Applications of subfactors and fusion categories to mathematical physics
UC Davis Mathematical Physics & Probability Seminar

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What is a subfactor?

**Definition**
A factor is a von Neumann algebra with trivial center. A subfactor is an inclusion \( A \subset B \) of factors.

- Our factors are type \( \text{II}_1 \), which means they are infinite dimensional with a trace.

**Remark**
Von Neumann algebras come in pairs \((M, M')\). Subfactors do too: \((A \subset B, B' \subset A')\).
Where do subfactors come from?

Some examples include:

- Groups – from $G \ltimes R$, we get $R^G \subset R$ and $R \subset R \rtimes_\alpha G$.
- finite dimensional unitary Hopf/Kac algebras
- Quantum groups – $\text{Rep}(U_q(g))$
- Conformal field theory
- endomorphisms of Cuntz C*-algebras
- tinkering with known subfactors (orbifolds, composites, ...)

However, there are certain possible infinite families without uniform constructions.
Finite index and the standard representation

Definition

$A \subset B$ has finite index iff $B$ is a finitely generated projective $A$-module.

The bimodule $A B_B$ is the standard representation of $A \subset B$.

A finite index subfactor $A \subset B$ comes with canonical maps:

\[
\begin{align*}
\text{Inclusion:} & \quad A B_B \otimes_B B A \\
\text{Evaluation:} & \quad B B_A \otimes A B_B
\end{align*}
\]

Since $A, B$ are analytical objects, these maps also have adjoints.
The Temperley-Lieb algebras

Definition

The Temperley-Lieb algebra \( TL_n(\delta) \) is the complex \(*\)-algebra spanned by diagrams with \( n \) upper and lower boundary points, connected by non-crossing strings.

\[
TL_3(\delta) = \text{span}_\mathbb{C} \left\{ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1}\\
\text{Diagram 2}\\
\text{Diagram 3}\\
\text{Diagram 4}\\
\text{Diagram 5}
\end{array}
\end{array} \right\}.
\]

- Multiplication is stacking of diagrams, but we trade closed loops for multiplicative factors of \( \delta \):

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 1}\\
\text{Diagram 2}
\end{array}
\end{array} \cdot \begin{array}{c}
\begin{array}{c}
\text{Diagram 3}\\
\text{Diagram 4}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 5}
\end{array}
\end{array} = \delta \begin{array}{c}
\begin{array}{c}
\text{Diagram 6}
\end{array}
\end{array}.
\]

- The involution \(*\) is given by vertical reflection:

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 7}
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\text{Diagram 8}
\end{array}
\end{array}.
\]
Jones’ index rigidity theorem

- The trace is given by capping off on the right

\[ \text{Tr}_n = \begin{array}{c} \cdots \\ \text{TL}_n(\delta) \rightarrow \text{TL}_0(\delta) \cong \mathbb{C} \end{array} \]

- There is a sesquilinear form given by \( \langle x, y \rangle_n = \text{Tr}_n(y^*x) \).

**Theorem (Jones)**

A finite index subfactor gives a positive-definite \(^*\)-representation of the Temperley-Lieb algebra \( \text{TL}_n(\delta) \) for \( \delta^2 = [B : A] \) and all \( n \geq 0 \). This is possible iff \( \delta \in \{2 \cos(\pi/k) \mid k \geq 3\} \cup [2, \infty) \).
Jones’ index rigidity theorem

- The trace is given by capping off on the right

\[ \text{Tr}_n = \begin{array}{c}
\end{array} \quad : TL_n(\delta) \to TL_0(\delta) \cong \mathbb{C} \]

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Jones’ index rigidity theorem

The trace is given by capping off on the right

\[ \text{Tr}_n = \begin{array}{c}
\quad \\
\quad \quad \\
\quad \quad \\
\quad \quad \\
\end{array} : TL_n(\delta) \to TL_0(\delta) \cong \mathbb{C} \]

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Temperley-Lieb and braid groups, part 1

$\text{TL}_n(\delta)$ has generators $E_i = \begin{array}{c|c|c|c|c|c}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}$ for $1 \leq i \leq n - 1$, and relations

- $E_i^2 = \delta E_i = \delta E_i^*$,
- $E_i E_j = E_j E_i$ if $|i - j| > 1$,
- $E_i E_{i \pm 1} E_i = E_i$. 
The braid group $B_n$ has generators $\sigma_i = \begin{bmatrix} \cdots & \cdots \\ i & n \end{bmatrix}$ for $1 \leq i \leq n - 1$, and relations

\[ \sigma_i \sigma_j = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \sigma_j \sigma_i \text{ for } |i - j| > 1, \text{ and }$

\[ \sigma_i \sigma_{i+1} \sigma_i = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
Knots and braids

Given a link, we can always write it as the closure of a braid.

\[
\text{Tr} \left( \begin{array}{c}
\left( \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array} \right)
\end{array} \right) = \begin{array}{c}
\left( \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array} \right)
\end{array}, \text{ a trefoil knot.}
\]

We have an algebra homomorphism \( \Phi : \mathbb{C}[B_n] \to TL_n(\delta) \) by

\[
\Phi \left( \begin{array}{c}
\left( \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array} \right)
\end{array} \right) = iq^{1/2} \left| \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array} \right| - iq^{-1/2} \left. \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array} \right|
\]

where \( \delta = q + q^{-1} \).
The Jones polynomial/Kauffman bracket

To get the framed Jones polynomial or Kauffman bracket of a link \( \ell \), first write \( \ell = \text{Tr}(b) \) for a braid \( b \). Then

\[
\langle \ell \rangle = \frac{1}{\delta} \text{Tr} \circ \Phi(b)
\]

is independent of the choice of braid representing the knot.

**Example**

\[
\langle \text{ } \rangle = \frac{1}{q + q^{-1}} \left( (i q^{1/2})^3 \begin{array}{c}
\hline
\end{array} \\
\right) + 3(i q^{1/2})^2 (-i q^{-1/2}) \begin{array}{c}
\hline
\end{array} + 3(i q^{1/2})(-i q^{-1/2})^2 \begin{array}{c}
\hline
\end{array} + (-i q^{-1/2})^3 \begin{array}{c}
\hline
\end{array}
\right)
\]

\[
= i(q^{-7/2} - q^{-3/2} - q^{5/2})
\]
Definition
The representation 2-category of $A \subset B$ is given by

1. 0-morphisms: $\{A, B\}$
2. 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$
3. 2-morphisms: bimodule intertwiners

- This 2-category is semi-simple, unitary, rigid (duals are well behaved), pivotal, sometimes spherical (iff $A \subset B$ extremal).
- The $A \otimes A$ bimodules form a rigid $C^*$-tensor category called the ‘principal even part’.
- The $B \otimes B$ bimodules form the ‘dual even part’.
- The principal even and dual even parts are Morita equivalent via the $A \otimes B$ bimodules.
Theorem (Popa [Pop94])
There is a Tannaka-Krein like duality between (strongly) amenable subfactors and their representation 2-categories.

Theorem (many authors)
Subfactors correspond to Frobenius algebra objects in rigid C*-tensor categories.
Fusion categories

If there are only finitely many isomorphism classes of simple $A - A$ bimodules, the principal even part is a unitary fusion category.

- Subfactors are a vital source of interesting fusion categories.

**Definition**

A fusion category is a semisimple, rigid tensor category with finitely many isomorphism classes of simple objects.

**Fact**

An $X \in \mathcal{C}$ with quantum dimension $\delta$ gives a representation

$$TL\cdot(\delta) \to \text{End}(\underbrace{X \otimes X \otimes \cdots}_{n \text{ alternating copies}}).$$

If $\mathcal{C}$ is unitary, the representation is positive definite.
Examples

Let $G$ be a finite group.

Example

$\text{Rep}(G)$, category of finite dimensional $\mathbb{C}$-representations.

Example

$\text{Vec}(G, \omega)$, $G$-graded vector spaces, $\omega \in H^3(G, \mathbb{C}^\times)$.

- Simple objects $V_g \cong \mathbb{C}$ for each $g \in G$.
- $V_g \otimes V_h = V_{gh}$
- The 3-cocycle gives the associator natural isomorphism:

\[ \alpha_{g,h,k} : (V_g \otimes V_h) \otimes V_k \xrightarrow{\omega_{g,h,k}} V_g \otimes (V_h \otimes V_k). \]

The pentagon axiom is exactly the 3-cocycle condition.
Let $G$ be a finite group. Build the subfactor $R \subset R \rtimes G$.

Example
The representation 2-category $\text{Rep}(R \subset R \rtimes G)$ has:

- principal even part $\text{Vec}(G, 1)$ ($R-R$ bimodules)
- dual even part $\text{Rep}(G)$ ($R \rtimes G - R \rtimes G$ bimodules)
- only one simple $R-R \rtimes G$ bimodule: $R \rtimes G$.

We see $\text{Vec}(G, 1)$ and $\text{Rep}(G)$ are Morita equivalent.

Fact
The subfactor $R \subset R \rtimes G$ corresponds to the algebra object $\mathbb{C}[G] \in \text{Vec}(G)$. 
The Haagerup: an ‘exotic’ example

The Haagerup fusion category $\mathcal{H}$ has 6 simple objects $1, g, g^2, X, gX, g^2X$ satisfying the following fusion rules:

- $\langle g \rangle \cong \mathbb{Z}/3$,
- $Xg \cong g^{-1}X$, and
- $X^2 \cong 1 \oplus X \oplus gX \oplus g^2X$ (the quadratic relation).

(Vec($\mathbb{Z}/3$) $\subset \mathcal{H}$ has trivial associator.)

The algebra object $1 \oplus X$ gives an ‘exotic’ subfactor with index

$$\frac{5 + \sqrt{13}}{2} \approx 4.30278.$$  

$\mathcal{H}$ has only been constructed by brute force.

- It appears $\mathcal{H}$ belongs to an infinite family, but only examples up to $\mathbb{Z}/19$ have been constructed [EG11].
Braided fusion categories

Definition
A fusion category is braided if it has natural isomorphisms

\[ c_{X,Y} : X \otimes Y \to Y \otimes X \]

satisfying the braid relations and a compatibility requirement.

Example
Vec is a symmetric braided fusion category, i.e., \( c_{b,a} \circ c_{a,b} = \text{id}_{a \otimes b} \)
for all \( a, b \in \text{Vec} \).

Facts
If \( \mathcal{C} \) is braided, an \( X \in \mathcal{C} \) gives a representation \( B_n \to \text{End}(X^\otimes n) \).
If \( \mathcal{C} \) is unitary, the representation is also.
If \( \mathcal{C} \) is symmetric, the representation factors through \( S_n \).
Modular tensor categories

**Definition**

A modular tensor category is a braided spherical fusion category (and more axioms...) such that the $S$ matrix ($S_{a,b}$) is invertible.

\[
S_{a,b} = \text{Tr}(c_{b,a} \circ c_{a,b}) = \begin{array}{c}
b \\
a
\end{array} = \begin{array}{c}
b \\
a
\end{array}
\]

**Example**

If $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$, then the quantum double $\mathcal{Z}(\mathcal{C})$ is a modular tensor category. If $\mathcal{C}$ is unitary, then so is $\mathcal{Z}(\mathcal{C})$.

**Theorem (Bruillard-Ng-Rowell-Wang [BNRW13])**

For a fixed $n$, there are only finitely many modular tensor categories with rank $n$.

- Rank finiteness not yet known for fusion categories.
Classification of fusion categories

Question (Hard!)
Can we classify all fusion categories with $n$ objects for $n$ small?

Examples

- Rank 2 was classified by Ostrik [Ost03]:
  - $\text{Vec}(\mathbb{Z}/2, \omega)$ for $\omega \in H^3(\mathbb{Z}/2, \mathbb{C}^\times)$
  - $\text{Fib} = \langle 1, \tau | \tau \otimes \tau \cong 1 \oplus \tau \rangle$ and Galois conjugate

- Rank 3 (pivotal) was classified by Ostrik [Ost13]:
  - $\text{Vec}(\mathbb{Z}/3, \omega)$ for $\omega \in H^3(\mathbb{Z}/3, \mathbb{C}^\times)$
  - $\text{Rep}(S_3)$ and twisted versions
  - Ising category (even part of $\mathfrak{sl}_2$ at 6th root of unity) and conjugates
    - even part of $\mathfrak{sl}_2$ at 7th root of unity and conjugates
    - even part of $E_6$ subfactor and conjugate

- Rank 4 (pseudo unitary) with a dual pair of objects $(1, X, Y, \bar{Y})$ was classified by Larson [Lar14].
  - New examples of Liu-Morrison-P [LMP14]
Topological quantum field theories (TQFTs)

Definition (Atiyah)
An $n$-dimensional TQFT is a symmetric monoidal functor

$$
\bigg(\bigg(n-1\bigg)\text{Bord, II}\bigg) \to (\text{Vec, } \otimes)
$$

Each $n - 1$ manifold is assigned a vector space, and each bordism is assigned a linear operator.

Examples for $n = 3$

- Turaev-Viro associated to a spherical fusion category
- Reshetikhin-Turaev associated to a modular tensor category

In fact, $TV(C) \cong RT(Z(C))$. 
Extended topological field theories

Definition
An \((n, n - 1, \ldots, d)\)-TFT is a symmetric monoidal functor
\[
\begin{pmatrix}
  n \\
  \vdots \\
  d
\end{pmatrix}
\text{Bord} \rightarrow (n - d) - \text{Vec}
\]
for an appropriate choice of \(n - d\) category \((n - d) - \text{Vec}\).

Examples
- Turaev-Viro is a \((3, 2, 1, 0)\)-TFT (fully extended)
- Reshetikhin-Turaev is a \((3, 2, 1)\)-TFT

The double construction relates these two.
Extended topological field theories

Definition
An \((n, n - 1, \ldots, d)\)-TFT is a symmetric monoidal functor

\[
\begin{pmatrix} n \\ \vdots \\ d \end{pmatrix} : \text{Bord} \longrightarrow (n - d) - \text{Vec}
\]

for an appropriate choice of \(n - d\) category \((n - d) - \text{Vec}\).

Examples

- \((1)\)-TFTs \(\leftrightarrow\) a dualizable object in a symmetric \(\otimes\)-category
- \((3, 2, 1, 0)\)-TFTs \(\leftrightarrow\) fusion categories in 3-category of \(\otimes\)-categories
  (recent work of Douglas-Schommer-Pries-Snyder [DSPS13])
Segal conformal field theory (CFT)

Definition (Segal)

A 2d-conformal field theory is a symmetric monoidal functor

\[
\binom{2}{1} \text{ConfBord} \longrightarrow \text{Hilb}
\]

This consists of:

- a Hilbert space \( H_S \) assigned to each compact, connected oriented 1-manifold \( S \)
- a unitary \( u_f : H_{S_1} \to H_{S_2} \) to every orientation preserving diffeomorphism \( f : S_1 \to S_2 \)
  (an anti-unitary for an orientation reversing diffeomorphism)
- a map \( g_\Sigma : \bigotimes H_{S_{\text{in}}} \to \bigotimes H_{S_{\text{out}}} \), to each cobordism \( \Sigma \) with a complex structure, where orientation is reversed for each \( S_{\text{out}} \).

Conformal welding allows for gluing along diffeomorphisms.
Conformal nets (algebraic quantum field theory)

Definition
A conformal net is a functor from intervals $I \subset S^1$ to von Neumann algebras in $B(H)$,

$$I \mapsto \mathcal{A}(I) \subset B(H),$$

satisfying axioms, like

- $I \subset J \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)$
- "locality": $I \cap J = \emptyset \Rightarrow [\mathcal{A}(I), \mathcal{A}(J)] = 0$.

The net is irreducible if each $\mathcal{A}(I)$ is a factor.

- Disjoint intervals give subfactors: $I \cap J = \emptyset \Rightarrow \mathcal{A}(I) \subset \mathcal{A}(J)'$. 
Modular tensor categories from conformal nets

Definition
A representation of the net $\mathcal{A}$ is a family of representations $\pi_I : \mathcal{A}(I) \to B(K)$ for a fixed Hilbert space $K$, such that if $I \subset J$, then $\pi_J|_{\mathcal{A}(I)} = \pi_I$.

Theorem (Kawahigashi-Longo-Müger [KLM01])
Consider the partition of $S^1$ into 4 disjoint intervals:

If $\mathcal{A}(I_1 \cup I_3) \subset \mathcal{A}(I_2 \cup I_4)'$ has finite index, then $\text{Rep}(\mathcal{A})$ is a unitary modular tensor category.
Modular categories $\xrightarrow{?} \text{CFT}$

Conjecture (Kawahigashi)
The quantum double of every unitary fusion category arises as the representation category of some conformal net.

Conjecture (Evans-Gannon [EG11])
There should be a CFT realizing the double of the Haagerup fusion category. In particular, there should be a conformal subalgebra of the central charge $c = 8$ vertex operator algebra corresponding to the root lattice $E_6 \oplus A_2$.

- The modular data of the double of Haagerup is ‘graft’ of the double of $S_3$ and $\mathfrak{so}(13)_2$.
- They compute possible character vectors for the VOA, and show they have non-negative integral Fourier coefficients.
Work in progress: conformal planar algebras

- Subfactors and CFT are related via conformal nets.
- Tannaka-Krein duality $A \subset B \leftrightarrow \text{Rep}(A \subset B)$ (Popa)
- $\text{Rep}(A \subset B)$ axiomatized as a planar algebra (Jones)

In joint work with Henriques and Tener, we expect a connection between genus zero Segal CFT (many-to-one genus zero Riemann surfaces) with topological defect strings and planar algebras.
Classifying small index subfactors

To each finite group $G$, there is a dual pair of subfactors $R \subset R \rtimes G$ and $R^G \subset R$.

Thus, one cannot hope to classify all subfactors. We need to restrict our search space. One way to do this is to look at small index subfactors.

Recall:
The representation 2-category of $A \subset B$ is given by

(0) 0-morphisms: $\{A, B\}$
(1) 1-morphisms: bimodule summands of $\bigotimes_A^k B$ for some $k \geq 0$
(2) 2-morphisms: bimodule intertwiners
Principal graphs

Definition
The principal (induction) graph $\Gamma_+$ has one vertex for each isomorphism class of simple $A^P A$ and $A^Q B$. There are

$$\dim(\text{Hom}_{A^B}(P \otimes_A B, Q))$$

edges from $P$ to $Q$.

The dual principal (restriction) graph $\Gamma_-$ is defined similarly using $B^B - B$ and $B - A$ bimodules.

- $\Gamma_\pm$ is pointed, where the base point is $A^A A$, $B^B B$ respectively.
- Duality is given by contragredient, which is always at the same depth, since $B$ is a $*$-algebra. However, duals at odd depths of $\Gamma_\pm$ are on $\Gamma_\mp$. 
Examples of principal graphs

- index < 4: $A_n, D_{2n}, E_6, E_8$. No $D_{\text{odd}}$ or $E_7$.
- index = 4: $A_{2n-1}^{(1)}, D_{n+2}^{(1)}, E_{6}^{(1)}, E_{7}^{(1)}, E_{8}^{(1)}, A_{\infty}, A_{\infty}^{(1)}, D_{\infty}$
- Graphs for $R \subset R \ltimes G$ obtained from $G$ and $\text{Rep}(G)$.

\[
\begin{pmatrix}
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\end{array}
\end{pmatrix},
\begin{array}{c}
2 \\
\end{array}
\]

$G = S_3$

- Principal graph for $R^G \subset R^H$ is the induction-restriction graph for $H \subset G$:

\[
\text{\begin{center}
\begin{tikzpicture}
\path (0,0) -- (2,0) -- (4,0) -- (6,0) -- (6,2) -- (4,2) -- (2,2) -- (0,2) -- cycle;
\end{tikzpicture}
\end{center}}
\]

$S_5 \subset S_6$

- First graph is principal, second is dual principal.
- Leftmost vertex corresponds to base points $AA_A, BB_B$.
- Red tags for duality of even vertices $(A_P A \leftrightarrow A_P A)$.
- Duality of odd vertices by depth and height
Supertransitivity

Definition
A principal graph is \( n \)-supertransitive if has an initial segment with \( n \) edges before branching.

Examples

\[ \text{is 1-supertransitive} \]

\[ \text{is 2-supertransitive} \]

\[ \text{is 3-supertransitive} \]
Small index subfactor classification program

Steps of subfactor classifications:

1. Enumerate graph pairs which survive obstructions.
2. Construct examples when graphs survive.

Fact (Popa [Pop94])

For a subfactor $A \subset B$, $[B : A] \geq \|\Gamma_+\|^2 = \|\Gamma_-\|^2$.

If we enumerate all graph pairs with norm at most $r$, we have found all principal graphs of subfactors with index at most $r^2$. 
Known small index subfactors, 2009

- Quantum groups and their quantum subgroups
- Composites
- Haagerup’s exotic subfactor and classification to $3 + \sqrt{3}$
- Izumi’s Cuntz algebra examples ($2221$, $3^n$)
Known small index subfactors, today

- Classification to 5 [MS12, MPPS12, IJMS12, PT12, IMP⁺14]
- Examples at $3 + \sqrt{5}$ [MP13, PP13, IMP13, MP14]
- 1-supertransitive to $6\frac{1}{5}$ and examples at $3 + 2\sqrt{2}$ [LMP14]
Theorem (Afzaly-Morrison-P)

We know all subfactor standard invariants up to index $5\frac{1}{4}$ (with at most finitely many exceptions).
Thank you for listening!

Slides available at


