^{-1}(xy) = \begin{cases} x_1 \otimes \cdots \otimes x_k E_{M_0}(x_{k+1} E_{M_0}(x_{k+2}(\cdots) y_{k-1}) y_k) \otimes y_{k+1} \otimes \cdots \otimes y_{2k} & n = 2k \\ x_1 \otimes \cdots \otimes x_{k+1} E_{M_0}(x_{k+2} E_{M_0}(x_{k+3}(\cdots) y_{k-1}) y_k) y_{k+1} \otimes \cdots \otimes y_{2k+1} & n = 2k+1. \end{cases}$$

Remark 3.2.42. If x, y as above are in $M_i' \cap M_n$ where $i \in \{0, 1\}$, then

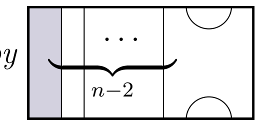
$\theta_n^{-1}(xy) =$ 

where the shading depends on i and the parity of n.

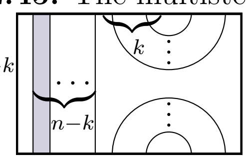
Proposition 3.2.43 ($*$-Structure). *Suppose $x \in M_n$ such that $\theta_n^{-1}(x) = x_1 \otimes \cdots \otimes x_n$. Then $\theta_n^{-1}(x^*) = x_n^* \otimes \cdots \otimes x_1^*$.*

Proposition 3.2.44 (Jones Projections). *(1) For $n \geq 1$, the Jones projection $E_n \in P_{n+1,+}$ is given by* 

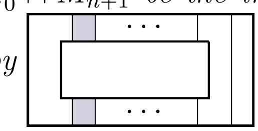
.

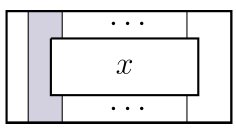
(2) For $n \geq 2$, the Jones projection $E_n \in P_{n,-}$ is given by 

.

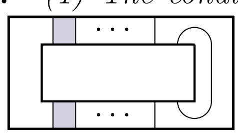
Remark 3.2.45. The multistep basic construction projection of Proposition 3.2.20 is given by $f^n_{n-k} = d^{-k}$ 

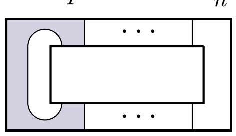
.

Proposition 3.2.46 (Inclusions). *(1) Let $i_n \colon M_0' \cap M_n \to M_0' \cap M_{n+1}$ be the inclusion. Then the inclusion $\theta_{n+1}^{-1} \circ i_n \circ \theta_n \colon P_{n,\pm} \to P_{n+1,\pm}$ is given by* 
.

(2) If $x \in P_{n,-}$, then 

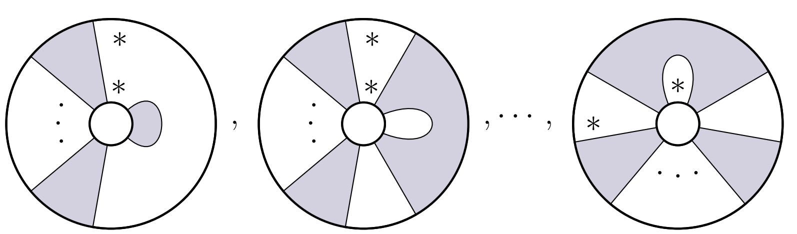
$= x \in P_{n+1,+}$.

Proposition 3.2.47 (Conditional Expectations). *(1) The conditional expectation $\theta_{n-1}^{-1} \circ E_{M_{n-1}} \circ \theta_n \colon P_{n,+} \to P_{n-1,+}$ is given by $d^{-1} \cdot$* 
.

(2) The conditional expectation $\theta_n^{-1} \circ E^{M_0'}_{M_1'} \circ \theta_n \colon P_{n,+} \to P_{n-1,-}$ (see Proposition 3.2.24) is given by $d^{-1} \cdot$ 
.

Notation 3.2.48. We use the notation from [Pen12a]:

(1) Denote the annular capping maps $P_{n,+} \to P_{n-1,+}$ by α_j as shown:



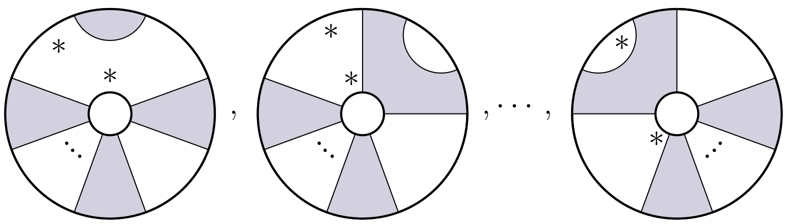
i.e., numbering the boundary points clockwise from $*$, the i^{th} and $(i+1)^{\text{th}}$ (modulo $2n$) internal boundary points are joined by a string and all other internal boundary points are connected to external boundary points such that

(i) If $i = 1$, then the first external point is connected to the third internal point.

(ii) If $1 < i < 2n$, then the first external point is connected to the first internal point.

(iii) If $i = 2n$, then the first external point is connected to the $(2n-1)^{\text{th}}$ internal point.

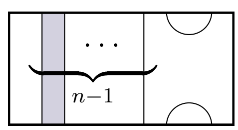
(2) Denote the annular cupping maps $P_{n-1,+} \to P_{n,+}$ by β_j as shown:



i.e., β_j is α_j turned inside out.

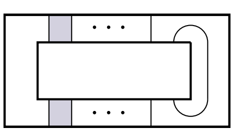
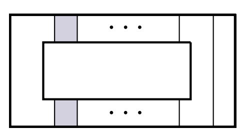
The following lemma is similar to a result in [KS04]:

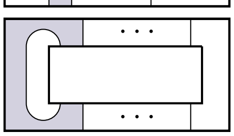
Lemma 3.2.49. *Suppose $P_\bullet$ is a planar $*$-algebra with modulus $d \neq 0$ and $Q_{n,\pm} \subset P_{n,\pm}$ are $*$-subalgebras which are closed under the following operations:*

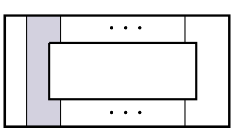
(1) left and right multiplication by tangles $E_n =$ 

$\in P_{n+1,+}$ *for* $n \in \mathbb{N}$*;*

(2) The maps from $P_{n,+}$ as follows:

$\alpha_n =$  $: P_{n,+} \to P_{n-1,+}$, $\beta_{n+1} =$  $: P_{n,+} \to P_{n+1,+}$,

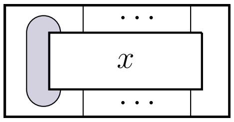
$\gamma_n^+ =$  $: P_{n,+} \to P_{n-1,-}$; *and*

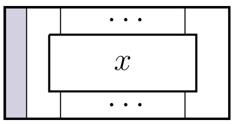
(3) the map $i_n^- =$  $: P_{n,-} \to P_{n+1,+}$.

Then the $Q_{n,\pm}$ define a planar $*$-subalgebra $Q_\bullet \subset P_\bullet$.

Proof. As $Q_{n,\pm}$ is closed under multiplication and $*$, it suffices to show $Q_\bullet$ is closed under all annular maps. To show this, it suffices to show all α_j's, all β_j's, and both rotations by 1 preserve $Q_\bullet$.

First, note that the maps $\gamma_n^- \colon P_{n,-} \to P_{n-1,+}$ and $i_n^+ \colon P_{n,+} \to P_{n+1,-}$ given by

$\gamma_n^-(x) =$  $= \frac{1}{d}\alpha_{n+2}(E_n E_{n-1} \cdots E_1 \cdot \beta_{n+2}(i_n^- x)) \cdot E_1 E_2 \cdots E_n)$ and

$i_n^+(x) =$  $= \gamma_{n+2}^+((E_1 E_2 \cdots E_n) \cdot \beta_{n+2}\beta_{n+1}(x) \cdot (E_{n+1} E_n \cdots E_1))$

preserve $Q_\bullet$.

We show all α_j's preserve $Q_\bullet$. For $j < n$ and $x \in Q_n$,

$$\alpha_j(x) = \frac{1}{d}\alpha_n \alpha_{n+1}((E_n E_{n-1} \cdots E_j) \cdot \beta_{n+1}(x) \cdot (E_n))).$$

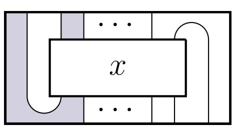
The case $n < j < 2n$ is similar. It is clear $\alpha_{2n}(x) = \alpha_{2n-1}(i_{n-1}^-(\gamma_n^+(x)))$.

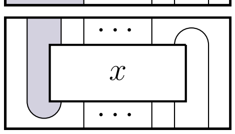
We show all β_j's preserve $Q_\bullet$. If $j < n+1$, we have

$$\beta_j(x) = (E_j E_{j-1} \cdots E_n) \cdot \beta_{n+1}(x).$$

The case $n+1 < j < 2n+2$ is similar. It is clear $\beta_{2n+2}(x) = \alpha_2 \gamma_{n+1}^- \gamma_n^+(x)$.

We show both rotations by 1 preserve $Q_\bullet$. We have

 $= \frac{1}{d}\gamma_{n+1}^+ \alpha_{2n+2} i_{n+1}^- i_n^+ \alpha_n \beta_{n+1}(x)$ and

 $= \alpha_{n+1}\beta_{n+2}\alpha_{2n+1} i_n^-(x).$

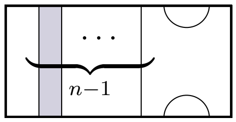
□

Theorem 3.2.50. *Given a strongly Markov inclusion $M_0 \subset (M_1, \mathrm{tr}_1)$, there is a unique planar $*$-algebra $P_\bullet$ of modulus $d = [M_1 \colon M_0]^{1/2}$ where*

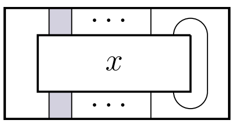
$$P_{n,+} = \theta_n^{-1}(M_0' \cap M_n) \text{ and } P_{n,-} = \theta_{n+1}^{-1}(M_1' \cap M_{n+1})$$

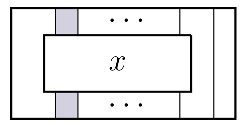
such that the multiplication is given by Remark 3.2.42,

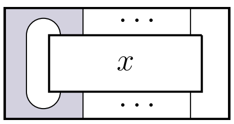
(0) for all tangles T with n input disks, $T(\xi_1^*, \cdots, \xi_n^*) = T^*(\xi_1, \cdots, \xi_n)^*$ where for $\xi_i \in P_{n_i, \pm_i}$, ξ_i^* is as in Proposition 3.2.43 and T^* is the mirror image of T;

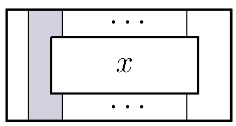
(1) for $n \in \mathbb{N}$, $E_n =$  $\in P_{n+1,+}$;

(2) for $x \in P_{n,+}$ *and* B *a Pimsner-Popa basis for* M_1 *over* M_0,

 $= dE_{M_{n-1}}(x)$,

 $= x \in P_{n+1,+}$, *and*

 $= dE^{M_0'}_{M_1'}(x) = d^{-1} \sum_{b \in B} bxb^*$; *and*

(3) for $x \in P_{n,-}$,  $= x \in P_{n+1,+}$.

Proof. Uniqueness follows from Lemma 3.2.49. Existence follows from the existence of the canonical planar $*$-algebra associated to $M_0 \subset (M_1, \mathrm{tr}_1)$. □

Corollary 3.2.51. *The canonical planar $*$-algebra associated to an extremal, finite index II_1-subfactor is the subfactor planar algebra constructed in [Jon99].*

3.3 The planar algebra isomorphism for finite dimensional C*-algebras

We now restrict our attention to a connected unital inclusion $M_0 \subset M_1$ of finite dimensional C*-algebras with the Markov trace. We show that in this case, the canonical planar $*$-algebra of Theorem 3.2.50 is isomorphic to the bipartite graph planar algebra [Jon00] of the Bratteli diagram.

Many of the results in this section can be found in [GdlHJ89],[JS97],[EK98], but we present them here for completeness and for the reader's convenience.

Loop algebras

We define loop algebras in the spirit of [Jon00] which are another description of Evans, Ocneanu, and Sunder's path algebras [GdlHJ89],[JS97],[EK98], with a more GNS (rather than spatial) flavor.

Notation 3.3.1. For this section, let Γ be a finite, connected, bipartite multi-graph. Let $\mathcal{V}_\pm$ denote the set of even/odd vertices of Γ, and let $\mathcal{E}$ denote the edge set of Γ. Usually we will denote edges by ε and ξ. All edges will be directed from even to odd vertices, so we have source and target functions $s\colon \mathcal{E} \to \mathcal{V}_+$ and $t\colon \mathcal{E} \to \mathcal{V}_-$. We will write ε^* to denote an edge ε traversed from an odd vertex to an even vertex, and we define source and target functions $s\colon \mathcal{E}^* = \{\varepsilon^* | \varepsilon \in \mathcal{E}\} \to \mathcal{V}_-$ and $t\colon \mathcal{E}^* \to \mathcal{V}_+$ by $s(\varepsilon^*) = t(\varepsilon)$ and $t(\varepsilon^*) = s(\varepsilon)$. Let $m_+\colon \mathcal{V}_+ \to \mathbb{N}$ be a dimension (row) vector for the even vertices. For $v \in \mathcal{V}_-$, define the dimension (row) vector for the odd vertices by

$$m_-(v) = \sum_{t(\varepsilon)=v} m_+(s(\varepsilon)).$$

Let Λ be the bipartite adjacency matrix for Γ ($\Lambda_{i,j}$ is the number of times the i^{th} vertex in $\mathcal{V}_+$ is connected to the j^{th} vertex in $\mathcal{V}_-$).

Remark 3.3.2. Given (Γ, m_+), we can associate a connected unital inclusion of finite dimensional C*-algebras $M_0 \subset M_1$. We set

$$M_0 = \bigoplus_{v\in\mathcal{V}_+} M_{m_+(v)}(\mathbb{C}) \text{ and } M_1 = \bigoplus_{v\in\mathcal{V}_-} M_{m_-}(\mathbb{C}),$$

and the inclusion is such that Γ is the Bratteli diagram for the inclusion, and Λ is the inclusion matrix ($\Lambda_{i,j}$ is the number of times the i^{th} simple summand of M_0 is contained in the j^{th} simple summand of M_1). Conversely, given such an inclusion, we get a finite, connected, bipartite multi-graph (the Bratteli diagram) and a dimension vector m_+ (corresponding to the simple summands of M_0).

Definition 3.3.3. Let $G_{0,\pm}$ be the complex vector space with basis $\mathcal{V}_\pm$ respectively. For $n \in \mathbb{N}$, $G_{n,\pm}$ will denote the complex vector space with basis loops of length $2n$ on Γ based at a vertex in $\mathcal{V}_\pm$ respectively.

We discuss the vector spaces $G_{n,+}$. The spaces $G_{n,-}$ are similar, and it is clear what the corresponding notation should be and how they will behave.

Notation 3.3.4. Loops in $G_{n,+}$ will be denoted $[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*]$. Any time we write such a loop, it is implied that

(i) $t(\varepsilon_i) = s(\varepsilon_{i+1}^*) = t(\varepsilon_{i+1})$ for all odd $i < 2n$,

(ii) $t(\varepsilon_i^*) = s(\varepsilon_i) = s(\varepsilon_{i+1})$ for all even $i < 2n$, and

(iii) $t(\varepsilon_{2n}^*) = s(\varepsilon_{2n}) = s(\varepsilon_1)$.

For a loop $\ell = [\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*] \in G_{n,+}$ and $1 \leq k \leq 2n$, we define the following paths in ℓ:

$$\ell_{[1,k]} = \begin{cases} \varepsilon_1\varepsilon_2^* \cdots \varepsilon_{k-1}\varepsilon_k^* & k \text{ even} \\ \varepsilon_1\varepsilon_2^* \cdots \varepsilon_{k-1}^*\varepsilon_k & k \text{ odd} \end{cases}$$

$$\ell_{[k,2n]} = \begin{cases} \varepsilon_k\varepsilon_{k+1}^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^* & k \text{ odd} \\ \varepsilon_k^*\varepsilon_{k+1} \cdots \varepsilon_{2n-1}\varepsilon_{2n}^* & k \text{ even.} \end{cases}$$

Definition 3.3.5. Define an antilinear map $*$ on $G_{n,+}$ by the antilinear extension of the map

$$[\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*]^* = [\varepsilon_{2n}\varepsilon_{2n-1}^* \cdots \varepsilon_2\varepsilon_1^*].$$

There is also an obvious notion of taking $*$ of a path $\gamma_{[j,k]}(\ell)$ for a loop $\ell \in G_{n,+}$. We define a multiplication on $G_{n,+}$ by

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[n+1,2n]},(\ell_2)_{[1,n]}}[(\ell_1)_{[1,n]}(\ell_2)_{[n+1,2n]}].$$

It is clear that $*$ is an involution, i.e., an anti-automorphism of period 2, for $G_{n,+}$ under this multiplication.

Remark 3.3.6. We can think of a loop in $G_{n,+}$ as a path up and down the multi-graph Γ_n corresponding to the Bratteli diagram for the inclusions

$$M_0 \subset M_1 \subset \cdots \subset M_n,$$

which is obtained by reflecting Γ a total of $n-1$ times, as the inclusion matrix of $M_j \subset M_{j+1}$ is given by Λ or Λ^T if j is even or odd, respectively [Jon83].

Definition 3.3.7. Let $\widetilde{\Gamma}$ be the augmentation of the bipartite graph Γ by adding a distinguished vertex $\star$ which is connected to each $v \in \mathcal{V}_+$ by $m_+(v)$ distinct edges. These edges are oriented so they begin at $\star$. We will denote these added edges by $\eta' s$ (and ζ's and κ's when necessary).

Definition 3.3.8. For $n \in \mathbb{Z}_{\geq 0}$, let A_n be the algebra defined as follows: a basis of A_n will consist of loops of length $2n+2$ on $\widetilde{\Gamma}$ of the form

$$[\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*]$$

i.e., the loops start and end at $\star$, but remain in Γ otherwise. Note that we have an obvious $*$-structure on each A_n. Multiplication will be given as follows: if one defines the similar path notation as in Notation 3.3.4, then we have

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[n+2,2n+2]},(\ell_2)_{[1,n+1]}}[(\ell_1)_{[1,n+1]}(\ell_2)_{[n+2,2n+2]}].$$

Remark 3.3.9. We can think of a loop in A_n as a path up and down the multi-graph $\widetilde{\Gamma}_n$ corresponding to the Bratteli diagram for the inclusions

$$\mathbb{C} \subset M_0 \subset M_1 \subset \cdots \subset M_n.$$

Definition 3.3.10 (Inclusions)**.** The inclusion $A_n \to A_{n+1}$ is given by the linear extension of

$$[\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \longmapsto \begin{cases} \displaystyle\sum_{s(\varepsilon)=s(\varepsilon_n)} [\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_n^*\varepsilon\varepsilon^*\varepsilon_{n+1} \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ even} \\ \displaystyle\sum_{s(\varepsilon)=t(\varepsilon_n)} [\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_n\varepsilon^*\varepsilon\varepsilon_{n+1}^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ odd.} \end{cases}$$

We identify A_n with its image in A_{n+1}.

Remark 3.3.11. The inclusion identifications allow us to define a multiplication $A_m \times A_n \to A_{\max\{m,n\}}$ by including A_m, A_n into $A_{\max\{m,n\}}$ and using the regular multiplication. Explicitly, if $\ell_1 \in A_m$ and $\ell_2 \in A_n$ with $m \leq n$, then

$$\ell_1 \cdot \ell_2 = \delta_{(\ell_1)^*_{[m+2,2m+2]},(\ell_2)_{[1,m+1]}}[(\ell_1)_{[1,m+1]}(\ell_2)_{[m+2,2n+2]}].$$

The case $m \geq n$ is similar.

Towers of loop algebras

We provide an isomorphism of the tower $(M_n)_{n\geq 0}$ coming from a connected unital inclusion of finite dimensional C*-algebras with the Markov trace and the corresponding tower $(A_n)_{n\geq 0}$ of loop algebras. Assume the notation of Subsection 3.3.

For $n \geq 0$, if S_i is the i^{th} simple summand of of M_n, then loops ℓ in A_n for which $\ell_{[1,n+1]}$ ends at the corresponding vertex of $\widetilde{\Gamma}_n$ form a system of matrix units for a simple algebra isomorphic to S_i. Hence for $n \in \mathbb{Z}_{\geq 0}$, there is a $*$-algebra isomorphism $A_n \cong M_n$, and $\dim(A_n) = \dim(M_n)$.

At this point, we only choose such isomorphisms $\varphi_n \colon A_n \to M_n$ for $n = 0, 1$ which respects the inclusion given in Definition 3.3.10. In Proposition 3.3.17, we will inductively define isomorphisms $\varphi_n \colon A_n \to M_n$ for $n \geq 2$ to identify the Jones projections.

Definition 3.3.12. Following [Jon83], let λ_i be the Markov trace (column) vector for M_i for $i = 0, 1$ such that

$$m_+\lambda_0 = 1 = m_-\lambda_1,$$

so λ_i gives the traces of minimal projections in the simple summands of M_i for $i = 0, 1$. In order for the trace on M_1 to restrict to the trace on M_0, we must have $\Lambda\lambda_1 = \lambda_0$.

Recall that the inclusion matrix for $M_n \subset M_{n+1}$ is given by Λ if n is even and Λ^T if n is odd. This means that to extend the trace, we must have $\Lambda\Lambda^T\lambda_0 = d^{-2}\lambda_0$, $\Lambda^T\Lambda\lambda_1 = d^{-2}\lambda_1$, and $\lambda_n = d^{-2}\lambda_{n-2}$ for all $n \geq 2$, where λ_n is the Markov trace vector for M_n and $d = \sqrt{\|\Lambda^T\Lambda\|} = \sqrt{\|\Lambda\Lambda^T\|}$.

Definition 3.3.13. Let $\lambda = \begin{pmatrix} \lambda_0 \\ d\lambda_1 \end{pmatrix}$, a Frobenius-Perron eigenvector for $\begin{pmatrix} 0 & \Lambda \\ \Lambda^T & 0 \end{pmatrix}$.

Definition 3.3.14 (Traces)**.** We define a trace on A_0 by

$$\operatorname{tr}_0([\eta_1\eta_2^*]) = \begin{cases} \lambda(t(\eta_1)) = \lambda_0(t(\eta_1)) & \text{if } \eta_1 = \eta_2 \\ 0 & \text{else.} \end{cases}$$

Suppose $\ell = [\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \in A_n$ with $n \geq 1$. We define a trace on A_n by

$$\operatorname{tr}_n(\ell) = \begin{cases} d^{-n}\lambda(s(\varepsilon_n)) & \text{if } n \text{ is even and } \ell = \ell^* \\ d^{-n}\lambda(t(\varepsilon_n)) & \text{if } n \text{ is odd and } \ell = \ell^* \\ 0 & \text{if } \ell \neq \ell^*. \end{cases}$$

Remark 3.3.15. The isomorphisms φ_n for $n = 0, 1$ preserve the trace. Moreover, $\operatorname{tr}_{n+1}|_{A_n} = \operatorname{tr}_n$ for all $n \in \mathbb{N}$ as λ is a Frobenius-Perron eigenvector.

Proposition 3.3.16 (Conditional Expectations)**.** *If $\ell = [\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \in A_n$, the conditional expectation $A_n \to A_{n-1}$ is given by*

$$E_{A_{n-1}}(\ell) = \begin{cases} d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\left(\dfrac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}\right)[\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{n-1}\varepsilon_{n+2}^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ even} \\ d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\left(\dfrac{\lambda(t(\varepsilon_n))}{\lambda(s(\varepsilon_n))}\right)[\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{n-1}^*\varepsilon_{n+2} \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] & n \text{ odd.} \end{cases}$$

Proof. We consider the case n even. The case n odd is similar. We must show $\operatorname{tr}_n(xy) = \operatorname{tr}_{n-1}(E_{A_{n-1}}(x)y)$ for all $x \in A_n$ and $y \in A_{n-1}$. It suffices to check when x, y are loops. If

$$x = [\eta_1\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta_2^*] \text{ and } y = [\eta_3\xi_1\xi_2^* \cdots \xi_{2n-3}\xi_{2n-2}^*\eta_4^*],$$

using the formula above, we have

$$\begin{aligned} \operatorname{tr}_{n-1}(E_{A_{n-1}}(x)y) &= d^{-1}\delta_{\varepsilon_n,\varepsilon_{n+1}}\delta_{y_{[1,n]},x^*_{[n+2,2n+2]}}\frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}\operatorname{tr}_{n-1}([\eta_1\varepsilon_1 \cdots \varepsilon_{n-1}\xi_n^*\xi_{n+1} \cdots \xi_{2n-2}^*\eta_4^*]) \\ &= d^{-n}\delta_{y_{[1,n]},x^*_{[n+2,2n+2]}}\delta_{\varepsilon_n,\varepsilon_{n+1}}\delta_{x_{[1,n]},y^*_{[n+1,2n-2]}}\lambda(s(\varepsilon_n)) = \operatorname{tr}_n(xy). \end{aligned}$$

□

Definition 3.3.17 (Jones Projections)**.** For $n \in \mathbb{N}$, define distinguished elements of A_{n+1} as follows: if n is odd, define

$$F_n = \sum_{\vec{i}}\sum_{t(\eta)=s(\varepsilon_{i_1})}\frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))}[\eta\varepsilon_{i_1}\varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}}^*\varepsilon_{i_n}\varepsilon_{i_n}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}^*\eta^*]$$

where the sum is taken over all vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$ such that

$$[\varepsilon_{i_1}\varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}}^*\varepsilon_{i_n}\varepsilon_{i_n}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}^*] \in G_{n+1,+}$$

If n is even, then define

$$F_n = \sum_{\vec{i}}\sum_{t(\eta)=s(\varepsilon_{i_1})}\frac{[\lambda(s(\varepsilon_{i_n}))\lambda(s(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(t(\varepsilon_{i_n}))}[\eta\varepsilon_{i_1}\varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}}\varepsilon_{i_n}^*\varepsilon_{i_n}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n-1}}^* \cdots \varepsilon_{i_2}\varepsilon_{i_1}^*\eta^*]$$

with a similar limitation on the vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$.

Lemma 3.3.18. *(1) $F_nxF_n = dE_{A_{n-1}}(x)F_n$ for all $x \in A_n$ and*

(2) $\operatorname{tr}_{n+1}(xF_n) = d^{-1}\operatorname{tr}_n(x)$ for all $x \in A_n$, i.e., $E_{A_n}(F_n) = d^{-1}$.

Proof. We prove the case n odd. The case n even is similar.

(1) If $x = [\zeta_1\xi_1\xi_2^* \cdots \xi_{n-1}\xi_n^* \cdots \xi_{2n-1}\xi_{2n}^*\zeta_2^*] \in A_n$, then

$$\begin{aligned}
F_n x F_n &= \sum_{\vec{i}} \sum_{t(\eta)=s(\varepsilon_{i_1})} \frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))}[\eta\varepsilon_{i_1}\varepsilon_{i_2}^* \cdots \varepsilon_{i_{n-1}}^*\varepsilon_{i_n}\varepsilon_{i_n}^*\varepsilon_{i_{n+1}}\varepsilon_{i_{n+1}}^*\varepsilon_{i_{n-1}} \cdots \varepsilon_{i_2}\varepsilon_{i_1}^*\eta^*]\times \\
&\quad x\sum_{\vec{j}} \sum_{t(\kappa)=s(\varepsilon_{j_1})} \frac{[\lambda(t(\varepsilon_{j_n}))\lambda(t(\varepsilon_{j_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{j_n}))}[\kappa\varepsilon_{j_1}\varepsilon_{j_2}^* \cdots \varepsilon_{j_{n-1}}^*\varepsilon_{j_n}\varepsilon_{j_n}^*\varepsilon_{j_{n+1}}\varepsilon_{j_{n+1}}^*\varepsilon_{j_{n-1}} \cdots \varepsilon_{j_2}\varepsilon_{j_1}^*\kappa^*] \\
&= \sum_{s(\xi)=s(\xi_{n-1})} \frac{[\lambda(t(\xi))\lambda(t(\xi_{n+1}))]^{1/2}}{\lambda(s(\xi))}[\zeta_1\xi_1\xi_2^* \cdots \xi_{n-1}^*\xi\xi^*\xi_n\xi_{n+1}^* \cdots \xi_{2n-1}\xi_{2n}^*\zeta_2^*]\times \\
&\quad \sum_{\vec{j}} \sum_{t(\kappa)=s(\varepsilon_{j_1})} \frac{[\lambda(t(\varepsilon_{j_n}))\lambda(t(\varepsilon_{j_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{j_n}))}[\kappa\varepsilon_{j_1}\varepsilon_{j_2}^* \cdots \varepsilon_{j_{n-1}}^*\varepsilon_{j_n}\varepsilon_{j_n}^*\varepsilon_{j_{n+1}}\varepsilon_{j_{n+1}}^*\varepsilon_{j_{n-1}} \cdots \varepsilon_{j_2}\varepsilon_{j_1}^*\kappa^*] \\
&= \delta_{\xi_n,\xi_{n+1}}\frac{\lambda(t(\xi_n))}{\lambda(s(\xi_n))} \sum_{\substack{s(\xi)=s(\xi_{n-1})\\ s(\varepsilon)=s(\xi_{n+2})}} \frac{[\lambda(t(\varepsilon))\lambda(t(\xi))]^{1/2}}{\lambda(s(\varepsilon))}[\zeta_1\xi_1\xi_2^* \cdots \xi_{n-1}^*\xi\xi^*\varepsilon\varepsilon^*\xi_{n+2} \cdots \xi_{2n-1}\xi_{2n}^*\zeta_2^*] \\
&= dE_{A_{n-1}}(x)F_n.
\end{aligned}$$

(2) Another straightforward calculation.

□

Proposition 3.3.19 (Basic Construction)**.** *For $n \in \mathbb{N}$, the inclusion*

$$A_{n-1} \subset A_n \subset (A_{n+1}, \mathrm{tr}_{n+1}, d^{-1}F_n)$$

is standard. Hence for all $k \geq 0$, there are isomorphisms $\varphi_k\colon A_k \to M_k$ preserving the trace such that $\varphi_{k+1}|_{A_k} = \varphi_k$ and $\varphi_m(F_n) = E_n$ for all $m > n$.

Proof. We construct the isomorphisms φ_n for $n \geq 1$ by induction on n. The base case is finished. Suppose we have constructed φ_n for $n \geq 1$. We know that $M_{n+1} = M_nE_nM_n$ and $A_n \cong M_n$ via φ_n. By Lemmata 3.2.15 and 3.3.18, there is an algebra isomorphism $h_{n+1}\colon M_{n+1} = M_nE_nM_n \to A_nF_nA_n \subseteq A_{n+1}$ such that $E_n \mapsto F_n$. But $\dim(M_{n+1}) = \dim(A_{n+1})$, so $A_{n+1} = A_nF_nA_n$, and we set $\varphi_{n+1} = h_{n+1}^{-1}$, which extends φ_n. Finally, note the φ_m's preserve the trace by construction and the uniqueness of the Markov trace. □

Relative commutants are isomorphic to loop algebras

We provide isomorphisms between the relative commutants of the tower $(A_n)_{n\geq 0}$ and the spaces $G_{n,\pm}$.

Proposition 3.3.20 (Central Vectors)**.** *A basis for the central vectors $A_0' \cap A_n$ is given by*

$$S_{0,n} = \left\{ \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*] \in A_n \,\middle|\, [\varepsilon_1\varepsilon_2^* \cdots \varepsilon_{2n-1}\varepsilon_{2n}^*] \in G_{n,+} \right\}.$$

A basis for the central vectors $A_1' \cap A_{n+1}$ is given by

$$S_{1,n+1} = \left\{ \sum_{\substack{t(\eta)=s(\varepsilon)\\ t(\varepsilon)=t(\varepsilon_1)}} [\eta\varepsilon\varepsilon_1^*\varepsilon_2\cdots\varepsilon_{2n-1}^*\varepsilon_{2n}\varepsilon^*\eta^*] \in A_{n+1} \middle| [\varepsilon_1^*\varepsilon_2\cdots\varepsilon_{2n-1}^*\varepsilon_{2n}] \in G_{n,-} \right\}.$$

Proof. Note that if $[\zeta_1\zeta_2^*] \in A_0$, then we have

$$\begin{aligned}
[\zeta_1\zeta_2^*]\cdot \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*] &= \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\zeta_2,\eta}[\zeta_1\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]\\
&= [\zeta_1\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\zeta_2^*] = \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\eta,\zeta_1}[\eta\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\zeta_2^*]\\
&= \left(\sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*]\right)\cdot[\zeta_1\zeta_2^*]
\end{aligned}$$

Hence $S_{0,n} \subset A_0' \cap A_n$. Similarly, $S_{1,n+1} \subset A_1' \cap A_n$.

Suppose now that $x \in A_0' \cap A_n$. Then since $1_{A_0} = \sum_\eta [\eta\eta^*]$, we have

$$x = \left(\sum_\eta [\eta\eta^*]\right) x = \left(\sum_\eta [\eta\eta^*]\cdot[\eta\eta^*]\right) x = \sum_\eta [\eta\eta^*]\cdot x\cdot[\eta\eta^*] \in \operatorname{span}(S_{0,n}).$$

Similarly, $A_1' \cap A_{n+1} \subseteq \operatorname{span}(S_{1,n+1})$. □

Corollary 3.3.21. *There are $*$-algebra isomorphisms*

$$\begin{aligned}
\phi_{n,+}&\colon G_{n,+} \longrightarrow A_0' \cap A_n \text{ and}\\
\phi_{n,-}&\colon G_{n,-} \longrightarrow A_1' \cap A_{n+1}.
\end{aligned}$$

If $n = 0$, the isomorphisms are given by

$$\phi_{0,+}(v_+) = \sum_{t(\eta)=v_+} [\eta\eta^*] \text{ and } \phi_{0,-}(v_-) = \sum_{t(\eta)=s(\varepsilon);t(\varepsilon)=v_-} [\eta\varepsilon\varepsilon^*\eta^*].$$

For $n \in \mathbb{N}$, the isomorphisms are given by

$$\begin{aligned}
\phi_{n,+}([\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*]) &= \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon_{2n}^*\eta^*] \text{ and}\\
\phi_{n,-}([\varepsilon_1^*\varepsilon_2\cdots\varepsilon_{2n-1}^*\varepsilon_{2n}]) &= \sum_{\substack{t(\eta)=s(\varepsilon)\\ t(\varepsilon)=t(\varepsilon_1)}} [\eta\varepsilon\varepsilon_1^*\varepsilon_2\cdots\varepsilon_{2n-1}^*\varepsilon_{2n}\varepsilon^*\eta^*].
\end{aligned}$$

It will be helpful to have an explicit Pimsner-Popa basis for A_1 over A_0:

Proposition 3.3.22 (Pimsner-Popa Bases). *For each $v_+ \in \mathcal{V}_+$, pick a distinguished η_{v_+} with $t(\eta_{v_+}) = v_+$. Set*

$$B_1 = \left\{ \left(\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \right)^{1/2} \sum_{t(\eta)=s(\varepsilon_1)} [\eta\varepsilon_1\varepsilon_2^*\eta^*] \middle| [\varepsilon_1\varepsilon_2^*] \in G_{1,+} \right\} \text{ and}$$
$$B_2 = \left\{ \left(\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \right)^{1/2} [\eta_1\varepsilon_1\varepsilon_2^*\eta_{s(\varepsilon_2)}^*] \middle| s(\varepsilon_1) \neq s(\varepsilon_2) \right\}.$$

Then $B = B_1 \amalg B_2$ is a Pimsner-Popa basis for A_1 over A_0.

Proof. Suppose $x = [\zeta_1\xi_1\xi_2^*\zeta_2^*] \in A_1$.

Case 1: Suppose that $s(\xi_1) = s(\xi_2)$, so $[\xi_1\xi_2^*] \in G_{1,+}$. If $b \in B_2$, then $E_{A_0}(b^*x) = 0$ as the formula will have delta functions $\delta_{\xi_i,\varepsilon_i}$ for $i = 1, 2$. Hence we have

$$\begin{aligned}
\sum_{b\in B} bE_{A_0}(b^*x) = \sum_{b\in B_1} bE_{A_0}(b^*x) &= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{\substack{t(\eta)=s(\varepsilon_1)\\ t(\zeta)=s(\varepsilon_1)}} [\eta\varepsilon_1\varepsilon_2^*\eta^*]E_{A_0}\big([\zeta\varepsilon_2\varepsilon_1^*\zeta^*]\cdot[\zeta_1\xi_1\xi_2^*\zeta_2^*]\big)\\
&= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\varepsilon_1)} \delta_{\zeta_1,\zeta}\delta_{\xi_1,\varepsilon_1}[\eta\varepsilon_1\varepsilon_2^*\eta^*]E_{A_0}([\zeta\varepsilon_2\xi_2^*\zeta_2^*])\\
&= \sum_{b\in B_1} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))} \sum_{t(\eta)=s(\xi_1)} [\eta\xi_1\varepsilon_2^*\eta^*]E_{A_0}([\zeta_1\varepsilon_2\xi_2^*\zeta_2^*])\\
&= \sum_{b\in B_1} \sum_{t(\eta)=s(\xi_1)} \delta_{\xi_2,\varepsilon_2}[\eta\xi_1\varepsilon_2^*\eta^*]\cdot[\zeta_1\zeta_2^*] = [\zeta_1\xi_1\xi_2^*\zeta_2^*] = x.
\end{aligned}$$

Case 2: Suppose that $s(\xi_1) \neq s(\xi_2)$. If $b \in B_1$, then similarly, $E_{A_0}(b^*x) = 0$. Hence

$$\begin{aligned}
\sum_{b\in B} bE_{A_0}(b^*x) = \sum_{b\in B_2} bE_{A_0}(b^*x) &= \sum_{b\in B_2} \frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}[\eta_1\varepsilon_1\varepsilon_2^*\eta_{s(\varepsilon_2)}^*]E_{A_0}\big([\eta_{s(\varepsilon_2)}\varepsilon_2\varepsilon_1^*\eta_1^*]\cdot[\zeta_1\xi_1\xi_2^*\zeta_2^*]\big)\\
&= [\zeta_1\xi_1\xi_2^*\eta_{s(\xi_2)}^*]\cdot[\eta_{s(\xi_2)}\zeta_2^*] = [\zeta_1\xi_1\xi_2^*\zeta_2] = x.
\end{aligned}$$

□

Remark 3.3.23. One could also take

$$B_2 = \left\{ \left(\frac{d\lambda(s(\varepsilon_2))}{m_+(s(\varepsilon_2))\lambda(t(\varepsilon_2))} \right)^{1/2} [\eta_1\varepsilon_1\varepsilon_2^*\eta_2^*] \middle| s(\varepsilon_1) \neq s(\varepsilon_2) \right\}.$$

Corollary 3.3.24 (Commutant Conditional Expectations). *If*

$$x = \sum_{t(\zeta)=s(\xi_1)} [\zeta\xi_1\xi_2^* \cdots \xi_{2n-1}\xi_{2n}^*\zeta^*] \in A_0' \cap A_n,$$

the conditional expectation $A_0' \cap A_n \to A_1' \cap A_n$ *is given by*

$$E_{A_1'}^{A_0'}(x) = d^{-1}\delta_{\xi_1,\xi_{2n}} \left(\frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\right) \sum_{t(\zeta)=s(\varepsilon);t(\varepsilon)=t(\xi_2)} [\eta\varepsilon\xi_2^*\xi_3\cdots\xi_{2n-2}^*\xi_{2n-1}\varepsilon^*\eta^*].$$

Proof. Let B be as in Proposition 3.3.22. By Proposition 3.2.24, we have

$$d^2 E_{A_1'}^{A_0'}(x) = \sum_{b\in B} bxb^* = \sum_{b\in B_1} bxb^* + \sum_{b\in B_2} bxb^*.$$

We treat each sum separately:

$$\begin{aligned}\sum_{b\in B_1} bxb^* &= \sum_{b\in B_1} \left(\frac{d\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\right) \sum_{\substack{t(\eta)=s(\varepsilon_1)=t(\kappa)\\ t(\zeta)=s(\xi_1)}} [\eta\varepsilon_1\varepsilon_2^*\eta^*]\cdot[\zeta\xi_1\xi_2^*\cdots\xi_{2n-1}\xi_{2n}^*\zeta^*]\cdot[\kappa\varepsilon_2\varepsilon_1^*\kappa^*]\\ &= d\sum_{\substack{s(\varepsilon)=s(\varepsilon_2)\\ t(\varepsilon)=t(\xi_2)}} \left(\frac{\lambda(s(\varepsilon_2))}{\lambda(t(\varepsilon_2))}\right) \sum_{\substack{t(\eta)=s(\varepsilon_1)=t(\kappa)\\ t(\zeta)=s(\xi_1)}} \delta_{\eta,\zeta}\delta_{\zeta,\kappa}\delta_{\varepsilon_2,\xi_1}\delta_{\varepsilon_2,\xi_{2n}}[\eta\varepsilon\xi_2^*\cdots\xi_{2n-1}\varepsilon^*\kappa^*]\\ &= d\sum_{\substack{t(\eta)=s(\varepsilon)=s(\xi_1)\\ t(\varepsilon)=t(\xi_2)}} \left(\frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\right)\delta_{\xi_1,\xi_{2n}}[\eta\varepsilon\xi_2^*\cdots\xi_{2n-1}\varepsilon^*\eta^*].\end{aligned}$$

Similarly, we have

$$\sum_{b\in B_2} bxb^* = d\sum_{\substack{t(\eta)=s(\varepsilon)\neq s(\xi_1)\\ t(\varepsilon)=t(\xi_2)}} \left(\frac{\lambda(s(\xi_1))}{\lambda(t(\xi_1))}\right)\delta_{\xi_1,\xi_{2n}}[\eta\varepsilon\xi_2^*\cdots\xi_{2n-1}\varepsilon^*\eta^*].$$

Putting these two together, we get the desired formula for $E_{A_1'}^{A_0'}$. □

The bipartite graph planar algebra and the isomorphism

We refer the reader to [Jon00] for the full definition of the planar algebra of a bipartite graph.

Let $G_\bullet$ be the planar algebra of the bipartite graph Γ with spin vector λ as in Subsections 3.3 and 3.3. We briefly recall the action of tangles on the $G_{n,\pm}$, and we calculate some necessary examples.

A state σ of a tangle T is a way of assigning the regions and strings of T with compatible vertices and edges of Γ respectively, i.e., if a string S of T partitions the unshaded region R_+ from the shaded region R_-, then for $\sigma(S)\in\mathcal{E}$, $s(\sigma(S)) = \sigma(R_+)\in\mathcal{V}_+$ and $t(\sigma(S)) = \sigma(R_-)\in\mathcal{V}_-$.

Define the output loop ℓ_σ as the loop obtained by reading clockwise around the outer boundary of T once it has been labeled by σ.

Suppose now that T has n input disks, and $\ell = \ell_1 \otimes \cdots \otimes \ell_n$ is a simple tensor of loops where ℓ_i is a loop in $G_{n_i,\pm_i}$. Then the action of T on ℓ is given by

$$T(\ell) = \sum_{\text{states } \sigma} c(\sigma, \ell)\ell_\sigma,$$

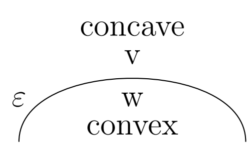
where $c(\sigma, \ell)$ is a correction factor defined as follows:

(1) First, label the regions and strings of T adjacent to the input disks with the edges and vertices which compose the ℓ_i's. If the labeling contradicts σ, then $c(\sigma, \ell) = 0$.

(2) If the labels agree, put the tangle in a standard form similar to Section 3.2, where the only difference is that the half the strings emanate from the top of the input rectangles, and half the strings emanate down, but the $*$ is still on the left side. Let $E(T)$ be the set of local extrema of the strings of the standard form of the tangle. For each $e \in E(T)$, let conv(e) be the vertex assigned by σ to the convex region of the extrema, and let conc(e) be the vertex assigned to the concave region. Set

$$k_e = \sqrt{\frac{\lambda(\text{conv}(e))}{\lambda(\text{conc}(e))}}.$$

Below is an example of an extrema e on a string S with $\sigma(S) = \varepsilon$, connecting vertices w, v:



$$\longrightarrow k_e = \sqrt{\frac{\lambda(w)}{\lambda(v)}}.$$

Note that conv(e) may be in either $\mathcal{V}_+$ or $\mathcal{V}_-$. Finally, set

$$c(\sigma, \ell) = \prod_{e \in E(T)} k_e.$$

The $*$-structure on the bipartite graph planar algebra is given as follows: if T, ℓ are as above, then

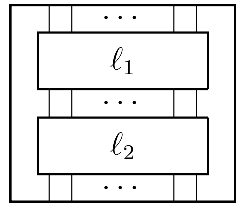
$$T(\ell_1^* \otimes \cdots \otimes \ell_n^*) = T^*(\ell_1 \otimes \cdots \otimes \ell_n)^*$$

where T^* is the mirror image of T, and the adjoint of a loop is the loop traversed backwards as in Definition 3.3.5.

Remark 3.3.25. Contractible loops are traded for a multiplicative factor of d as λ is a Frobenius-Perron eigenvector (see Definition 3.3.13).

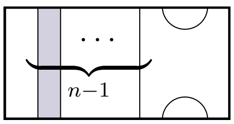
Remark 3.3.26. Note from Corollary 3.3.21 that there is a natural inclusion identification $G_{n,-} \to G_{n+1,+}$ given by

$$[\varepsilon_1^* \varepsilon_2 \cdots \varepsilon_{2n-1}^* \varepsilon_{2n}] \longmapsto \sum_{t(\varepsilon) = s(\varepsilon_1)} [\varepsilon \varepsilon_1^* \varepsilon_2 \cdots \varepsilon_{2n-1}^* \varepsilon_{2n} \varepsilon^*].$$

Examples 3.3.27. (0) If $\ell_1, \ell_2 \in G_{n,\pm}$, then $\ell_1 \cdot \ell_2 =$ 

, the shading depending on $n, \pm$.

(1) For $n \in \mathbb{N}$ odd,



$$= \sum_{\vec{i}} \frac{[\lambda(t(\varepsilon_{i_n}))\lambda(t(\varepsilon_{i_{n+1}}))]^{1/2}}{\lambda(s(\varepsilon_{i_n}))}[\varepsilon_{i_1}\cdots\varepsilon^*_{i_{n-1}}\varepsilon_{i_n}\varepsilon^*_{i_n}\varepsilon_{i_{n+1}}\varepsilon^*_{i_{n+1}}\varepsilon_{i_{n-1}}\cdots\varepsilon^*_{i_1}],$$

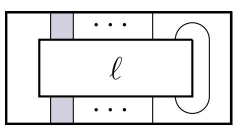
where the sum is taken over all vectors $\vec{i} = (i_1, i_2, \ldots, i_{n+1})$ such that

$$[\varepsilon_{i_1}\varepsilon^*_{i_2}\cdots\varepsilon^*_{i_{n-1}}\varepsilon_{i_n}\varepsilon^*_{i_n}\varepsilon_{i_{n+1}}\varepsilon^*_{i_{n+1}}\varepsilon_{i_{n-1}}\cdots\varepsilon_{i_2}\varepsilon^*_{i_1}] \in G_{n+1,+}.$$

There is a similar formula for n even. (Compare with Definition 3.3.17.)

(2) Suppose $\ell = [\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{2n-1}\varepsilon^*_{2n}] \in G_{n,+}$.

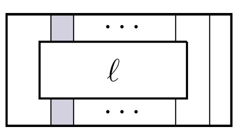
(i) If n is even, then



$$= \delta_{\varepsilon_n,\varepsilon_{n+1}} \frac{\lambda(s(\varepsilon_n))}{\lambda(t(\varepsilon_n))}[\varepsilon_1\varepsilon_2^*\cdots\varepsilon_{n-1}\varepsilon^*_{n+2}\varepsilon_{2n-1}\cdots\varepsilon^*_{2n}],$$

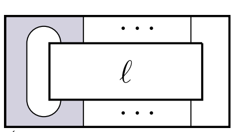
with a similar formula for n odd. (Compare with Proposition 3.3.16.)

(ii) If n is even, then



$$= \sum_{s(\varepsilon)=s(\varepsilon_n)} [\varepsilon_1\varepsilon_2^*\cdots\varepsilon_n^*\varepsilon\varepsilon^*\varepsilon_{n+1}\cdots\varepsilon_{2n-1}\varepsilon^*_{2n}],$$

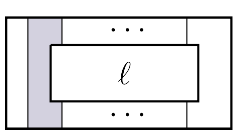
with a similar formula for n odd. (Compare with Definition 3.3.10.)

(iii) 

$$= \delta_{\varepsilon_1,\varepsilon_{2n}} \frac{\lambda(s(\varepsilon_1))}{\lambda(t(\varepsilon_1))}[\varepsilon_2^*\varepsilon_3\cdots\varepsilon^*_{2n-2}\varepsilon_{2n-1}].$$

(Compare with Proposition 3.3.24 and Remark 3.3.26.)

(3) If $\ell = [\varepsilon_1^*\varepsilon_2\cdots\varepsilon^*_{2n-1}\varepsilon_{2n}] \in G_{n,-}$, then



$$= \sum_{t(\varepsilon)=s(\varepsilon_1)} [\varepsilon\varepsilon_1^*\varepsilon_2\cdots\varepsilon^*_{2n-1}\varepsilon_{2n}\varepsilon^*],$$

which may be identified with $\ell \in G_{n+1,+}$ by Remark 3.3.26.

Theorem 3.3.28. *The canonical planar $*$-algebra $P_\bullet$ associated to $M_0 \subset (M_1, \mathrm{tr}_1)$ is isomorphic to the bipartite graph planar $*$-algebra $G_\bullet$ of the Bratteli diagram Γ for the inclusion.*

Proof. To show that the $*$-algebra isomorphisms

$$G_{n,+} \xrightarrow{\phi_{n,+}} A_0' \cap A_n \xrightarrow{\varphi_n|_{A_0'\cap A_n}} M_0' \cap M_n \xrightarrow{\theta_n^{-1}|_{M_0'\cap M_n}} P_{n,+}$$

$$G_{n,-} \xrightarrow{\phi_{n,-}} A_1' \cap A_{n+1} \xrightarrow{\varphi_{n+1}|_{A_1'\cap A_{n+1}}} M_1' \cap M_{n+1} \xrightarrow{\theta_{n+1}^{-1}|_{M_1'\cap M_{n+1}}} P_{n,-}$$

give an isomorphism of planar $*$-algebras $G_\bullet \to P_\bullet$, we must check that

(1) they map Jones projections in $G_\bullet$ to those in $P_\bullet$, and

(2) they preserve the action of annular tangles.

Both follow immediately from Examples 3.3.27 and the proof of Lemma 3.2.49. □

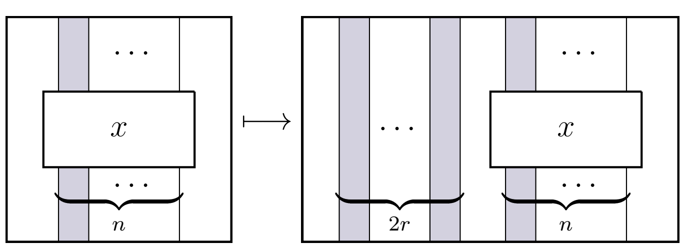
3.4 The Embedding Theorem

Let $Q_\bullet$ be a finite depth subfactor planar algebra of modulus d. Pick $r \geq 0$ minimal such that $Q_{2r,+} \subset Q_{2r+1,+} \subset (Q_{2r+2,+}, e_{2r+1})$ is standard (with the usual trace). Note this is possible if and only if $Q_\bullet$ has finite depth. In fact, $Q_{k,+} \subset Q_{k+1,+} \subset (Q_{k+2,+}, e_{k+1})$ is standard for all $k \geq 2r$. For $n \geq 0$, set $M_n = Q_{2r+n,+}$ and $F_{n+1} = E_{2r+n+1}$ (shifted Jones projections). Let $P_\bullet$ be the canonical planar $*$-algebra associated to the inclusion $M_0 \subset M_1$, i.e.,

$$\begin{aligned} P_{n,+} &= M_0' \cap M_n = Q_{2r,+}' \cap Q_{2r+n,+} \text{ and} \\ P_{n,-} &= M_1' \cap M_{n+1} = Q_{2r+1,+}' \cap Q_{2r+n+1,+}, \end{aligned}$$

where we suppress the isomorphisms θ_n with the tensor products of $Q_{2r+1,+}$ over $Q_{2r,+}$.

Theorem 3.4.1. *Define $\Phi\colon Q_\bullet \to P_\bullet$ by adding $2r$ strings to the left for $x \in Q_{n,+}$ and adding $2r+1$ strings to the left for $x \in Q_{n,-}$.*



Then Φ is an inclusion of planar $*$-algebras.

Proof. We use Lemma 3.2.49. Note that $\Phi(x^*) = \Phi(x)^*$ and $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in Q_{n,\pm}$.

(1) Since $\Phi(E_j) = E_{2r+j} = F_j$ for all $j \in \mathbb{N}$, we have $\Phi(E_j x) = F_j\Phi(x)$ and $\Phi(xE_j) = \Phi(x)F_j$ for all $x \in Q_{n,\pm}$ and all $j \in \mathbb{N}$.

(2) Note that

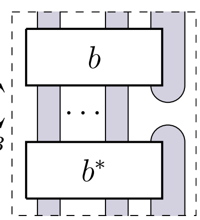
(i) For $n \in \mathbb{N}$, $\Phi(E_{Q_{n-1,+}}(x)) = E_{P_{n-1,+}}(\Phi(x))$ since

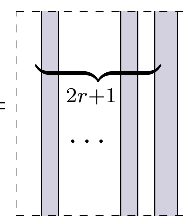
$$E_{Q_{2r+n-1,+}}|_{Q'_{2r,+}\cap Q_{2r+n,+}} = E_{Q_{2r+n-1,+}}|_{P_{n,+}} = E_{P_{n-1,+}}$$

(since $Q_{2r,+} \subset Q_{2r+n-1,+}$, we have that $E_{Q_{2r+n-1,+}}$ preserves $Q_{2r,+}$-central vectors as it is $Q_{2r+n-1,+}$-bilinear).

(ii) $\Phi(\beta_{n+1}(x)) = \beta_{n+1}(\Phi(x))$ for all $x \in Q_{n,+}$ since the inclusion $P_{n,+} \to P_{n+1,+}$ is the restriction of the inclusion $Q_{2r+n,+} \to Q_{2r+n+1,+}$.

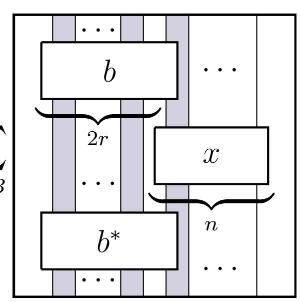
(iii) Let $B = \{b\}$ be a Pimsner-Popa basis for $M_1 = Q_{2r+1,+}$ over $M_0 = Q_{2r,+}$. Since each $b \in B$ is an $(2r+1,+)$-box in $Q_{2r+1,+}$,

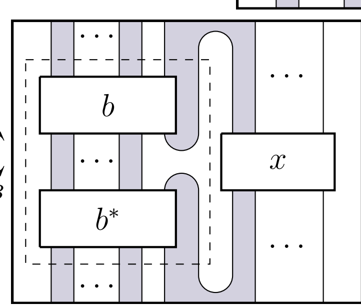
$$\frac{1}{d}\sum_{b\in B}$$ 

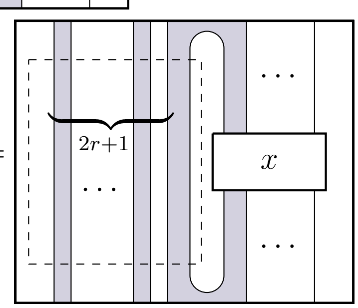
$$= \sum_{b\in B} be_{2r+1}b^* = 1_{P_{2r+2,+}} =$$ 

.

Then by Proposition 3.2.24 and Theorem 3.2.50, for all $x \in Q_{n,+}$,

$$\gamma_n^+(\Phi(x)) = \frac{1}{d}\sum_{b\in B} b\Phi(x)b^* = \frac{1}{d}\sum_{b\in B}$$ 

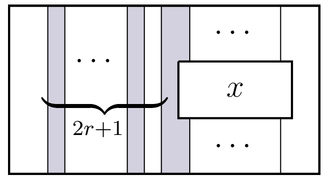
$$= \frac{1}{d}\sum_{b\in B}$$ 

$$=$$ 

$$= \Phi(\gamma_n^+(x)).$$

(3) The inclusion $i_n^-\colon P_{n,-} \to P_{n+1,+}$ is the identity in the canonical planar $*$-algebra. If $x \in Q_{n,-}$, then we have

$$i_n^-(\Phi(x)) = \Phi(x) =$$



$$= \Phi(i_n^-(x)).$$

□

Corollary 3.4.2. *Let $N \subset M$ be a finite index, finite depth II_1-subfactor, and let $P_\bullet$ be the associated canonical subfactor planar algebra. Let Γ be the principal graph of $N \subset M$, and let $G_\bullet$ be the bipartite graph planar algebra of Γ. Then there is an embedding of planar algebras $P_\bullet \to G_\bullet$.*

Chapter 4

A planar calculus for infinite index subfactors

4.1 Introduction

Jones initiated the modern theory of subfactors in [Jon83]. Given a finite index II_1-subfactor $A_0 \subseteq A_1$, he used the basic construction to obtain the Jones tower $(A_n)_{n\geq 0}$, obtained iteratively by adding the Jones projections $(e_n)_{n\geq 1}$ which satisfy the Temperley-Lieb relations. Jones used this structure to show the index lies in the range $\{4\cos^2(\pi/n)|n \geq 3\} \cup [4, \infty)$, and he found an example for each value.

Much initial subfactor research classified hyperfinite subfactors of small index ($[A_1 \colon A_0] \leq 4$) by studying the standard invariant, i.e., the two towers of higher relative commutants $(A_i' \cap A_j)_{i=0,1;j\geq 0}$ [Ocn88, GdlHJ89, Izu91, Pop94]. This combinatorial data was axiomatized in three slightly different structures: paragroups [Ocn88], λ-lattices [Pop95], and planar algebras [Jon99]. When combined, these viewpoints produce strong results, e.g., standard invariants with index in $(4, 5)$ are completely classified, excluding the A_∞ standard invariant at each index value [Pop93] (see [MS11, MPPS12, IJMS11, PT12] for more details).

Some finite index results generalize to infinite index subfactors, such as discrete, irreducible, "depth 2" subfactors correspond to outer (cocylce) actions of Kac algebras [HO89, EN96], and the classical Galois correspondence still holds for outer actions of infinite discrete groups and minimal actions of compact groups [ILP98].

In his Ph.D. thesis [Bur03], Burns studied rotations and extremality for infinite index, since the key to isotopy invariance of Jones' planar calculus in [Jon99] is the rotation operator (also known to Ocneanu). Burns' essential observation for finite index was that the centralizer algebras $A_0' \cap A_n$ coincide with the central L^2-vectors:

$$A_0' \cap L^2(A_n) = \left\{\zeta \in L^2(A_n) \middle| a\zeta = \zeta a \text{ for all } a \in A_0\right\}.$$

Burns found an elegant formula for the rotation on $P_{n,+} = A_0' \cap \bigotimes_{A_0}^n L^2(A_1)$:

$$\rho = \sum_\beta L_\beta R_\beta^*$$

where $\{\beta\}$ is a Pimsner-Popa basis for A_1 over A_0, L_β is the left creation operator, and R_β^* is the right annihilation operator (see Definition 4.2.4). This approach was generalized in [JP11] to define a canonical planar $*$-algebra associated to a strongly Markov inclusion of finite von Neumann algebras. Burns adapted his formula to infinite index, and he showed existence of the rotation on the central L^2-vectors is equivalent to approximate extremality of the subfactor.

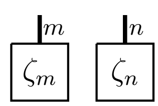
In infinite index, $A_0' \cap A_n$ and $A_0' \cap L^2(A_n)$ do not coincide. One naturally asks:

Question 4.1.1. *What is a suitable standard invariant for infinite index subfactors?*

A definitive answer to Question 4.1.1 is not yet known. On one hand, we have the two towers of centralizer algebras $(A_i' \cap A_j)_{i=0,1;j\geq 0}$ in which we can multiply (the shift isomorphisms $A_i' \cap A_j \cong A_{i+2}' \cap A_{j+2}$ still hold by [EN96]). On the other hand, we have the central L^2-vectors on which we have Burns' rotation (in the approximately extremal case) and graded multiplication in the sense of [GJS10] (tensoring of central vectors). However, the operator valued weights which replace the conditional expectations do not preserve these spaces and may not be well-defined. All this structure is necessary for a good planar calculus. We ask:

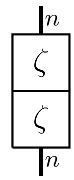
Question 4.1.2. *What is the strongest planar calculus we can define for infinite index subfactors?*

In this paper, we propose an answer to Question 4.1.2 using both centralizer algebras and central L^2-vectors. We do so in more generality, starting with a bimodule ${}_AH_A$ over a II_1-factor A (one recovers the subfactor case when $A = A_0$ and $H = L^2(A_1)$). First, we set $H^n = \bigotimes_A^n H$, $Q_n = A' \cap (A^{\text{op}})' \cap B(H^n)$ (the centralizer algebras), and $P_n = A' \cap H^n = \{\zeta \in H^n | a\zeta = \zeta a \text{ for all } a \in A\}$ (the central L^2-vectors). As mentioned above, the P_n's naturally form a graded algebra $P_\bullet$ in the sense of [GJS10] under relative tensor product. We represent central vectors in P_n as in [GJS10] by boxes with n strings emanating from the top, and we denote graded multiplication (relative tensor product) of $\zeta_m \in P_m$ and $\zeta_n \in P_n$ by

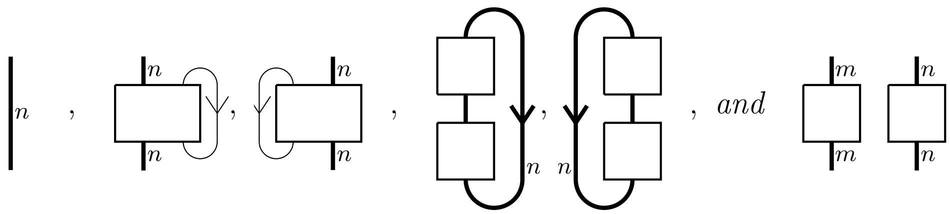
$\zeta_m \otimes \zeta_n =$ 

$\in P_{m+n}.$

We represent elements of Q_n as boxes with strings emanating from top and bottom. For $\zeta \in P_n$, note that the creation-annihilation operator $L(\zeta)L(\zeta)^* = R(\zeta)R(\zeta)^*$ lies in Q_n, which we represent as

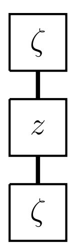
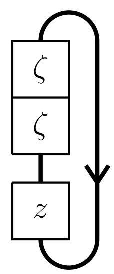
$$L(\zeta)L(\zeta)^* =$$  $$\in Q_n.$$

Theorem 4.1.3. *The extended positive cones* $\widehat{Q_n^+}$ *(in the sense of [Haa79]) naturally form an algebra* $\widehat{Q_\bullet^+}$ *over the operad* $\mathbb{BP}$ *generated by the oriented tangles*



for $m, n \geq 0$ *up to planar isotopy. (We suppress external disks, draw one thick string labelled* n *for* n *individual strings, and orient all strings upward unless otherwise specified.)*

Moreover, the $\mathbb{BP}$*-algebra* $\widehat{Q_\bullet^+}$ *and graded algebra* $P_\bullet$ *are compatible: if* $z \in \widehat{Q_n^+}$ *and* $\zeta \in P_n$*, then*

$$z(\omega_\zeta) =$$  $$=$$  $$= \mathrm{Tr}_n(L(\zeta)L(\zeta)^* \cdot z)$$

where Tr_n *is the canonical trace on* Q_n *coming from the right* A*-action on* H^n*. (Note that the multiplication tangle only makes sense once we take the trace by [Haa79]. See Theorem 4.2.14 for more details.)*

We generalize to bimodules Burns' work on rotations: an operator ρ on the central L^2-vectors P_n is a <u>Burns rotation</u> if for all left and right bounded vectors $b_1, \ldots, b_n \in H$, (omitting the subscript A on the tensors,)

$$\langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes b_1 \rangle.$$

Note this equation implies the uniqueness and periodicity of ρ if it exists. We generalize Burns' notion of (approximate) extremality, and we prove the following theorem:

Theorem 4.1.4. *Consider the following statements (include all or none of the parenthetical statements):*

(1) H^n *is (approximately) extremal for some* $n \geq 1$*,*

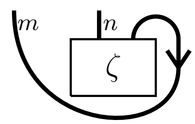
(2) H^n is (approximately) extremal for all $n \geq 1$,

(3) The (possibly non-)unitary ρ exists on P_{2n} for all $n \geq 1$, and

(4) The (possibly non-)unitary ρ exists on P_{2n} for some $n \geq 1$.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). *If H is symmetric, then* (4) $\Rightarrow$ (1).

When ρ exists, we represent it diagrammatically by

$\rho^m(\zeta) =$  *for* $\zeta \in P_{m+n}$,

(well-defined by Corollary 4.4.16) and these diagrams are compatible with the diagrams above in the sense of Theorem 4.4.17.

Interestingly, we find our planar structure without the use of Jones' basic construction and resulting Jones projections!

Outline:

In Section 4.2, we give a brief introduction to modules, the relative tensor product, extended positive cones, and operator valued weights. Subsections 4.2 and 4.2 provide some helpful, well-known results for the convenience of the reader.

In Subsection 4.3, starting with our $A-A$ bimodule H, we introduce H^n along with two towers of algebras C_n, C_n^{op}, a tower of centralizer algebras $Q_n = C_n \cap C_n^{\text{op}}$, and the central L^2-vectors P_n. We then compute formulas for the various canonical maps associated with these towers. In Subsection 4.3, we show the extended positive cones (in the sense of [Haa79]) of the centralizer algebras $\widehat{Q_n^+}$ naturally form an algebra over an operad $\mathbb{BP}$ (we use positive cones so we can "conditionally expect" using operator valued weights). In Subsection 4.3, we show that the vectors in $P_\bullet$ are left and right A-bounded and form a graded algebra in the sense of [GJS10]. We then show the compatibility of $\widehat{Q_\bullet^+}$ and $P_\bullet$ in Subsection 4.3.

Subsection 4.4 defines extremality for bimodules and Burns rotations. In Subsection 4.4, we show how the Burns rotation fits in our planar calculus, and in Subsection 4.4, we show that (approximate) extremality implies the existence of the Burns rotation (Theorem 4.4.20). A converse of this theorem for symmetric bimodules is obtained in Subsection 4.4, which finishes the proof of Theorem 4.1.4.

Subsection 4.5 discusses centralizer algebras Q_n and central L^2-vectors P_n for some basic examples, including the infinite index group-subgroup subfactor, and Subsection 4.5 determines if the examples are (approximately) extremal. In particular, Corollaries 4.5.9, 4.5.11, and 4.5.20 give an extremal infinite index subfactor for which $\dim(Q_n) < \infty$ and $\dim(P_n) = 1$ for all $n \in \mathbb{N}$. This example contrasts Burns' example of an infinite index subfactor with a type III summand in a higher relative commutant [Bur03].

Throughout the paper, we need some technical results which have been included in the last few sections. Section 4.6 shows that the relative tensor product of extended positive cones is well-defined and associative, which is necessary for our planar calculus. Section 4.7 discusses the operad $\mathbb{BP}$ which acts on the positive cones $\widehat{Q_n^+}$, including results on generating sets of tangles, standard form of tangles, and that the action is well-defined. In Section 4.8, we axiomatize the notion of extended positive cone to make rigorous the idea of a planar algebra over such objects. The main intricacy is that we must make multiplication by $\infty_{\mathbb{R}}$ well-defined.

Future research:

The annular Temperley-Lieb category, especially the rotation, played an important role in the construction of certain exotic finite index subfactors [Pet10, BMPS09]. In a future paper with Jones, we will incorporate the odd Jones projections for infinite index (see [Bur03]) into the planar calculus, and we will give the analog of the annular Tempeley-Lieb category for infinite index. We hope this viewpoint will be as fruitful as in the finite index case.

The results of this paper should generalize to bimodules over an arbitrary finite von Neumann algebra. As it requires substantial calculations while obscuring the main new ideas presented here, this generalization will appear in a future paper.

Finally, it would be interesting to try to connect Connes' results on self-dual positive cones [Con74] to the extended positive cones axiomatized in Section 4.8.

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4.2 Preliminaries

Notation 4.2.1.

- Throughout this paper, a trace on a finite von Neumann algebra means a faithful, normal, tracial state unless otherwise specified.
- A will always denote a finite von Neumann algebra with trace tr_A.
- We use the notation $\widehat{a}$ to denote the image of $a \in A$ in $L^2(A, \operatorname{tr}_A)$.

- For a semifinite von Neumann algebra M with normal, faithful, semifinite (n.f.s.) trace Tr_M, we write

$$\mathfrak{n}_{\mathrm{Tr}_M} = \{x \in M | \mathrm{Tr}_M(x^*x) < \infty\} \text{ and}$$
$$\mathfrak{m}_{\mathrm{Tr}_M} = \mathfrak{n}_{\mathrm{Tr}_M}^* \mathfrak{n}_{\mathrm{Tr}_M} = \mathrm{span}\left\{x^*y | x, y \in \mathfrak{n}_{\mathrm{Tr}_M}\right\}.$$

Modules and the relative tensor product

This exposition follows [Con80, Sau83, Pop94, EN96, Bis97, EV00, Bur03].

Definition 4.2.2 (Left modules)**.** If ${}_AK$ is a left Hilbert A-module, then the set of left A-bounded vectors is given by

$$D({}_AK) = \{\eta \in K | \|a\eta\|_2 \leq \lambda \|a\|_2 \text{ for some } \lambda \geq 0\},$$

and each $\eta \in D({}_AK)$ gives a bounded map $R(\eta)\colon L^2(A) \to H$ by the extension of $\widehat{a} \mapsto a\eta$.

For $\eta_1, \eta_2 \in D({}_AK)$, we have an A-valued inner product given by

$${}_A\langle \eta_1, \eta_2 \rangle = JR(\eta_1)^* R(\eta_2) J \in A$$

satisfying

(1) ${}_A\langle a\eta_1 + \eta_2, \eta_3 \rangle = a\,{}_A\langle \eta_1, \eta_3 \rangle + {}_A\langle \eta_2, \eta_3 \rangle$,

(2) ${}_A\langle \eta_1, \eta_2 \rangle^* = {}_A\langle \eta_2, \eta_1 \rangle$, and

(3) ${}_A\langle x\eta_1, \eta_2 \rangle = {}_A\langle \eta, x^*\eta_2 \rangle$

for all $a \in A$, $x \in A' \cap B(K)$, and $\eta_1, \eta_2, \eta_3 \in D({}_AK)$ (note $x\eta_i \in D({}_AK)$).

An ${}_AK$-basis is a set of vectors $\{\alpha\} \subset D({}_AK)$ such that

$$\sum_\alpha R(\alpha)R(\alpha)^* = 1_K \iff \sum_\alpha {}_A\langle \eta, \alpha \rangle \alpha = \eta \text{ for all } \eta \in D({}_AK).$$

${}_AK$-bases exist by [Con80].

The canonical trace on $A' \cap B(K)$ is given by $\mathrm{Tr}_{A' \cap B(K)}(x) = \sum_\alpha \langle x\alpha, \alpha \rangle$ where $\{\alpha\}$ is any ${}_AK$ basis.

If $\eta \in D({}_AK)$, then $\mathrm{Tr}_{A' \cap B(K)}(R(\eta)R(\eta)^*) = \mathrm{tr}_A({}_A\langle \eta, \eta \rangle) = \|\eta\|_2^2$.

Definition 4.2.3 (Right modules)**.** A right Hilbert A-module is the same as a left Hilbert A^{op}-module. If H_A is a right Hilbert A-module, we write ξa for $a^{\mathrm{op}}\xi$ for all $a^{\mathrm{op}} \in A^{\mathrm{op}}$. We get parallel definitions:

The set of right A-bounded vectors is given by

$$D(H_A) = \{\xi \in H | \|\xi a\|_2 \leq \lambda \|a\|_2 \text{ for some } \lambda \geq 0\}.$$

Each $\xi \in D(H_A)$ defines a bounded map $L(\xi)\colon L^2(A) \to H$ by the extension of $\widehat{a} \mapsto \xi a$.

For $\xi_1, \xi_2 \in D(H_A)$, we have an A-valued inner product given by

$$\langle \xi_1 | \xi_2 \rangle_A = L(\xi_1)^* L(\xi_2) \in A$$

satisfying

(1) $\langle \xi_1 | \xi_2 a + \xi_3 \rangle_A = \langle \xi_1 | \xi_2 \rangle_A a + \langle \xi_1 | \xi_3 \rangle_A$,

(2) $\langle \xi_1 | \xi_2 \rangle_A^* = \langle \xi_2 | \xi_1 \rangle_A$, and

(3) $\langle x\xi_1 | \xi_2 \rangle_A = \langle \xi_1 | x^* \xi_2 \rangle_A$

for all $a \in A$, $x \in (A^{\mathrm{op}})' \cap B(H)$, and $\xi_1, \xi_2, \xi_3 \in D(H_A)$ (note $x\xi_i \in D(H_A)$).

An H_A-basis is a set of vectors $\{\beta\} \subset D(H_A)$ such that

$$\sum_\beta L(\beta)L(\beta)^* = 1_H \Longleftrightarrow \sum_\beta \beta \langle \beta | \xi \rangle_A = \xi \text{ for all } \xi \in D(H_A).$$

H_A-bases exist by [Con80].

The canonical trace on on $(A^{\mathrm{op}})' \cap B(H)$ is given by $\mathrm{Tr}_{(A^{\mathrm{op}})' \cap B(H)}(x) = \sum_\beta \langle x\beta, \beta \rangle$ where $\{\beta\}$ is any H_A basis.

If $\xi \in D(H_A)$, then $\mathrm{Tr}_{(A^{\mathrm{op}})' \cap B(H)}(L(\xi)L(\xi)^*) = \mathrm{tr}_A(\langle \xi | \xi \rangle_A) = \|\xi\|_2^2$.

Definition 4.2.4 (Relative tensor product)**.** The relative tensor product $H \otimes_A K$ is given by one of the three equivalent definitions:

(1) the completion of the algebraic tensor product $D(H_A) \odot_A K$ under the pseudo-norm induced by the sesquilinear form $\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle = \langle \langle \xi_2 | \xi_1 \rangle_A \eta_1, \eta_2 \rangle$,

(2) the completion of the algebraic tensor product $H \odot_A D({}_AK)$ under the pseudo-norm induced by the sesquilinear form $\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle = \langle \xi_1 {}_A\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_H$, or

(3) the completion of the algebraic tensor product $D(H_A) \odot_A D({}_AK)$ under the pseudo-norm induced by the sesquilinear form

$$\langle \xi_1 \odot \eta_1, \xi_2 \odot \eta_2 \rangle = \langle \xi_1 {}_A\langle \eta_1, \eta_2 \rangle, \xi_2 \rangle_H = \langle \langle \xi_2 | \xi_1 \rangle_A \eta_1, \eta_2 \rangle_K.$$

The image of $\xi \odot \eta$ in $H \otimes_A K$ is denoted $\xi \otimes \eta$. (This notation avoids confusion with the operators $x \otimes_A y$ as in Lemma 4.6.4.)

Given $\xi \in D(H_A)$ and $\eta \in D({}_AK)$, we get bounded creation operators $L_\xi\colon K \to H \otimes_A K$ by $\eta' \mapsto \xi \otimes \eta'$ and $R_\eta\colon H \to H \otimes_A K$ by $\xi' \mapsto \xi' \otimes \eta$, whose adjoints are the annihilation operators given by $L_\xi^*(\xi' \otimes \eta') = \langle \xi | \xi' \rangle_A \eta'$ and $R_\eta^*(\xi' \otimes \eta') = \xi' {}_A\langle \eta', \eta \rangle$.

Definition 4.2.5 (Fiber product, [Sau85, EV00]). Suppose $A^{\mathrm{op}} \subset M_1 \subset B(H)$ and $A \subset M_2 \subset B(K)$. Then we define

$$M_1' \otimes_A M_2' = \{x \otimes_A y | x \in M_1' \text{ and } y \in M_2'\} \subset B(H \otimes_A K)$$

(see Section 4.6 and Lemma 4.6.4), and the fiber product of M_1 and M_2 over A is given by $M_1 \star_A M_2 = (M_1' \otimes_A M_2')'$. The fiber product satisfies:

- $(M_1 \star_A M_2) \cap (N_1 \star_A N_2) = (M_1 \cap N_1) \star_A (M_2 \cap N_2)$ and
- $M_1 \star_A A = ((A^{\mathrm{op}})' \cap M_1) \otimes_A 1_K$ and $A^{\mathrm{op}} \star_A M_2 = 1_H \otimes_A (A' \cap M_2)$.

In particular,

$$(B(H) \star_A A)' = ((A^{\mathrm{op}})' \otimes_A 1_K)' = A^{\mathrm{op}} \star_A B(K) = 1_H \otimes_A A'.$$

Some easy facts about the relative tensor product

The following are well-known to experts, but we reproduce them here for the sake of completeness and the reader's convenience. For this subsection, H_A is a right Hilbert A-module, and ${}_AK$ is a left Hilbert A-module unless otherwise stated.

Lemma 4.2.6. *Suppose $\{\beta\}$ is an H_A-basis. Then if $u \in U((A^{\mathrm{op}})' \cap B(H))$, $\{u\beta\}$ is another H_A-basis. If $v \in U(A)$, then $\{\beta v\}$ is also an H_A-basis. A similar result holds for left modules.*

Proof. For $u \in (A^{\mathrm{op}})' \cap B(H)$, $L(u\beta)L(u\beta)^* = uL(\beta)L(\beta)^*u^*$. Thus

$$\sum_{u\beta} L(u\beta)L(u\beta)^* = u\left(\sum_{\beta} L(\beta)L(\beta)^*\right)u^* = 1_H.$$

If $v \in U(A)$, then $L(\beta v)L(\beta v^*) = L(\beta)vv^*L(\beta)^* = L(\beta)L(\beta)^*$, and the result follows. □

Lemma 4.2.7. *Let $\xi_1, \xi_2 \in D(H_A)$ and $\eta_1, \eta_2 \in D({}_AK)$. Then $L_{\xi_1}^* L_{\xi_2} \in B(K)$ is left multiplication by $\langle \xi_1 | \xi_2 \rangle_A$ and $R_{\eta_1}^* R_{\eta_2} \in B(H)$ is right multiplication by ${}_A\langle \eta_1, \eta_2 \rangle$.*

Proof. $\langle L_{\xi_1}^* L_{\xi_2}\eta_1, \eta_2 \rangle = \langle \xi_2 \otimes \eta_1, \xi_1 \otimes \eta_2 \rangle = \langle \langle \xi_1 | \xi_2 \rangle_A \eta_1, \eta_2 \rangle$. The other is as trivial. □

Lemma 4.2.8. *If $\{\beta\}$ is an H_A-basis, then $\sum_\beta L_\beta L_\beta^* = 1_{H \otimes_A K}$. Similarly, if $\{\alpha\}$ is an ${}_AH$-basis, then $\sum_\alpha R_\alpha R_\alpha^* = 1_{H \otimes_A K}$.*

Proof. We prove the first statement. Suppose $\xi \in D(H_A)$ and $\eta \in D({}_AK)$. Then

$$\sum_\beta L_\beta L_\beta^*(\xi \otimes \eta) = \sum_\beta L_\beta(L_\beta^* L_\xi)\eta = \sum_\beta \beta \langle \beta | \xi \rangle_A \otimes \eta = \xi \otimes \eta.$$

□

Lemma 4.2.9. *Suppose $\eta \in {}_AK$ and $\eta' \in D({}_AK)$. Then there is a unique ${}_A\langle \eta', \eta\rangle \in L^2(A) \subset L^1(A)$ such that $\langle a\eta, \eta'\rangle_K = \langle a, {}_A\langle \eta', \eta\rangle\rangle_{L^2(A)}$ for all $a \in A$. A similar result holds for right modules.*

Proof. If $\xi \in D({}_AK)$, this is just the usual Radon-Nikodym derivative, and

$$\begin{aligned}\|{}_A\langle \eta', \eta\rangle\|_2 &= \sup_{a\in A, \|\widehat{a}\|_2\leq 1} |\langle \widehat{a}, {}_A\langle \eta', \eta\rangle\widehat{\ }\rangle_{L^2(A)}| = \sup_{a\in A, \|\widehat{a}\|_2\leq 1} \operatorname{tr}({}_A\langle \eta, \eta'\rangle a)\\ &= \sup_{a\in A, \|\widehat{a}\|_2\leq 1} |\langle a\eta, \eta'\rangle_K| \leq \left(\sup_{a\in A, \|\widehat{a}\|_2\leq 1} \|a^*\eta'\|_2\right) \|\eta\|_2 \leq \lambda\|\eta\|_2\end{aligned}$$

for some $\lambda > 0$ depending only on η' as $\eta' \in D({}_AK)$. Now if $\eta \notin D({}_AK)$, take $\eta_n \in D({}_AK)$ with $\eta_n \to \eta$ in $\|\cdot\|_2$, and define

$${}_A\langle \eta', \eta\rangle = \lim_n {}_A\langle \eta', \eta_n\rangle$$

which exists by the above estimate. Now $\langle a\eta, \eta'\rangle_K = \langle \widehat{a}, {}_A\langle \eta', \eta\rangle\rangle_{L^2(A)}$ for all $a \in A$ by construction. □

Corollary 4.2.10. *Each $\eta \in {}_AK$ gives a closable operator $R(\eta)^0 \colon \widehat{A} \to {}_AK$ by $\widehat{a} \mapsto a\eta$. A similar result holds for right modules.*

Proof. We need only show its adjoint is densely defined. If $\eta' \in D({}_AK)$, then

$$\langle R(\eta)^0\widehat{a}, \eta'\rangle_K = \langle a\eta, \eta'\rangle_K = \langle \widehat{A}, {}_A\langle \eta', \eta\rangle\rangle_{L^2(A)}$$

by Lemma 4.2.9, and the result follows as $D({}_AK)$ is dense in K. □

Corollary 4.2.11. *Each $\eta \in {}_AK$ gives a closable unbounded operator $R^0_\eta \colon D(H_A) \to H \otimes_A K$ by $\xi \mapsto \xi \otimes \eta$. A similar result holds for each $\xi' \in H_A$.*

Proof. Once again, we show its adjoint is densely defined. If $\xi' \in D(H_A)$ and $\eta' \in D({}_AK)$, then by Lemma 4.2.9,

$$\begin{aligned}\langle R^0_\eta\xi, \xi' \otimes \eta'\rangle_{H\otimes_AK} &= \langle \xi \otimes \eta, \xi' \otimes \eta'\rangle_{H\otimes_AK} = \langle \langle \xi'|\xi\rangle_A\eta, \eta'\rangle_K = \langle \langle \xi'|\xi\rangle\widehat{\ }_A, {}_A\langle \eta', \eta\rangle\widehat{\ }\rangle_{L^2(A)}\\ &= \langle L(\xi')^*\xi, {}_A\langle \eta', \eta\rangle\widehat{\ }\rangle_{L^2(A)} = \langle \xi, L(\xi')_A\langle \eta', \eta\rangle\widehat{\ }\rangle_H.\end{aligned}$$

The result now follows as $D(H_A) \otimes_A D({}_AK)$ is dense in $H \otimes_A K$. □

Haagerup's extended positive cones and operator valued weights

For this subsection, M is a von Neumann algebra acting on a Hilbert space H.

Definition 4.2.12 (Section 1 of [Haa79])**.** The extended positive cone of M, denoted $\widehat{M^+}$, is the set of weights on the predual of M, i.e., maps $m \colon M_*^+ \to [0, \infty]$ such that

(1) $m(\lambda\phi + \psi) = \lambda m(\phi) + m(\psi)$ for all $\lambda \geq 0$ and $\phi, \psi \in M_*^+$, and

(2) m is lower semicontinuous.

The extended positive cone has additional structure:

- There is a natural inclusion $M^+ \to \widehat{M^+}$ by $m \mapsto (\phi \mapsto \phi(m))$.
- For $m \in \widehat{M^+}$ and $a \in M$, we define $a^*ma \in \widehat{M^+}$ by

$$a^*ma(\phi) = m(a\phi a^*) = m(\phi(a^* \cdot a)).$$

  We write λm for $\lambda^{1/2} m \lambda^{1/2}$ for $\lambda \geq 0$.

- There is a natural partial ordering on $\widehat{M^+}$ given by $m_1 \leq m_2$ if $m_1(\phi) \leq m_2(\phi)$ for all $\phi \in M_*^+$.
- If I is a directed set, we say $(m_i)_{i\in I} \subset \widehat{M^+}$ increases to $m \in \widehat{M^+}$ if $i \leq j$ implies $m_i \leq m_j$ and $\sup_i m_i(\phi) = m(\phi)$ for all $\phi \in M_*^+$. Hence we can define the sum of elements of $\widehat{M^+}$ pointwise.
- Each $\phi \in M_*^+$ extends uniquely to a map $\widehat{M^+} \to [0, \infty]$ by $\phi(m) = m(\phi)$.

Remark 4.2.13 (Section 1 of [Haa79]). There are equivalent definitions of $\widehat{M^+}$:

- Given a projection $p \in P(M)$ and a densely-defined positive, self-adjoint operator S in $K = pH$ affiliated with M, we can define

$$m_{(K,S)}(\omega_\xi) = \begin{cases} \|S^{1/2}\xi\| & \text{if } \xi \in D(S^{1/2}) \\ \infty & \text{else} \end{cases} \tag{4.1}$$

  where $\omega_\xi = \langle \cdot\, \xi, \xi \rangle$. Conversely, given $m \in \widehat{M^+}$, there are unique (K, S) such that Equation (4.1) holds. In the sequel, we will write $m = (K, S)$ when we use this bijective correspondence.

- Each $m \in \widehat{M^+}$ has a unique spectral resolution

$$m(\phi) = \int_0^\infty \lambda d\phi(e_\lambda) + \infty\phi(p)$$

  where $\{e_\lambda\}_{\lambda\in[0,\infty)}$ are increasing family of projections in M such that:

  (1) $\lambda \mapsto e_\lambda$ is strongly continuous from the right, and

  (2) $p = 1 - \lim_{\lambda\to\infty} e_\lambda$

Moreover,

$$e_0 = 0 \iff m(\phi) > 0 \text{ for all } \phi \in M_*^+ \setminus \{0\}$$
$$p = 0 \iff \left\{\phi \in M_*^+ \middle| m(\phi) < \infty\right\} \text{ is dense in } M_*^+.$$

- Every $m \in \widehat{M^+}$ is a pointwise limit of an increasing sequence of operators in M^+.
- $\widehat{M^+}$ is the set of all $m \in \widehat{B(H)^+}$ affiliated to M ($umu^* = m$ for all $u \in U(M')$).

Theorem 4.2.14 ([Haa79], Proposition 1.11, Theorem 1.12)**.** *Suppose M is a semifinite von Neumann algebra with n.f.s. trace Tr_M. For $x, y \in M^+$, let $\mathrm{Tr}_M(x \cdot y) = \mathrm{Tr}_M(x^{1/2}yx^{1/2})$. Then the map $(x, y) \mapsto \mathrm{Tr}_M(x \cdot y)$ has a unique extension to $\widehat{M^+} \times \widehat{M^+}$ such that*

- $\mathrm{Tr}_M(x \cdot y) = \mathrm{Tr}_M(y \cdot x)$ *for all* $x, y \in \widehat{M^+}$,
- Tr_M *is additive and homogeneous in both variables,*
- *if* $(x_i), (y_j) \subset \widehat{M^+}$ *with* $x_i \nearrow x$ *and* $y_j \nearrow y$, *then* $\mathrm{Tr}_M(x_i \cdot y_j) \nearrow \mathrm{Tr}_M(x \cdot y)$, *and*
- $\mathrm{Tr}_M((a^*xa) \cdot y) = \mathrm{Tr}_M(x \cdot (aya^*))$ *for all* $x, y \in \widehat{M^+}$ *and* $a \in M$.

Moreover

- *The map* $x \mapsto \mathrm{Tr}(x\,\cdot\,)$ *is a homogeneous, additive bijection from* $\widehat{M^+}$ *onto the set of normal weights of* M,
- $x \leq y \iff \mathrm{Tr}(x\,\cdot\,) \leq \mathrm{Tr}(y\,\cdot\,)$ *and* $x_i \nearrow x \iff \mathrm{Tr}(x_i\,\cdot\,) \nearrow \mathrm{Tr}(x\,\cdot\,)$, *and*
- *If* $x = \int_0^\infty \lambda\, de_\lambda + \infty p$, *then* $\mathrm{Tr}(x\,\cdot\,)$ *is faithful if and only if* $e_0 = 0$ *and semifinite if and only if* $p = 0$.

Definition 4.2.15 ([Haa79], Definitions 2.1 and 2.2)**.** Let M and N be von Neumann algebras $N \subseteq M$. An operator valued weight from $M \to N$ is a map $T\colon M^+ \to \widehat{N^+}$ which satisfies the following conditions:

(1) $T(\lambda x + y) = \lambda T(x) + T(y)$ for all $\lambda \geq 0$ and $x, y \in M^+$, and

(2) $T(a^*xa) = a^*T(x)a$ for all $x \in M^+$ and $a \in N$.

As in the case of ordinary weights, we set

$$\mathfrak{n}_T = \left\{x \in M \middle| T(x^*x) \in N^+\right\} \text{ and}$$
$$\mathfrak{m}_T = \mathfrak{n}_T^*\mathfrak{n}_T = \operatorname{span}\left\{x^*y | x, y \in \mathfrak{n}_T\right\}.$$

Moreover, we say T is:

- normal if $x_i \nearrow x \Rightarrow T(x_i) \nearrow T(x)$ for all $x_i, x \in M^+$,
- faithful if $T(x^*x) = 0 \Rightarrow x = 0$ for all $x \in M^+$, and
- semifinite if $\mathfrak{n}_T$ is σ-weakly dense in M.

We will abbreviate normal, faithful, semifinite by the acronym n.f.s.

Remarks 4.2.16. (1) T is a conditional expectation if and only if $T(1) = 1$.

(2) If T is normal, it has a unique extension to $\widehat{M^+}$ satisfying (1) and (2).

(3) $\mathfrak{n}_T$ is a left-ideal and $\mathfrak{n}_T, \mathfrak{m}_T$ are algebraic $N - N$ bimodules. By polarization, T extends to a map $T\colon \mathfrak{m}_T \to N$, and $T(axb) = aT(x)b$ for all $x \in \mathfrak{m}_T$ and $a, b \in N$.

Theorem 4.2.17 ([Haa79], Theorem 2.7)**.** *Given an inclusion $N \subseteq M$ of semifinite von Neumann algebras with n.f.s. traces $\mathrm{Tr}_N, \mathrm{Tr}_M$ respectively. Then there is a unique n.f.s. trace-preserving operator valued weight $T\colon M^+ \to \widehat{N^+}$. Moreover, if $x \in M^+$, $T(x)$ is the unique element of $\widehat{N^+}$ such that*

$$\mathrm{Tr}_M(y \cdot x) = \mathrm{Tr}_N(y \cdot T(x)) \text{ for all } y \in N^+ \tag{4.2}$$

(where we also write Tr_N for the unique extension of Tr_N to $\widehat{N^+}$).

Definition 4.2.18. For $N \subseteq M$ an inclusion of von Neumann algebras, we write

- $\mathcal{P}(M, N)$ for the set of n.f.s. operator valued weights $M^+ \to \widehat{N^+}$, and
- $\mathcal{P}_0(M, N) \subseteq \mathcal{P}(M, N)$ for the set of operator valued weights whose restriction to $N' \cap M$ is semifinite.

Lemma 4.2.19 ([ILP98], Lemma 2.5 and Proposition 2.8, [Yam94], Corollary 28)**.** *Let $N \subset M$ be an inclusion of semifinite von Neumann algebras.*

(1) There is a unique central projection $z \in N' \cap M$ such that

- $\mathcal{P}_0(pMp, pN) = \emptyset$ *for all* $p \in N' \cap M$, $p \leq (1 - z)$ *and*
- $\mathcal{P}_0(zMz, zN) = \mathcal{P}(zMz, zN)$.

Moreover, for all $T \in \mathcal{P}(M, N)$,

- $(1 - z)(N' \cap M) \cap \mathfrak{m}_T = \{0\}$, *and*
- $T|_{z(N' \cap M)}$ *is semifinite.*

(2) If $\mathcal{P}_0(M, N) \neq \emptyset$ and $\mathcal{P}_0(N', M') \neq \emptyset$, then $N' \cap M$ is a direct sum of type I factors, and $pN \subset pMp$ has finite index for every finite rank $p \in N' \cap M$.

Useful lemmata on extended positive cones

For this subsection, M is a von Neumann algebra acting on a Hilbert space H.

Lemma 4.2.20. *For $m \in \widehat{M^+}$ and $\eta, \xi \in H$, the parallelogram identity holds:*

$$m(\omega_{\eta+\xi}) + m(\omega_{\eta-\xi}) = 2m(\omega_\eta) + 2m(\omega_\xi).$$

Proof. Take $(x_i) \subset M^+$ with x_i increasing to m. Then

$$\begin{aligned} m(\omega_{\eta+\xi}) + m(\omega_{\eta-\xi}) &= \sup_{i,j} \Big(x_i(\omega_{\eta+\xi}) + x_j(\omega_{\eta-\xi}) \Big) \\ &\leq \sup_{i,j} \left(\sup_{k \geq i,j} \Big(x_k(\omega_{\eta+\xi}) + x_k(\omega_{\eta-\xi}) \Big) \right) \\ &= \sup_{i,j} \left(\sup_{k \geq i,j} \Big(2x_k(\omega_\eta) + 2x_k(\omega_\xi) \Big) \right) \\ &\leq \sup_{i',j'} \Big(2x_{i'}(\omega_\eta) + 2x_{j'}(\omega_\xi) \Big) = 2m(\omega_\eta) + 2m(\omega_\xi). \end{aligned}$$

The other inequality is proved similarly. □

Lemma 4.2.21. *(1) $m_1 \leq m_2$ if and only if $m_1(\omega_\xi) \leq m_2(\omega_\xi)$ for all $\xi \in H$.*

(2) $(m_i)_{i \in I}$ increases to m if and only if $i \leq j$ implies $m_i \leq m_j$ and $\sup_i m_i(\omega_\xi) = m(\omega_\xi)$ for all $\xi \in H$.

(3) If $(m_i)_{i \in I}$ increases to m and $a \in M^+$, then $a^* m_i a$ increases to $a^* m a$.

Proof. First, note every $\phi \in M_*^+$ is a sum of functionals $\omega_{\xi_k} = \langle \cdot\, \xi_k, \xi_k \rangle$ for $\xi_i \in H$.

(1) Follows immediately by lower semicontinuity of $m \in \widehat{M^+}$.

(2) Suppose $\phi = \sum_k \omega_{\xi_k}$. By lower semicontinuity,

$$\begin{aligned} m(\phi) &= \sum_k m(\omega_{\xi_k}) = \sum_k \sup_i m_i(\omega_{\xi_k}) \\ &\geq \sup_i \sum_k m_i(\omega_{\xi_k}) = \sup_i m_i \left(\sum_k \omega_k \right) = \sup_i m_i(\phi). \end{aligned}$$

There are two cases:

<u>Case 1:</u> Suppose $m(\phi) = \infty$. Then there is a $\varepsilon > 0$ such that $\sup_i m_i(\omega_{\xi_k}) > \varepsilon$ for infinitely many k, say (k_n). Let $N > 0$, and let $M > 0$ such that $M\varepsilon > N$. Choose

$j_1 \in I$ such that $i \geq j_1$ implies $m_i(\omega_{k_1}) > \varepsilon$. For $n = 2, \ldots, M$, inductively choose $j_n > j_{n-1}$ such that $i \geq j_n$ implies $m_i(\omega_{k_n}) > \varepsilon$. Then for all $i > j_M$,

$$\sum_k m_i(\omega_{\xi_k}) \geq \sum_{n=1}^{M} m_i(\omega_{\xi_{k_n}}) \geq \sum_{n=1}^{M} \varepsilon = M\varepsilon > N.$$

Since N was arbitrary, we must have

$$\sup_i m_i(\phi) = \sup_i m_i\left(\omega_k\right) = \sup_i \sum_k m_i(\omega_k) = \infty.$$

Case 2: Suppose $m(\phi) < \infty$. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $\sum_{k>N} m(\omega_{\xi_k}) < \varepsilon$. Now as in the proof of Lemma 4.2.20,

$$m(\phi) - \varepsilon < \sum_{k=1}^{N} \sup_i m_i(\omega_{\xi_k}) = \sup_i \sum_{k=1}^{N} m_i(\omega_{\xi_k}) \leq \sup_i \sum_k m_i(\omega_k) = \sup_i m_i(\phi),$$

and the result follows as ε was arbitrary.

(3) We use (2). Let $\xi \in H$.

$$a^* m_i a(\omega_\xi) = m_i(\omega_{a\xi}) \leq m_j(\omega_{a\xi}) = a^* m_j a(\omega_\xi) \text{ for all } i \leq j \text{ and}$$
$$\sup_i a^* m_i a(\omega_\xi) = \sup_i m_i(\omega_{a\xi}) = m(\omega_{a\xi}) = a^* m a(\omega_\xi).$$

□

Remark 4.2.22. Suppose $(x_i)_{i\in I}, (y_i)_{i\in I} \subset M^+$ are directed families and $\lambda \geq 0$. Then by Lemma 4.2.21 and techniques similar to those used in the proof of Lemma 4.2.20,

$$\sup_i(\lambda x_i + y_i) = \lambda \sup_i x_i + \sup_j y_j.$$

Lemma 4.2.23. *Suppose $F \subset \widehat{M^+}$ is a directed family, i.e., if $x, y \in F$, then there is a $z \in F$ with $z \geq x$ and $z \geq y$. Then there is a unique $m_F = (K_F, S_F) \in \widehat{M^+}$ with $K_F = \overline{\operatorname{Dom}(S_F^{1/2})}$ such that*

$$m_F(\omega_\xi) = \langle S_F^{1/2}\xi, S_F^{1/2}\xi\rangle = \sup_{x\in F} x(\omega_\xi) \text{ for all}$$
$$\xi \in \operatorname{Dom}(S_F^{1/2}) = \left\{\xi \in H \,\middle|\, \sup_{x\in F} x(\omega_\xi) < \infty\right\}.$$

We denote m_F by $\sup_{x\in F} x$.

Proof. As in [Haa79, Con80, Tak03], one checks that the extended quadratic form $s_F\colon H \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ given by $s_F(\xi) = \sup_{x\in F} x(\omega_\xi)$ satisfies

(1) $s_F(\lambda\xi) = |\lambda|^2 s_F(\xi)$,

(2) $s_F(\eta+\xi) + s_F(\eta-\xi) = 2s_F(\eta) + 2s_F(\xi)$,

(3) s_F is lower semicontinuous, and

(4) $s_F(u\xi) = s_F(\xi)$ for all $u \in M'$.

(1) and (4) are trivial. (3) follows as sups of lower semicontinuous maps are lower semicontinuous. (2) is similar to the proof of Lemma 4.2.20. □

Definition 4.2.24. Suppose M is a semifinite von Neumann algebra with n.f.s. trace Tr_M acting on the right of H. Let $\xi \in D(H_M)$, and suppose $(x_i) \in (M' \cap B(H))^+$ with $x_i \nearrow x \in (\widehat{M' \cap B(H)})^+$. Then each $L(\xi)^* x_i L(\xi) \in M^+$ as it commutes with the right M-action on $L^2(M, \mathrm{Tr}_M)$, so we define

$$L(\xi)^* x L(\xi) = \sup_i L(\xi)^* x_i L(\xi) \in \widehat{M^+}.$$

Note that if $\kappa \in L^2(M, \mathrm{Tr}_M)$, then

$$\Big(L(\xi)^* x L(\xi)\Big)(\omega_\kappa) = \sup_i \Big(L(\xi)^* x_i L(\xi)\Big)(\omega_\kappa) = \sup_i x_i(\omega_{\xi\otimes\kappa}) = x(\omega_{\xi\otimes\kappa}),$$

which is independent of the choice of (x_i). Hence $L(\xi)^* x L(\xi)$ is well-defined by Lemma 4.2.21. Similarly, we may define operators of the form $R(\eta)^* y R(\eta)$, $L_\xi^* x L_\xi$, and $R_\eta^* y R_\eta$.

4.3 Planar calculus for bimodules

For this section, let A be a II_1-factor, and let ${}_AH_A$ be an $A-A$ Hilbert bimodule, i.e., H has commuting actions of A and A^{op}.

Centralizer algebras, central L^2-vectors, and canonical maps

Definition 4.3.1. For an $A-A$ bimodule K (algebraic or Hilbert), we define

$$A' \cap K = \{\xi \in K | a\xi = \xi a \text{ for all } a \in A\}.$$

Notation 4.3.2. For $n \geq 0$, let

- $H^n = \bigotimes_A^n H$, with the convention that $H^0 = L^2(A)$,

- $B^n = D({}_AH^n) \cap D(H^n_A)$, which is dense in H^n by Lemma 1.2.2 of [Pop86]. We also use the convention $B = B^1$. Note $B^0 = A$.
- $\{\alpha\} \subset B$ be an ${}_AH$ basis (possible due to the density of B in H), with

$$\{\alpha^n\} = \{\alpha_1 \otimes \cdots \otimes \alpha_n | \alpha_i \in \{\alpha\} \text{ for all } i = 1, \ldots, n\} \subset B^n$$

  the corresponding ${}_AH^n$ basis (as $R_{\alpha_1 \otimes \cdots \otimes \alpha_n} = R_{\alpha_1} \cdots R_{\alpha_n}$). We let $\{\beta\} \subset B$ be an H_A basis, with $\{\beta^n\} \subset B^n$ the corresponding H^n_A basis.
- (central L^2-vectors) $P_n = A' \cap H^n$. Note $P_0 = A' \cap L^2(A) = \mathbb{C}\widehat{1}$.
- $C_n = (A^{\text{op}})' \cap B(H^n)$ (the commutant of the right A-action on H^n) with canonical trace $\text{Tr}_n = \sum_{\beta^n} \langle \cdot \beta^n, \beta^n \rangle$,
- $C_n^{\text{op}} = A' \cap B(H^n)$ with canonical trace $\text{Tr}_n^{\text{op}} = \sum_{\alpha^n} \langle \cdot \alpha^n, \alpha^n \rangle$,
- (centralizer algebras) $Q_n = C_n \cap C_n^{\text{op}}$.

Remark 4.3.3. Note that $A \subset C_n$ and $A^{\text{op}} \subset C_n^{\text{op}}$.

Definition 4.3.4. *H is called symmetric if there is a conjugate-linear isomorphism $J : H \to H$ such that $J(a\xi b) = b^*(J\xi)a^*$ for all $a, b \in A$ and $\xi \in H$ and $J^2 = \text{id}_H$.*

Remark 4.3.5. If H is symmetric, then for $n \geq 1$, H^n is symmetric with conjugate-linear isomorphism $J_n : H^n \to H^n$ given by the extension of

$$J_n(\xi_1 \otimes \cdots \otimes \xi_n) = (J\xi_1) \otimes \cdots \otimes (J\xi_n).$$

for $\xi_i \in B$ for all i. Note that $J_n A J_n = A^{\text{op}}$, $J_n C_n J_n = C_n^{\text{op}}$, and $J_n B^n = B^n$. On $B(H^n)$, we define j_n by $j_n(x) = J_n x^* J_n$. Note that $j_n^2 = \text{id}$ and $\text{Tr}_n = \text{Tr}_n^{\text{op}} \circ j_n$.

If H is not symmetric, then in general, C_n^{op} is not the opposite algebra of C_n, e.g. ${}_{R\otimes 1}L^2(R \otimes R)_{R \otimes R}$ where R is the hyperfinite II_1-factor.

Remark 4.3.6. It is clear that B^n is an $A - A$ bimodule. If $\eta \in B^n$ and $c \in C_n$, then $c\xi \in D(H^n_A)$, but in general, $c\xi \notin D({}_AH^n)$. However, if $c \in Q_n$, then clearly $c\xi \in B^n$.

Proposition 4.3.7. *We have natural inclusions:*

$$i_n : C_n \to C_{n+1} \text{ by } x \mapsto x \otimes_A \text{id}_H = (\eta \otimes \xi \mapsto (x\eta) \otimes \xi \text{ for } \eta \in B^n \text{ and } \xi \in B) \text{ and}$$
$$i_n^{\text{op}} : C_n^{\text{op}} \to C_{n+1}^{\text{op}} \text{ by } y \mapsto \text{id}_H \otimes_A y = (\xi \otimes \eta \mapsto \xi \otimes (y\eta) \text{ for } \xi \in B \text{ and } \eta \in B^n).$$

Both maps include $Q_n \to Q_{n+1}$.

Proof. If $z \in Q_n$, then $i_n(z) \in Q_{n+1}$ as for all $a, b \in A$,

$$(z \otimes_A \text{id}_H)[a(\xi \otimes \eta)b] = (z(a\xi)) \otimes (\eta b) = (a(z\xi)) \otimes (\eta b) = a[(z\eta) \otimes \xi]b.$$

The result is similar for i_n^{op}. □

Proposition 4.3.8. *If $x \in C_n$, then $i_n(x) = \sum_\alpha R_\alpha x R_\alpha^*$. If $y \in C_n^{\mathrm{op}}$, then $i_n^{\mathrm{op}}(y) = \sum_\beta L_\beta y L_\beta^*$.*

Proof. We prove the first statement. If $\xi_1, \dots, \xi_{n+1} \in B$, we have

$$\begin{aligned}\left(\sum_\alpha R_\alpha x R_\alpha^*\right) \xi_1 \otimes \cdots \otimes \xi_n &= \sum_\alpha R_\alpha x(\xi_1 \otimes \cdots \otimes \xi_{n-1\,A}\langle \xi_n, \alpha\rangle) \\ &= \sum_\alpha \big(x(\xi_1 \otimes \cdots \otimes \xi_{n-1\,A}\langle \xi_n, \alpha\rangle) \otimes \alpha\big) \\ &= \sum_\alpha \big(x(\xi_1 \otimes \cdots \otimes \xi_{n-1})\big) \otimes {}_A\langle \xi_n, \alpha\rangle \alpha \\ &= [x(\xi_1 \otimes \cdots \otimes \xi_{n-1})] \otimes \xi_n = i_n(x)(\xi_1 \otimes \cdots \otimes \xi_n).\end{aligned}$$

□

Remark 4.3.9. By Definition 4.2.5, $(C_k \otimes_A \mathrm{id}_{n-k})' \cap B(H^n) = \mathrm{id}_k \otimes_A C_{n-k}^{\mathrm{op}}$.

Lemma 4.3.10. *Suppose $\xi \in H^n$ and $y \in (C_{n+1}^{\mathrm{op}})^+$. Recall the operator $R_\xi^0 \colon B \to H^{n+1}$ by $\eta \mapsto \eta \otimes \xi$ is closable by Corollary 4.2.11. Then $y^{1/2} R_\xi^0 \colon B \to H^{n+1}$ is also closable.*

Proof. Let p be the range/kernel perp projection of $y^{1/2}$. By the spectral theorem, there are projections $p_k \in C_{n+1}^{\mathrm{op}}$ such that $y^{1/2} p_k = p_k y^{1/2}$ is invertible on $p_k H^{n+1}$ and $p_k \nearrow p$ (strongly). Fix $k \geq 0$. Vectors of the form $\zeta = \sum_{i=1}^j \sigma_i \otimes \kappa_i \in p_k H^{n+1}$ where $\sigma_1, \dots, \sigma_j \in B$ and $\kappa_1, \dots, \kappa_j \in B^n$ are dense in $p_k H^{n+1}$ by the density of $B \otimes_A B^n \subset H^{n+1}$. Then for such ζ and all $\eta \in B$,

$$\langle y^{1/2} R_\xi^0 \eta, y^{-1/2} p_k \zeta\rangle = \sum_{i=1}^j \langle \eta \otimes \xi, \sigma_i \otimes \kappa_i\rangle = \sum_{i=1}^j \langle \eta, L_{\sigma_i}({}_A\langle \kappa_i, \xi\rangle)\rangle = \left\langle \eta, \sum_{i=1}^j L_{\sigma_i}({}_A\langle \kappa_i, \xi\rangle)\right\rangle$$

(see Corollary 4.2.11). Finally, the span of vectors of the form $y^{-1/2} p_k \zeta$ where ζ is as above and $k \geq 0$ is dense in pH^{n+1}. □

The following proposition and its proof are similar to Theorem 3.2.26 and Proposition 3.2.27 of [Bur03].

Proposition 4.3.11. *Recall from Proposition 4.3.7 that $i_n(C_n) \subset C_{n+1}$ and $i_n^{\mathrm{op}}(C_n^{\mathrm{op}}) \subset C_{n+1}^{\mathrm{op}}$. The unique trace-preserving operator valued weight*

$$T_{n+1} \colon (C_{n+1}^+, \mathrm{Tr}_{n+1}) \to (\widehat{C_n^+}, \mathrm{Tr}_n) \text{ is given by } x \mapsto \sum_\beta R_\beta^* x R_\beta.$$

The unique trace-preserving operator valued weight

$$T_{n+1}^{\mathrm{op}} \colon \left((C_{n+1}^{\mathrm{op}})^+, \mathrm{Tr}_{n+1}^{\mathrm{op}}\right) \to \left(\widehat{(C_n^{\mathrm{op}})^+}, \mathrm{Tr}_n^{\mathrm{op}}\right) \text{ is given by } y \mapsto \sum_\alpha L_\alpha^* y L_\alpha.$$

In particular, T_{n+1} and T_{n+1}^{op} are independent of the choice of basis.

Proof. We prove the result for the second statement.

Suppose $y \in (C_{n+1}^{\text{op}})^+$ and $\xi \in H^n$. By Lemma 4.3.10, $y^{1/2}R_\xi^0$ is closable, so we set $S = (y^{1/2}R_\xi^0)^*\overline{y^{1/2}R_\xi^0}$, which is affiliated with C_1^{op}, and define $m_S \in \widehat{(C_1^{\text{op}})^+}$ as in Equation (4.1) by

$$m_S(\omega_\eta) = \begin{cases} \|S^{1/2}\eta\| & \text{if } \eta \in D(S^{1/2}) \supset B \\ \infty & \text{else.} \end{cases}$$

Now we calculate that

$$\begin{aligned}\operatorname{Tr}_1^{\text{op}}(m_S) &= \sum_\alpha m_S(\omega_\alpha) = \sum_\alpha \|S^{1/2}\alpha\|_2^2 = \sum_\alpha \|y^{1/2}R_\xi^0\alpha\|_2^2 \\ &= \sum_\alpha \langle y(\alpha\otimes\xi), (\alpha\otimes\xi)\rangle_{H^{n+1}} = \left\langle \left(\sum_\alpha L_\alpha^* y L_\alpha\right)\xi, \xi\right\rangle_{H^n} = T_{n+1}^{\text{op}}(y)(\omega_\xi).\end{aligned}$$

As all elements of $B(H)_*^+$ are sums $\sum_i \omega_{\xi_i}$, T_{n+1}^{op} is well-defined and independent of the choice of $\{\alpha\}$.

Note that $T_{n+1}^{\text{op}}((C_{n+1}^{\text{op}})^+) \subset \widehat{(C_n^{\text{op}})^+}$ as if $y \in (C_{n+1}^{\text{op}})^+$, $\xi \in H^n$, and $u \in U(A)$, then

$$\begin{aligned}\sum_\alpha L_\alpha^* y L_\alpha(\omega_{u\xi}) &= \sum_\alpha \langle y(\alpha\otimes u\xi), \alpha\otimes u\xi\rangle = \sum_\alpha \langle y(\alpha u\otimes\xi), \alpha u\otimes\xi\rangle \\ &= \sum_\alpha L_{\alpha u}^* y L_{\alpha u}(\omega_\xi) = \sum_\alpha L_\alpha^* y L_\alpha(\omega_\xi)\end{aligned}$$

as $\{\alpha u\}$ is another ${}_AH$ basis by Lemma 4.2.6.

Finally, if $x \in (C_n^{\text{op}})^+$ and $y \in (C_{n+1}^{\text{op}})^+$, then

$$\begin{aligned}\operatorname{Tr}_{n+1}^{\text{op}}\left([i_n^{\text{op}}(x^{1/2})]y[i_n^{\text{op}}(x^{1/2})]\right) &= \sum_{\alpha^{n+1}} \left\langle [i_n^{\text{op}}(x^{1/2})]y[i_n^{\text{op}}(x^{1/2})]\alpha^{n+1}, \alpha^{n+1}\right\rangle \\ &= \sum_{\alpha,\alpha^n} \left\langle y(\alpha\otimes(x^{1/2}\alpha^n)), (\alpha\otimes(x^{1/2}\alpha^n))\right\rangle \\ &= \sum_{\alpha^n} \left\langle \sum_\alpha L_\alpha^* y L_\alpha(x^{1/2}\alpha^n), (x^{1/2}\alpha^n)\right\rangle \\ &= \operatorname{Tr}_n^{\text{op}}\left(x^{1/2}T_{n+1}^{\text{op}}(y)x^{1/2}\right),\end{aligned}$$

so T_{n+1}^{op} is the unique trace-preserving operator valued weight by Equation (4.2) in Theorem 4.2.17. □

Remark 4.3.12. If $z \in Q_{n+1}^+$, then $T_{n+1}^{\text{op}}(z) \in \widehat{Q_n^+}$ as if $\xi \in H^n$ and $u \in U(A)$,

$$\sum_\alpha L_\alpha^* z L_\alpha(\omega_{\xi u}) = \sum_\alpha \langle z(\alpha\otimes\xi u), \alpha\otimes\xi u\rangle = \sum_\alpha \langle (z(\alpha\otimes\xi))uu^*, \alpha\otimes\xi\rangle = \sum_\alpha L_\alpha^* z L_\alpha(\omega_\xi).$$

A similar result holds for T_{n+1}.

Corollary 4.3.13. *If $z \in Q_1^+$, then $\sum_\alpha L(\alpha)^* z L(\alpha) = \operatorname{Tr}_1^{\mathrm{op}}(z) 1_{L^2(A)}$.*
Similarly, $\sum_\alpha R(\beta)^* z R(\beta) = \operatorname{Tr}_1(z) 1_{L^2(A)}$.

Proof. We prove the first formula. First, $\sum_\alpha L(\alpha)^* z L(\alpha) \in \widehat{Q_0^+} = [0, \infty]$. Now

$$\left(\sum_\alpha L(\alpha)^* z L(\alpha)\right)(\omega_{\widehat{1}}) = \sum_\alpha \langle L(\alpha)^* z L(\alpha)\widehat{1}, \widehat{1}\rangle = \sum_\alpha \langle z\alpha, \alpha\rangle = \operatorname{Tr}_1^{\mathrm{op}}(z).$$

□

Proposition 4.3.14. *The unique trace-preserving operator valued weight*

$$\widetilde{T_{n+1}} \colon (Q_{n+1}^+, \operatorname{Tr}_{n+1}) \to (i_n^{\mathrm{op}}(\widehat{Q_n^+}), \operatorname{Tr}_n) \text{ is given by } x \mapsto \sum_\beta L_\beta^* x L_\beta.$$

The unique trace-preserving operator valued weight

$$\widetilde{T_{n+1}^{\mathrm{op}}} \colon \left(Q_{n+1}^+, \operatorname{Tr}_{n+1}^{\mathrm{op}}\right) \to \left(i_n(\widehat{Q_n^+}), \operatorname{Tr}_n^{\mathrm{op}}\right) \text{ is given by } y \mapsto \sum_\alpha R_\alpha^* y R_\alpha.$$

In particular, $\widetilde{T_{n+1}}$ and $\widetilde{T_{n+1}^{\mathrm{op}}}$ are independent of the choice of basis.

Proof. Similar to the proof of Proposition 4.3.11 using Remark 4.3.12. Note that if $u \in U(A)$, then $\{u\alpha\}, \{\beta u\}$ are also ${}_AH, H_A$-bases respectively by Lemma 4.2.6. □

Planar algebra over extended positive cones of centralizer algebras

In this subsection, we define an operad $\mathbb{BP}$, and describe a $\mathbb{BP}$-algebra of extended positive cones $\widehat{Q_\bullet^+}$. The proof that the action is well-defined is deferred to Section 4.7 as it is quite technical. The relations given in the next theorem will be important in our approach.

Theorem 4.3.15. *The following relations hold among the maps $i_n, i_n^{\mathrm{op}}, T_n, T_n^{\mathrm{op}}, \otimes_A, \operatorname{Tr}_n, \operatorname{Tr}_n^{\mathrm{op}}$ for $m, n \geq 1$ (compare with Theorem 4.7.2, Remark 4.7.8, and the proof of Theorem 4.7.13):*

(1) $T_n T_{n+1}^{\mathrm{op}}(z) = T_n^{\mathrm{op}} T_{n+1}(z)$ for all $z \in \widehat{Q_{n+1}^+}$,

(2) $z_1 \otimes_A (z_2 \otimes_A z_3) = (z_1 \otimes_A z_2) \otimes_A z_3$ for all $z_i \in \widehat{Q_{n_i}^+}$, $i = 1, 2, 3$,

(3) $T_{m+n}(z_1 \otimes z_2) = z_1 \otimes_A (T_n z_2)$ and $T_{m+n}^{\mathrm{op}}(z_1 \otimes z_2) = (T_m^{\mathrm{op}} z_1) \otimes_A z_2$ for all $z_1 \in \widehat{Q_m^+}$ and $z_2 \in \widehat{Q_n^+}$,

(4) $\operatorname{Tr}_n(z_1 \cdot z_2) = \operatorname{Tr}_n(z_2 \cdot z_1)$ for all $z_1, z_2 \in \widehat{Q_n^+}$, and similarly for $\operatorname{Tr}_n^{\mathrm{op}}$, and

(5) $\operatorname{Tr}_{n+1}(z_1 \cdot i_n(z_2)) = \operatorname{Tr}_n(T_{n+1}(z_1) \cdot z_2)$ for all $z_1 \in \widehat{Q_{n+1}^+}$ and $z_2 \in \widehat{Q_n^+}$, and a similar statement holds with ${}^{\mathrm{op}}$.

Proof. (1) For all $\xi \in H^n$ and $z \in \widehat{Q_{n+1}^+}$,

$$\begin{aligned}\left(T_n T_{n+1}^{\mathrm{op}}(z)\right)(\omega_\xi) &= \left(\sum_\beta R_\beta^* \left(\sum_\alpha L_\alpha^* z L_\alpha\right) R_\beta\right)(\omega_\xi) = \sum_{\alpha,\beta} (R_\beta^* L_\alpha^* z L_\alpha R_\beta)(\omega_\xi) \\ &= \sum_{\alpha,\beta} z(\omega_{\alpha\otimes\xi\otimes\beta}) = \left(\sum_\alpha L_\alpha^* \left(\sum_\beta R_\beta^* z R_\beta\right) L_\alpha\right)(\omega_\xi) \\ &= \left(T_n^{\mathrm{op}} T_{n+1}(z)\right)(\omega_\xi).\end{aligned}$$

(2) This is Corollary 4.6.14.

(3) Suppose $z_{1,j} \in Q_m^+$ increases to z_1 and $z_{2,k} \in Q_n^+$ increases to z_2. Then

$$\begin{aligned}T_{m+n}(z_{1,j} \otimes_A z_{2,k}) &= \sum_\beta R_\beta^*(z_{1,j} \otimes_A z_{2,k}) R_\beta = \sum_\beta z_{1,j} \otimes_A \left(R_\beta^* z_{2,k} R_\beta\right) \\ &= z_{1,j} \otimes_A \left(\sum_\beta R_\beta^* z_{2,k} R_\beta\right) = z_{1,j} \otimes_A (T_n z_{2,k}).\end{aligned}$$

Now $T_n z_{2,k}$ increases to $T_n z_2$, and we are finished by Theorem 4.6.16. The other equality is similar.

(4) This is Theorem 4.2.14.

(5) This is Proposition 4.8.11.

□

Corollary 4.3.16. *The following relations also hold:*

(1) $i_{n+1} i_n^{\mathrm{op}}(z) = i_{n+1}^{\mathrm{op}} i_n(z)$ *for all* $z \in \widehat{Q_n^+}$.

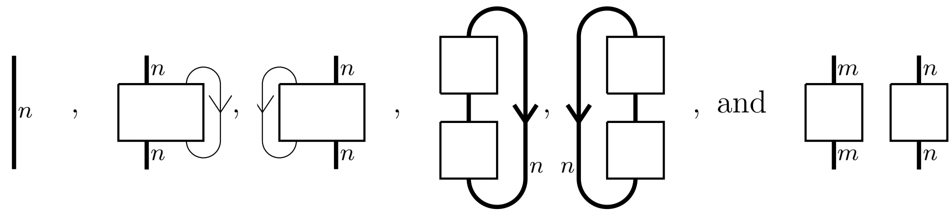
(2) $i_{m+n}(z_1 \otimes_A z_n) = z_1 \otimes_A i_n(z_2)$ *and* $i_{m+n}^{\mathrm{op}}(z_1 \otimes_A z_2) = i_m^{\mathrm{op}}(z_1) \otimes_A z_2$ *for all* $z_1 \in \widehat{Q_m^+}$ *and* $z_2 \in \widehat{Q_n^+}$,

(3) $i_{n-1}^{\mathrm{op}} T_n(z) = T_{n+1} i_n^{\mathrm{op}}(z)$ *and* $i_{n-1} T_n^{\mathrm{op}}(z) = T_{n+1}^{\mathrm{op}} i_n(z)$ *for all* $z \in \widehat{Q_n^+}$,

(4) $(T_{n+1} \circ \cdots \circ T_{m+n})(z_1 \otimes_A z_2) = \mathrm{Tr}_n(z_2) z_1$ *for all* $z_1 \in \widehat{Q_m^+}$ *and* $z_2 \in \widehat{Q_n^+}$, *and a similar statement holds with* $^{\mathrm{op}}$. *In particular,* $\mathrm{Tr}_{m+n}(z_1 \otimes z_2) = \mathrm{Tr}_m(z_1)\,\mathrm{Tr}_n(z_2)$ *and* $\mathrm{Tr}_{m+n}^{\mathrm{op}}(z_1 \otimes z_2) = \mathrm{Tr}_m^{\mathrm{op}}(z_1)\,\mathrm{Tr}_n^{\mathrm{op}}(z_2)$.

(5) $\mathrm{Tr}_{m+n}((z_1 \otimes_A z_2) \cdot (z_3 \otimes_A z_4)) = \mathrm{Tr}_m(z_1 \cdot z_3)\,\mathrm{Tr}_n(z_2 \cdot z_4)$ *for all* $z_1, z_3 \in \widehat{Q_m^+}$ *and* $z_2, z_4 \in \widehat{Q_n^+}$. *A similar statement holds for* $\mathrm{Tr}_n^{\mathrm{op}}$.

Definition 4.3.17. The bimodule planar operad $\mathbb{BP}$ is the operad of oriented, unshaded planar tangles (up to planar isotopy) generated by



for $m, n \geq 0$ up to planar isotopy. (We draw all disks as boxes, suppress external disks, draw one thick string labelled n for n individual strings, and orient all strings upward unless otherwise specified.) A topological characterization of $\mathbb{BP}$ tangles is given in Theorem 4.7.9.

A $\mathbb{BP}$-algebra (of extended positive cones) $V_\bullet$ is a sequence $\{V_n\}_{n\geq 0}$ of extended positive cones (defined in Section 4.8) and an action by multilinear maps

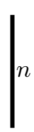
$$Z\colon \mathbb{BP} \to ML\{V_n\}$$

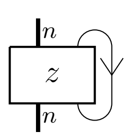
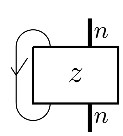
(Z is the partition function) which is well-behaved under composition.

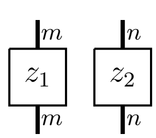
A $\mathbb{BP}$-algebra is called:

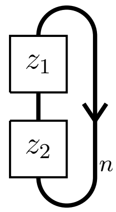
- central if $V_0 = [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$,
- normal if $Z(\mathcal{T})$ is normal for all $\mathcal{T} \in \mathbb{BP}$, and
- self-dual if V_n is self-dual for all n, and for all annular tangles $\mathcal{T} \in \mathbb{BP}$, flipping it inside out gives the adjoint map (see Definitions 4.8.8 and 4.8.10).

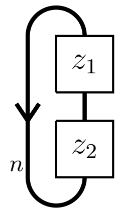
Theorem 4.3.18. *Given an $A - A$ bimodule H, the extended positive cones $\widehat{Q_n^+}$ form a central, normal, self-dual $\mathbb{BP}$-algebra $\widehat{Q_\bullet^+}$ such that:*

(1) $\mathrm{id}_{H^n} = \mathrm{id}_n =$  ,

(2) $T_{n+1}(z) =$  *and* $T_{n+1}^{\mathrm{op}}(z) =$  *for all* $z \in \widehat{Q_{n+1}^+}$,

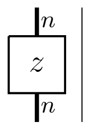
(3) $z_1 \otimes_A z_2 =$  *(defined in Section 4.6) for all* $z_1 \in \widehat{Q_m^+}$ *and* $z_2 \in \widehat{Q_n^+}$, *and*

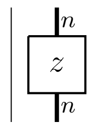
(4) $\mathrm{Tr}_n(z_1 \cdot z_2) =$ 

and $\mathrm{Tr}_n^{\mathrm{op}}(z_1 \cdot z_2) =$ 

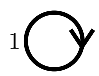
for all $z_1, z_2 \in \widehat{Q_n^+}$.

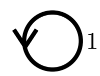
Moreover, the following hold:

(5) $i_n(z) =$ 

and $i_n^{\mathrm{op}}(z) =$ 

for all $z \in \widehat{Q_n^+}$ *and*

(6) $\dim_{-A}(H) = T_1(1) =$ 

and $\dim_{A-}(H) = T_1^{\mathrm{op}}(1) =$ 

.

Note that for Z to be well-defined, any closed diagram must count for a multiplicative factor in $\widehat{Q_0^+} = \widehat{Z(A)^+} = [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$.

We call $\widehat{Q_\bullet^+}$ the canonical $\mathbb{BP}$-algebra associated to H.

Proof. We will show (1)-(4) uniquely determine the action of any $\mathbb{BP}$-tangle. We defer this technical proof to Section 4.7 (Theorem 4.7.13), which uses the important relations given in Theorem 4.3.15 and Corollary 4.3.16. Note that $\widehat{Q_\bullet^+}$ is central since $\widehat{Q_0^+} = \widehat{Z(A)^+} = [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, normal by Theorem 4.2.14 and Remark 4.8.7, and self-dual by Proposition 4.8.11. □

Remark 4.3.19. Given some operad $\mathbb{P}$ of (shaded, unshaded, oriented, disoriented, etc.) planar tangles, it is not always possible to define an (extended) positive cone planar algebra over $\mathbb{P}$. For example, the rotation does not always map positive elements to positive elements in a subfactor planar algebra.

Graded algebra of central L^2-vectors

In this subsection, we define a graded algebra $P_\bullet$ of central L^2-vectors.

Lemma 4.3.20. *Suppose K is a Hilbert $A-A$ bimodule. Then $A' \cap K \subseteq D(_AK) \cap D(K_A)$.*

Proof. Suppose $\zeta \in A' \cap K$, $\zeta \neq 0$. Define $\varphi\colon A_+ \to \mathbb{C}$ by $a \mapsto \langle a\zeta, \zeta\rangle$. Note that φ is traicial as

$$\varphi(a^*a) = \langle a^*a\zeta, \zeta\rangle = \langle a^*\zeta a, \zeta\rangle = \langle a^*\zeta, \zeta a^*\rangle = \langle a^*\zeta, a^*\zeta\rangle = \langle aa^*\zeta, \zeta\rangle = \varphi(aa^*).$$

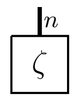
Hence there is a $\lambda \geq 0$ such that $\varphi = \lambda \operatorname{tr}_A$ by the uniqueness of the trace on a II_1-factor. Now for all $a \in A$,

$$\|a\zeta\|_2^2 = \|\zeta a\|_2^2 = \varphi(a^*a) = \lambda \operatorname{tr}_A(a^*a) = \lambda\|a\|_2^2,$$

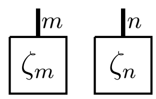
and ζ is left and right A-bounded. □

Remark 4.3.21. In the sequel, we will confuse elements $\zeta \in P_n$ and the operators $L(\zeta) = R(\zeta)\colon L^2(A) \to H^n$. We will omit $R(\zeta)$ and only write $L(\zeta)$.

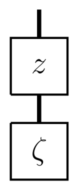
Definition 4.3.22. We represent elements $\zeta \in P_n$ by boxes with n strings emanating from the top

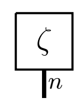
ζ or $L(\zeta) =$  .

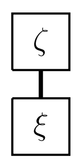
By Lemma 4.3.20, the P_n's form a graded algebra $P_\bullet$ in the sense of [GJS10] where the graded multiplication is given by relative tensor product (over A) of central vectors. We denote the product of $\zeta_m \in P_m$ and $\zeta_n \in P_n$ by

$\zeta_m \otimes \zeta_n =$  $\in P_{m+n}$.

If $z \in Q_n$ and $\zeta \in P_n$, then $z\zeta \in P_n$, which we denote as:

$z\zeta$ or $L(z\zeta) =$  .

The reflections of these diagrams denote the functionals $\langle \cdot, \zeta\rangle$ or adjoints $L(\zeta)^* =$  .

The inner product $\langle \cdot, \cdot \rangle\colon P_n \times P_n^* \to \mathbb{C}$ is given by $\langle \xi, \zeta\rangle =$  (see Lemma 4.3.23 (2)).

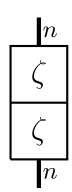
Compatibility

We now show how the $\mathbb{BP}$-algebra $\widehat{Q_\bullet^+}$ and the graded algebra $P_\bullet$ are compatible.

Lemma 4.3.23. *(1) If $\zeta \in P_n$ and $\xi \in B^n$, then ${}_A\langle \zeta, \xi\rangle = \langle \xi|\zeta\rangle_A$.*

(2) If $\zeta, \xi \in P_n$, ${}_A\langle \zeta, \xi\rangle = \langle \xi|\zeta\rangle_A = \langle \zeta, \xi\rangle 1_{L^2(A)} \in \mathbb{C}1_{L^2(A)}$.

(3) For $\zeta \in P_n$, $L(\zeta)L(\zeta)^* = R(\zeta)R(\zeta)^* \in Q_n^+$. We denote the common operator as:

 $\in Q_n^+$.

(4) If $\zeta \in P_n$ and $\|\zeta\|_2 = 1$, $L(\zeta)L(\zeta)^*|_{P_n} = p_\zeta$, the projection onto $\mathbb{C}\zeta$.

Proof. (1) Suppose $a_1, a_2 \in A$. Then

$$\begin{aligned}\langle {}_A\langle \zeta, \xi\rangle \widehat{a_1}, \widehat{a_2}\rangle &= \langle JR(\zeta)^* R(\xi) J\widehat{a_1}, \widehat{a_2}\rangle = \langle \widehat{a_2^*}, R(\zeta)^* R(\xi)\widehat{a_1^*}\rangle = \langle a_2^*\zeta, a_1^*\xi\rangle \\ &= \langle \zeta a_2^*, a_1^*\xi\rangle = \langle a_1\zeta, \xi a_2\rangle = \langle \zeta a_1, \xi a_2\rangle = \langle L(\zeta)\widehat{a_1}, L(\xi)\widehat{a_2}\rangle \\ &= \langle \langle \xi|\zeta\rangle_A \widehat{a_1}, \widehat{a_2}\rangle.\end{aligned}$$

(2) Since $\zeta, \xi \in P_n$, for all $a, b, a_1, a_2 \in A$,

$$\begin{aligned}\langle \langle \xi|\zeta\rangle_A (a\widehat{a_1}b), \widehat{a_2}\rangle &= \langle \zeta a a_1 b, \xi a_2\rangle = \langle \zeta a_1, \xi a^* a_2 b^*\rangle \\ &= \langle \langle \xi|\zeta\rangle_A \widehat{a_1}, a^*\widehat{a_2}b^*\rangle = \langle a(\langle \xi|\zeta\rangle_A \widehat{a_1})b, \widehat{a_2}\rangle,\end{aligned}$$

so $\langle \xi|\zeta\rangle_A \in Z(A) = \mathbb{C}1_A$. Now setting $a = b = a_1 = a_2 = 1_A$ gives the result.

(3) For $\xi \in B^n$, by (1),

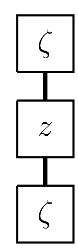
$$L(\zeta)L(\zeta)^*\xi = \zeta\langle \zeta|\xi\rangle_A = \langle \zeta|\xi\rangle_A \zeta = {}_A\langle \xi, \zeta\rangle\zeta = R(\zeta)R(\zeta)^*\xi,$$

so the two are equal on H^n. We have $C_n \ni L(\zeta)L(\zeta)^* = R(\zeta)R(\zeta)^* \in C_n^{\mathrm{op}}$, so $L(\zeta)L(\zeta)^* \in Q_n^+$.

(4) Trivial from (2) and (3).

□

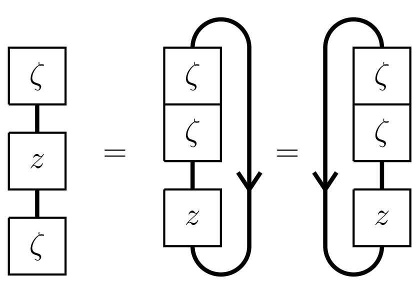
Theorem 4.3.24. *Suppose $\zeta \in P_n$ and $z \in \widehat{Q_n^+}$.*

(1) $L(\zeta)^* z L(\zeta) = z(\omega_\zeta)1_{L^2(A)} = R(\zeta)^* z R(\zeta)$. We denote this diagrammatically by 

.

(2) In the notation of Theorem 4.2.14,

$$\begin{aligned}z(\omega_\zeta) &= \mathrm{tr}_A(L(\zeta)^* z L(\zeta)) = \mathrm{Tr}_n(L(\zeta)L(\zeta)^* \cdot z) \\ &= \mathrm{tr}_{A^{\mathrm{op}}}(R(\zeta)^* z R(\zeta)) = \mathrm{Tr}_n^{\mathrm{op}}(z \cdot R(\zeta)R(\zeta)^*).\end{aligned}$$

In diagrams, 

.

Proof. (1) We show the first equality. If $z \in Q_n^+$, this is just (2) of Lemma 4.3.23 with $\zeta_1 = \zeta_2 = z^{1/2}\zeta$. Now for $z \in \widehat{Q_n^+}$, pick $(z_m) \subset Q_n^+$ with $z_m \nearrow z$ to get

$$L(\zeta)^* z L(\zeta) = \lim_{m\to\infty} L(\zeta)^* z_m L(\zeta) = \lim_{m\to\infty} z_m(\omega_\zeta) 1_{L^2(A)} = z(\omega_\zeta)1_{L^2(A)}.$$

The second equality is similar.

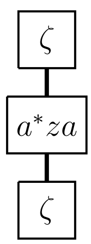
(2) We show the second equality. We may assume $z \in Q_n^+$, after which we may take sups to get the full result. Then as $z^{1/2}\zeta \in P_n$, we have

$$\begin{aligned}\mathrm{Tr}_n(z \cdot L(\zeta)L(\zeta)^*) &= \mathrm{Tr}_n(z^{1/2}L(\zeta)L(\zeta)^* z^{1/2}) = \mathrm{Tr}_n(L(z^{1/2}\zeta)L(z^{1/2}\zeta)^*)\\ &= \mathrm{tr}_A(L(z^{1/2}\zeta)^* L(z^{1/2}\zeta)) = \mathrm{tr}_A(L(\zeta)^* z L(\zeta)).\end{aligned}$$

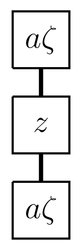
The other equality is similar.

□

Remark 4.3.25. If $a \in Q_n$, $z \in \widehat{Q_n^+}$, and $\zeta \in P_n$,

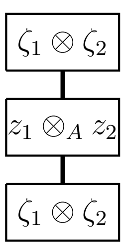


$$= (a^* z a)(\omega_\zeta) = z(\omega_{a\zeta}) =$$

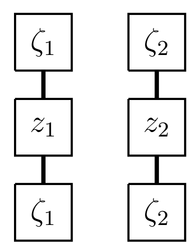


.

Corollary 4.3.26. *If $\zeta_1 \in P_m$, $\zeta_2 \in P_n$, $z_1 \in Q_m^+$, and $z_2 \in Q_n^+$, then*

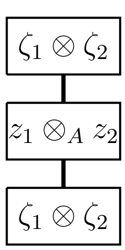


$$= \langle (z_1 \otimes_A z_2)(\zeta_1 \otimes \zeta_2), (\zeta_1 \otimes \zeta_2)\rangle = \langle z_1\zeta_1, \zeta_1\rangle\langle z_2\zeta_2, \zeta_2\rangle =$$

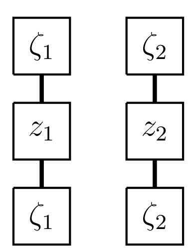


.

For $z_1 \in \widehat{Q_m^+}$, and $z_2 \in \widehat{Q_n^+}$, taking sups gives

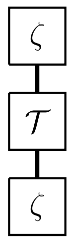


$$= (z_1 \otimes_A z_2)(\omega_{\zeta_1 \otimes \zeta_2}) = z_1(\omega_{\zeta_1}) z_2(\omega_{\zeta_2}) =$$



.

Theorem 4.3.27 ($P_\bullet$ acts on $\widehat{Q_\bullet^+}$). *Given a tangle $\mathcal{T} \in \mathbb{BP}$ with $2n$ boundary points and a $\zeta \in P_n$, we have*



$$:= \operatorname{ev}_{\omega_\zeta} \circ \mathcal{T} : V_{i_1} \times \cdots \times V_{i_k} \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}].$$

In this sense, we say $P_\bullet$ acts as weights on $\widehat{Q_\bullet^+}$. By Theorems 4.3.15 and 4.3.24 and Corollary 4.3.26, we may remove closed subdiagrams and multiply by the appropriate scalar in $[0_{\mathbb{R}}, \infty_{\mathbb{R}}]$.

Remark 4.3.28. If $A \subset (B, \operatorname{tr}_B)$ is an inclusion of II_1-factors and $H = L^2(B)$, then one can also define a shaded bimodule planar operad which works similarly to the above construction. This will be explored in a future paper.

4.4 Extremality and rotations

For this section, A is a II_1-factor. Assume the notation of the last section.

Extremality

Definition 4.4.1. H is approximately extremal with constant $\lambda \geq 1$ if on Q_1^+,

$$\lambda^{-1} \operatorname{Tr}_1 \leq \operatorname{Tr}_1^{\mathrm{op}} \leq \lambda \operatorname{Tr}_1 .$$

H is extremal if $\operatorname{Tr}_1 = \operatorname{Tr}_1^{\mathrm{op}}$ on Q_1^+.

The following proposition is almost identical to Proposition 2.8 in [ILP98].

Proposition 4.4.2 (Structure of Q_n). *$Q_n = \mathfrak{a}_n \oplus \mathfrak{b}_n \oplus \mathfrak{b}_n^{\mathrm{op}} \oplus \mathfrak{c}_n$ such that*

- *$\mathfrak{a}_n$ is a direct sum of type I factors, and for each finite rank $p \in \mathfrak{a}_n$, $pA \subset pC_np$ has finite index.*
- *$\operatorname{Tr}_n |_{\mathfrak{a}_n \oplus \mathfrak{b}_n}$ and $\operatorname{Tr}_n^{\mathrm{op}} |_{\mathfrak{a}_n \oplus \mathfrak{b}_n^{\mathrm{op}}}$ are semifinite,*
- *$\mathfrak{b}_n^{\mathrm{op}} \oplus \mathfrak{c}_n \cap \mathfrak{m}_{\operatorname{Tr}_n} = \{0\} = \mathfrak{b}_n \oplus \mathfrak{c}_n \cap \mathfrak{m}_{\operatorname{Tr}_n^{\mathrm{op}}}$, and*
- *If H^n is symmetric, then j_n fixes $\mathfrak{a}_n, \mathfrak{c}_n$ and $j_n(\mathfrak{b}_n) = \mathfrak{b}_n^{\mathrm{op}}$.*

Proof. By Lemma 4.2.19, let $z_n, z_n^{\mathrm{op}} \in Q_n$ be the unique central projections corresponding to $A \subset C_n$ and $A^{\mathrm{op}} \subset C_n^{\mathrm{op}}$. Set

$$\begin{aligned} \mathfrak{a}_n &= z_n z_n^{\mathrm{op}} Q_n & \mathfrak{b}_n &= z_n(1 - z_n^{\mathrm{op}}) Q_n \\ \mathfrak{b}_n^{\mathrm{op}} &= (1 - z_n) z_n^{\mathrm{op}} Q_n & \mathfrak{c}_n &= (1 - z_n)(1 - z_n^{\mathrm{op}}) Q_n, \end{aligned}$$

and the rest follows immediately. □

Proposition 4.4.3. *Let $Q_1 = \mathfrak{a}_1 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_1^{\mathrm{op}} \oplus \mathfrak{c}_1$ as in Proposition 4.4.2. The following are equivalent:*

(1) H is approximately extremal with constant $\lambda \geq 1$, and

(2) $\mathfrak{b}_1 = \mathfrak{b}_1^{\mathrm{op}} = \{0\}$ and there is a $\lambda \geq 1$ such that on $Q_1^+ \cap \mathfrak{a}_1$, $\lambda^{-1} \operatorname{Tr}_1 \leq \operatorname{Tr}_1^{\mathrm{op}} \leq \lambda \operatorname{Tr}_1$.

A similar result holds for the extremal case.

Proof.
(1) ⇒ (2): Suppose H is approximately extremal. We show $\mathfrak{b}_1 = \{0\}$. As $\operatorname{Tr}_1|_{\mathfrak{a}_1 \oplus \mathfrak{b}_1}$ is semifinite by Proposition 4.4.2, we choose $z \in \mathfrak{b}_1$ such that $z \geq 0$ and $z \in \mathfrak{m}_{\operatorname{Tr}_1}$. Then $z \in \mathfrak{m}_{\operatorname{Tr}_1^{\mathrm{op}}}$, but $\mathfrak{b}_1 \cap \mathfrak{m}_{\operatorname{Tr}_1^{\mathrm{op}}} = \{0\}$. Similarly $\mathfrak{b}_1^{\mathrm{op}} = \{0\}$.
(2) ⇒ (1): $\operatorname{Tr}_1|_{\mathfrak{c}_1 \cap Q_1^+} = \operatorname{Tr}_1^{\mathrm{op}}|_{\mathfrak{c}_1 \cap Q_1^+} = \infty$. □

Corollary 4.4.4. *H is extremal if and only if for each Hilbert $A - A$ bimodule $K \subset H$, the left and right von Neumann dimensions agree.*

Remark 4.4.5. If H has a two-sided basis $\{\gamma\}$, then H is extremal as

$$\operatorname{Tr}_1 = \sum_{\gamma} \langle \cdot \gamma, \gamma \rangle = \operatorname{Tr}_1^{\mathrm{op}}.$$

Remark 4.4.6. If H is approximately extremal, then there is a $\lambda \geq 1$ such that for all $z \in \widehat{Q_1^+}$,

$$\lambda^{-1} \sum_{\beta} z(\omega_\beta) \leq \sum_{\alpha} z(\omega_\alpha) \leq \lambda \sum_{\beta} z(\omega_\beta).$$

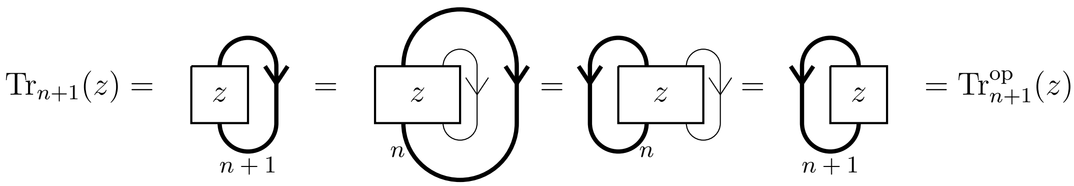
If H is extremal, then $\lambda = 1$ works.

Theorem 4.4.7. *(1) If H is (approximately) extremal (with constant $\lambda \geq 1$), then H^n is (approximately) extremal for all $n \geq 1$ (with constant λ^n).*

(2) If H^n is (approximately) extremal for some $n \geq 1$, then H is (approximately) extremal.

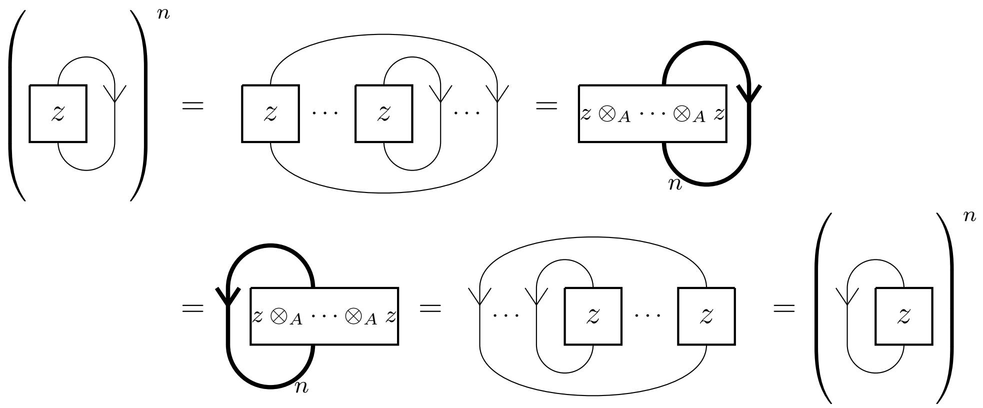
Proof. We prove the extremal case, and the approximately extremal case is similar.

(1) We use strong induction on n. Suppose H^1 and H^n are extremal. If $z \in Q_{n+1}^+$,



Hence H^{n+1} is extremal.

(2) Suppose H^n is extremal and $z \in Q_1^+$. Then $z \otimes_A \cdots \otimes_A z \in Q_n^+$. By the bimodule planar calculus,



In equations:

$$\mathrm{Tr}_1(z)^n = \mathrm{Tr}_n(z \otimes_A \cdots \otimes_A z) = \mathrm{Tr}_n^{\mathrm{op}}(z \otimes_A \cdots \otimes_A z) = \mathrm{Tr}_1^{\mathrm{op}}(z)^n.$$

Taking n^{th} roots gives the desired result.

□

Proposition 4.4.8. *If H is extremal and $z \in \widehat{Q_n^+}$, then $\sum_\beta R_\beta^* z R_\beta = \sum_\alpha R_\alpha^* z R_\alpha$ and $\sum_\alpha L_\alpha^* z L_\alpha = \sum_\beta L_\beta^* z L_\beta$.*

Proof. Immediate from Propositions 4.3.11 and 4.3.14. □

Rotations

Definition 4.4.9 (Inspired by [Bur03]). A Burns rotation is a map $\rho\colon P_n \to P_n$ such that for all $\zeta \in P_n$ and $b_1, \ldots, b_n \in B$,

$$\langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes b_1 \rangle. \tag{4.3}$$

An opposite Burns rotation is defined similarly:

$$\langle \rho^{\mathrm{op}}(\zeta), b_1 \otimes \cdots \otimes b_n \rangle = \langle \zeta, b_n \otimes b_1 \otimes \cdots \otimes b_{n-1} \rangle.$$

Remark 4.4.10. Note that if such a ρ exists, it is unique, and $\rho^n = \mathrm{id}_{P_n}$. In this case, $\rho^{\mathrm{op}} = \rho^{-1}$.

Theorem 4.4.11 (Essentially due to [Bur03])**.** *If $\rho = \sum_\beta L_\beta R_\beta^*$ converges strongly on P_n, then ρ is a Burns rotation. Similarly, if $\rho^{\mathrm{op}} = \sum_\alpha R_\alpha L_\alpha^*$ converges strongly on P_n, then ρ^{op} is an opposite Burns rotation.*

Proof. We must show that ρ preserves P_n and that ρ satisfies Equation (4.3). The latter follows from:

$$\begin{aligned}\langle \rho(\zeta), b_1 \otimes \cdots \otimes b_n\rangle &= \sum_\beta \langle \zeta, R_\beta L_\beta^*(b_1 \otimes \cdots \otimes b_n)\rangle \\ &= \sum_\beta \langle \zeta, \langle \beta | b_1\rangle_A b_2 \otimes \cdots \otimes b_n \otimes \beta\rangle \\ &= \sum_\beta \langle \langle \beta | b_1\rangle_A^* \zeta, b_2 \otimes \cdots \otimes b_n \otimes \beta\rangle \\ &= \sum_\beta \langle \zeta \langle \beta | b_1\rangle_A^*, b_2 \otimes \cdots \otimes b_n \otimes \beta\rangle \\ &= \sum_\beta \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes \beta \langle \beta | b_1\rangle_A\rangle \\ &= \langle \zeta, b_2 \otimes \cdots \otimes b_n \otimes b_1\rangle.\end{aligned}$$

Now ρ is independent of the choice of $\{\beta\}$. In particular, for any $u \in U(A)$, $\{u\beta\}$ is an H_A-basis, and

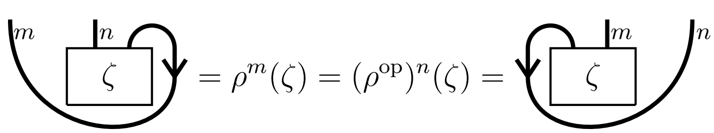
$$u\rho(\zeta)u^* = u\left(\sum_\beta L_\beta R_\beta^* \zeta\right)u^* = \sum_\beta L_{u\beta} R_{u\beta}^* \zeta = \rho(\zeta) \in P_n.$$

□

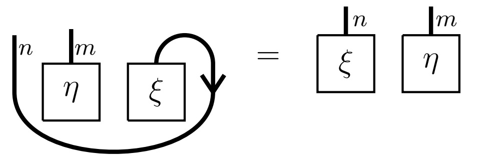
Diagrammatic representation of the Burns rotation

For this section, we assume the Burns rotation ρ exists on P_n for all $n \geq 0$. Recall for all $k \geq 0$, $\rho^{-k} = (\rho^{\mathrm{op}})^k$.

Notation 4.4.12. For $\zeta \in P_{m+n}$, we denote $\rho^m(\zeta) = (\rho^{\mathrm{op}})^n(\zeta) \in P_{m+n}$ by moving m strings around the bottom counterclockwise or by moving n strings around the bottom clockwise:



Proposition 4.4.13. *If $\eta \in P_m$ and $\xi \in P_n$, then $\rho^n(\eta \otimes \xi) = \xi \otimes \eta$:*



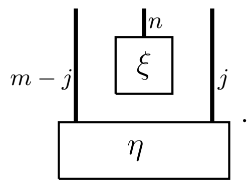
Proof. Suppose $\alpha \in B^m$ and $\beta \in B^n$. Then by (1) of Lemma 4.3.23,

$$\langle \rho^n(\eta \otimes \xi), \beta \otimes \alpha\rangle = \langle \eta \otimes \xi, \alpha \otimes \beta\rangle = \langle \langle \alpha|\eta\rangle_A \xi, \beta\rangle = \langle \xi_A\langle \eta, \alpha\rangle, \beta\rangle = \langle \xi \otimes \eta, \beta \otimes \alpha\rangle.$$

□

Definition 4.4.14. For $0 \le j < m$, define $\mu_j \colon P_m \times P_n \to P_{m+n}$ by $\mu_j(\eta, \xi) = \rho^{-j}(\rho^j(\eta) \otimes \xi)$. We represent μ_j diagrammatically as follows:

$\mu_j(\eta, \xi) =$



That this diagram is well-defined relies on the following proposition.

Proposition 4.4.15. *The μ_i's are associative, i.e., if $\sigma \in P_\ell$, $\eta \in P_m$, and $\xi \in P_n$, and $i \le \ell$, $j \le m$, then*

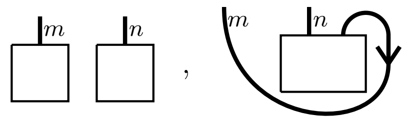
$$\mu_i(\kappa, \mu_j(\eta, \xi)) = \mu_{i+j}(\mu_i(\kappa, \eta), \xi).$$

Proof. Suppose $\alpha \in B^{\ell - i}$, $\beta \in B^{m-j}$, $\gamma \in B^n$, $\delta \in B^j$, and $\varepsilon \in B^i$. Then

$$\begin{aligned}
\langle \mu_i(\kappa, \mu_j(\eta, \xi)), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon\rangle &= \left\langle \rho^{-i}\left(\rho^i(\kappa) \otimes \rho^{-j}(\rho^j(\eta) \otimes \xi)\right), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon\right\rangle \\
&= \left\langle \rho^i(\kappa) \otimes \rho^{-j}\left(\rho^j(\eta) \otimes \xi\right), \varepsilon \otimes \alpha \otimes \beta \otimes \gamma \otimes \delta\right\rangle \\
&= \left\langle \rho^{-j}\left(\rho^j(\eta) \otimes \xi\right), \langle \rho^i(\kappa)|\varepsilon\rangle_A \alpha \otimes \beta \otimes \gamma \otimes \delta\right\rangle \\
&= \left\langle \rho^j(\eta) \otimes \xi, \delta \otimes \langle \rho^i(\kappa)|\varepsilon\rangle_A \alpha \otimes \beta \otimes \gamma\right\rangle \\
&= \left\langle \rho^j(\eta), \delta \otimes \langle \rho^i(\kappa)|\varepsilon\rangle_A \alpha \otimes \beta_A\langle \gamma, \xi\rangle\right\rangle \\
&= \left\langle \eta, \langle \rho^i(\kappa)|\varepsilon\rangle_A \alpha \otimes \beta_A\langle \gamma, \xi\rangle \otimes \delta\right\rangle \\
&= \left\langle \rho^i(\kappa) \otimes \eta, \varepsilon \otimes \alpha \otimes \beta_A\langle \gamma, \xi\rangle \otimes \delta\right\rangle \\
&= \left\langle \rho^j\left(\rho^i(\kappa) \otimes \eta\right), \delta \otimes \varepsilon \otimes \alpha \otimes \beta_A\langle \gamma, \xi\rangle\right\rangle \\
&= \left\langle \rho^j\left(\rho^i(\kappa) \otimes \eta\right) \otimes \xi, \delta \otimes \varepsilon \otimes \alpha \otimes \beta \otimes \gamma\right\rangle \\
&= \left\langle \rho^{-i-j}\left(\rho^{i+j}(\rho^{-i}(\rho^i(\kappa) \otimes \eta)) \otimes \xi\right), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon\right\rangle \\
&= \langle \mu_{i+j}(\mu_i(\kappa, \eta), \xi), \alpha \otimes \beta \otimes \gamma \otimes \delta \otimes \varepsilon\rangle.
\end{aligned}$$

□

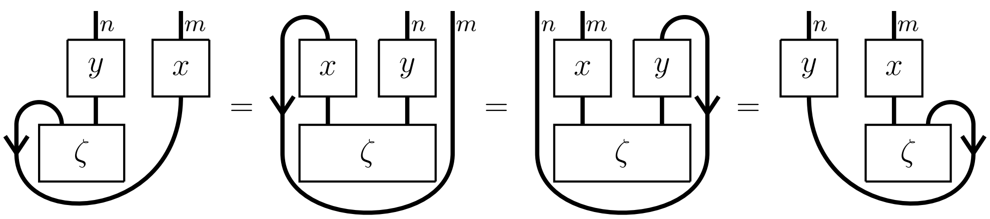
Corollary 4.4.16. *$P_\bullet$ naturally forms an algebra over the operad generated by the unshaded, oriented tangles*



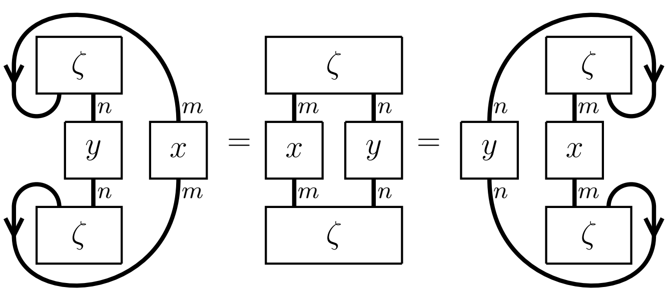
for $m, n \geq 0$ up to planar isotopy.

The Burns rotation is also compatible with the $\mathbb{BP}$-algebra $\widehat{Q_\bullet^+}$.

Theorem 4.4.17. *(1) For all $\zeta \in P_{m+n}$ and $x \in Q_m$, and $y \in Q_n$, $\rho^n((x \otimes_A y)\zeta) = (y \otimes_A x)\rho^n(\zeta)$:*



(2) If ρ is unitary, then for all $\zeta \in P_{m+n}$ and $x \in \widehat{Q_m^+}$, and $y \in \widehat{Q_n^+}$, $(y \otimes_A x)(\omega_{\rho^n\zeta}) = (x \otimes_A y)(\omega_\zeta)$:



Proof. (1) For $\eta \in B^n$ and $\xi \in B^m$,

$$\begin{aligned}\langle \rho^n((x \otimes_A y)\zeta), \eta \otimes \xi\rangle &= \langle (x \otimes_A y)\zeta, \xi \otimes \eta\rangle = \langle \zeta, (x^* \otimes_A y^*)(\xi \otimes \eta)\rangle \\ &= \langle \zeta, (x^*\xi) \otimes (y^*\eta)\rangle = \langle \rho^n(\zeta), (y^*\eta) \otimes (x^*\xi)\rangle \\ &= \langle (y \otimes_A x)\rho^n(\zeta), \eta \otimes \xi\rangle.\end{aligned}$$

(2) Pick $(x_i) \subset Q_m^+$ and $(y_j) \subset Q_n^+$ with $x_i \nearrow x$ and $y_j \nearrow y$. Then by (1), for all i,

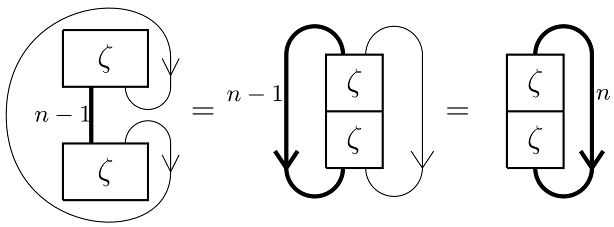
$$\begin{aligned}(y_j \otimes_A x_i)(\omega_{\rho^n\zeta}) &= \|(y_j^{1/2} \otimes_A x_i^{1/2})\rho^n\zeta\|_2^2 = \|\rho^n((x_i^{1/2} \otimes_A y_j^{1/2})\zeta)\|_2^2 \\ &= \|(x_i^{1/2} \otimes_A y_j^{1/2})\zeta\|_2^2 = (x_i \otimes_A y_j)(\omega_\zeta).\end{aligned}$$

We are finished by Theorem 4.6.13, since $x_i \otimes_A y_j \nearrow x \otimes_A y$ and $y_j \otimes_A x_i \nearrow y \otimes_A x$. □

Remark 4.4.18. When the operads for $P_\bullet$ and $\widehat{Q_\bullet^+}$ interact as in Theorem 4.3.27, we may remove closed subdiagrams and multiply by the appropriate scalar in $[0_\mathbb{R}, \infty_\mathbb{R}]$ by Corollary 4.4.16 and Theorem 4.4.17.

Extremality implies the existence of the Burns rotation

We will show in the next lemma and theorem that (approximate) extremality implies the existence of the Burns rotation. The intuition comes from the bimodule planar calculus. In diagrams, for the extremal case, we have:



Although these diagrams are not yet well-defined, they tell us how to proceed. They become well-defined after the Burns rotation exists by Theorems 4.3.24 and 4.4.17.

Lemma 4.4.19. *Let p_n be the projection in $B(H^n)$ with range P_n.*

(1) If H is approximately extremal with constant $\lambda \geq 1$, then

$$\left(\sum_\beta p_n R_\beta R_\beta^* p_n\right) \leq \lambda^{n-1} p_n \text{ and } \left(\sum_\alpha p_n L_\alpha L_\alpha^* p_n\right) \leq \lambda^{n-1} p_n.$$

(2) If H is extremal, then on P_n, $\sum_\beta p_n R_\beta R_\beta^* p_n = p_n = \sum_\alpha p_n L_\alpha L_\alpha^* p_n$.

Proof. (1) We prove the first inequality. Note that $R_\beta^* \zeta \in D({}_A H^{n-1})$, and $R(R_\beta^*\zeta) = R_\beta^* R(\zeta) \colon L^2(A) \to H^{n-1}$. Since H is (approximately) extremal, so is H^{n-1} with constant λ^{n-1}, and

$$\begin{aligned}\left\langle \left(\sum_\beta p_n R_\beta R_\beta^* p_n\right)\zeta, \zeta \right\rangle_{P_n} &= \sum_\beta \|R_\beta^*\zeta\|_2^2 = \sum_\beta \operatorname{tr}_A\left({}_A\langle R_\beta^*\zeta, R_\beta^*\zeta\rangle\right) \\ &= \sum_\beta \operatorname{Tr}_{n-1}^{\mathrm{op}}\left(R_\beta^* R(\zeta) R(\zeta)^* R_\beta\right) = \operatorname{Tr}_{n-1}^{\mathrm{op}} T_{n-1}(R(\zeta)R(\zeta)^*) \\ &\leq \lambda^{n-1} \operatorname{Tr}_{n-1} T_{n-1}(L(\zeta)L(\zeta)^*) = \lambda^{n-1} \operatorname{Tr}_n(L(\zeta)L(\zeta)^*) \\ &= \lambda^{n-1}\|\zeta\|_2^2 = \langle(\lambda^{n-1}p_n)\zeta, \zeta\rangle_{P_n}.\end{aligned}$$

(2) As $\lambda = 1$, by (1),

$$\left\langle \left(\sum_\beta p_n R_\beta R_\beta^* p_n\right)\zeta, \zeta \right\rangle = \langle \zeta, \zeta\rangle$$

for all $\zeta \in P_n$, and the result follows from polarization.

□

Theorem 4.4.20. *Suppose H is approximately extremal. Then $\rho = \sum_\beta L_\beta R_\beta^*$ converges strongly on P_n. Moreover if H is extremal, ρ is unitary. A similar result holds for $\rho^{\mathrm{op}} = \sum_\alpha R_\alpha L_\alpha^*$.*

Proof. We begin as in the proof of Proposition 3.3.19 of [Bur03], but as we do not have Jones projections, we use Lemma 4.4.19.

Suppose $\zeta \in P_n$, and enumerate $\{\beta\} = \{\beta_i\}_{i\in\mathbb{N}}$. We will show

$$\left\| \sum_{i=r}^{s} L_{\beta_i} R_{\beta_i}^* \zeta \right\|_2^2 \to 0 \text{ as } r, s \to \infty.$$

First note that the infinite matrix $(L_{\beta_j}^* L_{\beta_i})$ is a projection, so it is dominated by $1 = \delta_{i,j}$. Hence each corner $(L_{\beta_j}^* L_{\beta_i})_{i,j=r}^s$ is dominated by $1 = \delta_{i,j}$, and

$$\left\| \sum_{i=r}^{s} L_{\beta_i} R_{\beta_i}^* \zeta \right\|_2^2 = \sum_{i,j=r}^{s} \left\langle (L_{\beta_j}^* L_{\beta_i}) R_{\beta_i}^* \zeta, R_{\beta_j}^* \zeta \right\rangle \leq \sum_{i=r}^{s} \langle R_{\beta_i}^* \zeta, R_{\beta_i}^* \zeta \rangle.$$

We need to show that the right hand side tends to zero, which is certainly true if the infinite sum $\sum_\beta \|R_\beta^* \zeta\|_2^2$ converges. But this follows immediately from Lemma 4.4.19. Hence ρ converges and $\|\rho\| \leq \sqrt{\lambda^{n-1}}$ (where λ is the approximate extremality constant). If $\lambda = 1$, then $\|\rho\| \leq 1$ and $\rho^n = \mathrm{id}_{P_n}$, so ρ is necessarily isometric and thus unitary. □

Symmetric bimodules and a converse of Theorem 4.4.20

We prove a converse of Theorem 4.4.20, with some additional structure on H.

Remark 4.4.21. For the rest of this section, we assume H is symmetric (see Remark 4.3.5).

Lemma 4.4.22. *For all $\eta, \xi \in B^n$, $\langle \eta | \xi \rangle_A = {}_A\langle J\eta, J\xi \rangle$.*

Proof. Suppose $a_1, a_2 \in A$. Then

$$\begin{aligned} \left\langle {}_A\langle J\eta, J\xi\rangle \widehat{a_1}, \widehat{a_2} \right\rangle &= \langle JR(J\eta)^* R(J\xi) J\widehat{a_1}, \widehat{a_2}\rangle = \langle \widehat{a_2^*}, R(J\eta)^* R(J\xi)\widehat{a_1^*}\rangle = \langle a_2^* J\eta, a_1^* J\xi\rangle \\ &= \langle J(\eta a_2), J(\xi a_1)\rangle = \langle \xi a_1, \eta a_2\rangle = \left\langle \langle \eta|\xi\rangle_A \widehat{a_1}, \widehat{a_2}\right\rangle. \end{aligned}$$

□

Definition 4.4.23. Using Lemma 4.4.22, we define an algebra structure on $B^n \otimes_A B^n$ as follows: if $\eta_1, \eta_2, \xi_1, \xi_2 \in B^n$, then

$$(\eta_1 \otimes \xi_1)(\eta_2 \otimes \xi_2) = \eta_1 \langle J\xi_1 | \eta_2\rangle_A \otimes \xi_2 = \eta_1 {}_A\langle \xi_1, J\eta_2\rangle \otimes \xi_2.$$

Proposition 4.4.24 ([Sau83, HO89])**.** *The map $B^n \otimes_A B^n \to C_n$ by $\eta \otimes J_n\xi \mapsto L(\eta)L(\xi)^*$ gives a $*$-algebra isomorphism onto its image, and it extends to a $C_n - C_n$ bimodule isomorphism $\theta_n \colon H^{2n} \to L^2(C_n, \mathrm{Tr}_n)$. The same result holds swapping ${}^{\mathrm{op}}$.*

Proof. The map is well defined as it is A-middle linear:

$$\eta a \otimes J_n\xi \mapsto L(\eta a)L(\xi)^* = L(\eta)aL(\xi)^* = L(\eta)L(\xi a^*)^* \text{ and}$$
$$\eta \otimes aJ_n\xi \mapsto L(\eta)L(J_n(aJ_n\xi))^* = L(\eta)L(\xi a^*)^*.$$

The map clearly preserves the multiplicative structure and is isometric by construction. If $\eta_1, \eta_2, \xi_1, \xi_2 \in B^n$, then

$$\begin{aligned}\langle L(\eta_1)L(\xi_1)^*, L(\eta_2)L(\xi_2)^*\rangle_{L^2(C_n,\mathrm{Tr}_n)} &= \mathrm{Tr}_n\left(L(\xi_2)L(\eta_2)^*L(\eta_1)L(\xi_1)^*\right)\\ &= \mathrm{Tr}_n\left(L(\xi_2)\langle\eta_2|\eta_1\rangle_A L(\xi_1)^*\right)\\ &= \mathrm{Tr}_n\left(L(\xi_2\langle\eta_2|\eta_1\rangle_A)L(\xi_1)^*\right)\\ &= \langle \xi_2\langle\eta_2|\eta_1\rangle_A, \xi_1\rangle_{H^n}\\ &= \langle J_n\xi_1, J_n(\xi_2\langle\eta_2|\eta_1\rangle_A)\rangle_{H^n}\\ &= \langle J_n\xi_1, \langle\eta_1|\eta_2\rangle_A J_n\xi_2\rangle_{H^n}\\ &= \langle \eta_1\otimes J_n\xi_1, \eta_2\otimes J_n\xi_2\rangle_{H^{2n}}.\end{aligned}$$

Hence it clearly extends to a $C_n - C_n$ bilinear bimodule isomorphism. □

Corollary 4.4.25. *$C_{n-k} \subseteq C_n \subseteq C_{n+k}$ is standard (isomorphic to the basic construction) for all $n, k \geq 0$.*

Proof. By Remark 4.3.9 and Proposition 4.4.24,

$$J_{2n}(C_{n-k}\otimes_A \mathrm{id}_{n+k})'J_{2n} = J_{2n}(\mathrm{id}_{n-k}\otimes_A C^{\mathrm{op}}_{n+k})J_{2n} = C_{n+k}\otimes_A \mathrm{id}_{n-k}\,.$$

□

Lemma 4.4.26 ([Bur03], Theorem 3.3.13)**.** *Let N be a von Neumann subalgebra of a semifinite von Neumann algebra M with n.f.s. trace Tr_M. Then*

(1) $N' \cap L^2(M) = \overline{N'\cap \mathfrak{n}_{\mathrm{Tr}_M}}^{\|\cdot\|_2}$

(2) $(N'\cap L^2(M))^\perp = \overline{[N, \mathfrak{n}_{\mathrm{Tr}_M}]}^{\|\cdot\|_2}$, the closure of the span of the commutators in $L^2(M)$.

Remark 4.4.27. By Proposition 4.4.24 and Lemma 4.4.26, θ_n yields an isomorphsim

$$P_{2n} = A'\cap H^{2n} \cong A'\cap L^2(C_n,\mathrm{Tr}_n) = \overline{A'\cap \mathfrak{n}_{\mathrm{Tr}_n}}^{\|\cdot\|_2} = \overline{C_n^{\mathrm{op}}\cap \mathfrak{n}_{\mathrm{Tr}_n}}^{\|\cdot\|_2} = L^2(Q_n,\mathrm{Tr}_n)$$

of $Q_n - Q_n$ bimodules. A similar result holds swapping $^{\mathrm{op}}$.

Theorem 4.4.28. *If ρ exists on P_{2n}, then H^n is approximately extremal. If ρ is unitary, then H^n is extremal.*

Proof. The main step is to show the following lemma, whose proof is essentially the same as in [Bur03].

Lemma 4.4.29 (3.3.21.(ii) of [Bur03])**.** *If ρ exists on P_{2n}, then for all $x \in C_n^{\mathrm{op}} \cap \mathfrak{n}_{\mathrm{Tr}_n}$, $\rho^n(\theta_n^{-1}(\widehat{x})) = \theta_n^{-1}(\widehat{j_n(x)}) \in C_n^{\mathrm{op}} \cap \mathfrak{n}_{\mathrm{Tr}_n}$. In particular, $C_n^{\mathrm{op}} \cap \mathfrak{n}_{\mathrm{Tr}_n} = \mathfrak{n}_{\mathrm{Tr}_n^{\mathrm{op}}} \cap \mathfrak{n}_{\mathrm{Tr}_n}$. A similar result holds swapping ${}^{\mathrm{op}}$.*

Using this lemma, Burns' proof shows $\mathrm{Tr}_n^{\mathrm{op}} \leq \|\rho^n\| \,\mathrm{Tr}_n$ on Q_n^+. Suppose $z \in Q_n$. If $\mathrm{Tr}_n(z^*z) = \infty$, we are finished. Otherwise, $z \in C_n^{\mathrm{op}} \cap \mathfrak{n}_{\mathrm{Tr}_n} = \mathfrak{n}_{\mathrm{Tr}_n^{\mathrm{op}}} \cap \mathfrak{n}_{\mathrm{Tr}_n}$, and

$$\begin{aligned}
\mathrm{Tr}_n^{\mathrm{op}}(z^*z) &= \mathrm{Tr}_n \circ j_n(z^*z) = \mathrm{Tr}_n(j_n(z)^* j_n(z)) = \left\langle \widehat{j_n(z)}, \widehat{j_n(z)} \right\rangle_{L^2(Q_n, \mathrm{Tr}_n)} \\
&= \left\langle \theta_n^{-1}(\widehat{j_n(z)}), \theta_n^{-1}(\widehat{j_n(z)}) \right\rangle_{P_n} = \left\langle \rho^n(\theta_n^{-1}(\widehat{z})), \rho^n(\theta_n^{-1}(\widehat{z})) \right\rangle_{P_n} \\
&= \|\rho^n(\theta_n^{-1}(\widehat{z}))\|_{P_n}^2 \leq \|\rho^n\|^2 \|\theta_n^{-1}(\widehat{z})\|_{P_n}^2 = \|\rho^n\|^2 \|\widehat{z}\|_{L^2(Q_n, \mathrm{Tr}_n)}^2 \\
&= \|\rho^n\|^2 \,\mathrm{Tr}_n(z^*z).
\end{aligned}$$

Similarly $\mathrm{Tr}_n \leq \|\rho^n\|^2 \,\mathrm{Tr}_n^{\mathrm{op}}$ on Q_n^+, and H^n is approximately extremal. In particular, if $\|\rho\| = 1$, H^n is extremal. □

Remark 4.4.30. Theorem 4.1.4 now follows immediately from Theorems 4.4.7, 4.4.20, and 4.4.28.

4.5 Examples

Centralizer algebras and central L^2-vectors

Example 4.5.1 (Bifinite bimodules)**.** In the case that H is a symmetric, bifinite $A - A$ bimodule, then the $\mathbb{BP}$-algebra structure encodes the C^*-tensor category whose objects are the sub-bimodules of H^n for some n and whose morphisms are intertwiners.

Example 4.5.2. Suppose $A_0 = A \subset B = A_1$ is an infinite index inclusion of II_1-factors. Then $H = L^2(B)$ gives an $A - A$ bimodule. In this case, letting A_{n+1} be the n^{th} iterated basic construction of $A_{n-1} \subset A_n$, we have

- $H^n \cong L^2(A_n, \mathrm{Tr}_n)$,
- C_n, C_n^{op} is the left,right action respectively of A_{2n}, and
- $Q_n = A_0' \cap A_{2n}$.

Theorem 4.1.4 was proven for this case by [Bur03].

Example 4.5.3. Suppose A is a II_1-factor, and $\sigma \in \mathrm{Aut}(A)$. Define $H_\sigma = {}_A L^2(A)_{\sigma(A)}$ by $a\widehat{b}c = \widehat{ab\sigma(c)}$ for all $a, b, c, \in A$. Suppose that σ is outer and not periodic, and σ^n is outer for all $n \in \mathbb{N}$. Then $H_\sigma^n \cong H_{\sigma^n}$ is extremal and $P_n = (0)$ for all $n \geq 1$.

Example 4.5.4 (Group actions)**.** Suppose G is a countable i.c.c. group, and $\pi\colon G \to U(K)$ is a unitary representation. We can define two bimodules:

(1) $H = K \otimes_{\mathbb{C}} \ell^2(G)$ where the left action is given by the diagonal action $\pi \otimes \lambda$ and the right action is given by $1 \otimes \rho$ where λ, ρ are the left,right regular representation of G on $\ell^2(G)$. Hence $K \otimes_{\mathbb{C}} \ell^2(G)$ gives an $A - A$ bimodule where $A = LG$. Then we may identify

$$H^n = K^n \otimes_{\mathbb{C}} \ell^2(G)$$

where we write $K^n = K^{\otimes_{\mathbb{C}} n}$, and the left action is the diagonal action $\pi^n \otimes \lambda$ and the right action is $1_n \otimes \rho$. It is clear that projections in Q_n correspond to $LG - LG$ invariant subspaces of H^n. Every G-invariant subspace of K^n yields such a subspace, but in general, they do not exhaust all possible subspaces.

(2) To fix this problem, we use an idea of Richard Burstein and add a copy of the hyperfinite II_1-factor R. Suppose $\alpha\colon G \to \mathrm{Aut}(R)$ is an outer action, so $A = R \rtimes_\alpha G$ is a II_1-factor. Set $H = K \otimes_{\mathbb{C}} L^2(R) \otimes_{\mathbb{C}} \ell^2(G)$, and consider the left and right actions where

$$\begin{aligned} r_1(k \otimes \widehat{r_2} \otimes \delta_g) r_3 &= k \otimes \widehat{r_1 r_2 \alpha_g(r_3)} \otimes \delta_g \\ g_1(k \otimes \widehat{r} \otimes \delta_{g_2}) g_3 &= (\pi_{g_1} k) \otimes \widehat{\alpha_{g_1}(r)} \otimes \delta_{g_1 g_2 g_3} \end{aligned}$$

for $r, r_i \in R$ and $g, g_i \in G$ for $i = 1, 2, 3$. Hence $g \in G$ acts on the left by $\pi_g \otimes \alpha_g \otimes \lambda_g$ and on the right by $1 \otimes 1 \otimes \rho_g$. Then similarly we may identify

$$H^n = K^n \otimes_{\mathbb{C}} L^2(R) \otimes_{\mathbb{C}} \ell^2(G).$$

Theorem 4.5.5. *For $A = R \rtimes_\alpha G$ and H^n as above, $A - A$ invariant subspaces of H^n correspond to G-invariant subspaces of K^n.*

Proof. First, if $L_0 \subset K^n$ is a G-invariant subspace, then $L_0 \otimes L^2(A)$ is an $A - A$ invariant subspace of H^n.

Now suppose $L \subset H^n$ is an $A - A$ invariant subspace, and let $p \in Q_n$ be the projection onto L. Note that

$$\begin{aligned} p &\in \Big(1_{K^n} \otimes R\Big)' \cap \Big(1_{K^n} \otimes A^{\mathrm{op}}\Big)' \\ &= \Big(B(K^n) \otimes (R' \cap B(L^2(A)))\Big) \cap \Big(B(K^n) \otimes A\Big) \\ &= B(K^n) \otimes (R' \cap A) = B(K^n) \otimes 1_{L^2(A)}. \end{aligned}$$

Hence there is a $q \in B(K^n)$ such that $p = q \otimes 1_{L^2(A)}$. But since q commutes with the left G-action on H^n, we have $q \in \pi(G)' \cap B(K^n)$. □

Corollary 4.5.6. *$A - A$ invariant vectors of H^n correspond to G-invariant vectors of K^n.*

Example 4.5.7 (Group-subgroup)**.** Suppose $G_0 \subseteq G_1$ is an inclusion of countable i.c.c. groups, and let $K = \ell^2(G_1/G_0)$. As in Example 4.5.4, we consider two cases:

(1) $A_0 = LG_0$, $A_1 = LG_1$, and $H = K \otimes_{\mathbb{C}} \ell^2(G_1)$.

(2) $A_0 = R \rtimes G_0$, $A_1 = R \rtimes G_1$, and $H = K \otimes_{\mathbb{C}} L^2(R) \otimes \ell^2(G_1)$.

Note that in either case, $H^n \cong L^2(A_{n+1})$, where $A_{n+1} = J_n A'_{n-1} J_n$ is the basic construction of $A_{n-1} \subset A_n$. As in the usual subfactor treatment, we can consider H^n as an $A_i - A_j$ bimodule for $i, j \in \{0, 1\}$.

Theorem 4.5.8. *Let $G_1 = S_\infty$, the group of finite permutations of $\mathbb{N}$, and let $G_0 = \mathrm{Stab}(1)$ be the permutations which fix 1. Let $A_0 = R \rtimes G_0$ and $A_1 = R \rtimes G_1$, and let $H = K \otimes_{\mathbb{C}} L^2(R) \otimes \ell^2(G_1)$ as in (2) of Example 4.5.7. Then considering H^n as an $A_0 - A_0$ or as an $A_1 - A_1$ bimodule, we have that $\dim(Q_n) < \infty$ for all $n \in \mathbb{N}$.*

Proof. Since $A'_i \cap A_j \cong A'_{i+2} \cap A_{j+2}$ for all $i, j \geq 0$ by [EN96], it suffices to show that $\dim(A'_1 \cap A_{2n+1}) < \infty$ for all $n \geq 0$. Also by [EN96],

$$A'_1 \cap A_{2n+1} \cong \mathrm{End}_{A_1 - A_1}(L^2(A_{n+1})) \cong \mathrm{End}_{A_1 - A_1}(H^n).$$

By Theorem 4.5.5, $A_1 - A_1$ invariant subspaces of H^n correspond to G_1-invariant subspaces of K^n. The result now follows by [Lie72]. □

Corollary 4.5.9. *The infinite index II_1-subfactor $R \rtimes G_0 \subset R \rtimes G_1$ for $G_0 = \mathrm{Stab}(1) \subset S_\infty = G_1$ has finite dimensional higher relative commutants.*

Theorem 4.5.10. *Suppose $G_0 \subset G_1$ and K are as in Example 4.5.7 such that $[G_1 \colon G_0] = \infty$ and $\# G_0 \backslash G_1 / G_0 = 2$. Then*

(1) the space of G_0-invariant vectors in K^n is one dimensional, and

(2) zero is the only G_1-invariant vector in K^n.

Proof. Let $\{g_i\}_{i \geq 0}$ be a set of coset representatives for G_1/G_0 with $g_0 = e$. Since $\# G_0 \backslash G_1 / G_0 = 2$, for $i, j \geq 1$, there are $h_{i,j} \in G_0$ such that $h_{i,j} g_i G_0 = g_j G_0$.

(1) Suppose
$$\xi = \sum_{i_1, \dots, i_n} \lambda_{i_1, \dots, i_n} \delta_{g_{i_1} G_0} \otimes \cdots \otimes \delta_{g_{i_n} G_0} \in K^n$$
is G_0-invariant. Then since $\pi_{h_{i,j}} \xi = \xi$ for all $i, j \geq 1$, we must have $\lambda_{i_1, \dots, i_n} = 0$ unless $i_j = 0$ for all $j = 1, \dots, n$. (Otherwise, there would be infinitely many coefficients which would be nonzero and equal, a contradiction to $\xi \in K^n \cong \ell^2((G_1/G_0)^n)$.) Hence $\xi \in \mathrm{span}\{\delta_{G_0} \otimes \cdots \otimes \delta_{G_0}\}$.

(2) Since $\delta_{G_0} \otimes \cdots \otimes \delta_{G_0}$ is not G_1-invariant, the result follows from (1).

□

Corollary 4.5.11. *Let $G_0 = \mathrm{Stab}(1) \subset S_\infty = G_1$. Let $A_i = R \rtimes G_i$ for $i = 0, 1$, and let $K = \ell^2(G_1/G_0)$.*

(1) When we consider $H = K \otimes_C L^2(R) \otimes_{\mathbb{C}} \ell^2(G_1)$ as an $A_1 - A_1$ bimodule, $P_n = (0)$.

(2) When we consider $H = L^2(A_1) = L^2(R) \otimes_{\mathbb{C}} \ell^2(G_1)$ as an $A_0 - A_0$ bimodule,

$$H^n \cong L^2(A_n) \cong K^{n-1} \otimes_{\mathbb{C}} L^2(R) \otimes_{\mathbb{C}} \ell^2(G_1),$$

and for all $n \geq 0$, P_n is one-dimensional and spanned by

$$\widehat{1} \otimes \cdots \otimes \widehat{1} \in \bigotimes_{A_0}^{n} L^2(A_1) \cong L^2(A_n).$$

In joint work with Steven Deprez, we have shown an even stronger result:

Theorem 4.5.12. *The algebras Q_n for the bimodules in (1) and (2) in Example 4.5.7 are finite dimensional, and the dimensions grow super-factorially.*

Corollary 4.5.13. *The infinite index II_1-subfactor $LG_0 \subset LG_1$ where $G_0 = \mathrm{Stab}(1) \subset S_\infty = G_1$ has finite dimensional higher relative commutants.*

(Approximate) Extremality

Example 4.5.14. If ${}_AH_A$ is a bifinite bimodule (e.g., as in Example 4.5.1), then $\dim(Q_1) < \infty$ by [Jon83]. Since any two faithful traces on a finite dimensional von Neumann algebra are comparable, H is approximately extremal.

In the case that $H = L^2(A_1)$ and $A = A_0$ where $A_0 \subset A_1$ is a finite index (not necessarily extremal) II_1-subfactor, rotations for H^n were constructed in [JP11].

Example 4.5.15. To get an example of an infinite index approximately extremal bimodule, take any bifinite bimodule ${}_AH_A$ and tensor it with ℓ^2 over $\mathbb{C}$.

In the subfactor setting, this is equivalent to looking at the infinite index subfactor $A_0 \otimes 1 \subset A_1 \otimes R$ where $A_0 \subset A_1$ is finite index. To get an example which is approximately extremal and not extremal, just take $A_0 \subset A_1$ non-extremal (such examples with principal graph $A_{-\infty,\infty}$ are given in [Jon83]).

Example 4.5.16. The bimodules in Example 4.5.3 and Theorem 4.5.10 (2) are trivially extremal, and the rotation is trivial.

We will now derive necessary and sufficient conditions for the (approximate) extremality for the infinite index group-subgroup subfactor as in Example 4.5.7. For the rest of this subsection, Suppose $G_0 \subset G_1$ is an inclusion of countable groups with $[G_1 \colon G_0] = \infty$, and $\alpha \colon G_1 \to \mathrm{Aut}(R)$ is an outer action. Set $A_0 = R \rtimes_\alpha G_0 \subset R \rtimes_\alpha G_1 = A_1$ and $H = L^2(A_1)$, and note that $A_0 \subset A_1$ is an irreducible inclusion of II_1-factors, i.e., $A_0' \cap A_1 = \mathbb{C}1$.

Example 4.5.17 (Two-sided bases)**.** As stated in Remark 4.4.5, any time H has a two-sided basis, H is extremal. For example, if $G_0 = \{e\}$ is trivial, then $H = L^2(A_1) \cong L^2(R) \otimes \ell^2(G_1)$ is extremal, since $\left\{\widehat{1} \otimes \delta_g \middle| g \in G_1\right\}$ is a two-sided basis.

In fact, an H_A-basis is obtained from a set of left coset representatives for G_1/G_0, and an ${}_AH$-basis is obtained from a set of right coset representatives. Hence if G_1 has a set of simultaneous left and right coset representatives, then H is extremal by Remark 4.4.5. For example, if $G_0 = \operatorname{Stab}(1) \subset S_\infty = G_1$, then such a set of representatives is given by the transpositions $\{(1\ n) | n \in \mathbb{N}\}$.

Proposition 4.5.18 (Similar to [ILP98], Example 3.5)**.** *For $g \in G_1$, let $|\mathcal{O}_{gG_0}|$ denote the size of the orbit of gG_0 in the G_0-set G_1/G_0. Then*

(1) $Q_1 \cong \ell^\infty(G_0 \backslash G_1 / G_0)$, where we denote the minimal projection onto $\mathbb{C}\delta_{G_0 g G_0}$ by p_g for $g \in G_1$.

(2) $\operatorname{Tr}_1(p_g) = |\mathcal{O}_{gG_0}| = [G_0\colon G_0 \cap gG_0g^{-1}]$, and

(3) Since $j_1(p_g) = p_{g^{-1}}$,

$$\operatorname{Tr}_1^{\mathrm{op}}(p_g) = |\mathcal{O}_{g^{-1}G_0}| = [G_0\colon G_0 \cap g^{-1}G_0 g] = [gG_0g^{-1}\colon G_0 \cap gG_0g^{-1}].$$

Theorem 4.5.19. *Assume the notation of Proposition 4.5.18. Then exactly one of the following occurs:*

(1) $|\mathcal{O}_{gG_0}| = |\mathcal{O}_{g^{-1}G_0}|$ for all $g \in G_1$ and H is extremal, or

(2) there is a $g \in G_1$ for which $|\mathcal{O}_{gG_0}| \neq |\mathcal{O}_{g^{-1}G_0}|$, and H is not approximately extremal.

Proof. If there is a $g \in G$ where exactly one of $|\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}|$ is finite, then H is not approximately extremal. Hence we must only consider the case where for all $g \in G$, both $|\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}|$ are finite or infinite. Recall that the commensurator

$$\operatorname{Comm}_{G_1}(G_0) = \{g \in G_1 | |\mathcal{O}_{gG_0}|, |\mathcal{O}_{g^{-1}G_0}| < \infty\}$$

is a subgroup of G_1, and the map $\varphi\colon \operatorname{Comm}_{G_1}(G_0) \to \mathbb{Q}_{>0}$ by

$$g \mapsto \frac{|\mathcal{O}_{gG_0}|}{|\mathcal{O}_{g^{-1}G_0}|}$$

is a homomorphism. Hence if there is a $g \in \operatorname{Comm}_{G_1}(G_0)$ with $\varphi(g) > 1$, then for each $n \in \mathbb{N}$, there is a $k_n \in \mathbb{N}$ such that

$$n < \varphi(g)^{k_n} = \varphi(g^{k_n}) = \frac{|\mathcal{O}_{g_n^k G_0}|}{|\mathcal{O}_{g^{-k_n}G_0}|} = \frac{\operatorname{Tr}_1(p_{g^{k_n}})}{\operatorname{Tr}_1^{\mathrm{op}}(p_{g^{k_n}})},$$

and H is not approximately extremal. $\square$

Corollary 4.5.20. *(1) If H is approximately extremal, then H is extremal.*

(2) If $\#G_0\backslash G_1/G_0 = 2$, then H is extremal.

(3) If there is a $g \in G_1$ such that $gG_0g^{-1} \subsetneq G_0$, then H is not approximately extremal.

Remark 4.5.21. In [ILP98], Izumi, Longo, and Popa give an example of $G_0 \subset G_1$ where there is a $g \in G_1$ such that $gG_0g^{-1} \subsetneq G_0$ (so $|\mathcal{O}_{g^{-1}G_0}| = 1$) and $|\mathcal{O}_{gG_0}| = \infty$. Thus they give an example of an irreducible infinite index subfactor which is not approximately extremal.

Finally, we leave the reader with an open question:

Question 4.5.22. *Is there an irreducible infinite index II_1-subfactor which is approximately extremal and not extremal?*

4.6 Relative tensor products of extended positive cones

Notation 4.6.1. For this section, let H_A be a right Hilbert A-module, ${}_AK_B$ be a Hilbert $A - B$ bimodule, and ${}_BL$ be a left Hilbert B-module where A, B are finite von Neumann algebras. We write:

- $X = (A^{\text{op}})' \cap B(H)$,
- ${}_AK$ when we ignore the right B-action,
- $Y_0 = A' \cap B(K)$,
- $Y = A' \cap (B^{\text{op}})' \cap B(K)$,
- $Z = B' \cap B(L)$,
- $X \otimes_A Y_0 = \{x \otimes_A y | x \in X \text{ and } y \in Y_0\}''$, and
- $X \otimes_A Y \otimes_B Z = \{x \otimes_A y \otimes_B z | x \in X,\ y \in Y, \text{ and } z \in Z\}''$.

The goal of this section is to define the operator $x \otimes_A y \in (\widehat{X \otimes_A Y_0})^+$ for $x \in \widehat{X^+}$ and $y \in \widehat{Y_0^+}$ such that certain properties, e.g., associativity, are satisfied.

The next three lemmata are straightforward, but we include some proofs for completeness and for the convenience of the reader.

Lemma 4.6.2. *Suppose $x \in M^+$ and $(x_i)_{i \in I} \subset M^+$ is a directed net, with $x_i \leq x$ for all $i \in I$. The following are equivalent:*

(1) $x_i \to x$ strongly (if and only if σ-strongly as $\|x_i\|_\infty \leq \|x\|_\infty$ for all i)

(2) $x_i \to x$ *weakly (if and only if* σ*-weakly as* $\|x_i\|_\infty \leq \|x\|_\infty$ *for all* i*)*

(3) $x_i \nearrow x$*, i.e.,* $x_i(\omega_\xi) \nearrow x(\omega_\xi)$ *for all* $\xi \in H$*,*

(4) $x_i(\omega_\xi) \nearrow x(\omega_\xi)$ *for all* ξ *in a dense subspace* D *of* H*.*

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(3) $\Rightarrow$ (1): Suppose $(x-x_i)(\omega_\xi) \to 0$ for all $\xi \in H$. Then $\|\sqrt{x-x_i}\xi\|_2 \to 0$, so $\sqrt{x-x_i} \to 0$ strongly. Hence $x_i \to x$ strongly as multiplication is strongly continuous on bounded sets.

(4) $\Rightarrow$ (3): Choose an orthonormal basis $\{e_n\}_{n\geq 1} \subset D$ for H. Suppose $\xi = \sum_n \lambda_n e_n \in H \backslash \{0\}$, and let $\varepsilon > 0$. Then there is an $N > 0$ such that

$$\xi_N := \sum_{n>N} \lambda_n e_n \Longrightarrow \|\xi_N\|_2^2 = \sum_{n>N} |\lambda_n|^2 < \frac{\varepsilon^2}{16\|x\|_\infty^2 \|\xi\|_2^2}.$$

For $n = 1, \ldots, N$, there are $i_n \in I$ such that $i > i_n$ implies

$$|\langle (x-x_i)\lambda_n e_n, \xi\rangle| \leq \|(x-x_i)\lambda_n e_n\|_2 \|\xi\|_2 < \frac{\varepsilon}{2^{n+1}}.$$

Now choose $i' > i_n$ for all $n = 1, \ldots, N$. We calculate that for $i > i'$,

$$\begin{aligned}
(x-x_i)(\omega_\xi) &= \langle (x-x_i)\xi, \xi\rangle \\
&\leq \sum_{n=1}^N |\langle (x-x_i)\lambda_n e_n, \xi\rangle| + |\langle (x-x_i)\xi_N, \xi\rangle| \\
&\leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} + |\langle x\xi_N, \xi\rangle| + |\langle x_i \xi_N, \xi\rangle| \\
&\leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} + 2\|x\|_\infty \|\xi_N\|_2 \|\xi\|_2 \\
&< \frac{\varepsilon}{2} + 2\|x\|_\infty \frac{\varepsilon}{4\|x\|_\infty \|\xi\|_2} \|\xi\|_2 = \varepsilon.
\end{aligned}$$

As ε was arbitrary, we are finished. □

Lemma 4.6.3. *If* $x, y \in M^+$*, and* $(x_i)_{i\in I}, (y_j)_{j\in J} \subset M^+$ *are directed nets of increasing operators such that*

- *any two elements in* $\{x,y\} \cup \{x_i | i \in I\} \cup \{y_j | j \in J\}$ *commute and*
- $x_i \nearrow x$ *and* $y_j \nearrow y$*,*

then $x_i y_j \nearrow xy$ *(and Lemma 4.6.2 applies).*

Lemma 4.6.4. *Suppose $x \in X$ and $y \in Y_0$. Then $x \otimes_A y \colon H \otimes_A K \to H \otimes_A K$ given by the unique extension of $\xi \otimes \eta \mapsto (x\xi) \otimes (y\eta)$ where $\xi \in D(H_A)$ and $\eta \in D({}_AK)$ is well-defined and bounded, and $\|x \otimes_A y\|_\infty \leq \|x\|_\infty \|y\|_\infty$. Hence the $*$-algebra map $x \odot_{\mathbb{C}} y \mapsto x \otimes_A y$ is a binormal representation of $X \odot_{\mathbb{C}} Y_0$ on $H \otimes_A K$.*

Proof. (1) Fix $\xi_1, \ldots, \xi_k \in D(H_A)$ and $\eta_1, \ldots, \eta_k \in D({}_AK)$, and let $\xi = (\xi_1, \ldots, \xi_k)$ and $\eta = (\eta_, \ldots, \eta_k)$. Since the matrices $m = ({}_A\langle y\eta_i, y\eta_j\rangle)_{i,j}, n = (\langle \xi_j, \xi_i\rangle_A)_{i,j} \in M_k(A)$ are positive (see Lemma 1.8 of [Bis97]), we have

$$\begin{aligned}
\left\| \sum_{i=1}^k (x\xi_i) \otimes (y\eta_i) \right\|_2^2 &= \sum_{i,j=1}^k \langle (x\xi_i) \otimes (y\eta_i), (x\xi_j) \otimes (y\eta_j)\rangle \\
&= \sum_{i,j=1}^k \langle (x\xi_i)_A\langle y\eta_i, y\eta_j\rangle, (x\xi_j)\rangle = \langle (x\xi)n, (x\xi)\rangle \\
&= \|(x\xi)n^{1/2}\|_2^2 = \|x(\xi n^{1/2})\|_2^2 \\
&\leq \|x\|_\infty^2 \|\xi n^{1/2}\|_2^2 = \|x\|_\infty^2 \sum_{i,j=1}^k \langle \xi_{iA}\langle y\eta_i, y\eta_j\rangle, \xi_j\rangle \\
&= \|x\|_\infty^2 \sum_{i,j=1}^k \langle \langle \xi_j, \xi_i\rangle_A (y\eta_i), (y\eta_j)\rangle = \|x\|_\infty^2 \|m^{1/2}(y\eta)\|_2^2 \\
&= \|x\|_\infty^2 \|y(m^{1/2}\eta)\|_2^2 \leq \|x\|_\infty^2 \|y\|_\infty^2 \|m^{1/2}\eta\|_2^2 \\
&= \|x\|_\infty^2 \|y\|_\infty^2 \left\| \sum_{i=1}^k \xi_i \otimes \eta_i \right\|_2^2.
\end{aligned}$$

(2) That $x \mapsto x \otimes_A 1_K$ is a normal representation of X follows from the density of $D(H_A) \otimes_A K$ and (4) of Lemma 4.6.2. Similar for $y \mapsto 1_H \otimes_A y$.

□

Notation 4.6.5. Let $\mathcal{B}$ be the Borel σ-algebra of subsets of $[0_{\mathbb{R}}, \infty_{\mathbb{R}}]$. For a spectral measure $E \colon \mathcal{B} \to P(H)$, we use the conventions $E_\lambda = E([0, \lambda])$, so $E_\infty = 1$, and $E^\infty = E(\{\infty\})$ (in general, our spectral measures on $\mathcal{B}$ have non-trivial mass at ∞).

Lemma 4.6.6. *Suppose $E \colon \mathcal{B} \to P(X) \subset B(H_A)$ is a spectral measure. Suppose $f \colon [0, \infty] \to [0, \infty)$ is a bounded Borel-measurable function, and (φ_n) is a sequence of positive simple functions increasing pointwise to f. Then*

$$\int_0^\infty f(\lambda)\, dE_\lambda := \sup_n \int_0^\infty \varphi_n(\lambda)\, dE_\lambda$$

is well-defined.

Proof. Suppose $\xi \in H$. Then as ω_ξ is normal, $\omega_\xi \circ E$ is a Borel measure, and

$$\int_0^\infty f(\lambda)\, d(\omega_\xi(E_\lambda)) = \sup_n \int_0^\infty \varphi_n(\lambda)\, d(\omega_\xi(E_\lambda))$$

is independent of the choice of positive simple functions φ_n increasing to f. □

Proposition 4.6.7. *Suppose*

$$\begin{aligned} E &\colon \mathcal{B} \longrightarrow P(X) \subset B(H_A) \text{ and} \\ F &\colon \mathcal{B} \longrightarrow P(Y_0) \subset B({}_AK) \end{aligned}$$

are spectral measures.

(1) The map $E \otimes_A F \colon \mathcal{B} \otimes \mathcal{B} \longrightarrow P(X \otimes_A Y_0)$ *by*

$$(I_1, I_2) \longmapsto \int_{I_1 \times I_2} d(E_\lambda \otimes_A F_\mu) := E(I_1) \otimes_A F(I_2)$$

extends uniquely to a spectral measure by countable additivity.

(2) If $\varphi, \psi \colon [0, \infty] \to [0, \infty)$ *are positive simple functions, then*

$$\int_0^\infty \int_0^\infty \varphi(\lambda)\psi(\mu)\, d(E_\lambda \otimes_A F_\mu) = \left(\int_0^\infty \varphi(\lambda)\, dE_\lambda\right) \otimes_A \left(\int_0^\infty \psi(\mu)\, dF_\mu\right) \in X \otimes_A Y_0.$$

(3) If f, g *are bounded,* $\mathcal{B}$*-measurable functions and* $(\varphi_m), (\psi_n)$ *are sequences of positive simple functions increasing to* f, g*, then*

$$\sup_{m,n} \int_0^\infty \int_0^\infty \varphi_m(\lambda)\psi_n(\mu)\, d(E_\lambda \otimes_A F_\mu) = \left(\int_0^\infty f(\lambda)\, dE_\lambda\right) \otimes_A \left(\int_0^\infty g(\mu)\, dF_\mu\right) \in X \otimes_A Y_0.$$

Proof. (1) One simply needs to check countable additivity (pointwise on $H \otimes_A K$), which follows from countably additivity on products of intervals, which is straightforward.

(2) Obvious.

(3) Immediate from (2) together with Lemmas 4.6.3 and 4.6.6.

□

Lemma 4.6.8. *The relative tensor product of spectral measures as in Proposition 4.6.7 is associative, i.e., if*

$$\begin{aligned} E &\colon \mathcal{B} \longrightarrow P(X) \subset B(H_A), \\ F &\colon \mathcal{B} \longrightarrow P(Y) \subset B({}_AK_B), \text{ and} \\ G &\colon \mathcal{B} \longrightarrow P(Z) \subset B({}_BL) \end{aligned}$$

are spectral measures on $\mathcal{B}$*, then* $(E \otimes_A F) \otimes_B G = E \otimes_A (F \otimes_B G)$*. Moreover, if* $f, g, h\colon [0,\infty] \to [0,\infty)$ *are bounded* $\mathcal{B}$*-measurable functions, and* $(\varphi_m), (\psi_n), (\gamma_k)$ *are positive simple functions increasing to* f, g, h *respectively, then*

$$\sup_{m,n,k} \int_0^\infty \int_0^\infty \int_0^\infty \varphi_m(\lambda)\psi_n(\mu)\gamma_\ell(\nu)\, d(E_\lambda \otimes_A F_\mu \otimes_B G_\nu) =$$
$$= \left(\int_0^\infty f(\lambda)\, dE_\lambda\right) \otimes_A \left(\int_0^\infty g(\mu)\, dF_\mu\right) \otimes_B \left(\int_0^\infty h(\nu)\, dG_\nu\right) \in X \otimes_A Y \otimes_B Z.$$

Proof. Immediate from associativity of the relative tensor product and Proposition 4.6.7. □

Definition 4.6.9. Suppose $x \in \widehat{X^+}$ and $y \in \widehat{Y_0^+}$ have spectral resolutions

$$x = \int_{[0,\infty)} \lambda\, dE_\lambda + \infty E^\infty \text{ and } y = \int_{[0,\infty)} \mu\, dF_\mu + \infty F^\infty$$

(recall Notation 4.6.5). Then

$$E\colon \mathcal{B} \longrightarrow P(X) \subset B(H_A) \text{ and}$$
$$F\colon \mathcal{B} \longrightarrow P(Y_0) \subset B({}_AK)$$

are two spectral measures as in Proposition 4.6.7. For $m, n \in \mathbb{N}$, set

$$x_m = \int_{[0,m]} \lambda\, dE_\lambda + mE^\infty \text{ and } y_n = \int_{[0,n]} \mu\, dF_\mu + nF^\infty.$$

Applying Lemma 4.2.23 to the directed set

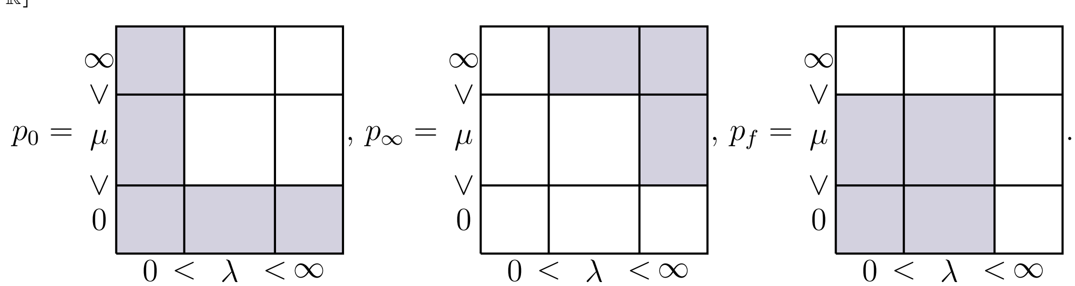
$$\mathcal{F} = \{x_m \otimes_A y_n | m, n \in \mathbb{N}\} \subset (X \otimes_A Y_0)^+,$$

we get a positive, self-adjoint operator affiliated to $X \otimes_A Y_0$ and densely-defined in an affiliated subspace of $X \otimes_A Y_0$. We denote this operator as $x \otimes_A y$.

Remark 4.6.10. Assume the notation of Definition 4.6.9. When we work with $x \otimes_A y$, it helps to consider the following 3 projections:

$$p_0 = (E_0 \otimes_A 1_K) \vee (1_H \otimes F_0),$$
$$p_\infty = \Big((1 - E_0) \otimes_A F^\infty\Big) + \Big(E^\infty \otimes_A (1 - F_0)\Big) + E^\infty \otimes_A F^\infty, \text{ and}$$
$$p_f = \sup_{\lambda,\mu<\infty} E_\lambda \otimes_A F_\mu = (1 - E^\infty) \otimes_A (1 - F^\infty),$$

which we should think of as having the following "supports" given by the shaded areas in $[0_\mathbb{R}, \infty_\mathbb{R}]^2$ below:



- These three projections commute with $x \otimes_A y$.
- $\operatorname{Dom}((x \otimes_A y)^{1/2}) \subset (1-p_\infty)(H \otimes_A K)$, and $(x \otimes_A y)(1-p_\infty)$ is densely defined on $(1-p_\infty)(H \otimes_A K)$.
- $(x \otimes_A y)p_f = \sup_{m,n<\infty} \int_{[0,m]} \int_{[0,n]} \lambda\mu \, d(E_\lambda \otimes_A F_\mu)$.
- $(x \otimes_A y)p_0 = 0$.

Lemma 4.6.11. *Let $x \in \widehat{X^+}$ and $y \in \widehat{Y_0^+}$, and assume the notation of Definition 4.6.9 and Remark 4.6.10. Suppose $x' \in X^+$, $y' \in Y_0^+$ with $x' \leq x$ and $y' \leq y$. Then*

(1) $(x' \otimes_A y')p_0 = p_0(x' \otimes_A y') = 0$,

(2) for all $\xi \in H \otimes_A K$, $(x \otimes_A y)(\omega_\xi) = (x \otimes_A y)(\omega_{(1-p_0)\xi})$, and

(3) $x' \otimes_A y' \leq x \otimes_A y$.

Proof. (1) Suppose $\eta \in D((E_0H)_A)$ and $\kappa \in D({}_AK)$ (recall $E_0 \in X$ and $F^\infty \in Y_0$). Then since $x' \leq x$, we must have

$$\|(x')^{1/2}\eta\|_H^2 = \langle x'\eta, \eta\rangle = x'(\omega_\eta) \leq x(\omega_\eta) = x(\omega_{E_0\eta}) = xE_0(\omega_\eta) = 0.$$

But this implies $x'\eta = 0$. Hence we have

$$(x' \otimes_A y')(\eta \otimes \kappa) = 0.$$

Similarly, for all $\eta \in D(H_A)$ and $\kappa \in D({}_A(F_0K))$, $(x' \otimes_A y')(\eta \otimes \kappa) = 0$. By density of $D(H_A) \otimes_A D({}_AK)$, we have $(x' \otimes_A y')p_0 = 0$. Taking adjoints gives $p_0(x' \otimes_A y') = 0$.

(2) By (1), for all $m, n > 0$, $p_0(x_m \otimes_A y_n) = (x_m \otimes_A y_n)p_0 = 0$, so

$$\begin{aligned}
(x \otimes_A y)(\omega_\xi) &= \sup_{m,n}(x_m \otimes_A y_n)(\omega_\xi) \\
&= \sup_{m,n}\Big((x_m \otimes_A y_n)(\omega_{(1-p_0)\xi}) + \langle (x_m \otimes_A y_n)p_0\xi, p_0\xi\rangle \\
&\qquad + \langle (x_m \otimes_A y_n)p_0\xi, \xi\rangle + \langle (x_m \otimes_A y_n)\xi, p_0\xi\rangle\Big) \\
&= \sup_{m,n}(x_m \otimes_A y_n)(\omega_{(1-p_0)\xi}) = (x \otimes_A y)(\omega_{(1-p_0)\xi}).
\end{aligned}$$

(3) By (2), it suffices to show that for all $\xi \in \operatorname{Dom}((x \otimes_A y)^{1/2})$ with $\xi = p_f\xi$,

$$\Big(p_f(x' \otimes_A y')p_f\Big)(\omega_\xi) = (x' \otimes_A y')(\omega_\xi) \leq (x \otimes_A y)(\omega_\xi) = \Big(p_f(x \otimes_A y)p_f\Big)(\omega_\xi).$$

Fix such a ξ, and let $\varepsilon > 0$. As $E_\lambda \otimes_A F_\mu \to p_f$ strongly as $\lambda, \mu \to \infty$ from below, there is an $N > 0$ such that for all $\lambda, \mu > N$,

$$\Big(p_f(x' \otimes_A y')p_f - (E_\lambda x' E_\lambda \otimes_A F_\mu y' F_\mu)\Big)(\omega_\xi) < \varepsilon.$$

Since $x' \le x$ and $y' \le y$, we have $E_N x' E_N \le x E_N$, $F_N y' F_N \le y F_N$ by Lemma 4.2.21, so $E_N x' E_N \otimes_A F_N y' F_N \le x E_N \otimes_A y F_N$ as all these operators mutually commute. Hence

$$\begin{aligned}\Big(p_f(x' \otimes y')p_f\Big)(\omega_\xi) &= \Big(p_0(x_m \otimes_A y_n)p_0 - (E_N x' E_N \otimes_A F_N y' F_N)\Big)(\omega_\xi)\\ &\qquad + (E_N x' E_N \otimes_A F_N y' F_N)(\omega_\xi)\\ &< \varepsilon + (x E_N \otimes_A y F_N)(\omega_\xi) \le \varepsilon + (x \otimes_A y)(\omega_\xi).\end{aligned}$$

Since ε was arbitrary, the result follows.

□

Lemma 4.6.12. *Suppose $(x'_j)_{j\in J} \subset \widehat{X^+}$ increases to $x \in \widehat{X^+}$. Suppose $p, q \in P(X)$ are spectral projections of x such that $p + q = 1$. Then $\langle x'_j p\xi, q\xi\rangle \to 0$ for all $\xi \in \mathrm{Dom}(x^{1/2})$.*

Proof. For $k = 0, 1, 2, 3$, $p\xi + i^k q\xi \in \mathrm{Dom}(x^{1/2}) \subseteq \mathrm{Dom}((x'_j)^{1/2})$ for all $j \in J$. Since x'_j increases to x, by polarization

$$\begin{aligned}\lim_{j\in J}\langle (x'_j)^{1/2}p\xi, (x'_j)^{1/2}q\xi\rangle &= \lim_{j\in J}\frac{1}{4}\sum_{k=0}^{3} i^k x'_j(\omega_{p\xi+i^k q\xi}) = \frac{1}{4}\sum_{k=0}^{3} i^k x(\omega_{p\xi+i^k q\xi})\\ &= \langle x^{1/2}p\xi, x^{1/2}q\xi\rangle = 0\end{aligned}$$

as p, q commute with $x^{1/2}$. □

Theorem 4.6.13. *Let $x \in \widehat{X^+}$ and $y \in \widehat{Y_0^+}$, and assume the notation of Definition 4.6.9 and Remark 4.6.10. Suppose there are sequences $(x'_m) \subset X^+$, $(y'_n) \subset Y_0^+$ which increase to x, y respectively. Then $x'_m \otimes_A y'_n$ increases to $x \otimes_A y$.*

Proof.

Case 1: Suppose $\xi \notin \mathrm{Dom}((x \otimes_A y)^{1/2})$ and $M > 0$. Since $\sup_{m,n} x_m \otimes_A y_n = x \otimes_A y$, there is an $N_0 \in \mathbb{N}$ such that for all $m, n \ge N_0$, $(x_m \otimes_A y_n)(\omega_\xi) > M$. Since $p_0\xi \ne \xi$ by Lemma 4.6.11, we must have

$$(1_H \otimes_A (1_K - F_0))\xi \ne 0 \text{ and } ((1_H - E_0) \otimes_A 1_K)\xi \ne 0.$$

Claim: There is an $N_1 > N_0$ such that $(x'_m \otimes 1_K)\xi \ne 0 \ne (1_H \otimes_A y'_n)\xi$ for all $m, n > N_1$.

Proof. We prove the second non-equality. Suppose not. Then for each $n > 0$, there is an $k > n$ such that $(1 \otimes_A y'_k)\xi = 0$. But then

$$(1_H \otimes_A y'_n)(\omega_\xi) \leq (1_H \otimes_A y'_k)(\omega_\xi) = 0,$$

so $(1_H \otimes_A y'_n)\xi = 0$ for all $n \in \mathbb{N}$. Since $(1_H \otimes_A (1 - F_0))\xi \neq 0$, and $D(H_A) \otimes_A D({}_A((1-F_0)K))$ is dense in $H \otimes_A ((1_K - F_0)K)$, there is an $\eta \in D(H_A)$ such that $L^*_\eta \xi \in ((1_K - F_0)K) \setminus \{0\}$ and $L_\eta L^*_\eta \leq 1_H \otimes_A 1_K$. Now since y'_n increases to y, and $y(\omega_{L^*_\eta \xi}) > 0$, there is an $N' > 0$ such that for all $n > N'$,

$$0 < y'_n(\omega_{L^*_\eta \xi}) = (L_\eta y_n L^*_\eta)(\omega_\xi) = \Big(L_\eta L^*_\eta (1_H \otimes_A y'_n)\Big)(\omega_\xi) \leq (1_H \otimes_A y'_n)(\omega_\xi) = 0,$$

a contradiction. □

Choose N_1 as in the claim, and suppose $n > N_1$. Let $\{\alpha_i\} \subset D({}_AK)$ be an ${}_AK$-basis, and let $\eta = (1_H \otimes_A (y_{N_1})^{1/2})\xi \neq 0$, and note $(x_{N_1} \otimes_A 1_K)(\omega_\eta) > M$. Then

$$M < (x_{N_1} \otimes 1_K)(\omega_\eta) = \left((x_{N_1} \otimes_A 1_K)\left(\sum_i R_{\alpha_i} R^*_{\alpha_i}\right)\right)(\omega_\eta) = \sum_i (R_{\alpha_i}(x_{N_1}) R^*_{\alpha_i})(\omega_\eta),$$

so there is an $N_2 > 0$ such that

$$M < \sum_{i=1}^{N_2} (R_{\alpha_i} x_{N_1} R^*_{\alpha_i})(\omega_\eta) = \sum_{i=1}^{N_2} x_{N_1}(\omega_{R^*_\alpha \eta}) \leq \sum_{i=1}^{N_2} x(\omega_{R^*_\alpha \eta}).$$

Now as x'_m increases to x, there is an $N_3 > N_1$ such that $m > N_3$ implies

$$\begin{aligned} M &< \sum_{i=1}^{N_2} x'_m(\omega_{R^*_\alpha \eta}) = \sum_{i=1}^{N_2} (R_{\alpha_i} x'_m R^*_{\alpha_i})(\omega_\eta) \leq \sum_i (R_{\alpha_i} x'_m R^*_{\alpha_i})(\omega_\eta) \\ &= \left((x'_m \otimes_A 1_K)\left(\sum_i R_{\alpha_i} R^*_{\alpha_i}\right)\right)(\omega_\eta) = (x'_m \otimes y_{N_1})(\omega_\xi). \end{aligned}$$

Repeating the above argument for y'_n yields an N_4 such that $m, n > N_4$ implies $M < (x'_m \otimes_A y'_n)(\omega_\xi)$.

<u>Case 2:</u> Suppose $\xi \in \mathrm{Dom}((x \otimes_A y)^{1/2})$. Then $\xi = (1 - p_\infty)\xi$. We want to show

$$\sup_{m,n} (x'_m \otimes_A y'_n)(\omega_\xi) = (x \otimes_A y)(\omega_\xi) = \sup_{m,n} (x_m \otimes_A y_n)(\omega_\xi),$$

so by Lemma 4.6.11, we may assume $\xi = (1 - p_0)\xi$, and thus $\xi = p_f \xi$. Let $\varepsilon > 0$. Since

$$p_f (x \otimes_A y) p_f = \sup_{\lambda, \mu < \infty} x E_\lambda \otimes_A y F_\mu,$$

there is an $N_0 \in \mathbb{N}$ such that for all $\lambda, \mu \geq N_0$,

$$\Big((x \otimes_A y) - (xE_\lambda \otimes_A yF_\mu)\Big)(\omega_\xi) < \frac{\varepsilon}{4}.$$

By Lemma 4.6.11, $x'_m \otimes_A y'_n \leq x \otimes_A y$ for all m, n, so using Lemma 4.2.21, we have

$$\Big((x'_m \otimes_A y'_n) - (E_{N_0} x'_m E_{N_0}) \otimes_A (F_{N_0} y'_n F_{N_0})]\Big) \leq \Big((x \otimes_A y) - (xE_{N_0} \otimes_A yF_{N_0})\Big) \text{ and}$$
$$E_{N_0} x'_m E_{N_0} \otimes_A F_{N_0} y'_n F_{N_0} \leq xE_{N_0} \otimes_A yF_{N_0}$$

by multiplying on either side by $1_{H \otimes_A K} - (E_{N_0} \otimes_A F_{N_0})$ and $E_{N_0} \otimes_A F_{N_0}$ respectively. Now since x'_m, y'_n increase to x, y respectively, by Lemma 4.2.21, $E_{N_0} x'_m E_{N_0}, F_{N_0} y'_n F_{N_0}$ increases to xE_{N_0}, yF_{N_0} respectively. Thus $E_{N_0} x'_m E_{N_0} \otimes_A F_{N_0} y'_n F_{N_0}$ increases to $xE_{N_0} \otimes_A yF_{N_0}$ by Lemma 4.6.3, and there is an $N_1 > N_0$ such that for all $m, n \geq N_1$,

$$\Big((xE_{N_0} \otimes_A yF_{N_0}) - (E_{N_0} x'_m E_{N_0} \otimes_A F_{N_0} y'_n F_{N_0})\Big)(\omega_\xi) < \frac{\varepsilon}{4}.$$

By Lemma 4.6.12, there is an $N_2 > N_1$ such that for all $m, n > N_2$,

$$\left|\Big\langle (x'_m \otimes y'_n)(1_{H \otimes_A K} - E_{N_0} \otimes_A F_{N_0})\xi, (E_{N_0} \otimes_A F_{N_0})\xi \Big\rangle\right| < \frac{\varepsilon}{4}.$$

Now we calculate that for all $m, n > N_2$,

$$\begin{aligned}
(x \otimes_A y - x'_m \otimes y'_n)(\omega_\xi) &= (1 - E_{N_0} \otimes_A F_{N_0})(x \otimes_A y - x'_m \otimes y'_n)(1 - E_{N_0} \otimes_A F_{N_0})(\omega_\xi) \\
&\quad + (1_{H \otimes_A K} - E_{N_0} \otimes_A F_{N_0})(x \otimes_A y - x'_m \otimes y'_n)(E_{N_0} \otimes_A F_{N_0})(\omega_\xi) \\
&\quad + (E_{N_0} \otimes_A F_{N_0})(x \otimes_A y - x'_m \otimes y'_n)(1_{H \otimes_A K} - E_{N_0} \otimes_A F_{N_0})(\omega_\xi) \\
&\quad + (E_{N_0} \otimes_A F_{N_0})(x \otimes_A y - x'_m \otimes y'_n)(E_{N_0} \otimes_A F_{N_0})(\omega_\xi) \\
&\leq \Big((x \otimes_A y) - (xE_{N_0} \otimes_A yF_{N_0})\Big)(\omega_\xi) \\
&\quad + |((1_{H \otimes_A K} - E_{N_0} \otimes_A F_{N_0})(x'_m \otimes_A y'_n)(E_{N_0} \otimes_A F_{N_0})(\omega_\xi)| \\
&\quad + |(E_{N_0} \otimes_A F_{N_0})(x'_m \otimes_A y'_n)(1 - E_{N_0} \otimes_A F_{N_0})(\omega_\xi)| \\
&\quad + \Big((xE_{N_0} \otimes_A yF_{N_0}) - (E_{N_0} x'_m E_{N_0} \otimes_A F_{N_0} y'_n F_{N_0})\Big)(\omega_\xi) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

□

Corollary 4.6.14. *If $x \in \widehat{X^+}$, $y \in \widehat{Y^+}$, and $z \in \widehat{Z^+}$, then $(x \otimes_A y) \otimes_B z = x \otimes_A (y \otimes_B z)$.*

Proof. Take sequences $(x_m) \subset X^+$, $(y_n) \subset Y^+$, and $(z_\ell) \subset Z^+$ which increase to x, y, z respectively. Then

$$(x \otimes_A y) \otimes_B z = \sup_{m,n,\ell} (x_m \otimes_A y_n) \otimes_B z_\ell = \sup_{m,n,\ell} x_m \otimes_A (y_n \otimes_B z_\ell) = x \otimes_A (y \otimes_B z).$$

□

Corollary 4.6.15. *If $x, w \in \widehat{X^+}$, $y \in \widehat{Y_0^+}$, and $\lambda \in [0, \infty]$, then $(\lambda x + w) \otimes_A y = \lambda(x \otimes_A y) + (w \otimes_A y)$.*

Proof. Choose $X^+ \ni x_m, w_n \nearrow x, w \in \widehat{X^+}$ respectively and $\widehat{Y_0^+} \ni y_\ell \nearrow y \in \widehat{Y_0^+}$. Then $(\lambda x_m + w_n) \otimes_A y_\ell = \lambda(x_m \otimes_A y_\ell) + (w_n \otimes_A y_\ell)$, and the result follows by Remark 4.2.22 and Theorem 4.6.13. □

By taking sups appropriately, and with a little more care, Lemma 4.6.11 and Theorem 4.6.13 can be generalized to prove:

Theorem 4.6.16. *Let $x \in \widehat{X^+}$ and $y \in \widehat{Y_0^+}$. Suppose there are nets $(x_i)_{i\in I} \subset \widehat{X^+}$, $(y_j)_{j\in J} \subset \widehat{Y_0^+}$ which increase to x, y respectively. Then $x_i \otimes_A y_j \nearrow x \otimes_A y$.*

4.7 The action of $\mathbb{BP}$ is well-defined

In this section, we show the action of $\mathbb{BP}$ is well-defined in Theorem 4.3.18. We do so in two steps. First, we define a sub-operad $\mathbb{BP}_1 \subset \mathbb{BP}$, define the action of $\mathbb{BP}_1$ on the extended positive cones $\widehat{Q_n^+}$, and show the action is well-defined. We show that each connected tangle (see Definition 4.7.1) has a unique standard form (see Algorithm 4.7.4) that behaves well under composition, analogous to the methods of [Pen12a]. Second, we extend the action to $\mathbb{BP}$ and show it is well-defined by considering the possibilities that occur when inserting connected $\mathbb{BP}_1$-tangles into the quadratic pairing tangle τ_n or τ_n^{op} (see Definition 4.7.7).

The operad $\mathbb{BP}_1$

Definition 4.7.1. We will define $\mathbb{BP}_1$, an operad of unshaded, oriented tangles up to planar isotopy. First, we require for tangles $\mathcal{T} \in \mathbb{BP}_1$:

(1) $\mathcal{T}$ has an external disk D_0 and internal disks $D_1, \dots, D_s$, each with an even number $2k_i$ of market boundary points and a distinguished interval marked $*$. The boundary points of D_i are numbered $1, \dots, 2k_i$ clockwise from $*$, and and we use the convention that for $1 \le n \le 2k_i$, the $-n^{\text{th}}$ boundary point is the point numbered $2k_i - n + 1$.

(2) Each boundary point of $\mathcal{T}$ is connected to exactly one oriented string. Each oriented string is either a closed loop, or it is attached to two distinct boundary points.

(3) For $i = 1, \ldots, s$, reading counter-clockwise from $*$, the strings attached to the first k_i boundary points of D_i are oriented away from D_i, and the second k_i strings are oriented toward D_i,

(4) Reading counter-clockwise from $*$, the strings attached to the first k_0 boundary points of D_0 are oriented toward D_0, and the second k_0 strings are oriented away from D_0,

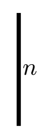
When we draw such a tangle, we draw all disks D_i $(0 \leq i \leq s)$ as rectangles with k_i strings connected to the top and bottom, we suppress the external disk, we draw one thick string labelled n for n individual strings, and we orient all strings upward unless otherwise specified.

A tangle with disks $\{D_i\}_{i=0}^{s}$ and strings $\{S_j\}_{j=1}^{t}$ satisfying (1)-(4) is called:

- connected if $\{D_i\}_{i=0}^{s} \cup \{S_j\}_{j=1}^{t}$ is connected in $\mathbb{R}^2$, and
- internally connected if $\mathcal{T}$ has no external boundary points and $\{D_i\}_{i=1}^{s} \cup \{S_j\}_{j=1}^{t}$ is connected in $\mathbb{R}^2$.

Let $\mathbb{BP}_1$ be the operad generated by the following tangles:

Temperley-Lieb: For $n \geq 0$, the "Temperley-Lieb" tangle 1_n with no inputs and $2n$ boundary points:

$1_n =$ 

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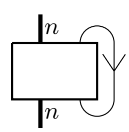
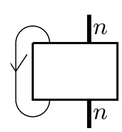
Note that 1_0 is the empty tangle.

Partial trace: For $n \geq 0$, the tangles $t_{n+1}, t_{n+1}^{\mathrm{op}}$ with $2n+2$ internal boundary points and $2n$ external boundary points and only one right, left cap respectively:

$t_{n+1} =$ 

and $t_{n+1}^{\mathrm{op}} =$ 

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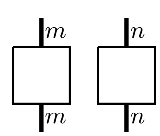
Tensoring: For $m, n \geq 0$, the tangles $\otimes_{m,n}$ with internal disks D_1, D_2 with $2m, 2n$ internal boundary points and $2(m+n)$ external boundary points as follows:

$\otimes_{m,n} =$ 

.

Theorem 4.7.2. *The following relations hold in $\mathbb{BP}_1$ for $m, n \geq 0$ (compare with (1)-(3) in Theorem 4.3.15):*

(1) $t_m t_{m+1}^{\mathrm{op}} = t_m^{\mathrm{op}} t_{m+1}$,

(2) $\otimes_{\ell,m+n}(-, \otimes_{m,n}(-,-)) = \otimes_{\ell+m,n}(\otimes_{\ell,m}(-,-),-)$, *and*

(3) $t_{m+n}(\otimes_{m,n}(-,-)) = \otimes_{m,n-1}(-,t_n(-))$ *and* $t^{\mathrm{op}}_{m+n}(\otimes_{m,n}(-,-)) = \otimes_{m-1,n}(t^{\mathrm{op}}_m(-),-)$.

Proof. Clear by drawing pictures. □

Theorem 4.7.3. *Suppose $\mathcal{T}$ is an unshaded, oriented tangle which satisfies requirements (1)-(4) in Definition 4.7.1. Then*

- *(BP0) If boundary points m and n of D_0 are connected by a string, then $m = -n$ (recall the convention $-n = 2k_i - n + 1$ from (1) of Definition 4.7.1).*

The tangle $\mathcal{T}$ is in $\mathbb{BP}_1$ if and only if the following conditions are satisfied:

- *(BP1) No string may connect the input disks D_i and D_j for $i \neq j$.*
- *(BP2) If the string S connects the n^{th} boundary point of D_i to the m^{th} boundary point of D_0, then there is a string S' connecting the $-n^{th}$ boundary point of D_i to the the $-m^{th}$ boundary point of D_0, and any other string connected to D_i must only be connected to D_i or D_0.*

If (BP1) and (BP2) hold, then the following condition also holds:

- *(BP3) If the string S connects boundary points m and n of D_i, then $m = -n$. Such a string is called an i-cap of $\mathcal{T}$. We call the i-cap a left i-cap if when we connect boundary points n and $-n$ by an imaginary string S' inside D_i, the loop $S \cup S'$ contains the distinguished interval of D_i. The i-cap is a right i-cap otherwise.*

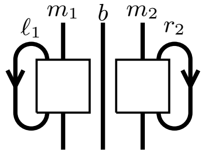
Proof. (BP0) follows from (1)-(4) in Definition 4.7.1 by a simple counting argument. Similarly, (BP3) follows from (BP0)-(BP2). Clearly tangles in $\mathbb{BP}_1$ satisfy (BP1) and (BP2), since these properties are preserved under composition of the tangles which generate $\mathbb{BP}_1$.

Now suppose $\mathcal{T}$ satisfies (BP0)-(BP3). If $\mathcal{T}$ is internally connected, then either $\mathcal{T}$ is a closed loop, or $\mathcal{T}$ has only one input disk D_1, and we may write $\mathcal{T}$ uniquely as

$$\mathcal{T} = t^{\mathrm{op}}_1 \cdots t^{\mathrm{op}}_\ell t_{\ell+1} t_{\ell+2} \cdots t_{\ell+r} \tag{4.4}$$

where ℓ is the number of left caps and r is the number of right caps of D_1 of $\mathcal{T}$. Hence, we may reduce to the case that $\mathcal{T}$ is connected. Now Algorithm 4.7.4 expresses the connected tangle $\mathcal{T}$ in a standard form as a composite of generators of $\mathbb{BP}_1$. □

Algorithm 4.7.4 (Standard form of connected $\mathbb{BP}_1$-tangles)**.** Suppose $\mathcal{T}$ satisfies (1)-(4) of Definition 4.7.1 and (BP1)-(BP3) in Theorem 4.7.3, and suppose $\mathcal{T}$ is connected. Then we can use $\otimes_{m,n}$ to "parenthesize" the D_i's ($i \geq 0$) and groups of through strings 1_b from right to left. Before we give the algorithm we give an example:



$$= \otimes_{m_1,t+m_2}(t^{\mathrm{op}}_{m_1+1} \cdots t^{\mathrm{op}}_{m_1+\ell_1}, \otimes_{b,m_2}(1_b, t_{m_2+1} \cdots t_{m_2+r_2})).$$

The following algorithm expresses $\mathcal{T}$ in a standard form as a composite of generators of $\mathbb{BP}_1$:

(0) If $\mathcal{T}$ is the empty tangle, break.

(1) Start at $*$ on the external boundary. Going clockwise along D_0, denote the strings oriented toward D_0 by $S_1, \ldots, S_{k_0}$ (note $k_0 > 0$). Set:

- $a = k_0$ (a is the number of strings $S_1, \ldots, S_{k_0}$ remaining to be examined) and
- $n = 0$ (S_{n+1} is the string we are currently examining).

Record a place holder ? to be replaced.

(2) If S_{n+1} connects D_0 to D_0, find b maximal such that $S_{n+1}, \ldots, S_{n+b}$ all connect D_0 to D_0. Set $a = a - b$.

(2a) If $a = 0$, replace the last ? with 1_b and break.

(2b) If $a > 0$ and $b > 0$, replace the last ? with $\otimes_{b,a}(1_b, ?)$, where ? will be replaced later, and set $n = n + b$.

(3) Now $a > 0$, and S_{n+1} is the first string connecting D_0 to some input disk D_i. Find m_i maximal such that $S_{n+1}, \ldots, S_{n+m_i}$ connect D_0 to D_i. Set $a = a - m_i$, let ℓ_i be the number of left caps of D_i, and let r_i be the number of right caps of D_i.

(3b) If $a = 0$, replace the last ? with $t^{\mathrm{op}}_{m_i+1} \cdots t^{\mathrm{op}}_{m_i+\ell_i} t_{m_i+\ell_i+1} \cdots t_{m_i+\ell_i+r_i}$ and break.

(3a) If $a > 0$, replace the last ? with $\otimes_{m_i,a}(t^{\mathrm{op}}_{m_i+1} \cdots t^{\mathrm{op}}_{m_i+\ell_i} t_{m_i+\ell_i+1} \cdots t_{m_i+\ell_i+r_i}, ?)$, where ? will be replaced later, set $n = n + m_i$, and go to (2).

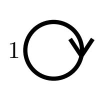
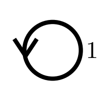
Definition 4.7.5 (Action of tangles in $\mathbb{BP}_1$)**.** We may now describe the action of a tangle $\mathcal{T} \in \mathbb{BP}$ on a tuple

$$(z_1, \ldots, z_s) \in \prod_{i=1}^{s} \widehat{Q^+_{n_i}}.$$

If $\mathcal{T}$ is connected, we put $\mathcal{T}$ in the standard form afforded by Algorithm 4.7.4, label the inputs with the z_i's, and replace 1_n with id_{H_n}; t_n, t^{op}_n with T_n, T^{op}_n; and $\otimes_{m,n}$ with $\otimes_A$.

If $\mathcal{T}$ is not connected, then there are internally connected subtangles which are either closed loops, or which can be uniquely written as in Equation (4.4). These subtangles will act as scalars in $\widehat{Q^+_0} = \widehat{Z(A)^+} = [0_{\mathbb{R}}, \infty_R]$, and the order of scalar multiplication does not matter, so it suffices to define the scalar given by a single internally connected subtangle.

First, closed loops count for a multiplicative factor:

$\dim_{-A}(H) = T_1(1) =$  and $\dim_{A-}(H) = T^{\mathrm{op}}_1(1) =$  .

Suppose $\mathcal{S}$ is a closed, internally connected subtangle of $\mathcal{T}$ with only one input disk. Then we may write $\mathcal{S}$ uniquely as in Equation (4.4), label the tangle by z_i, and replace t_n, t^{op}_n with T_n, T^{op}_n.

Theorem 4.7.6. *Definition 4.7.5 gives a well-defined action of* $\mathbb{BP}_1$.

Proof. The methods of [Pen12a] show that the standard forms of connected and internally connected tangles given in Algorithm 4.7.4 and Equation (4.4) and the maps given in Subsection 4.3 behave the same under composition by Theorems 4.3.15 and 4.7.2. We briefly sketch such an argument.

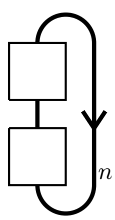
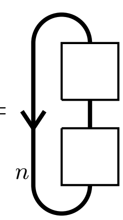
We need only consider the composites $\mathcal{R} \circ \mathcal{S}$ and $\mathcal{S} \circ_i \mathcal{T}$ where $\mathcal{R}, \mathcal{S}, \mathcal{T} \in \mathbb{BP}_1$ such that $\mathcal{R}$ is internally connected with 1 input disk and $\mathcal{S}, \mathcal{T}$ are connected. That the action is well-defined follows from using the relations in Theorems 4.3.15 and 4.7.2 and (4) in Corollary 4.3.16 to get the standard form of the composite from the composite of the standard forms (push all $\otimes_{m,n}, \otimes_A$ as far to the left as possible, and push all left caps $t^{\text{op}}, T^{\text{op}}$ to the left of the right caps t, T). Once again, since internally connected tangles act as scalars in $[0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, the order in which we remove them and multiply by the scalar does not matter. □

The operad $\mathbb{BP}$

We now include the pairing tangles to get the operad $\mathbb{BP}$ and show its action is well-defined.

Definition 4.7.7. Let $\mathbb{BP}$ be the operad generated by $\mathbb{BP}_1$ and the following tangles:

Pairing: For $n \geq 1$, the tangles $\tau_n, \tau_n^{\text{op}}$ with two input disks, each with $2n$ internal boundary points, and no external boundary points such that boundary point m of input disk D_1 is connected to boundary point $2n - m + 1$ of input disk D_2 for each $m = 1, \ldots, 2n$ as follows:

$\tau_n =$  and $\tau_n^{\text{op}} =$  .

There are similar notions of connectivity and internal connectivity for tangles $\mathcal{T} \in \mathbb{BP}$.

Remark 4.7.8. $\tau_n(\mathcal{T}_1(-), \mathcal{T}_2(-)) = \tau_n(\mathcal{T}_2(-), \mathcal{T}_1(-))$ and similarly for τ_n^{op} for all $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}$ up to reindexing internal disks.

Theorem 4.7.9. *Suppose $\mathcal{T}$ is an unshaded, oriented, internally connected tangle which satisfies (1)-(4) in Definition 4.7.1. Then $\mathcal{T} \in \mathbb{BP}$ if and only if conditions (BP0), (BP2), and (BP3) from Theorem 4.7.3 are satisfied (we now exclude (BP1)) along with the following conditions:*

- *(BP4) If the string S connects boundary point m of D_i to boundary point n of D_j where $1 \leq i < j \leq s$, then*
  - *(i) no string of D_i or D_j connects to D_0, and*

(ii) there is another string S' connecting boundary points $-m$ of D_i and $-n$ of D_j.

We call $S \cup S'$ an <u>*i, j-cap*</u> *of $\mathcal{T}$. In this case, if we connect boundary points m and $-m$ of D_i and boundary points n and $2k_j - n + 1$ of D_j by imaginary strings S_i, S_j inside D_i, D_j respectively, then the loop $S \cup S' \cup S_i \cup S_j$ either*

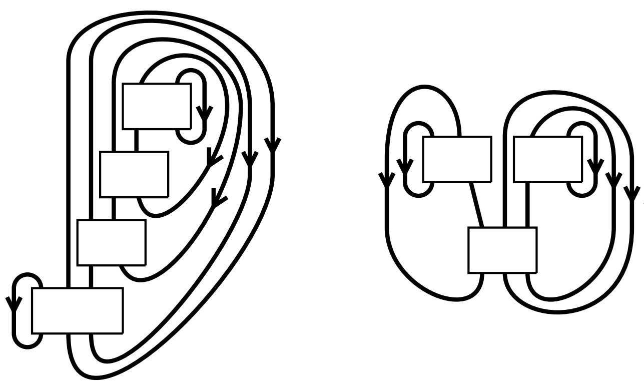
(i) contains the $*$'d intervals of D_i and D_j, and the i, j-cap is a <u>*left i, j-cap*</u>*, or*

(ii) does not contain the $*$'d intervals, and the i, j-cap is a <u>*right i, j-cap*</u>*.*

- *(BP5) The i, j-caps of $\mathcal{T}$ are either all right or all left caps, and they form concentric circles.*

Proof. Once again, it is clear that any tangle in $\mathbb{BP}$ satisfies the desired properties, since these properties are preserved under composition of tangles (the total number of i, j-caps can only decrease under composition of connected and internally connected tangles), and the generating tangles satisfy these properties. The other direction follows from Algorithm 4.7.11, which shows how to 'comb' the tangle into a unique standard form. □

Example 4.7.10. The tangle on the left is in $\mathbb{BP}$ (see Algorithm 4.7.11), but the tangle on the right is not:



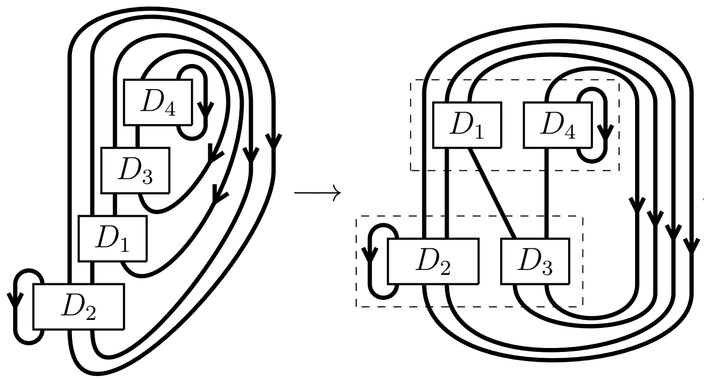
Algorithm 4.7.11. Suppose $\mathcal{T}$ is an internally connected tangle which satisfies (1)-(4) of Definition 4.7.1 and (BP0),(BP2),(BP3) in Theorem 4.7.3 and (BP4),(BP5) in Theorem 4.7.9. Suppose further that $\mathcal{T}$ has at least two input disks, so there is an i, j-cap. Let C_1 be the outermost i, j-cap of $\mathcal{T}$. Then there is a unique smallest $n \in \mathbb{N}$ and two unique connected tangles $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}_1$ up to swapping such that:

<u>Right:</u> if C_1 is a right i, j-cap, $\mathcal{T} = \tau_n(\mathcal{T}_1(-), \mathcal{T}_2(-))$, and

<u>Left:</u> if C_1 is a left i, j-cap, $\mathcal{T} = \tau_n^{\mathrm{op}}(\mathcal{T}_1(-), \mathcal{T}_2(-))$.

We give an algorithm for the right-cap case, and the left-cap case is similar. We will build $\mathcal{T}_1$ and $\mathcal{T}_2$ by partitioning the internal disks of $\mathcal{T}$ into two sets U and L, standing for "upper" and "lower." All i, j-caps of $\mathcal{T}$ will be between a $D_i \in U$ and a $D_j \in L$. We form $\mathcal{T}_1$ by putting a box around the $D_i \in U$ together with all "contractible" i-caps, and we form $\mathcal{T}_2$ by doing the same to the $D_j \in L$.

Before we describe the algorithm, we give an example:



(1) Start at the $*$ on the external boundary. Set $U = L = \emptyset$. Let c be the number of i, j-caps of $\mathcal{T}$.

(2) If $c = 0$, then go to (4).

(3) Find the next outermost i, j-cap C in $\mathcal{T}$, where $i < j$. Set $c = c - 1$.

(3a) If $U = L = \emptyset$, then set $U = \{D_i\}$ and $L = \{D_j\}$.

(3b) If D_i or D_j is not in $U \cup L$ (note that at least one of D_i, D_j is in $U \cup L$), put the missing one where the other one is not, e.g., if $D_i \notin U \cup L$ and $D_j \in L$, then set $U = U \cup \{D_i\}$. (There are 4 cases here.)

(3c) Isotope the tangle so that

- all disks in U and L appear on the same horizontal levels, with L below U,
- any string connecting a disk $D_u \in U$ to a disk $D_\ell \in L$ travels upward from D_ℓ to D_u with no critical points, or travels in a large arc from D_u to D_ℓ with only two critical points,
- all k-caps which enclose the i, j-cap C are large arcs with only two critical points,
- all k-caps for $D_k \in U \cup L$ which do not enclose an a, b-cap are close to D_k.

(3d) Go to (2).

(4) Put boxes around the disks and caps in U, L as desired. We have $\tau_n(\mathcal{T}_1(-), \mathcal{T}_2(-))$ for some $n \in \mathbb{N}$ and some connected tangles $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}_1$.

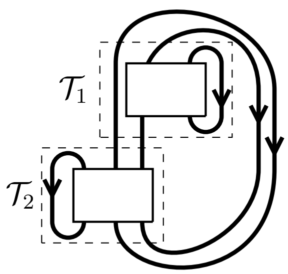
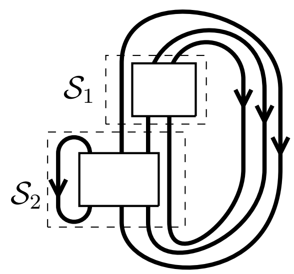
Note that the n is determined by the i, j-caps and the k-caps which enclose an i, j-cap, and this n is minimal when all other ℓ-caps are contracted so they are close to D_ℓ. Moreover, the only choice we made was the initial choice $U = \{D_i\}$ and $L = \{D_j\}$ with $i < j$, but if we swapped U and L, we would have ended up with $\tau_n(\mathcal{T}_2(-), \mathcal{T}_1(-))$. Hence $\mathcal{T}_1, \mathcal{T}_2$ are unique up to swapping.

Definition 4.7.12 (Action of tangles in $\mathbb{BP}$)**.** We extend the action of $\mathbb{BP}_1$ to an action of $\mathbb{BP}$. Note that it suffices to define the action of an internally connected tangle with at least 2 input disks (so there is necessarily an i, j-cap), and any such tangle can be written uniquely as $\tau_n(\mathcal{T}_1, \mathcal{T}_2)$ (or τ_n^{op}) with n minimal and $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}_1$ unique up to swapping by Algorithm 4.7.11. Simply use the action prescribed by Definition 4.7.5 for $\mathcal{T}_1$ and $\mathcal{T}_2$, and then the action of $\tau_n, \tau_n^{\text{op}}$ is given by replacing it with $\text{Tr}_n, \text{Tr}_n^{\text{op}}$.

Theorem 4.7.13. *Definition 4.7.12 gives a well-defined action of* $\mathbb{BP}$.

Proof. We show that for any connected $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{BP}_1$ and $m \in \mathbb{N}$, that the action of the composite tangle $\tau_m(\mathcal{S}_1, \mathcal{S}_2)$ is the same as the composite of the actions of τ_m and the actions of the tangles $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{BP}_1$. A similar result holds for τ_m^{op}.

First, note that (4) and (5) of Corollary 4.3.16 allow us to reduce to the case where $\tau_m(\mathcal{S}_1, \mathcal{S}_2)$ is internally connected. If $\tau_m(\mathcal{S}_1, \mathcal{S}_2)$ is internally connected, then Algorithm 4.7.11 gives a standard form $\tau_n(\mathcal{T}_1, \mathcal{T}_2) = \tau_m(\mathcal{S}_1, \mathcal{S}_2)$ where $n \in \mathbb{N}$ is minimal and $\mathcal{T}_1, \mathcal{T}_2 \in \mathbb{BP}_1$ are unique connected tangles up to swapping. If $m > n$, then setting $b = m - n$, we must have (up to swapping) that $\mathcal{T}_1 = t_{n+1} \cdots t_{n+b}(\mathcal{S}_1)$ and $\mathcal{S}_2 = \otimes_{n,b}(\mathcal{T}_2, 1_b)$. A similar statement holds for τ_m^{op} using t^{op}'s and $\otimes_{b,n}(1_b, -)$.

$\tau_2(\mathcal{T}_1, \mathcal{T}_2) =$  $=$  $= \tau_3(\mathcal{S}_1, \mathcal{S}_2)$

Now the result follows from (5) in Theorem 4.3.15 (which is also Proposition 4.8.11). ☐

4.8 Extended positive cones

For the bimodule planar calculus, we need to make multiplication by $\infty_{\mathbb{R}}$ rigorous. We do so by generalizing the notion of an extended positive cone.

Definition 4.8.1. An extended positive cone is a set V together with a partial order $\leq$, an addition $+\colon V \times V \to V$, and a scalar multiplication $\cdot\colon [0_{\mathbb{R}}, \infty_{\mathbb{R}}] \times V \to V$ such that

Additivity axioms:

- (Zero) There is a $0_V \in V$ such that $0_V + v = v + 0_V = v$ for all $v \in V$.
- (Infinity) There is an $\infty_V \in V \setminus \{0\}$ such that $v + \infty_V = \infty_V + v = \infty_V$ for all $v \in V$.
- (Associativity) $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$ for all $v_1, v_2, v_3 \in V$.

- (Commutativity) $v_1 + v_2 = v_2 + v_1$ for all $v_1, v_2 \in V$.

Multiplicative axioms:

- (Unit) $1_{\mathbb{R}} v = v$ for all $v \in V$.
- (Associativity) $(\lambda\mu)v = \lambda(\mu v)$ for all $\lambda, \mu \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ and $v \in V$.
- (Zero) $0_{\mathbb{R}} v = 0_V$ for all $v \in V$.
- (Infinity) $\lambda \infty_V = \infty_V$ for all $\lambda > 0_{\mathbb{R}}$.

Distributivity:

- (Scalars distribute) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$ for all $\lambda \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ and $v_1, v_2 \in V$.
- (V distributes) $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$ for all $\lambda_1, \lambda_2 \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ and $v \in V$.

Partial order axioms:

- (Non-degeneracy) $0_V \leq x \leq \infty_V$ for all $x \in V$.
- (Linearity) if $x_i \leq y_i$ for $i = 0, 1$ and $\lambda \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, then $\lambda x_0 + x_1 \leq \lambda y_0 + y_1$.

Remark 4.8.2. (1) $0_V, \infty_V \in V$ are unique.

(2) If $\lambda v = 0_V$, then $v = 0_V$ or $\lambda = 0_{\mathbb{R}}$.

Examples 4.8.3. (1) The set $[0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ with the usual ordering and the convention that $\lambda\infty_{\mathbb{R}} = \infty\lambda = \infty_{\mathbb{R}}$ for all $\lambda \in \mathbb{R}_{>0}$ and $0_{\mathbb{R}}\infty_{\mathbb{R}} = \infty_{\mathbb{R}} 0_{\mathbb{R}} = 0_{\mathbb{R}}$ is an extended positive cone.

(2) Let X be a nonempty set. The space of functions $\{f : X \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]\}$ is an extended positive cone with pointwise addition and scalar multiplication, where $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Similarly, the space of extended positive measurable functions on a measure space is an extended positive cone.

(3) If M is a von Neumann algebra, $\omega(M)$, the set of normal weights $\omega : M^+ \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, is an extended positive cone where $\infty_{\omega(M)}$ is the map which sends 0_M to $0_{\mathbb{R}}$ and all other elements of M^+ to $\infty_{\mathbb{R}}$, and $\varphi \leq \psi$ if $\varphi(x) \leq \psi(x)$ for all $x \in M^+$.

(4) If M is a von Neumann algebra, $\widehat{M^+}$ is an extended positive cone where $\infty_{\widehat{M^+}}$ is the unbounded operator affiliated to M with domain (0), and $m_1 \leq m_2$ if $m_1(\phi) \leq m_2(\phi)$ for all $\phi \in M_*^+$.

(5) If V, W are extended positive cones, then so is $V \times W$ where $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$, $\lambda(v_1, w_1) = (\lambda v_1, \lambda w_1)$, $0_{V\times W} = (0_V, 0_W)$, $\infty_{V\times W} = (\infty_V, \infty_W)$, and $(v_1, w_1) \leq (v_2, w_2)$ if $v_1 \leq v_2$ and $w_1 \leq w_2$.

Definition 4.8.4. Let V, W be extended positive cones. A function $T\colon V \to W$ is a linear map (of extended positive cones) if

- $T(\lambda u + v) = \lambda Tu + Tv$ for all $u, v \in V$ and $\lambda \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, and
- if $u, v \in V$ with $u \leq v$, then $Tu \leq Tv$.

We define a multi-linear map of extended positive cones $V_1 \times \cdots \times V_n \to V_0$ similarly.

Examples 4.8.5. (1) For a fixed scalar $\lambda \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, multiplication by λ is a map of extended positive cones.

(2) Suppose $\omega\colon M^+ \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ is a normal weight. Then its unique extension to a normal weight $\omega\colon \widehat{M^+} \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ is a map of extended positive cones.

(3) If $m \in \widehat{M^+}$, then $m\colon \omega(M) \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$ given by$\varphi \mapsto m(\varphi)$ is a map of extended positive cones.

(4) Suppose $N \subset M$ is an inclusion of von Neumann algebras, $i\colon \widehat{N^+} \to \widehat{M^+}$ is the inclusion (well-defined by Equation (4.1)), and $T\colon \widehat{M^+} \to \widehat{N^+}$ is the unique extension of an operator valued weight $M^+ \to \widehat{N^+}$. Then i, T are maps of extended positive cones.

(5) Using the notation of Section 4.6, the map $\widehat{X^+} \times \widehat{Y_0^+} \to \widehat{X \otimes_A Y_0^+}$ given by $(x, y) \mapsto x \otimes_A y$ is a multilinear map of extended positive cones by Lemma 4.6.15.

Definition 4.8.6. An increasing net $(x_i)_{i\in I} \subset V$ converges to $x \in V$ if x is the unique least upper bound for $(x_i)_{i\in I}$. We denote this convergence by $\sup_{i\in I} x_i = x$ or $x_i \nearrow x$.

- V is <u>complete</u> if each increasing net $(x_i)_{i\in I}$ has a unique least upper bound.
- A map $T\colon V \to W$ is <u>normal</u> if $x_i \nearrow x$ implies $Tx_i \nearrow Tx$.

Remark 4.8.7. The maps in Examples 4.8.5 are all normal.

Definition 4.8.8. The <u>dual space</u> of V, denoted V^*, is the set of all normal maps $V \to [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$. Note that V^* is a complete extended positive cone with

(1) $(\lambda\varphi + \psi)(v) = \lambda\varphi(v) + \psi(v)$ for all $v \in V$, $\lambda \in [0_{\mathbb{R}}, \infty_{\mathbb{R}}]$, and $\varphi, \psi \in V^*$, with the convention that $0_{\mathbb{R}} \cdot \infty_{\mathbb{R}} = 0_{\mathbb{R}}$,

(2) 0_{V^*} is the zero map,

(3) $\infty_{V^*}(v) = \begin{cases} 0 & \text{if } v = 0 \\ \infty_V & \text{else, and} \end{cases}$

(4) $(\sup_{i\in I} \varphi_i)(v) := \sup_{i\in I} \varphi_i(v)$.

- There is a natural inclusion $V \to V^{**}$ by $x \mapsto (\mathrm{ev}_x \colon \varphi \mapsto \varphi(x))$.
- The completion of V is the set of sups of increasing nets in the image of V in V^{**}.

Theorem 4.8.9. *Let M be a semifinite von Neumann algebra with n.f.s. trace Tr_M. Let $\omega(M)$ be the set of normal weights on M^+.*

(1) $\widehat{M^+}$ is the dual extended positive cone of $\omega(M)$ (the ordering on each is given in Examples 4.8.3).

(2) The map $\widehat{M^+} \ni x \mapsto \mathrm{Tr}_M(x\,\cdot\,) \in \omega(M)$ is a normal isomorphism of extended positive cones.

Proof. This is a rewording of Theorem 4.2.14 into the language of this subsection. □

Definition 4.8.10. If $T \colon V \to W$ is a normal map of extended positive cones, we get a map of dual spaces $T^* \colon W^* \to V^*$ by $T^*(\phi) = \phi \circ T$ for all $\phi \in W^*$. We can characterize it as the unique map satisfying

$$\langle T(v), \varphi\rangle_W = \varphi(T(v)) = \langle v, T^*(\varphi)\rangle_V$$

for all $v \in V$ and $\varphi \in W$.

Proposition 4.8.11. *Suppose $N \subset M$ is an inclusion of semifinite von Neumann algebras with n.f.s. traces $\mathrm{Tr}_N, \mathrm{Tr}_M$ respectively. Let $i \colon \omega(N) \cong \widehat{N^+} \to \widehat{M^+} \cong \omega(M)$ be the inclusion, and let $T \colon \widehat{M^+} \to \widehat{N^+}$ be the unique extension to $\widehat{M^+}$ of the unique trace-preserving operator valued weight. Then i, T are normal and $T = i^*$, $T^* = i$.*

Proof. Clearly i, T are normal. Suppose $n \in \widehat{N^+}$ and $m \in (\widehat{M^+})^* = \widehat{M^+}$. Then

$$\langle i(n), m\rangle_{\widehat{M^+}} = \mathrm{Tr}_M(m \cdot n) = \mathrm{Tr}_N(T(m) \cdot n) = \langle n, T(m)\rangle_{\widehat{N^+}},$$

so $T = i^*$. Since $\mathrm{Tr}_M(m \cdot n) = \mathrm{Tr}_M(n \cdot m)$, $i = T^*$. □

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